On the Convergence of Iterative Methods for Nonlinear Operator Equations

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Abstract

We study the convergence properties of a class of iterative methods applied to nonlinear operator equations in Banach spaces. Under suitable Lipschitz continuity assumptions on the Fréchet derivative, we establish both local and semi-local convergence results. Our main theorem generalises classical results of Kantorovich and provides sharper error bounds.

Keywords: nonlinear equations, iterative methods, Banach space, convergence analysis.
MSC 2020: 65J15, 47H10, 47J25.

1 Introduction

Let X and Y be Banach spaces and let $F\colon D \subseteq X \to Y$ be a nonlinear operator. We consider the problem of finding $x^* \in D$ such that

$$F(x^*) = 0. \tag{1}$$

Newton's method generates a sequence $\{x_n\}$ via

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \qquad n = 0, 1, 2, \ldots \tag{2}$$

where $F'(x)$ denotes the Fréchet derivative of F at x.

Greek letters appear naturally: the Lipschitz constant γ, step size α, tolerance $\varepsilon > 0$, and domain radius δ.

2 Preliminaries

Definition 2.1 (Lipschitz Condition)**.** We say $F'\colon D \to \mathcal{L}(X, Y)$ satisfies a *centre Lipschitz condition* at x_0 with constant $\omega > 0$ if

$$\|F'(x) - F'(x_0)\| \leq \omega \|x - x_0\|, \qquad \forall\, x \in D. \tag{3}$$

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Definition 2.2 (ω-Condition)**.** Operator F satisfies the *ω-condition* on D if there exist constants $\beta, \eta, \omega > 0$ such that

$$\|F'(x_0)^{-1}\| \le \beta, \tag{4}$$
$$\|F'(x_0)^{-1}F(x_0)\| \le \eta, \tag{5}$$
$$\|F'(x) - F'(y)\| \le \omega\|x - y\|, \quad \forall\, x, y \in D. \tag{6}$$

3 Main Results

Lemma 3.1. *Let F satisfy the ω-condition with parameters (β, η, ω) and set $h = \beta\omega\eta$. If $h \le \frac{1}{2}$, then for every $n \ge 0$,*

$$\|x_{n+1} - x_n\| \le \frac{1}{2^n}\|x_1 - x_0\|. \tag{7}$$

Proof. We proceed by induction on n. The base case $n = 0$ is immediate from the definition of η. Assume the bound holds for some $n \ge 0$. By the Newton step (2) and estimate (3), we obtain

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|F'(x_{n+1})^{-1}F(x_{n+1})\| \\
&\le \frac{\omega}{2}\|x_{n+1} - x_n\|^2 \cdot \|F'(x_{n+1})^{-1}\| \\
&\le \frac{1}{2}\|x_{n+1} - x_n\|, \qquad (8)
\end{aligned}$$

which completes the induction. □

Theorem 3.2 (Semi-local Convergence)**.** *Under the hypotheses of Lemma 3.1, the sequence $\{x_n\}$ defined by* (2) *converges to a solution x^* of $F(x^*) = 0$. Moreover, the error satisfies*

$$\|x_n - x^*\| \le \frac{2\eta}{2^n - 1}, \qquad n \ge 1. \tag{9}$$

Proof. Since $\{x_n\}$ is Cauchy by Lemma 3.1 and X is complete, the sequence converges to some $x^* \in \overline{B}(x_0, r)$ where

$$r = \frac{1 - \sqrt{1 - 2h}}{\omega\beta}.$$

Continuity of F then gives $F(x^*) = 0$. The error bound (9) follows by summing the geometric series from Lemma 3.1. □

Corollary 3.3. *If F' is Lipschitz continuous on D and $h < \frac{1}{2}$, then Newton's method converges* quadratically*:*

$$\|x_{n+1} - x^*\| \le C\|x_n - x^*\|^2, \tag{10}$$

where $C = \frac{\omega\beta}{2}$.

3.1 Matrix Notation Example

The Jacobian of a system $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ can be written as

$$J(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_n}{\partial x_1} & \cdots & \dfrac{\partial F_n}{\partial x_n} \end{pmatrix}. \tag{11}$$

3.2 Additional Notation

Fractions and subscripts:

$$a_n = \frac{n!}{\Gamma(n+\alpha+1)}, \qquad S_N = \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}.$$

Operators and Greek letters:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e, \qquad \int_0^\pi \sin\theta \, d\theta = 2.$$

4 Conclusion

We have established semi-local convergence of Newton's method under a centre Lipschitz condition, obtaining error bounds that improve upon the classical Kantorovich theory. Future work will explore higher-order methods and relaxed smoothness assumptions.

References

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