Advanced Point-Set Topology

Lecture Notes

Master M1 — 2025–2026

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“A mathematician is a machine for turning coffee into theorems.”
— Alfréd Rényi

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Contents

Preface

This course on *Advanced Point-Set Topology* is aimed at Master's and doctoral students in pure mathematics, as well as researchers in theoretical computer science wishing to deepen their understanding of the topological foundations of domain theory.

Motivations

General topology, as taught in undergraduate and early graduate programs, focuses almost exclusively on Hausdorff (separated) spaces. While natural for analysis and differential geometry, this restriction conceals a substantial portion of the theory. Indeed, many fundamental structures in mathematics and computer science fail to satisfy the T_2 separation axiom:

- The spectrum of a commutative ring, endowed with the Zariski topology, is generally a T_0-space that is not T_1.
- Scott domains, essential in denotational semantics, carry a T_0 topology that is not T_1.
- Locales, which generalize topological spaces by eliminating reference to points, provide a natural framework for constructive logic and topos theory.
- Quasi-metrics, where symmetry is abandoned, model asymmetric cost notions in computer science and optimization.

The goal of this course is to systematically develop topological theory in this broader setting, emphasizing the interactions between topology, order theory, and theoretical computer science.

Organization of the Course

The course is divided into ten chapters, organized into three thematic parts:

Part I: Foundations (Chapters 1–3). We begin with a systematic review of general topology enriched with categorical language (Chapter 1), then study T_0-spaces via the specialization order, establishing the fundamental bridge between topology and order theory (Chapter 2), before addressing sober spaces and the theory of locales providing the "pointless" framework of topology (Chapter 3).

Part II: Asymmetric Structures (Chapters 4–6). We develop the theory of quasi-metrics and asymmetric topology (Chapter 4), bitopological spaces capturing the duality between a topology and its conjugate (Chapter 5), and then Scott topology on ordered sets and domains, fundamental for denotational semantics (Chapter 6).

Part III: Convergence, Compactness, and Applications (Chapters 7–10). We study generalized convergence spaces and filter theory (Chapter 7), compactness beyond the Hausdorff setting with its multiple variants (Chapter 8), applications of asymmetric topology to theoretical computer science (Chapter 9), and finally topological fixed point theory with applications to domains and semantics (Chapter 10).

Prerequisites

The reader is assumed to be familiar with:

- Basic general topology: topological spaces, continuity, compactness, connectedness, separation axioms T_0–T_4 (advanced undergraduate/early graduate level).
- Elementary notions of ordered set theory: partial orders, lattices, completeness.
- The basics of categorical language: categories, functors, natural transformations, adjunctions.
- Elementary theory of metric spaces and convergence.

Conventions and Notation

Throughout this course, we adopt the following conventions:

- (X, τ) denotes a topological space, where τ is the topology (the collection of open sets). When the topology is clear from context, we write simply X.
- $\mathcal{O}(X)$ denotes the lattice of open sets of X.
- $\mathcal{C}(X)$ denotes the collection of closed sets of X.
- $\overline{A}$, $\mathrm{Int}(A)$ denote the closure and interior of $A \subseteq X$, respectively.
- $\leq_X$ or $\sqsubseteq$ denotes the specialization order on a T_0-space.
- $\downarrow x = \{y : y \leq x\}$ and $\uparrow x = \{y : x \leq y\}$.
- For a subset A, we write $\downarrow A = \bigcup_{a \in A} \downarrow a$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$.
- **Top** is the category of topological spaces and continuous maps.
- $\mathbf{Top}_0$ is the full subcategory of T_0-spaces.
- **Sob** is the full subcategory of sober spaces.
- **Loc** is the category of locales.
- **Frm** is the category of frames.
- **Dcpo** is the category of dcpos and Scott-continuous functions.

- **Dom** denotes the category of domains (context-dependent).
- ω denotes the first infinite ordinal, identified with $\mathbb{N}$.
- $\mathbb{S}$ denotes the Sierpiński space $\{0, 1\}$ with topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Pedagogical Approach

Each chapter contains:

- Precise **definitions** and **theorems** with complete proofs.
- Detailed **examples** and **counterexamples** illustrating the scope and limitations of results.
- **Exercises** of varying difficulty, some accompanied by hints.
- **Intuition** and **Warning** boxes to guide understanding.

References

This course draws primarily on the following works:

- Gierz, Hofmann, Keimel, Lawson, Mislove, Scott, *Continuous Lattices and Domains*, Cambridge University Press, 2003.
- Goubault-Larrecq, *Non-Hausdorff Topology and Domain Theory*, Cambridge University Press, 2013.
- Johnstone, *Stone Spaces*, Cambridge University Press, 1982.
- Vickers, *Topology via Logic*, Cambridge University Press, 1989.
- Abramsky, Jung, *Domain Theory*, Handbook of Logic in Computer Science, vol. 3, Oxford University Press, 1994.
- Kelly, *Bitopological Spaces*, Proc. London Math. Soc., 1963.
- Fletcher, Lindgren, *Quasi-Uniform Spaces*, Marcel Dekker, 1982.
- Smyth, *Topology*, Handbook of Logic in Computer Science, vol. 1, Oxford University Press, 1992.
- Escardó, *Synthetic Topology of Data Types and Classical Spaces*, ENTCS, 2004.
- Nachbin, *Topology and Order*, Van Nostrand, 1965.

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The author

Chapter 1

General Topology Review and Categorical Language

What is a space? The question sounds simple, almost naive, yet it forced generations of mathematicians to rethink the foundations of geometry. In the early twentieth century, Felix Hausdorff, Maurice Fréchet, and others forged the concept of a *topological space* — a structure so general that it encompasses familiar Euclidean spaces alongside exotic objects where our intuition falters.

This chapter is a starting point, but not a mere rehash of an introductory course. Our goal is to revisit the foundations with fresh eyes: those of category theory. We will reformulate classical constructions — continuity, products, quotients — in a language that reveals their deep structure. And above all, we will abandon a stubborn prejudice: the assumption that every "reasonable" space should be Hausdorff. Non-separated spaces lie at the heart of this course, and that is where things get truly interesting.

Intuition

This chapter reviews the fundamental notions of general topology, reformulated in categorical language. The goal is not to repeat an introductory course, but to set up the vocabulary and tools needed in subsequent chapters, emphasizing aspects often neglected: non-separated spaces, the role of the category **Top** and its properties, and the limitations of Hausdorff-based intuition.

1.1 Topological Spaces: Definitions and First Examples

Everything starts with three axioms. Three deceptively simple rules that determine which subsets of a set X deserve to be called "open." It is the choice of these open sets — not any notion of distance — that defines the geometry of a space. Hausdorff himself, in his *Grundzüge der Mengenlehre* of 1914, understood that topology is fundamentally about *neighborhoods*, not metrics.

Definition 1.1 (Topological space)**.** A *topological space* is a pair (X, τ) where X is a set and $\tau \subseteq \mathcal{P}(X)$ satisfies:

1. $\emptyset \in \tau$ and $X \in \tau$;

2. τ is closed under arbitrary unions: if $(U_i)_{i\in I} \subseteq \tau$, then $\bigcup_{i\in I} U_i \in \tau$;

3. τ is closed under finite intersections: if $U_1, \dots, U_n \in \tau$, then $U_1 \cap \cdots \cap U_n \in \tau$.

The elements of τ are called *open sets*. Complements of open sets are *closed sets*.

The asymmetry between unions (arbitrary) and intersections (finite only) is not a flaw in the definition: it reflects a deep reality. If we allowed infinite intersections of open sets, the topology would collapse on itself. In $\mathbb{R}$, the intersection of all intervals $(-1/n, 1/n)$ is just the singleton $\{0\}$, which is not open. Understanding why this asymmetry is necessary is already understanding topology.

To build intuition, let us consider the extreme cases.

Example 1.2 (Extremal topologies)**.** On any set X:

- The *discrete topology* $\tau = \mathcal{P}(X)$ makes every subset open.
- The *indiscrete topology* $\tau = \{\emptyset, X\}$ has only the trivial open sets.

Between these two extremes lies a tiny but remarkably rich space that will appear again and again throughout this course.

Example 1.3 (Sierpiński space)**.** The Sierpiński space is $\mathbb{S} = \{0,1\}$ with $\tau = \{\emptyset, \{1\}, \{0,1\}\}$. It is a T_0-space that is not T_1. The singleton $\{0\}$ is closed but not open, and $\{1\}$ is open but not closed. This space plays a fundamental role throughout the course.

This two-point space is in fact the "classifier of open sets": we will see in the exercises that the open sets of a space X are in bijection with the continuous maps from X to $\mathbb{S}$. This is the first manifestation of a recurring theme: small non-separated spaces encode information.

Example 1.4 (Cofinite topology)**.** On an infinite set X, the *cofinite topology* is $\tau = \{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is finite}\}$. This is a T_1-space that is not T_2.

1.2 Continuous Maps and Homeomorphisms

In topology, distance is absent, but "nearness" persists through open sets. Continuity, which in analysis requires ε–δ arguments, takes a purer form here: a map is continuous if it respects the open-set structure, meaning that the preimage of every open set is open.

Definition 1.5 (Continuous map)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f\colon X \to Y$ is *continuous* if for every $V \in \tau_Y$, we have $f^{-1}(V) \in \tau_X$.

This laconic yet elegant definition admits several equivalent reformulations, each useful in different contexts.

Proposition 1.6 (Characterizations of continuity)**.** Let $f\colon X \to Y$. The following are equivalent:

1. f is continuous.

2. For every closed set F of Y, $f^{-1}(F)$ is closed in X.

3. For every $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

4. For every $x \in X$ and every neighborhood V of $f(x)$, $f^{-1}(V)$ is a neighborhood of x.

Proof. (1) $\Leftrightarrow$ (2): Immediate by taking complements.

(1) $\Rightarrow$ (3): Let $A \subseteq X$ and $x \in \overline{A}$. Let V be an open set in Y containing $f(x)$. Then $f^{-1}(V)$ is open and contains x, so $f^{-1}(V) \cap A \neq \emptyset$, whence $V \cap f(A) \neq \emptyset$. Thus $f(x) \in \overline{f(A)}$.

(3) $\Rightarrow$ (2): If F is closed in Y and $A = f^{-1}(F)$, then $f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{F} = F$, so $\overline{A} \subseteq f^{-1}(F) = A$, showing A is closed.

(1) $\Leftrightarrow$ (4): Exercise. □

When two spaces share the same "topological shape," we say they are homeomorphic. This is the notion of equivalence in topology: the analogue of isometry in metric geometry, or isomorphism in algebra.

Definition 1.7 (Homeomorphism)**.** A map $f\colon X \to Y$ is a *homeomorphism* if f is bijective and both f and f^{-1} are continuous. We then say X and Y are *homeomorphic*, written $X \cong Y$.

1.3 Separation Axioms

Here is one of the central themes of this course. The separation axioms classify topological spaces according to their ability to "distinguish" points. In a standard course, one focuses on Hausdorff spaces (T_2) and treats the lower axioms as curiosities. But in this course, T_0 spaces that fail to be T_1 will be our daily companions — because they are precisely the spaces that arise in domain theory, algebraic geometry, and the study of spectral spaces.

Definition 1.8 (Axioms T_0, T_1, T_2)**.** Let (X, τ) be a topological space.

- X is T_0 (or Kolmogorov) if for all $x \neq y$ in X, there exists an open set containing one but not the other.

- X is T_1 if for all $x \neq y$, there exists an open set containing x but not y (and by symmetry, one containing y but not x).

- X is T_2 (or Hausdorff) if for all $x \neq y$, there exist disjoint open sets $U \ni x$ and $V \ni y$.

Proposition 1.9. We have the implications $T_2 \Rightarrow T_1 \Rightarrow T_0$, and no reverse implication holds in general.

The difference between T_0 and T_1 is subtle but fundamental. In a T_0 space, one can "distinguish" two points, but asymmetrically: one sees an open set the other does not, but not necessarily the reverse. The Sierpiński space is the archetypal example.

Remark 1.10. A space is T_1 if and only if every singleton is closed. This characterization does not extend to T_0-spaces: in the Sierpiński space, $\{0\}$ is closed but $\{1\}$ is not.

Beyond T_2, the hierarchy continues with regularity and normality axioms, which concern the separation of closed sets.

Definition 1.11 (Higher separation axioms)**.**

- X is *regular* if for every closed set F and every $x \notin F$, there exist disjoint open sets separating x and F. X is T_3 if it is regular and T_1.
- X is *normal* if for every pair of disjoint closed sets F_1, F_2, there exist disjoint open sets separating them. X is T_4 if it is normal and T_1.

1.4 Basic Topological Constructions

How do we build new spaces from existing ones? Three fundamental constructions — products, quotients, and subspaces — allow us to erect the entire edifice of topology. What is remarkable is that each of these constructions is characterized by a *universal property*, a theme we will develop fully in the categorical section.

Definition 1.12 (Product topology)**.** Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces. The *product topology* on $\prod_{i\in I} X_i$ is the coarsest topology making all projections $\pi_j\colon \prod_{i\in I} X_i \to X_j$ continuous. A base for this topology consists of sets $\prod_{i\in I} U_i$ where $U_i \in \tau_i$ and $U_i = X_i$ for all but finitely many indices.

Definition 1.13 (Quotient topology)**.** Let $f\colon X \to Y$ be a surjection where X is a topological space. The *quotient topology* on Y is $\tau_Y = \{V \subseteq Y : f^{-1}(V) \in \tau_X\}$. This is the finest topology making f continuous.

Definition 1.14 (Subspace)**.** If (X, τ) is a topological space and $A \subseteq X$, the *subspace topology* on A is $\tau_A = \{U \cap A : U \in \tau\}$.

1.5 Categorical Language

We now arrive at the tool that will transform the way we think about topology. Category theory, born in 1945 from the work of Samuel Eilenberg and Saunders Mac Lane, is often described as "the mathematics of mathematics": it provides a language for talking about mathematical structures and the relationships between them, abstracting away the specific nature of the objects under study.

Why introduce this formalism in a topology course? Because the topological constructions we have just seen — products, quotients, subspaces — are special cases of categorical *limits* and *colimits*. Phrasing them this way reveals why these constructions work, and more importantly, why certain others do not.

Definition 1.15 (Category)**.** A *category* $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}(\mathcal{C})$ of *objects*;
- for each pair of objects A, B, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms*;
- an associative composition law $\circ\colon \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$;
- for each object A, an identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, neutral for composition.

Here are the categories we will use throughout this course.

Example 1.16 (Fundamental topological categories)**.**

- **Set**: sets and functions.

- **Top**: topological spaces and continuous maps.
- $\mathbf{Top}_0$: full subcategory of **Top** consisting of T_0-spaces.
- **Pos**: partially ordered sets and monotone maps.
- **Lat**: lattices and lattice morphisms.

To relate categories to one another, we use *functors*: "translators" that preserve categorical structure.

Definition 1.17 (Functor)**.** A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between categories assigns:

- to each object A of $\mathcal{C}$, an object $F(A)$ of $\mathcal{D}$;
- to each morphism $f\colon A \to B$, a morphism $F(f)\colon F(A) \to F(B)$;

such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$. A functor is *contravariant* if it reverses the direction of arrows.

And to compare two functors, we use *natural transformations* — one of Eilenberg and Mac Lane's deepest ideas. They famously said they invented categories in order to define natural transformations.

Definition 1.18 (Natural transformation)**.** Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\eta\colon F \Rightarrow G$ is a family of morphisms $(\eta_A\colon F(A) \to G(A))_{A\in\mathrm{Ob}(\mathcal{C})}$ such that for every morphism $f\colon A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

The most important concept for this course is *adjunction*, which captures the duality between "building freely" and "forgetting structure."

Definition 1.19 (Adjunction)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. We say F is *left adjoint* to G (written $F \dashv G$) if there exists a natural bijection

$$\mathrm{Hom}_{\mathcal{D}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, G(B))$$

for all objects A of $\mathcal{C}$ and B of $\mathcal{D}$.

Adjunctions are everywhere in topology. Every time we encounter a topological construction, we will be able to understand it as a left or right adjoint. Here is why this is so powerful.

Theorem 1.20 (Universal property of adjunctions)**.** *If $F \dashv G$, then:*

1. *F preserves colimits.*
2. *G preserves limits.*
3. *There exist natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ (unit) and $\varepsilon\colon FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ (counit) satisfying the triangle identities.*

Proof. This is a classical result. The unit η_A corresponds to the image of $\mathrm{id}_{F(A)}$ under the bijection $\mathrm{Hom}_{\mathcal{D}}(F(A), F(A)) \cong \mathrm{Hom}_{\mathcal{C}}(A, GF(A))$. The counit ε_B corresponds to the image of $\mathrm{id}_{G(B)}$. Preservation of (co)limits follows because a left (resp. right) adjoint commutes with colimits (resp. limits), since Hom-functors convert colimits to limits. □

CHAPTER 1. GENERAL TOPOLOGY: REVIEW AND CATEGORICAL LANGUAGE

1.6 Top as a Category: Fundamental Properties

Armed with this language, what can we say about the category **Top** itself? The good news is that it is very rich: it admits all conceivable limits and colimits. The bad news — and this is where the trouble begins — is that it fails to be cartesian closed.

Theorem 1.21 (**Top** is complete and cocomplete)**.** *The category* **Top** *admits all small limits and all small colimits. In particular:*

- *Products exist (product topology).*
- *Equalizers exist (subspace topology).*
- *Coproducts exist (disjoint union topology).*
- *Coequalizers exist (quotient topology).*

The forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ — which "forgets" the topology and retains only the underlying set — is remarkably well-endowed: it has both a left and a right adjoint, which is relatively rare.

Proposition 1.22 (The forgetful functor)**.** The forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ has both a left adjoint (discrete topology) and a right adjoint (indiscrete topology). Denoting these by D and I:

$$D \dashv U \dashv I.$$

Proof. For any space Y and set S, a function $f\colon S \to U(Y)$ is automatically continuous when S carries the discrete topology, since every subset of S is open. Thus $\mathrm{Hom}_{\mathbf{Top}}(D(S), Y) \cong \mathrm{Hom}_{\mathbf{Set}}(S, U(Y))$, giving $D \dashv U$.

Similarly, any function $g\colon U(X) \to S$ is continuous when S carries the indiscrete topology, since the only preimages to check are $\emptyset$ and S. Hence $U \dashv I$. □

But here is the problem that will motivate a large part of this course: **Top** fails to be cartesian closed.

Theorem 1.23 (Is **Top** cartesian closed?)**.** *The category* **Top** *is* not *cartesian closed. Specifically, the compact-open topology on* $\mathrm{Hom}(X, Y)$ *does not generally yield an exponential functor adjoint to the product. However, restricting to locally compact Hausdorff spaces gives a cartesian closed subcategory.*

Warning

The failure of **Top** to be cartesian closed is a fundamental limitation that motivates the search for better-behaved topological categories, such as convergence spaces (Chapter 7) or compactly generated spaces.

1.7 The Functor $\mathcal{O}$ and the Lattice of Opens

Let us shift perspective. Instead of studying a space through its *points*, let us study it through its *open sets*. This seemingly innocuous idea will lead us to locale theory (Chapter 3) and constitutes one of the deepest bridges between topology and logic.

The set of open subsets of a topological space forms a lattice with a very particular structure: it is a *frame*.

Definition 1.24 (Lattice of open sets)**.** For a topological space X, the *lattice of open sets* $\mathcal{O}(X)$ is the set of open subsets of X, ordered by inclusion, with operations:

- $\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i$ (arbitrary joins);
- $U \wedge V = U \cap V$ (finite meets);
- $\bot = \emptyset$, $\top = X$.

This is a *frame*: a complete lattice satisfying the infinite distributivity law:

$$U \cap \bigcup_{i \in I} V_i = \bigcup_{i \in I} (U \cap V_i).$$

What makes this construction categorically interesting is that it is functorial: continuous maps induce frame morphisms, but in the reverse direction.

Proposition 1.25 (Functoriality of $\mathcal{O}$)**.** The assignment $X \mapsto \mathcal{O}(X)$ extends to a contravariant functor $\mathcal{O}\colon \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$. For a continuous map $f\colon X \to Y$, $\mathcal{O}(f) = f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is a frame morphism.

Proof. The inverse image f^{-1} preserves arbitrary unions and finite intersections, so it is indeed a frame morphism. Functoriality follows from $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

Remark 1.26. This functor lies at the heart of locale theory (Chapter 3), where the perspective is reversed: instead of studying a space via its points, one studies it via its lattice of open sets.

1.8 Initial and Final Topologies

To construct new topologies, two opposing strategies present themselves. The *initial* topology is the smallest topology making certain maps continuous: this is how products and subspaces are built. The *final* topology is the largest: it is the mechanism behind quotients and disjoint unions. These two constructions are dual, and their universal properties encapsulate everything one needs to know to work with them.

Definition 1.27 (Initial topology)**.** Let X be a set and $(f_i\colon X \to (Y_i, \tau_i))_{i \in I}$ a family of maps. The *initial topology* on X is the coarsest topology making all f_i continuous. It is generated by the subbase $\{f_i^{-1}(U) : i \in I, U \in \tau_i\}$.

Definition 1.28 (Final topology)**.** Let Y be a set and $(f_i\colon (X_i, \tau_i) \to Y)_{i \in I}$ a family of maps. The *final topology* on Y is the finest topology making all f_i continuous: $\tau_Y = \{V \subseteq Y : f_i^{-1}(V) \in \tau_i \text{ for all } i\}$.

The power of these constructions lies in their universal properties, which reduce continuity verification to a component-by-component test.

Theorem 1.29 (Universal property)**.** *1.* Initial topology: $g\colon Z \to X$ *is continuous if and only if* $f_i \circ g$ *is continuous for all* i.

2. Final topology: $g\colon Y \to Z$ *is continuous if and only if* $g \circ f_i$ *is continuous for all* i.

Proof. (1) If g is continuous, the $f_i \circ g$ are continuous as compositions of continuous maps. Conversely, for every $U \in \tau_i$, $(f_i \circ g)^{-1}(U) = g^{-1}(f_i^{-1}(U))$ is open in Z. Since the $f_i^{-1}(U)$ form a subbase for the initial topology, g is continuous.

(2) If g is continuous, the $g \circ f_i$ are continuous. Conversely, let V be open in Z. For every i, $f_i^{-1}(g^{-1}(V)) = (g \circ f_i)^{-1}(V)$ is open in X_i, so $g^{-1}(V)$ is open in the final topology. □

1.9 Kolmogorov Spaces and the T_0 Quotient

Let us return to the separation axioms for a construction that beautifully illustrates categorical thinking. Given an arbitrary topological space, can we "make it" T_0 in the most economical way possible? The answer is the *Kolmogorov quotient*, and it is an example of *adjunction* in action.

Definition 1.30 (Kolmogorov quotient)**.** Let (X, τ) be a topological space. Define the equivalence relation: $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. The *Kolmogorov quotient* is $X_0 = X/{\sim}$ with the quotient topology.

The idea is simple: we identify points that the topology cannot tell apart. The result is a T_0 space, and this construction is "the best possible" in the following precise sense.

Theorem 1.31 (T_0 reflection)**.** *The Kolmogorov quotient X_0 is a T_0-space, and the quotient map $q\colon X \to X_0$ is universal: for every continuous map $f\colon X \to Y$ with Y a T_0-space, there exists a unique continuous map $\tilde{f}\colon X_0 \to Y$ such that $f = \tilde{f} \circ q$. Categorically, the inclusion functor $\mathbf{Top}_0 \hookrightarrow \mathbf{Top}$ has a left adjoint.*

Proof. We show X_0 is T_0. Let $[x] \neq [y]$ in X_0. Then $\overline{\{x\}} \neq \overline{\{y\}}$, so there exists an open set U in X containing one but not the other. Since U is saturated for $\sim$ (if $z \in U$ and $\overline{\{z\}} = \overline{\{z'\}}$, then $z' \in U$), $q(U)$ is open in X_0 and separates $[x]$ from $[y]$.

For universality: if $f\colon X \to Y$ is continuous and Y is T_0, then $\overline{\{x\}} = \overline{\{y\}}$ implies $\overline{\{f(x)\}} = \overline{\{f(y)\}}$ (by continuity), hence $f(x) = f(y)$ since Y is T_0. Thus f factors through the quotient. □

1.10 Compactness and Variants

Let us close this review chapter with a crucial terminological clarification. The notion of compactness, so central in topology, is subject to a convention disagreement between the French and Anglo-Saxon traditions that can cause serious confusion.

Definition 1.32 (Compactness)**.** A space X is *compact* if every open cover of X has a finite subcover.

Warning

Unlike the Bourbaki convention, we do *not* assume that a compact space is Hausdorff. This convention, standard in domain theory and algebraic topology, is essential for this course. When we need a compact Hausdorff space, we will say so explicitly.

1.11 Exercises

Exercise 1.1. Show that the Sierpiński space $\mathbb{S}$ is the classifier of open sets: for every space X, there is a bijection between $\mathcal{O}(X)$ and the set of continuous maps $X \to \mathbb{S}$.

Exercise 1.2. Show that in the category **Top**, the equalizer of $f, g\colon X \rightrightarrows Y$ is the subspace $\{x \in X : f(x) = g(x)\}$ with the subspace topology.

Exercise 1.3. Verify that the functor $\mathcal{O}\colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$ is faithful if and only if one restricts to T_0-spaces. Provide a counterexample in the non-T_0 case.

Exercise 1.4. Let X be a finite ordered set. Show that the Alexandrov topology on X (whose open sets are the upward-closed subsets) is the finest topology whose specialization order is the given order.

Exercise 1.5. Show that the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ preserves neither coproducts nor coequalizers. Give explicit examples.

Exercise 1.6 (Limits in $\mathbf{Top}_0$)**.** Show that $\mathbf{Top}_0$ is a reflective subcategory of **Top** and deduce that it admits all limits and colimits. Do limits in $\mathbf{Top}_0$ coincide with those in **Top**?

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Chapter 2

T_0-Spaces and Specialization Order

One of the most fruitful discoveries in twentieth-century topology is that behind every topological space hides an *order*. This order, called the "specialization order," connects topology to partial order theory in a deep and unexpected way. In the world of T_0-spaces, this correspondence becomes an equivalence: the topology *is* the order, and the order *is* the topology. It is this duality that makes T_0-spaces the central objects of this course.

Intuition

The specialization order is the fundamental bridge between topology and order theory. In a T_0-space, each point occupies a position in a partial order determined by the topology. This chapter systematically explores this correspondence and its consequences.

2.1 The Specialization Order

Definition 2.1 (Specialization order)**.** Let (X, τ) be a topological space. The *specialization order* is the relation $\leq$ on X defined by:

$$x \leq y \quad \iff \quad x \in \overline{\{y\}} \quad \iff \quad \text{every open set containing } x \text{ also contains } y.$$

Proposition 2.2 (Basic properties)**.** Let (X, τ) be a topological space with specialization order $\leq$.

1. $\leq$ is a preorder (reflexive and transitive).
2. $\leq$ is a partial order (antisymmetric) if and only if X is T_0.
3. $x \leq y$ if and only if $\overline{\{y\}} \subseteq \overline{\{x\}}$ (note the reversal!).
4. Every open set U is upward-closed: if $x \in U$ and $x \leq y$, then $y \in U$.
5. Every closed set F is downward-closed: if $y \in F$ and $x \leq y$, then $x \in F$.

Proof. (1) Reflexivity: $x \in \overline{\{x\}}$ always. Transitivity: if $x \leq y$ and $y \leq z$, then for every open $U \ni x$, we have $y \in U$ (since $x \leq y$), then $z \in U$ (since $y \leq z$), so $x \leq z$.

(2) Antisymmetry: $x \leq y$ and $y \leq x$ mean $\overline{\{x\}} = \overline{\{y\}}$, i.e., x and y are topologically indistinguishable. This equals $x = y$ iff X is T_0.

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(4) Let U be open, $x \in U$, $x \le y$. Every open containing x contains y; in particular U does, so $y \in U$.

(5) By taking complements. $\square$

Warning

The ordering convention varies among authors. We follow the convention where $x \le y$ means "x specializes to y," i.e., $x \in \overline{\{y\}}$, or equivalently, y is "more generic" than x. Some authors use the opposite convention. In our convention, open sets are *upward-closed.*

2.2 Alexandrov Topology

Definition 2.3 (Alexandrov topology)**.** Let $(P, \le)$ be a preordered set. The *Alexandrov topology* on P is the topology whose open sets are exactly the upward-closed subsets:

$$U \in \tau_{\text{Alex}} \iff (x \in U \text{ and } x \le y \Rightarrow y \in U).$$

Theorem 2.4 (Characterization of Alexandrov topology)**.** *A topological space (X, τ) carries the Alexandrov topology of its specialization order if and only if τ is closed under arbitrary intersections.*

Proof. ($\Rightarrow$) Upward-closed subsets of a preorder are closed under arbitrary intersections.

($\Leftarrow$) If τ is closed under arbitrary intersections, then for every x, $U_x = \bigcap\{U \in \tau : x \in U\}$ is open. This is the smallest open set containing x. We have $y \in U_x$ iff every open containing x contains y, i.e., $x \le y$. Thus $U_x = \uparrow x$ and every open is a union of such sets, hence upward-closed. $\square$

Proposition 2.5 (Equivalence of categories)**.** The functor associating to a preordered set $(P, \le)$ the Alexandrov space (P, τ_{Alex}) establishes an equivalence of categories between **Preord** (preorders and monotone maps) and the full subcategory of **Top** consisting of Alexandrov spaces. Restricting to partial orders gives an equivalence with T_0 Alexandrov spaces.

Proof. A map $f : P \to Q$ is monotone if and only if the preimage of every upward-closed set is upward-closed, which is equivalent to continuity for the Alexandrov topologies. The inverse functor associates to an Alexandrov space its specialization order. $\square$

2.3 Closure and Order

Proposition 2.6 (Closure and downward-closed sets)**.** In any topological space (X, τ):

1. $\overline{\{x\}} = \downarrow x$ (for the specialization order).
2. For any $A \subseteq X$, $\overline{A} = \downarrow A$ if and only if X carries the Alexandrov topology.
3. In general, $\downarrow A \subseteq \overline{A}$, with equality for singletons.

Proof. (1) By definition, $y \in \overline{\{x\}}$ iff every open containing y contains x, i.e., $y \le x$, i.e., $y \in \downarrow x$.

(3) If $a \in A$ and $y \le a$, then $y \in \overline{\{a\}} \subseteq \overline{A}$, so $\downarrow A \subseteq \overline{A}$. $\square$

Definition 2.7 (Generic point)**.** Let F be an irreducible closed set of X (i.e., $F \neq \emptyset$ and if $F = F_1 \cup F_2$ with F_1, F_2 closed, then $F = F_1$ or $F = F_2$). A point $x \in F$ is a *generic point* of F if $\overline{\{x\}} = F$.

Remark 2.8. The existence and uniqueness of generic points for irreducible closed sets is at the heart of the notion of sober space (Chapter 3).

2.4 Continuity and Monotonicity

Theorem 2.9 (Continuity implies monotonicity)**.** *If $f : X \to Y$ is continuous, then f is monotone for the specialization orders: $x \leq_X y \Rightarrow f(x) \leq_Y f(y)$.*

Proof. Let V be an open set in Y containing $f(x)$. Then $f^{-1}(V)$ is open in X and contains x. Since $x \leq y$, we have $y \in f^{-1}(V)$, so $f(y) \in V$. Thus $f(x) \leq_Y f(y)$. □

Warning

The converse is false in general: a monotone function between T_0-spaces need not be continuous. Continuity is a *stronger* condition than monotonicity. However, for Alexandrov topologies, the two notions coincide.

2.5 The Specialization Functor

Definition 2.10 (Specialization functor)**.** The *specialization functor* $\Sigma\colon \mathbf{Top}_0 \to \mathbf{Pos}$ assigns to each T_0-space its underlying set with the specialization order, and to each continuous map the same function (which is monotone by the preceding theorem).

Theorem 2.11 (Left adjoint of the specialization functor)**.** *The Alexandrov functor* $\mathrm{Alex}\colon \mathbf{Pos} \to \mathbf{Top}_0$ *is left adjoint to the specialization functor Σ:*

$$\mathrm{Alex} \dashv \Sigma.$$

That is, for every poset P and every T_0-space X:

$$\mathrm{Hom}_{\mathbf{Top}_0}(\mathrm{Alex}(P), X) \cong \mathrm{Hom}_{\mathbf{Pos}}(P, \Sigma(X)).$$

Proof. A morphism $\mathrm{Alex}(P) \to X$ in $\mathbf{Top}_0$ is a continuous map, hence monotone from $(P, \leq)$ to $(\Sigma(X), \leq_X)$. Conversely, a monotone map $f\colon P \to \Sigma(X)$ is continuous $\mathrm{Alex}(P) \to X$ because the preimage of an open U of X (upward-closed for $\leq_X$) is upward-closed in P (since f is monotone), hence Alexandrov-open. □

2.6 d-Spaces

Definition 2.12 (d-space)**.** A topological space X is a *d-space* (or monotone convergence space) if for every directed set D in $(X, \leq)$ (specialization order) having a supremum $\bigvee D$ in $(X, \leq)$, every open set containing $\bigvee D$ contains an element of D. Equivalently, if the net along D converges to $\bigvee D$ in the topological sense.

Theorem 2.13. *Every sober space is a d-space. Every T_1-space is a d-space (trivially, since directed sets for the discrete order are singletons).*

Proof. Let X be sober and D a directed set with supremum $s = \bigvee D$. The closure $\overline{D}$ is a closed set, and it is downward-closed. We show $\overline{D}$ is irreducible. If $\overline{D} = F_1 \cup F_2$ with F_1, F_2 closed, then $D \subseteq F_1 \cup F_2$. Since D is directed and the F_i are downward-closed, if $D \not\subseteq F_1$ and $D \not\subseteq F_2$, pick $d_1 \in D \setminus F_1$ and $d_2 \in D \setminus F_2$. By directedness, there exists $d \geq d_1, d_2$ in D. Then $d \in F_1 \cup F_2$, say $d \in F_1$. Since F_1 is downward-closed and $d_1 \leq d$, we get $d_1 \in F_1$, contradiction. So $\overline{D}$ is irreducible. By sobriety, $\overline{D}$ has a generic point, which must be s. □

2.7 Upper Topology

Definition 2.14 (Upper topology)**.** Let $(P, \leq)$ be a partially ordered set. The *upper topology* is the topology generated by the complements of principal downward-closed sets:

$$\tau_{\text{up}} = \langle P \setminus \downarrow x : x \in P \rangle.$$

This is in general coarser than the Alexandrov topology.

Proposition 2.15. The upper topology has the same specialization order as the Alexandrov topology, namely the given order $\leq$. It is the coarsest topology with this specialization order.

Proof. Opens of τ_{up} are upward-closed, so the specialization order refines $\leq$. Conversely, if $x \leq y$, then every open $P \setminus \downarrow z$ containing x (i.e., $x \not\leq z$) satisfies $y \not\leq z$ (otherwise $x \leq y \leq z$, contradiction), so y is in this open. Thus the specialization order is exactly $\leq$. □

2.8 Saturated Sets

Definition 2.16 (Saturated set)**.** A subset A of a topological space is *saturated* if it equals the intersection of all open sets containing it: $A = \bigcap\{U \in \tau : A \subseteq U\}$.

Proposition 2.17. In any topological space:

1. A subset is saturated if and only if it is upward-closed for the specialization order.

2. Every open set is saturated.

3. The saturation of a set A is $\uparrow A$.

4. A compact saturated set is the correct analogue of a compact set in the non-Hausdorff setting.

Proof. (1) The intersection of all opens containing A is the set of points y such that every open containing A contains y. Now y belongs to every open containing $x \in A$ iff $x \leq y$. So this intersection is $\uparrow A$. □

2.9 Stably Compact Spaces (Preview)

Definition 2.18 (Stably compact space)**.** A T_0-space is *stably compact* if it is:

1. compact,
2. locally compact (every point has a base of compact saturated neighborhoods),
3. sober,
4. *coherent*: the intersection of two compact saturated sets is compact saturated.

Remark 2.19. Stably compact spaces play a central role in the theory, as they are precisely the compact T_0-spaces whose lattice of opens is a *continuous lattice*. We will return to them in Chapter 8.

2.10 Exercises

Exercise 2.1. Show that a space is T_1 if and only if its specialization order is equality.

Exercise 2.2. Let $X = \mathrm{Spec}(A)$ be the spectrum of a commutative ring with the Zariski topology. Describe the specialization order. Show that the generic point of X (if it exists) corresponds to the minimal prime ideal.

Exercise 2.3. Show that in a T_0-space, the closure of a singleton $\overline{\{x\}}$ is always an irreducible closed set.

Exercise 2.4. Construct a T_0-space containing an irreducible closed set that is not the closure of any singleton. (Hint: consider an ordered set with no greatest element, endowed with the upper topology.)

Exercise 2.5. Let $f\colon X \to Y$ be a continuous open map between T_0-spaces. Show that f is monotone and that $f^{-1}(\uparrow y) = \uparrow f^{-1}(y)$ for every $y \in Y$.

Exercise 2.6 (Alexandrov topology and direct limits)**.** Let $(P_i, f_{ij})_{i \le j}$ be a direct system of finite posets with monotone maps. Show that the direct limit in **Top** of the corresponding Alexandrov spaces is a space whose specialization order is the direct limit of the orders.

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Chapter 3

Sober Spaces and Locales

What if you could do topology without ever mentioning points? The idea sounds absurd: since Cantor and Hausdorff, a topological space is a set of *points* equipped with a collection of open sets. Yet starting in the 1960s, a tradition rooted in logic and theoretical computer science—championed by mathematicians like John Isbell and the category theorists—showed that the open sets themselves contain all the information. The lattice of opens, viewed as an algebraic structure, suffices to reconstruct the topology. Spaces where this reconstruction is faithful are called *sober*; the abstract algebraic structure is a *locale*.

This chapter explores the fascinating passage from the concrete (points) to the abstract (opens as autonomous entities), a journey that reveals deep connections between topology, order theory, and logic.

Intuition

Sober spaces constitute the most natural class of T_0-spaces: they are those where the topology completely determines the point structure, via the correspondence between points and irreducible closed sets. Locales formalize the idea of "pointless topology" and provide an algebraic framework for topology.

3.1 Irreducible Closed Sets

Definition 3.1 (Irreducible closed set)**.** A closed set F of a topological space X is *irreducible* if:

1. $F \neq \emptyset$;
2. if $F = F_1 \cup F_2$ with F_1, F_2 closed in F, then $F = F_1$ or $F = F_2$.

Equivalently, F is irreducible iff for all opens U_1, U_2 of X with $F \cap U_1 \neq \emptyset$ and $F \cap U_2 \neq \emptyset$, we have $F \cap U_1 \cap U_2 \neq \emptyset$.

Proposition 3.2 (Properties of irreducible closed sets)**.**

1. The closure of a singleton $\overline{\{x\}}$ is always irreducible.
2. The continuous image of an irreducible set is irreducible.
3. The closure of an irreducible set is irreducible.
4. If F is irreducible and $F \cap G \neq \emptyset$ with G open, then $F \cap G$ is irreducible in G.

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Proof. (1) If $\overline{\{x\}} = F_1 \cup F_2$ with F_i closed, then $x \in F_1$ or $x \in F_2$, say $x \in F_1$. Then $\overline{\{x\}} \subseteq F_1$, so $F_1 = \overline{\{x\}}$.

(2) Let $f\colon X \to Y$ be continuous and F irreducible in X. If $f(F) \subseteq G_1 \cup G_2$ with G_i closed, then $F \subseteq f^{-1}(G_1) \cup f^{-1}(G_2)$ with $f^{-1}(G_i)$ closed, so $F \subseteq f^{-1}(G_i)$ for some i, giving $f(F) \subseteq G_i$. The closure $\overline{f(F)}$ is irreducible by (3).

(3) If $\overline{F} = G_1 \cup G_2$, then $F \subseteq G_1 \cup G_2$, so $F = (F \cap G_1) \cup (F \cap G_2)$. By irreducibility, $F \subseteq G_i$ for some i, whence $\overline{F} \subseteq G_i$. □

3.2 Sober Spaces

Definition 3.3 (Sober space)**.** A topological space X is *sober* if every irreducible closed set has a unique generic point: for every irreducible closed F, there exists a unique $x \in X$ with $\overline{\{x\}} = F$.

Theorem 3.4 (Characterization of sober spaces)**.** *A space X is sober if and only if:*

1. *X is T_0, and*

2. *every irreducible closed set has a generic point.*

Proof. ($\Rightarrow$) Uniqueness of the generic point implies T_0: if $\overline{\{x\}} = \overline{\{y\}}$, then x and y are both generic points of the same irreducible closed set, so $x = y$.

($\Leftarrow$) It suffices to show uniqueness. If $\overline{\{x\}} = \overline{\{y\}} = F$, then since X is T_0, $x = y$. □

Example 3.5 (Examples of sober spaces)**.**
1. Every Hausdorff space is sober (irreducible closed sets are singletons in T_2 spaces).

2. The spectrum Spec(A) of a commutative ring is sober (a classical result in algebraic geometry).

3. The Sierpiński space $\mathbb{S}$ is sober.

4. Every complete lattice with the Scott topology is sober.

Example 3.6 (A T_0-space that is not sober)**.** Let $X = \mathbb{N}$ with the topology whose opens are $\emptyset$ and the cofinal sets $\{n, n+1, n+2, \ldots\}$. The specialization order is the usual order on $\mathbb{N}$. The closed set X itself is irreducible (any two nonempty opens intersect). But $X = \overline{\{n\}}$ is impossible for any n, since $\overline{\{n\}} = \{0, \ldots, n\}$. So X is not sober.

3.3 Sobrification

Definition 3.7 (Sobrification)**.** Let X be a topological space. The space of *irreducible closed sets* of X, denoted X^s (or Sob(X)), is the set of irreducible closed subsets of X, equipped with the topology whose opens are:

$$\tilde{U} = \{F \in X^s : F \cap U \neq \emptyset\}, \quad U \in \mathcal{O}(X).$$

The map $\eta_X\colon X \to X^s$ defined by $\eta_X(x) = \overline{\{x\}}$ is continuous.

Theorem 3.8 (Properties of sobrification)**.**
1. *X^s is sober.*

2. *The map $U \mapsto \tilde{U}$ is a frame isomorphism $\mathcal{O}(X) \cong \mathcal{O}(X^s)$.*

3. *η_X is universal: for every continuous map $f\colon X \to Y$ with Y sober, there exists a unique continuous map $\tilde{f}\colon X^s \to Y$ with $f = \tilde{f} \circ \eta_X$.*

4. *The sobrification functor* Sob: **Top** $\to$ **Sob** *is left adjoint to the inclusion* **Sob** $\hookrightarrow$ **Top**.

Proof. (1) We show X^s is T_0. If $F_1 \neq F_2$ are irreducible closed sets, there exists an open U with, say, $F_1 \cap U \neq \emptyset$ and $F_2 \cap U = \emptyset$. Then $F_1 \in \tilde{U}$ and $F_2 \notin \tilde{U}$.

Every irreducible closed set of X^s has a generic point. Let $\mathcal{F}$ be an irreducible closed subset of X^s. Set $F_0 = \overline{\bigcup_{F \in \mathcal{F}} F}$ in X. One checks that F_0 is irreducible in X and $\overline{\{F_0\}}^{X^s} = \mathcal{F}$.

(2) The map $U \mapsto \tilde{U}$ preserves arbitrary unions and finite intersections, and is injective since $\tilde{U_1} = \tilde{U_2}$ implies, for every irreducible closed F, that $F \cap U_1 \neq \emptyset \Leftrightarrow F \cap U_2 \neq \emptyset$. Since singletons $\overline{\{x\}}$ are irreducible closed, this gives $U_1 = U_2$.

(3) If $f\colon X \to Y$ is continuous and Y sober, define $\tilde{f}(F) =$ the unique generic point of $\overline{f(F)}$ in Y (which exists since $f(F)$ is irreducible and Y is sober). Continuity and uniqueness are verified directly. □

3.4 Frames and Locales

Definition 3.9 (Frame)**.** A *frame* is a complete lattice L satisfying the infinite distributivity law:

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$

for all $a \in L$ and every family $(b_i)_{i \in I}$ in L. A *frame morphism* $h\colon L \to M$ preserves arbitrary joins and finite meets (including $\top$ and $\bot$).

Definition 3.10 (Locale)**.** The category **Loc** of *locales* is the opposite category of the category of frames:

$$\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}.$$

An object of **Loc** is a frame viewed as a "pointless space," and a morphism of locales $f\colon L \to M$ is a frame morphism $f^*\colon M \to L$.

Remark 3.11. The reversal of arrows is motivated by the analogy with topological spaces: a continuous map $f\colon X \to Y$ induces $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$, which goes in the opposite direction.

3.5 The Functor pt and the Adjunction $\mathcal{O} \dashv \mathrm{pt}$

Definition 3.12 (Points of a locale)**.** Let L be a locale (i.e., a frame). A *point* of L is a frame morphism $p\colon L \to \{0, 1\}$ (where $\{0, 1\}$ is the two-element frame), or equivalently, a completely prime element $q \in L$ ($q \neq \top$ and $a \wedge b \leq q \Rightarrow a \leq q$ or $b \leq q$).

The space of points $\mathrm{pt}(L)$ is the set of points of L with the topology whose opens are $\Sigma_a = \{p : p(a) = 1\}$ for $a \in L$.

Theorem 3.13 (Adjunction $\mathcal{O} \dashv \mathrm{pt}$)**.** *The functors* $\mathcal{O}\colon$ **Top** $\to$ **Loc** *and* pt: **Loc** $\to$ **Top** *form an adjunction:*

$$\mathcal{O} \dashv \mathrm{pt}.$$

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The unit $\eta_X : X \to \mathrm{pt}(\mathcal{O}(X))$ *sends* x *to the point* $\eta_X(x)(U) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$.

Proof. For a space X and a locale L, we establish:

$$\mathrm{Hom}_{\mathbf{Top}}(X, \mathrm{pt}(L)) \cong \mathrm{Hom}_{\mathbf{Loc}}(\mathcal{O}(X), L).$$

The right side is $\mathrm{Hom}_{\mathbf{Frm}}(L, \mathcal{O}(X))$. A continuous map $f : X \to \mathrm{pt}(L)$ induces a frame morphism $L \to \mathcal{O}(X)$ sending $a \mapsto f^{-1}(\Sigma_a)$. Conversely, a frame morphism $h : L \to \mathcal{O}(X)$ induces a continuous map $X \to \mathrm{pt}(L)$ sending x to the point $a \mapsto [x \in h(a)]$. □

Theorem 3.14 (Characterization of sober spaces via the adjunction)**.** *A space* X *is sober if and only if the unit* $\eta_X : X \to \mathrm{pt}(\mathcal{O}(X))$ *is a homeomorphism.*

Proof. The map η_X is injective iff X is T_0 (distinct points give different "points" of the frame). It is surjective iff every point of $\mathcal{O}(X)$ arises from a point of X, which is equivalent to every irreducible closed set having a generic point. Finally, η_X is always a topological embedding. □

3.6 Spatial Locales

Definition 3.15 (Spatial locale)**.** A locale L is *spatial* if the canonical frame morphism $L \to \mathcal{O}(\mathrm{pt}(L))$ is an isomorphism, i.e., if L has "enough points."

Theorem 3.16. *The restrictions of the adjunction* $\mathcal{O} \dashv \mathrm{pt}$ *induce an equivalence of categories between the category of sober spaces and the category of spatial locales.*

Proposition 3.17 (Non-spatial locales exist)**.** There exist frames with no points at all: for example, the frame of regular open subsets of $\mathbb{R}$ modulo a certain ideal is a nontrivial frame without any points.

3.7 Sublocales and Nuclei

Definition 3.18 (Nucleus)**.** Let L be a frame. A *nucleus* on L is a map $j : L \to L$ such that:

1. $a \le j(a)$ (inflationary);
2. $j(j(a)) = j(a)$ (idempotent);
3. $j(a \wedge b) = j(a) \wedge j(b)$ (preserves finite meets).

Theorem 3.19 (Sublocales via nuclei)**.** *Sublocales of a locale* L *correspond bijectively to nuclei on* L*. If* j *is a nucleus, the image* $j(L) = \{a \in L : j(a) = a\}$ *is a frame (with the same meets as* L *and modified joins* $\bigvee^j a_i = j(\bigvee a_i)$*), and the frame morphism* $L \to j(L)$*,* $a \mapsto j(a)$*, defines a locale embedding* $j(L) \hookrightarrow L$*.*

Proof. We verify that $j(L)$ is a frame. It is ordered by the inherited order from L. For meets: if $a, b \in j(L)$, then $j(a \wedge b) = j(a) \wedge j(b) = a \wedge b$, so $a \wedge b \in j(L)$. For joins: $j(\bigvee a_i)$ is in $j(L)$ and is the smallest element of $j(L)$ above all a_i. Distributivity is verified using that of L. □

3.8 Stone Duality

Definition 3.20 (Bounded distributive lattice)**.** A *bounded distributive lattice* is a lattice D with a least element 0 and a greatest element 1, satisfying $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Definition 3.21 (Spectral space)**.** A topological space X is *spectral* if it is:

1. T_0 and sober;
2. compact;
3. the set of compact opens is closed under finite intersection and forms a base.

Theorem 3.22 (Stone duality for distributive lattices)**.** *There is an equivalence of categories between the category of bounded distributive lattices and the category of spectral spaces (with spectral continuous maps). The functor in one direction sends a lattice D to the space of its prime filters, and in the other, sends a spectral space X to the lattice of its compact open subsets.*

3.9 Exercises

Exercise 3.1. Show that the Sierpiński space $\mathbb{S}$ is sober and describe its irreducible closed sets.

Exercise 3.2. Show that a space is sober if and only if it is a d-space and every irreducible closed set has a generic point.

Exercise 3.3. Let L be a frame. Show that $\mathrm{pt}(L)$ is always a sober space.

Exercise 3.4. Verify that the frame $\mathcal{O}(\mathbb{R})$ is spatial (i.e., that $\mathbb{R}$ with the usual topology gives a spatial locale).

Exercise 3.5 (Open and closed sublocales)**.** Let L be a frame and $a \in L$. Show that:

1. The nucleus $j_a(x) = a \vee x$ defines a closed sublocale.
2. The nucleus $j^a(x) = a \Rightarrow x$ (Heyting implication) defines an open sublocale.

Exercise 3.6. Show that the sobrification of $\mathbb{N}$ (with cofinal open sets) is homeomorphic to $\mathbb{N} \cup \{\infty\}$ with the Alexandrov topology of the usual order.

Exercise 3.7 (Heyting algebra)**.** Show that every frame is a Heyting algebra: for all a, b, there exists a greatest element c with $a \wedge c \leq b$, denoted $a \Rightarrow b$. Show that $a \Rightarrow b = \bigvee\{c : a \wedge c \leq b\}$.

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Chapter 4

Quasi-metrics and Asymmetric Topology

Since Fréchet (1906), a distance satisfies three axioms: positivity, symmetry, and the triangle inequality. But many real-world situations are intrinsically asymmetric: travel time uphill differs from downhill, the computational cost of an approximation depends on the direction of refinement, and in theoretical computer science, Smyth's distance between programs measures a rapprochement that is not reciprocal. Dropping the symmetry axiom leads to *quasi-metrics*, a framework developed notably by Hans-Peter Künzi from the 1980s onward, revealing a rich and surprising topology: each quasi-metric generates *two* topologies, one forward and one backward.

Intuition

Quasi-metrics generalize metrics by dropping the symmetry axiom: the "distance" from x to y may differ from the distance from y to x. This asymmetry, far from being pathological, naturally captures notions of oriented cost, refinement, and complexity in computer science.

4.1 Fundamental Definitions

Definition 4.1 (Quasi-metric)**.** A *quasi-metric* on a set X is a function $d\colon X \times X \to [0, +\infty]$ satisfying:

1. $d(x, x) = 0$ for all $x \in X$;
2. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality).

If additionally $d(x, y) = 0 = d(y, x) \Rightarrow x = y$, we say d is *T_0-separated.* The pair (X, d) is a *quasi-metric space.*

Remark 4.2. We allow d to take the value $+\infty$, which is essential for certain constructions (infinite products, complexity domains).

Definition 4.3 (Conjugate and symmetrization)**.** Let (X, d) be a quasi-metric space.

- The *conjugate* of d is $d^{-1}(x, y) = d(y, x)$.
- The *symmetrization* of d is $d^s(x, y) = \max(d(x, y), d^{-1}(x, y))$.

- The *sum symmetrization* is $\hat{d}(x, y) = d(x, y) + d^{-1}(x, y)$.

The symmetrization d^s is a metric (possibly $[0, +\infty]$-valued) when d is T_0-separated.

4.2 Topology Associated to a Quasi-metric

Definition 4.4 (Open balls)**.** For a quasi-metric space (X, d), $x \in X$, and $\varepsilon > 0$:

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

is the *forward open ball* (or *direct ball*). The *forward topology* τ_d is the topology generated by these balls.

Proposition 4.5. The open balls $B_d(x, \varepsilon)$ form a base for τ_d. The space (X, τ_d) is T_0 if and only if d is T_0-separated.

Proof. Let $y \in B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$. Set $\delta = \min(\varepsilon_1 - d(x_1, y), \varepsilon_2 - d(x_2, y)) > 0$. For any $z \in B_d(y, \delta)$, $d(x_i, z) \leq d(x_i, y) + d(y, z) < d(x_i, y) + \delta \leq \varepsilon_i$. So $B_d(y, \delta) \subseteq B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$, showing the balls form a base.

For T_0 separation: if τ_d is T_0, then for $x \neq y$, there exists $\varepsilon > 0$ with, say, $y \notin B_d(x, \varepsilon)$, giving $d(x, y) \geq \varepsilon > 0$. Since at least one of $d(x, y)$, $d(y, x)$ is nonzero, d is T_0-separated. □

Definition 4.6 (Quasi-metric specialization order)**.** The specialization order of τ_d is:

$$x \leq_d y \iff d(x, y) = 0.$$

4.3 Fundamental Examples

Example 4.7 (Sorgenfrey quasi-metric)**.** On $\mathbb{R}$, define:

$$d_S(x, y) = \begin{cases} y - x & \text{if } y \geq x, \\ 1 & \text{if } y < x. \end{cases}$$

Then τ_{d_S} is the Sorgenfrey topology (base of half-open intervals $[a, b)$).

Example 4.8 (Baire quasi-metric)**.** On $\mathbb{N}^{\mathbb{N}}$ (infinite sequences of naturals), define:

$$d(f, g) = \begin{cases} 0 & \text{if } f = g, \\ 2^{-\min\{n : f(n) \neq g(n)\}} & \text{if } f \neq g \text{ and } f \sqsubseteq g, \\ 1 & \text{otherwise,} \end{cases}$$

where $f \sqsubseteq g$ means f is a prefix of g (in the sense that $f(k) \leq g(k)$ for the initial indices). This quasi-metric captures the idea of approximation: $d(f, g)$ is small when g extends f over a long prefix.

Example 4.9 (Complexity quasi-metric)**.** Let Σ^* be the set of words over an alphabet Σ. Define:

$$d(u, v) = \begin{cases} 0 & \text{if } u \text{ is a prefix of } v, \\ 2^{-|u \wedge v|} & \text{otherwise,} \end{cases}$$

where $|u \wedge v|$ is the length of the longest common prefix. This is a T_0-separated quasi-metric whose forward topology is the Scott topology on (partial) words.

4.4 Quasi-uniformities

Definition 4.10 (Quasi-uniformity). A *quasi-uniformity* on X is a filter $\mathcal{U}$ of subsets of $X \times X$ such that:

1. For every $U \in \mathcal{U}$, the diagonal $\Delta_X \subseteq U$.
2. For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V \circ V \subseteq U$ (where $V \circ V = \{(x, z) : \exists y,\ (x, y) \in V,\ (y, z) \in V\}$).

(We do *not* require $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$.)

Proposition 4.11. Every quasi-metric d generates a quasi-uniformity with basic entourages $U_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$.

Definition 4.12 (Pervin quasi-uniformity). For any topological space (X, τ), the *Pervin quasi-uniformity* is generated by the entourages:

$$S_U = (U \times U) \cup (U^c \times X), \quad U \in \tau.$$

Its associated topology is τ.

Theorem 4.13 (Quasi-uniformizability). *Every topological space is quasi-uniformizable (e.g., via the Pervin quasi-uniformity).*

4.5 Completion of Quasi-metric Spaces

Definition 4.14 (Forward Cauchy sequence). In a quasi-metric space (X, d), a sequence (x_n) is a *forward-Cauchy sequence* if for every $\varepsilon > 0$, there exists N such that for all $N \leq m \leq n$, $d(x_m, x_n) < \varepsilon$.

Definition 4.15 (Yoneda limit). A sequence (x_n) *d-converges* to ℓ (or is *Yoneda-convergent*) if for every $\varepsilon > 0$, there exists N such that for all $n \geq N$, $d(\ell, x_n) < \varepsilon$, *and* for every $y \in X$ with $\limsup_{n\to\infty} d(y, x_n) \leq r$, we have $d(y, \ell) \leq r$.

Definition 4.16 (Yoneda completion). A quasi-metric space is *Yoneda-complete* if every forward-Cauchy sequence has a Yoneda limit. The *Yoneda completion* of (X, d) is the smallest Yoneda-complete quasi-metric space containing (X, d) as a dense subspace.

Theorem 4.17 (Completion and domains). *Let (X, d) be a T_0-separated quasi-metric space. If d is finite-valued and satisfies $d(x, y) = 0 \Rightarrow x \leq y$, then the Yoneda completion of (X, d) is a dcpo equipped with a quasi-metric compatible with the Scott topology.*

4.6 Continuity and Quasi-metrics

Definition 4.18 (Non-expansive map). A map $f : (X, d_X) \to (Y, d_Y)$ is *non-expansive* if $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all x, y.

Proposition 4.19. Every non-expansive map is continuous for the forward topologies. The converse is false.

Theorem 4.20 (Category of quasi-metric spaces). *The category* **QMet** *of T_0-separated quasi-metric spaces and non-expansive maps is complete and cocomplete. The forgetful functor* **QMet** $\to$ **Top**$_0$ *preserves limits.*

4.7 Co-forward Topology and the Conjugate Pair

Definition 4.21 (Co-forward topology)**.** The *co-forward topology* (or conjugate topology) is $\tau_{d^{-1}}$, the topology generated by the balls of the conjugate quasi-metric d^{-1}.

Proposition 4.22. The topologies τ_d and $\tau_{d^{-1}}$ determine a bitopological space $(X, \tau_d, \tau_{d^{-1}})$. The join topology $\tau_d \vee \tau_{d^{-1}}$ coincides with the topology of the symmetrization τ_{d^s}.

Proof. The join topology is generated by the open balls of d and of d^{-1}. An open ball of d^s is $B_{d^s}(x, \varepsilon) = B_d(x, \varepsilon) \cap B_{d^{-1}}(x, \varepsilon)$, which generates the same topology. □

4.8 Weighted Quasi-metrics (Preview)

Definition 4.23 (Weighted quasi-metric)**.** A *weighted quasi-metric* (or *partial quasi-metric*) on X is a pair (d, w) where $d\colon X \times X \to [0, +\infty]$ is a quasi-metric and $w\colon X \to [0, +\infty)$ is a *weight* function satisfying:

$$d(x, y) + w(x) = d(y, x) + w(y)$$

for all $x, y \in X$. This generalizes the notion of partial metric.

Remark 4.24. Weighted quasi-metrics arise naturally in complexity analysis, where $w(x)$ measures the "self-distance" or "partiality" of x. They will reappear in Chapter 9.

4.9 Exercises

Exercise 4.1. Show that the specialization order of the forward topology of a quasi-metric d is given by $x \leq y \Leftrightarrow d(x, y) = 0$.

Exercise 4.2. Let (X, d) be a quasi-metric space. Show that d is a metric if and only if τ_d is T_1.

Exercise 4.3. Construct a quasi-metric on $\{0, 1\}$ whose forward topology is the Sierpiński topology.

Exercise 4.4. Show that the Pervin quasi-uniformity of a T_0-space is the smallest quasi-uniformity compatible with the topology.

Exercise 4.5 (Completion of $\mathbb{Q}$)**.** Let $d(x, y) = \max(y - x, 0)$ on $\mathbb{Q}$. Describe the forward topology, the specialization order, and the Yoneda completion of $(\mathbb{Q}, d)$.

Exercise 4.6. Show that for a finite-valued quasi-metric space (X, d), the following are equivalent:

1. (X, d) is Yoneda-complete.
2. Every forward Cauchy filter converges.
3. Every forward-Cauchy sequence has a Yoneda limit.

Exercise 4.7. Let D be a dcpo with the Scott topology. Construct a quasi-metric d on D such that τ_d is the Scott topology and $d(x, y) = 0 \Leftrightarrow x \sqsubseteq y$. (Hint: use a family of Scott-continuous functions into $[0, +\infty)$.)

Chapter 5

Bitopological Spaces

In 1963, John C. Kelly proposed a bold idea: equip the same set with *two* distinct topologies and study their interaction. The idea is far from gratuitous—it arises naturally from quasi-metrics, where the "forward" and "backward" topologies coexist, and from order theory, where upper and lower topologies on an ordered set capture complementary aspects. Nachbin's duality between compact ordered spaces and stably compact spaces shows that this bitopological structure is not an artefact, but a deep phenomenon.

Intuition

A bitopological space carries *two* topologies on the same set, capturing a natural duality. This structure arises naturally from quasi-metrics (forward and conjugate topologies), from order theory (upper and lower topologies), and in Nachbin's duality between compact ordered spaces and stably compact spaces.

5.1 Fundamental Definitions

Definition 5.1 (Bitopological space)**.** A *bitopological space* is a triple (X, τ_1, τ_2) where X is a set and τ_1, τ_2 are two topologies on X. We call τ_1 the *primary topology* and τ_2 the *secondary* (or *dual*) topology.

Definition 5.2 (Bicontinuous map)**.** Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A map $f : X \to Y$ is *bicontinuous* if f is continuous from (X, τ_1) to (Y, σ_1) and from (X, τ_2) to (Y, σ_2).

Definition 5.3 (Category **BiTop**)**.** The category **BiTop** has bitopological spaces as objects and bicontinuous maps as morphisms. A *bi-homeomorphism* is an isomorphism in **BiTop**.

5.2 Fundamental Examples

Example 5.4 (Bitopological space of a quasi-metric)**.** Let (X, d) be a quasi-metric space. The associated bitopological space is $(X, \tau_d, \tau_{d^{-1}})$, where τ_d is the forward topology and $\tau_{d^{-1}}$ is the conjugate topology.

Example 5.5 (Ordered spaces)**.** Let $(X, \leq)$ be a partially ordered set. The Alexandrov bitopological space is $(X, \tau_{\text{up}}, \tau_{\text{down}})$, where τ_{up} is the Alexandrov topology (opens = upward-closed sets) and τ_{down} is the dual topology (opens = downward-closed sets).

Example 5.6 (Ordered real line)**.** On $\mathbb{R}$, the topology of rightward open sets $\{(a, +\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and the topology of leftward open sets $\{(-\infty, b) : b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ define a bitopological space. Their join is the usual topology of $\mathbb{R}$.

5.3 Bitopological Separation Axioms

Definition 5.7 (Kelly separation)**.** A bitopological space (X, τ_1, τ_2) is:

- *Pairwise* T_0 if for all $x \neq y$, there exists an open in τ_1 or τ_2 containing one but not the other.
- *Pairwise* T_1 if for all $x \neq y$, there exists a τ_1-open containing x but not y, and a τ_2-open containing y but not x (or vice versa).
- *Pairwise* T_2 (or *pairwise Hausdorff*) if for all $x \neq y$, there exist a τ_1-open $U \ni x$ and a τ_2-open $V \ni y$ with $U \cap V = \emptyset$.

Theorem 5.8 (Pairwise T_2 and quasi-metrics)**.** *If (X, d) is a T_0-separated quasi-metric, then $(X, \tau_d, \tau_{d^{-1}})$ is pairwise T_2.*

Proof. Let $x \neq y$. Then $d(x, y) > 0$ or $d(y, x) > 0$. Suppose $d(x, y) > 0$ and set $\varepsilon = d(x, y)/2 > 0$. Then $U = B_d(x, \varepsilon)$ is a τ_d-open containing x and $V = B_{d^{-1}}(y, \varepsilon)$ is a $\tau_{d^{-1}}$-open containing y. If $z \in U \cap V$, then $d(x, z) < \varepsilon$ and $d^{-1}(y, z) = d(z, y) < \varepsilon$, so $d(x, y) \leq d(x, z) + d(z, y) < 2\varepsilon = d(x, y)$, a contradiction. Hence $U \cap V = \emptyset$. □

5.4 Pairwise Normality

Definition 5.9 (Pairwise normality)**.** A bitopological space (X, τ_1, τ_2) is *pairwise normal* if for every τ_1-closed F_1 and τ_2-closed F_2 with $F_1 \cap F_2 = \emptyset$, there exist a τ_2-open $U_2 \supseteq F_1$ and a τ_1-open $U_1 \supseteq F_2$ with $U_1 \cap U_2 = \emptyset$.

Theorem 5.10 (Bitopological Urysohn lemma)**.** *If (X, τ_1, τ_2) is pairwise normal and pairwise T_1, then for every disjoint pair of a τ_1-closed F_1 and a τ_2-closed F_2, there exists a function $f : X \to [0, 1]$ such that:*

- *$f \equiv 0$ on F_1 and $f \equiv 1$ on F_2;*
- *f is lower semicontinuous for τ_1 and upper semicontinuous for τ_2.*

Proof. The proof follows the classical Urysohn scheme. By induction on dyadic rationals $r \in [0, 1] \cap \mathbb{Z}[1/2]$, construct sets (A_r) such that:

- $F_1 \subseteq A_0$ and $A_1 = X \setminus F_2$;
- A_r is τ_2-open for $r < 1$;
- $r < s \Rightarrow \overline{A_r}^{\tau_1} \subseteq A_s$.

Existence follows from pairwise normality. Define $f(x) = \inf\{r : x \in A_r\}$ and verify the semicontinuity properties. □

5.5 Bitopological Compactness

Definition 5.11 (Joint compactness)**.** A bitopological space (X, τ_1, τ_2) is *jointly compact* if the space $(X, \tau_1 \vee \tau_2)$ is compact.

Definition 5.12 (Pairwise compactness)**.** A bitopological space is *pairwise compact* if every cover consisting of opens from τ_1 and τ_2 has a finite subcover.

Proposition 5.13. Pairwise compactness implies joint compactness. The converse holds when τ_1 and τ_2 form a "compatible pair."

5.6 Nachbin Duality

Definition 5.14 (Compact ordered space)**.** A *compact ordered space* (in the sense of Nachbin) is a triple $(X, \tau, \leq)$ where:

1. (X, τ) is a compact Hausdorff space;
2. $\leq$ is a partial order that is closed in $X \times X$ (for the product topology).

Theorem 5.15 (Nachbin duality)**.** *There is an equivalence of categories between:*

1. *the category of compact ordered spaces (with continuous monotone maps);*
2. *the category of* pairwise compact pairwise T_2 jointly T_1 *bitopological spaces (with bicontinuous maps).*

Proof sketch. ($\Rightarrow$) Given a compact ordered space $(X, \tau, \leq)$, define:

- $\tau_1 =$ the topology of upward-closed opens (upper topology within τ);
- $\tau_2 =$ the topology of downward-closed opens.

More precisely, τ_1 is generated by opens $U \in \tau$ with $U = \uparrow U$ and τ_2 by opens $V \in \tau$ with $V = \downarrow V$. Nachbin's theorem gives $\tau_1 \vee \tau_2 = \tau$.

($\Leftarrow$) Given a bitopological space (X, τ_1, τ_2) of the described type, set $\tau = \tau_1 \vee \tau_2$ and define $x \leq y \Leftrightarrow x \in \overline{\{y\}}^{\tau_1}$. ☐

5.7 Connection with Stably Compact Spaces

Theorem 5.16 (Stably compact and bitopological spaces)**.** *Let X be a stably compact space. Then:*

1. *The co-compact topology τ^d (generated by complements of compact saturated sets) determines a bitopological space (X, τ, τ^d).*
2. *The triple $(X, \tau \vee \tau^d, \leq)$ is a compact ordered space.*
3. *This construction establishes an equivalence between the category of stably compact spaces and a subcategory of compact ordered spaces.*

Remark 5.17. This is one of the deepest results in asymmetric topology: stably compact spaces, compact ordered spaces, and certain bitopological spaces are three facets of the same phenomenon.

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5.8 Functors Between BiTop and Top

Proposition 5.18 (Forgetful and symmetrization functors)**.** The following functors are available:

1. $U_1, U_2 \colon \mathbf{BiTop} \to \mathbf{Top}$, projection onto each topology.
2. $J \colon \mathbf{BiTop} \to \mathbf{Top}$, sending (X, τ_1, τ_2) to $(X, \tau_1 \vee \tau_2)$ (join topology).
3. $\Delta \colon \mathbf{Top} \to \mathbf{BiTop}$, sending (X, τ) to (X, τ, τ) (diagonal doubling).

The functor Δ is left and right adjoint to J in certain subcategories.

5.9 Bitopological Spaces and Lattices

Definition 5.19 (Biframe)**.** A *biframe* is a triple (L, L_1, L_2) where L is a frame and L_1, L_2 are subframes of L that generate L: every element of L is a join of finite meets of elements from $L_1 \cup L_2$.

Remark 5.20*.* Biframes are the "pointless" analogue of bitopological spaces, just as frames are the analogue of topological spaces. The theory of bilocales developed by Banaschewski, Brümmer, and Hardie provides a rich algebraic framework for bitopology.

5.10 Exercises

Exercise 5.1. Show that for any quasi-metric d, the bitopological space $(X, \tau_d, \tau_{d^{-1}})$ is pairwise T_0 if and only if d is T_0-separated.

Exercise 5.2. Let (X, τ_1, τ_2) be pairwise T_2. Show that the join topology $\tau_1 \vee \tau_2$ is Hausdorff.

Exercise 5.3. Show that every compact ordered space is pairwise normal.

Exercise 5.4. Let $X = [0, 1]$ with the usual order. Describe explicitly the topologies τ_{up}, τ_{down}, the join topology, and verify that $(X, \tau_{\text{up}} \vee \tau_{\text{down}}, \leq)$ is a compact ordered space.

Exercise 5.5. Show that the category **BiTop** is complete and cocomplete. Describe products and coproducts.

Exercise 5.6 (Priestley duality)**.** Let D be a bounded distributive lattice and X the associated Priestley space (compact ordered totally order-separated space). Describe the corresponding bitopological space via Nachbin duality and show that one recovers the spectral space of Stone duality.

Chapter 6

Scott Domains and Scott Topology

In 1970, Dana Scott was searching for a mathematical model for the untyped lambda calculus—a formal system where a function can take itself as an argument. Classical topological spaces would not do: what was needed was a space isomorphic to its own function space. Scott discovered that certain complete partial orders, equipped with a carefully chosen topology—the *Scott topology*—possess exactly this property. *Scott domains* thus became the foundation of denotational semantics for programming languages, revealing a deep connection between topology, logic, and computer science.

Intuition

The Scott topology is the natural topology on a directed-complete partial order (dcpo) that captures the notion of "upward convergence." It is the fundamental topology in denotational semantics, where programs are modeled as elements of domains and the denotation of a recursive program is the supremum of a chain of approximations.

6.1 Directed-Complete Partial Orders

Definition 6.1 (Directed set)**.** Let $(P, \leq)$ be a partially ordered set. A subset $D \subseteq P$ is *directed* if $D \neq \emptyset$ and for all $d_1, d_2 \in D$, there exists $d \in D$ with $d_1 \leq d$ and $d_2 \leq d$.

Definition 6.2 (Dcpo)**.** A *dcpo* (directed-complete partial order) is a poset $(D, \leq)$ in which every directed subset has a supremum. We write $\bigsqcup S$ or $\bigvee S$ for the supremum of a directed set S.

Example 6.3 (Examples of dcpos)**.**

1. Every complete lattice is a dcpo.
2. $(\mathcal{P}(X), \subseteq)$ is a dcpo (and a complete lattice).
3. $([0, 1], \leq)$ is a dcpo.
4. The set $\Sigma^{\leq\omega}$ of finite and infinite words over an alphabet Σ, ordered by the prefix relation, is a dcpo.
5. The set $X \rightharpoonup Y$ of partial functions from X to Y, ordered by graph extension, is a dcpo.

Definition 6.4 (Pointed dcpo)**.** A dcpo is *pointed* if it has a least element, denoted $\perp$.

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6.2 The Scott Topology

Definition 6.5 (Scott topology)**.** Let $(D, \leq)$ be a dcpo. A subset $U \subseteq D$ is *Scott-open* if:

1. U is upward-closed: $x \in U$ and $x \leq y \Rightarrow y \in U$.

2. U is *inaccessible by directed suprema*: if S is directed and $\bigsqcup S \in U$, then $S \cap U \neq \emptyset$.

The collection of Scott-open sets forms the *Scott topology* $\sigma(D)$.

Proposition 6.6 (The Scott topology is indeed a topology)**.** The Scott-open sets satisfy:

1. $\emptyset$ and D are Scott-open.

2. Any union of Scott-open sets is Scott-open.

3. Any finite intersection of Scott-open sets is Scott-open.

Proof. Only (3) requires verification. Let U_1, U_2 be Scott-open and S directed with $\bigsqcup S \in U_1 \cap U_2$. There exist $s_1 \in S \cap U_1$ and $s_2 \in S \cap U_2$. By directedness, there exists $s \in S$ with $s_1 \leq s$ and $s_2 \leq s$. Since U_1 and U_2 are upward-closed, $s \in U_1 \cap U_2$, so $S \cap (U_1 \cap U_2) \neq \emptyset$. $\square$

Proposition 6.7 (Specialization order)**.** The specialization order of the Scott topology coincides with the original order: $x \leq_\sigma y \Leftrightarrow x \leq y$.

6.3 Scott-Continuous Functions

Definition 6.8 (Scott-continuous function)**.** Let D and E be dcpos. A function $f : D \to E$ is *Scott-continuous* if it is continuous for the Scott topologies, equivalently:

1. f is monotone: $x \leq y \Rightarrow f(x) \leq f(y)$.

2. f preserves directed suprema: for every directed set S, $f(\bigsqcup S) = \bigsqcup f(S)$.

Theorem 6.9 (Characterization of Scott-continuity)**.** *For a function $f : D \to E$ between dcpos, the following are equivalent:*

1. *f is continuous for the Scott topologies.*

2. *f is monotone and preserves directed suprema.*

Proof. (1) $\Rightarrow$ (2): Continuity implies monotonicity (since the specialization order is the original order). Let S be directed in D. Since f is monotone, $f(\bigsqcup S)$ is an upper bound of $f(S)$. Let e be any upper bound of $f(S)$ and suppose $f(\bigsqcup S) \not\leq e$. Then there is a Scott-open $V \ni f(\bigsqcup S)$ with $e \notin V$. The preimage $f^{-1}(V)$ is Scott-open and contains $\bigsqcup S$, so some $s \in S$ is in $f^{-1}(V)$, giving $f(s) \in V$. Since $f(s) \leq e$ and V is upward-closed, $e \in V$, contradiction. So $f(\bigsqcup S) = \bigsqcup f(S)$.

(2) $\Rightarrow$ (1): Let V be Scott-open in E. Then $f^{-1}(V)$ is upward-closed since f is monotone. If S is directed with $\bigsqcup S \in f^{-1}(V)$, then $f(\bigsqcup S) = \bigsqcup f(S) \in V$. Since V is Scott-open and $f(S)$ is directed, some $s \in S$ has $f(s) \in V$, so $s \in f^{-1}(V)$. $\square$

6.4 The Category Dcpo

Theorem 6.10 (Properties of **Dcpo**). *1. The category* **Dcpo** *of dcpos and Scott-continuous functions is cartesian closed.*

2. *The product of dcpos is the cartesian product with the componentwise order.*

3. *The exponential $[D \to E]$ is the set of Scott-continuous functions from D to E, ordered pointwise.*

4. *$[D \to E]$ is a dcpo, and the Scott topology on $[D \to E]$ is the Isbell topology (which coincides with the compact-open topology for compact Scott-open sets).*

Proof sketch. For (3): let $(f_i)_{i \in I}$ be directed in $[D \to E]$. For every $d \in D$, $(f_i(d))_{i \in I}$ is directed in E (by pointwise monotonicity and directedness of (f_i)). Define $(\bigsqcup f_i)(d) = \bigsqcup_i f_i(d)$. One must verify that $\bigsqcup f_i$ is Scott-continuous, which uses the fact that directed suprema commute in a dcpo. □

6.5 Continuous Domains

Definition 6.11 (Way-below relation). In a dcpo D, we say x is *way below* y (written $x \ll y$) if for every directed set S with $y \leq \bigsqcup S$, there exists $s \in S$ with $x \leq s$.

Definition 6.12 (Basis of a dcpo). A subset $B \subseteq D$ is a *basis* for the dcpo D if for every $x \in D$, the set $\{b \in B : b \ll x\}$ is directed with supremum x.

Definition 6.13 (Continuous domain). A dcpo D is a *continuous domain* if for every $x \in D$, the set $\Downarrow x = \{y \in D : y \ll x\}$ is directed and $x = \bigsqcup(\Downarrow x)$.

Theorem 6.14 (Scott topology of a continuous domain). *If D is a continuous domain, then:*

1. *The sets $\Uparrow b = \{x \in D : b \ll x\}$ form a base for the Scott topology.*

2. *The Scott topology is sober.*

3. *The lattice $\sigma(D)$ is a continuous lattice.*

4. *$(D, \sigma(D))$ is locally compact.*

Proof. (1) Let U be Scott-open and $x \in U$. Since $x = \bigsqcup(\Downarrow x)$ and U is inaccessible, there exists $b \ll x$ with $b \in U$. Then $x \in \Uparrow b \subseteq \uparrow b \subseteq U$ since U is upward-closed.

(2) Let F be an irreducible closed set. The set $B_F = \{b : \exists x \in F,\, b \ll x\}$ is directed. Indeed, if $b_1 \ll x_1 \in F$ and $b_2 \ll x_2 \in F$, then $\Uparrow b_1$ and $\Uparrow b_2$ are opens meeting F, so by irreducibility, $F \cap \Uparrow b_1 \cap \Uparrow b_2 \neq \emptyset$, giving some $y \in F$ with $b_1 \ll y$ and $b_2 \ll y$. The supremum $\bigsqcup B_F$ lies in F (since F is Scott-closed) and is the generic point. □

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6.6 Algebraic Domains

Definition 6.15 (Compact element)**.** An element $k \in D$ is *compact* (or *finite*) if $k \ll k$. The set of compact elements is denoted $K(D)$.

Definition 6.16 (Algebraic domain)**.** A dcpo D is *algebraic* if for every $x \in D$, the set $\{k \in K(D) : k \leq x\}$ is directed and $x = \bigsqcup\{k \in K(D) : k \leq x\}$.

Proposition 6.17. Every algebraic domain is a continuous domain. The converse is false.

Example 6.18. $(\mathcal{P}(\mathbb{N}), \subseteq)$ is an algebraic domain: the compact elements are the finite subsets. The interval domain $\mathbf{I}\mathbb{R} = \{[a, b] : a \leq b, a, b \in \mathbb{R}\}$ ordered by reverse inclusion is a continuous domain that is not algebraic.

6.7 Representation Theorem

Theorem 6.19 (Algebraic domains and ideals)**.** *If D is an algebraic domain, then D is isomorphic (as a dcpo) to the set* $\mathrm{Idl}(K(D))$ *of ideals of* $(K(D), \leq)$ *(directed downward-closed subsets of $K(D)$), ordered by inclusion.*

Proof. The isomorphism sends $x \in D$ to $\downarrow x \cap K(D)$. This is an ideal: it is downward-closed in $K(D)$ by definition, and directed because D is algebraic. The inverse sends an ideal I to $\bigsqcup I$. The key point is that directed unions of ideals correspond to directed suprema in D. ☐

6.8 Exercises

Exercise 6.1. Show that the Scott topology on $(\mathbb{R}, \leq)$ consists of $\emptyset$, $\mathbb{R}$, and the intervals $(a, +\infty)$ for $a \in \mathbb{R}$.

Exercise 6.2. Show that the Scott topology on $(\mathcal{P}(\mathbb{N}), \subseteq)$ is not metrizable.

Exercise 6.3. Show that the product of two dcpos (with the product order) is a dcpo and that the Scott topology of the product is finer than the product of the Scott topologies. Give an example where the inclusion is strict.

Exercise 6.4. Let D be a pointed dcpo. Show that the function $\mathrm{fix}\colon [D \to D] \to D$ sending f to its least fixed point is Scott-continuous.

Exercise 6.5. Let D be a continuous domain. Show that $x \ll y$ if and only if $y \in \mathrm{Int}(\uparrow x)$ (Scott interior).

Exercise 6.6. Show that a dcpo D is continuous if and only if $\sigma(D)$ is a completely distributive lattice.

Exercise 6.7 (Step functions)**.** Let D be an algebraic domain and E a dcpo. Show that every Scott-continuous function $f\colon D \to E$ is the directed supremum of "step functions" (functions defined on compact elements of D and extended by directed suprema).

Chapter 7

Convergence Spaces and Filters

What is convergence, really? In a standard analysis course, we learn that a sequence (x_n) converges to ℓ if, for every neighbourhood of ℓ, the sequence eventually stays inside it. But this definition presupposes a topology. In the 1960s, mathematicians such as Gustave Choquet, H. J. Kowalsky, and C. H. Cook asked a bold question: what if convergence itself were the primitive notion? Rather than defining open sets first and deducing convergence, why not specify directly which filters converge to which points? This logical inversion gave birth to *convergence spaces*, a generalization of topological spaces that elegantly solves the cartesian closure problem — a technical defect of the category of topological spaces that complicated the study of function spaces.

Intuition

Convergence in a topological space is classically described via filters or nets. Convergence spaces generalize topological spaces by directly specifying which filters converge to which points, without requiring the convergence structure to arise from a topology. This generalization solves the cartesian closure problem and is particularly suited to function spaces.

7.1 Filters: Review and Complements

Definition 7.1 (Filter)**.** A *filter* on a set X is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:

1. $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;
2. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Definition 7.2 (Filter base)**.** A *filter base* is a nonempty family $\mathcal{B} \subseteq \mathcal{P}(X)$, not containing $\emptyset$, such that for all $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$. The generated filter is $\mathcal{F} = \{A \subseteq X : \exists B \in \mathcal{B},\ B \subseteq A\}$.

Definition 7.3 (Ultrafilter)**.** An *ultrafilter* is a maximal filter (under inclusion). Equivalently, $\mathcal{U}$ is an ultrafilter if for every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Theorem 7.4 (Existence of ultrafilters)**.** *Every filter is contained in an ultrafilter (by Zorn's lemma).*

Definition 7.5 (Image filter)**.** If $f\colon X \to Y$ and $\mathcal{F}$ is a filter on X, the image filter is $f(\mathcal{F}) = \{B \subseteq Y : f^{-1}(B) \in \mathcal{F}\}$.

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7.2 Convergence in Topological Spaces

Definition 7.6 (Neighborhood filter)**.** For $x \in X$, the *neighborhood filter* $\mathcal{V}(x)$ is the set of neighborhoods of x: $\mathcal{V}(x) = \{V \subseteq X : \exists U \in \tau,\, x \in U \subseteq V\}$.

Definition 7.7 (Filter convergence)**.** A filter $\mathcal{F}$ on a topological space X *converges* to x (written $\mathcal{F} \to x$) if $\mathcal{V}(x) \subseteq \mathcal{F}$.

Theorem 7.8 (Characterization of topology via filters)**.** *Let X be a topological space. Then:*

1. *U is open iff for every $x \in U$ and every filter $\mathcal{F} \to x$, $U \in \mathcal{F}$.*
2. *$f\colon X \to Y$ is continuous iff for every filter $\mathcal{F} \to x$, $f(\mathcal{F}) \to f(x)$.*
3. *X is compact iff every ultrafilter converges.*
4. *X is Hausdorff iff every filter has at most one limit.*

Proof. (3) ($\Rightarrow$) Let $\mathcal{U}$ be an ultrafilter. For each $x \in X$, if $\mathcal{U} \not\to x$, there exists $U_x \in \mathcal{V}(x)$ with $U_x \notin \mathcal{U}$, so $X \setminus U_x \in \mathcal{U}$. If no ultrafilter converges, the U_x cover X; extract a finite subcover $U_{x_1}, \ldots, U_{x_n}$; then $\bigcap_{i=1}^{n}(X \setminus U_{x_i}) = \emptyset \in \mathcal{U}$, contradiction.

($\Leftarrow$) Let (U_i) be an open cover with no finite subcover. The sets $X \setminus U_i$ have the finite intersection property, so they generate a filter contained in an ultrafilter $\mathcal{U}$. By hypothesis, $\mathcal{U} \to x$ for some $x \in U_j$, giving $U_j \in \mathcal{U}$ and $X \setminus U_j \in \mathcal{U}$, contradiction. □

7.3 Convergence Spaces

Definition 7.9 (Convergence space)**.** A *convergence space* is a pair $(X, \to)$ where X is a set and $\to$ is a relation between filters on X and points of X (called a *convergence structure*) satisfying:

1. For every $x \in X$, the principal filter $\dot{x} = \{A \subseteq X : x \in A\}$ converges to x: $\dot{x} \to x$.
2. If $\mathcal{F} \to x$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} \to x$.

Definition 7.10 (Pretopological convergence space)**.** A convergence space is *pretopological* if additionally:

3. If $\mathcal{F} \to x$ and $\mathcal{G} \to x$, then $\mathcal{F} \cap \mathcal{G} \to x$.

Definition 7.11 (Topological convergence space)**.** A convergence space is *topological* if additionally the neighborhood filter $\mathcal{V}(x) = \bigcap\{\mathcal{F} : \mathcal{F} \to x\}$ converges to x.

Theorem 7.12 (Hierarchy)**.** *We have strict inclusions of categories:*

$$\mathbf{Top} \subsetneq \mathbf{PrTop} \subsetneq \mathbf{Conv}$$

where **Conv** *is the category of convergence spaces and continuous maps (preserving convergence), and* **PrTop** *is the pretopological subcategory.*

7.4 Cartesian Closedness

Theorem 7.13 (**Conv** is cartesian closed)**.** *The category* **Conv** *is cartesian closed: for all convergence spaces X and Y, the space $[X, Y]$ of continuous maps with the continuous convergence structure is a convergence space, and:*

$$\mathrm{Hom}_{\mathbf{Conv}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{Conv}}(X, [Y, Z]).$$

Definition 7.14 (Continuous convergence)**.** On $[X, Y]$, a filter Φ *converges continuously* to f if for every $x \in X$ and every filter $\mathcal{F} \to x$ in X, the evaluation filter $\Phi[\mathcal{F}]$ (generated by $\{g(y) : g \in \phi, y \in F\}$ for $\phi \in \Phi$, $F \in \mathcal{F}$) converges to $f(x)$ in Y.

Remark 7.15. The fact that **Conv** is cartesian closed while **Top** is not is one of the main motivations for studying convergence spaces. In denotational semantics, this allows constructing function spaces without leaving the category.

7.5 Nets and Convergence

Definition 7.16 (Net)**.** A *net* in X is a function $\varphi\colon I \to X$ where $(I, \leq)$ is a directed set. We write $(x_i)_{i\in I}$.

Definition 7.17 (Net convergence)**.** A net $(x_i)_{i\in I}$ converges to x in a topological space if for every neighborhood V of x, there exists $i_0 \in I$ such that for all $i \geq i_0$, $x_i \in V$.

Theorem 7.18 (Filter-net equivalence)**.** *In a topological space, a set F is closed iff every limit of a net in F lies in F. Similarly, f is continuous iff it preserves net limits.*

Warning

In non-Hausdorff spaces, a net (or filter) may converge to multiple points simultaneously. The set of limits of a filter $\mathcal{F}$ in a T_0-space is a downward-closed set for the specialization order.

7.6 Scott Filters and Convergence

Definition 7.19 (Scott filter)**.** Let D be a dcpo. A filter $\mathcal{F}$ on D is a *Scott filter* if it contains a directed set S such that $\bigsqcup S$ exists and $\mathcal{F}$ converges to $\bigsqcup S$ in the Scott topology.

Proposition 7.20. In a dcpo D with the Scott topology, a filter $\mathcal{F}$ converges to x if and only if every Scott-open set containing x belongs to $\mathcal{F}$.

7.7 Choquet Spaces

Definition 7.21 (Choquet space)**.** A topological space X is a *Choquet space* (or has a favorable topology) if Player II has a winning strategy in the Choquet game: the players alternately choose nonempty opens $U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \ldots$, and Player II wins if $\bigcap_n V_n \neq \emptyset$.

Theorem 7.22. *Every complete metric space is a Choquet space. Every continuous dcpo with the Scott topology is a Choquet space.*

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7.8 Convergence and Domain Theory

Proposition 7.23 (Scott convergence and approximation)**.** In a continuous domain D, a filter $\mathcal{F}$ converges to x in the Scott topology iff for every $b \ll x$, $\uparrow b \in \mathcal{F}$. Convergence in a continuous domain is thus determined by the way-below relation $\ll$.

Theorem 7.24 (Convergence spaces and domains)**.** *The category of continuous domains with Scott-continuous functions is equivalent to a full subcategory of* **Conv** *consisting of convergence spaces satisfying additional axioms (directed convergence).*

7.9 Exercises

Exercise 7.1. Show that a topological space is T_0 iff in the associated convergence structure, the set of limits of every filter is a downward-closed set (for the specialization order) and two distinct points do not have the same convergent filters.

Exercise 7.2. Construct a convergence space that is not pretopological.

Exercise 7.3. Show that continuous convergence on $[X, Y]$ coincides with compact-open convergence when X is locally compact Hausdorff.

Exercise 7.4. Let D be a dcpo. Show that the filter generated by a directed set S converges to $\bigsqcup S$ in the Scott topology.

Exercise 7.5. Show that in a T_0-space, if a filter $\mathcal{F}$ converges to x and to y with $x \leq y$, then $\mathcal{F}$ also converges to every z with $x \leq z \leq y$.

Exercise 7.6 (Category **Conv** and limits)**.** Show that **Conv** is complete and cocomplete. Describe initial and final structures in terms of convergence.

Chapter 8

Compactness in Non-Hausdorff Spaces

Compactness is one of the most reassuring notions in classical topology: in a Hausdorff space, a compact set is closed, bounded (in a metric), and possesses all the finiteness properties one could wish for. But what happens when one abandons the Hausdorff axiom? Compact sets cease to be closed, limit points are no longer unique, and geometric intuition falters. This is no accident: in domain theory, the spaces that model computations are typically T_0 but not T_1. The notion of *saturated compact set* — a compact set equal to the intersection of its open neighbourhoods — then replaces that of closed compact set, and Smyth compactification plays the role that Alexandroff played in the Hausdorff setting. This chapter develops this refined theory.

Intuition

In Hausdorff spaces, compactness is well understood: compact sets are closed, intersections of compact sets are compact, and the Alexandroff one-point compactification yields a Hausdorff space if and only if the original space is locally compact and Hausdorff. In non-Hausdorff spaces, compactness presents fundamental subtleties. Compact sets need not be closed, and one must resort to the notion of *saturated compact set* to recover reasonable behavior. This chapter develops the theory of compactness adapted to T_0 spaces, with applications to domain theory.

8.1 Saturated Compact Sets

Definition 8.1 (Saturated set)**.** Let (X, τ) be a topological space. A subset $A \subseteq X$ is *saturated* if it equals the intersection of all open sets containing it:

$$A = \bigcap \{U \in \tau : A \subseteq U\}.$$

Equivalently, A is saturated if and only if A is an upper set for the specialization order: if $x \in A$ and $x \leq y$ (i.e., $x \in \overline{\{y\}}$), then $y \in A$.

Remark 8.2. In a T_1 space, every subset is saturated since the specialization order is equality. The notion of saturation is thus only relevant in non-T_1 spaces.

Proposition 8.3 (Saturation of a set)**.** For any subset A of a topological space X, the *saturation* of A is

$$\uparrow A = \{y \in X : \exists x \in A,\ x \leq y\}$$

where $\leq$ denotes the specialization order. It is the smallest saturated set containing A.

Definition 8.4 (Saturated compact set)**.** A subset Q of a topological space is a *saturated compact set* if it is both compact and saturated.

> **Attention**
>
> In a non-Hausdorff space, a compact set need not be closed. For example, in the Sierpiński space $\mathbb{S} = \{0, 1\}$ with $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$, the singleton $\{1\}$ is compact (finite) and open, but not closed. However, it is saturated.

Theorem 8.5 (Saturated compact sets and intersections)**.** *Let X be a topological space. The intersection of a saturated compact set Q and a closed compact set K is compact. More generally, the finite intersection of saturated compact sets is a saturated compact set.*

Proof. Let $(U_i)_{i\in I}$ be an open cover of $Q_1 \cap Q_2$, where Q_1 and Q_2 are two saturated compact sets. For every $x \in Q_1 \setminus Q_2$, since Q_2 is saturated, there exists an open set V_x containing Q_2 but not x. Then $(U_i)_{i\in I} \cup (X \setminus Q_2)$ covers Q_1, and by compactness of Q_1 we extract a finite subcover. The open sets from this subcover that are among the U_i cover $Q_1 \cap Q_2$. □

Example 8.6. Consider $\mathbb{N}$ equipped with the Alexandrov topology associated with the usual order. The open sets are the upper sets $\{n, n+1, n+2, \ldots\}$ and $\emptyset$. Every finite set $\{n\}$ is compact but not saturated (since $\uparrow\{n\} = \{n, n+1, \ldots\}$).

8.2 Alexandroff One-Point Compactification for Non-Hausdorff Spaces

Definition 8.7 (Alexandroff compactification)**.** Let (X, τ) be a topological space. The *one-point compactification* (or Alexandroff compactification) is the space $X^* = X \cup \{\infty\}$ equipped with the topology

$$\tau^* = \tau \cup \{X^* \setminus K : K \subseteq X \text{ compact and saturated}\}.$$

Theorem 8.8 (Properties of the Alexandroff compactification)**.** *Let X be a topological space.*

1. *X^* is compact.*
2. *The inclusion $X \hookrightarrow X^*$ is a topological (open) embedding.*
3. *If X is T_0, then X^* is T_0.*
4. *If X is sober, then X^* is sober if and only if X is locally compact.*

Proof. (1) Let $(U_i)_{i\in I}$ be an open cover of X^*. One of the U_i contains ∞, so it is of the form $X^* \setminus K$ for some saturated compact set K. The remaining open sets cover K, and by compactness of K we extract a finite subcover.

(2) The open sets of X in X^* are exactly the elements of τ, since $U \cap X = U$ for $U \in \tau$.

(3) If X is T_0 and $x \in X$, then $\{x\}$ and $\{\infty\}$ are separated by the open set X of X^*.

(4) Omitted (see the next section on locally compact sober spaces). □

Remark 8.9. Unlike the Hausdorff case, the Alexandroff compactification of a non-Hausdorff space is never Hausdorff (unless the original space is already Hausdorff and locally compact). Nevertheless, it preserves T_0 separation, which is the natural level of separation in domain theory.

8.3 Locally Compact Sober Spaces

Definition 8.10 (Locally compact space)**.** A topological space X is *locally compact* if for every $x \in X$ and every open set U containing x, there exist an open set V and a saturated compact set Q such that $x \in V \subseteq Q \subseteq U$.

Proposition 8.11. In a sober space, local compactness is equivalent to the following condition: the lattice of open sets $\mathcal{O}(X)$ is a continuous lattice.

Definition 8.12 (Way-below relation)**.** In a complete lattice L, we say that a is *way below* b, written $a \ll b$, if for every directed family D such that $b \leq \bigvee D$, there exists $d \in D$ with $a \leq d$.

Theorem 8.13 (Characterization of locally compact sober spaces)**.** *For a topological space X, the following conditions are equivalent:*

1. *X is sober and locally compact.*
2. *The lattice $\mathcal{O}(X)$ is a continuous lattice.*
3. *X is homeomorphic to the spectrum of a continuous lattice.*

Proof. (1) $\Rightarrow$ (2): If X is locally compact, for every open set U and every $x \in U$, there exist an open V and a saturated compact Q with $x \in V \subseteq Q \subseteq U$. One verifies that $V \ll U$ in $\mathcal{O}(X)$ and therefore $U = \bigvee\{V : V \ll U\}$.

(2) $\Rightarrow$ (3): By the representation theorem for continuous lattices, the spectrum of $\mathcal{O}(X)$ is sober and locally compact, and $\mathcal{O}(X)$ is isomorphic to its lattice of open sets.

(3) $\Rightarrow$ (1): The spectrum of a continuous lattice is sober by definition and locally compact because the way-below relation provides the required saturated compact neighborhoods. □

8.4 Stably Compact Spaces

Definition 8.14 (Stably compact space)**.** A topological space X is *stably compact* if it satisfies:

1. X is T_0;
2. X is compact;
3. X is locally compact;
4. X is *coherent*: the intersection of any two saturated compact sets is a saturated compact set;
5. X is sober.

Intuition

Stably compact spaces are the non-Hausdorff analogues of compact Hausdorff spaces. They arise naturally as spaces of models in denotational semantics (Scott domains) and in logic (generalized Stone spaces).

Example 8.15 (Spectrum of a distributive lattice)**.** If L is a bounded distributive lattice, its spectrum (the set of prime ideals equipped with the Hull-Kernel topology) is a stably compact space.

Theorem 8.16 (Characterization via frames)**.** *A T_0 space is stably compact if and only if its lattice of open sets $\mathcal{O}(X)$ is a continuous lattice and X is compact and coherent.*

Proposition 8.17 (Stability properties)**.** The class of stably compact spaces is closed under:

1. finite products;
2. closed saturated subspaces;
3. passage to the patch topology.

8.5 The Patch Topology

Definition 8.18 (Patch (or Lawson) topology)**.** Let X be a topological space. The *patch topology* on X is the topology generated by the open sets of X and the complements of saturated compact sets of X. We write X^{patch} for the resulting space.

Theorem 8.19 (Patch of a stably compact space)**.** *If X is stably compact, then:*

1. *X^{patch} is compact Hausdorff.*
2. *The open sets of X are exactly the open sets of X^{patch} that are upper sets for the specialization order of X.*
3. *The specialization order of X is a closed partial order in $(X^{\text{patch}})^2$.*

Proof. (1) The patch topology is finer than the original topology, hence separated (since saturated compact sets allow separation of points). Compactness follows from Alexander's subbasis theorem applied to the subbasis consisting of open sets and complements of saturated compact sets, using coherence of X.

(2) and (3) follow from the construction of the patch topology and the properties of the specialization order. □

Example 8.20. The interval $[0, 1]$ equipped with the Scott topology (open sets: $\emptyset$ and the intervals $(a, 1]$ for $a \in [0, 1)$, plus $[0, 1]$) is stably compact. Its patch topology is the usual topology on $[0, 1]$.

8.6 De Groot Duality

Definition 8.21 (Co-compact (de Groot dual) topology)**.** Let X be a topological space. The *de Groot dual topology* τ^d is the topology generated by the complements of saturated compact sets of (X, τ):

$$\tau^d = \langle \{X \setminus Q : Q \text{ saturated compact in } (X, \tau)\} \rangle.$$

We write $X^d = (X, \tau^d)$ for the dual space.

Theorem 8.22 (De Groot duality for stably compact spaces)**.** *If X is stably compact, then:*

1. *X^d is stably compact.*
2. *$(X^d)^d = X$ (the duality is involutive).*
3. *The specialization order of X^d is the reverse of the specialization order of X.*
4. *The patch topologies of X and X^d coincide.*

Proof. (1) The saturated compact sets of X^d are exactly the closed sets of X that are lower sets for the specialization order. One verifies the five axioms one by one.

(2) Follows from the fact that the patch topology determines both τ and τ^d.

(3) The specialization order of τ^d is determined by: $x \leq^d y$ iff for every saturated compact set Q of τ, $y \in Q \Rightarrow x \in Q$, which corresponds to $y \leq x$ in the specialization order of τ.

(4) The patch topology of X^d is generated by τ^d and complements of saturated compact sets of X^d, which are open sets of X. Hence it coincides with the patch topology of X. □

Summary: de Groot duality

	Space X	**Dual X^d**
Open sets	τ	Generated by $X \setminus Q$, Q saturated compact
Spec. order	$\leq$	$\geq$
Patch topology	τ^{patch}	$(\tau^d)^{\text{patch}} = \tau^{\text{patch}}$

8.7 Applications in Domain Theory

Theorem 8.23 (Domains as stably compact spaces)**.** *Let D be a continuous bounded-complete domain (i.e., a continuous dcpo in which every bounded subset has a supremum). Then D equipped with the Scott topology is stably compact.*

Corollary 8.24. *The interval domain $\mathbf{IR} = \{[a, b] : a \leq b,\ a, b \in \mathbb{R}\}$ ordered by reverse inclusion is stably compact for the Scott topology.*

Exercise 8.1. Show that in a T_0 space, a compact set is saturated if and only if it is an intersection of open sets.

Exercise 8.2. Let X be a locally compact sober space. Show that for every open set U of X, U is the union of the interiors of saturated compact sets contained in U.

Exercise 8.3. Let P be a finite ordered set equipped with the Alexandrov topology. Determine the saturated compact sets of P. Deduce the patch topology and the de Groot dual topology.

Exercise 8.4. Show that if X is stably compact and $f\colon X \to Y$ is continuous, surjective, and open, then Y is stably compact.

Exercise 8.5 (Generalized Stone duality)**.** Let L be a bounded distributive lattice and $X = \mathrm{Spec}(L)$ its spectrum. Show that X is stably compact and that L is isomorphic to the lattice of open saturated compact sets of X.

Exercise 8.6. Let X be a stably compact space and Q_1, Q_2 two saturated compact sets. Show that $Q_1 \cup Q_2$ and $Q_1 \cap Q_2$ are saturated compact sets, and that the set of saturated compact sets of X forms a distributive lattice.

Chapter 9

Asymmetric Topology in Theoretical CS

In 1969, Dana Scott, confronted with the problem of giving mathematical meaning to Church's lambda calculus, sought a space D isomorphic to its own space of continuous functions $D \to D$. Classical Hausdorff topological spaces would not do — such an isomorphism is impossible there for cardinality reasons. Scott discovered that the solution lies in *continuous lattices* equipped with their Scott topology, a resolutely non-Hausdorff topology where open sets represent the "observable" properties of a computation. This discovery founded *denotational semantics*, the bridge between program logic and continuous mathematics. This chapter shows how asymmetric topology — quasi-metrics, T_0 spaces, the Scott topology — constitutes the natural language of theoretical computer science.

Intuition

Domain theory, born from the work of Dana Scott in the 1960s–70s, provides a mathematical model for the denotational semantics of programming languages. The central idea is that elements of a domain represent partial "levels of information," and the order reflects information refinement. The Scott topology, which is non-Hausdorff, captures exactly the observability properties in a program. This chapter presents these applications of non-Hausdorff topology to theoretical computer science.

9.1 Scott Domains and Denotational Semantics

Definition 9.1 (Directed set and dcpo)**.** A subset D of a partially ordered set $(P, \leq)$ is *directed* if it is nonempty and for all $x, y \in D$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$. A *dcpo* (directed-complete partial order) is a partially ordered set in which every directed subset has a supremum.

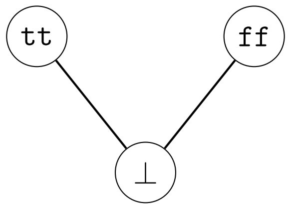
Definition 9.2 (Scott topology)**.** Let $(P, \leq)$ be a dcpo. A subset $U \subseteq P$ is *Scott-open* if:

1. U is an upper set: $x \in U$ and $x \leq y$ imply $y \in U$;
2. U is inaccessible by directed suprema: if D is directed and $\bigvee D \in U$, then $D \cap U \neq \emptyset$.

Proposition 9.3. The Scott-open sets on a dcpo P form a topology. This topology is T_0 and its specialization order coincides with the order $\leq$ of P.

Proof. The sets P and $\emptyset$ are Scott-open. The finite intersection of Scott-open sets is Scott-open: the upper set condition is preserved by intersection, and if $\bigvee D \in U_1 \cap U_2$, then $D \cap U_1$ is cofinal in D and $D \cap U_1 \cap U_2$ is nonempty. The arbitrary union of Scott-open sets is clearly Scott-open.

For the specialization order: $x \leq y$ in the order of P if and only if every open set containing x contains y, which is exactly the definition of the specialization order. □



Flat domain $\mathbb{B}_\bot = \{\bot, \mathtt{tt}, \mathtt{ff}\}$

Definition 9.4 (Scott-continuous function)**.** Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be dcpos. A function $f\colon P \to Q$ is *Scott-continuous* if it is monotone and preserves directed suprema: for every directed subset $D \subseteq P$,

$$f\Big(\bigvee D\Big) = \bigvee f(D).$$

Proposition 9.5. A function between dcpos is Scott-continuous if and only if it is continuous for the Scott topologies.

Theorem 9.6 (Category of dcpos)**.** *The category* **DCPO** *whose objects are dcpos and whose morphisms are Scott-continuous functions is Cartesian closed. The function space* $[P \to Q]$ *is the dcpo of Scott-continuous functions from* P *to* Q*, ordered pointwise.*

Proof sketch. The key step is showing that $[P \to Q]$ with the pointwise order ($f \leq g \iff \forall x,\ f(x) \leq g(x)$) is a dcpo. Given a directed family $(f_i)_{i \in I}$ in $[P \to Q]$, one defines $f = \bigvee f_i$ pointwise by $f(x) = \bigvee_i f_i(x)$; Scott-continuity of f follows from the interchange of directed suprema. For Cartesian closedness, one must show that the evaluation map $\mathrm{ev}\colon [P \to Q] \times P \to Q$ is Scott-continuous: if U is Scott-open in Q, then $\mathrm{ev}^{-1}(U)$ is Scott-open in $[P \to Q] \times P$. This holds because the preimage of a Scott-open set under evaluation is a union of rectangles of the form $\{f : f(x) \in U\} \times V$, each of which is Scott-open. See Abramsky–Jung, *Domain Theory*, Handbook of Logic in Computer Science, vol. 3, 1994. □

9.2 Domains for the Lambda Calculus

Definition 9.7 (Continuous domain)**.** A dcpo D is a *continuous domain* if for every $x \in D$,

$$\Downarrow\! x = \{a \in D : a \ll x\}$$

is directed and $x = \bigvee \Downarrow\! x$, where $a \ll x$ means that a is *way below* x (compact relative to x).

Theorem 9.8 (Scott's model of the untyped lambda calculus)**.** *There exists a continuous domain* D_∞ *such that*

$$D_\infty \cong [D_\infty \to D_\infty]$$

in the category of continuous domains with Scott-continuous functions. Such a domain provides a model for the denotational semantics of the untyped lambda calculus.

Proof sketch. One constructs D_∞ as a projective limit of a sequence $D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots$ where $D_0 = \{\bot\}$ and $D_{n+1} = [D_n \to D_n]$, with the arrows being retractions (projection/section pairs). The projective limit in **DCPO** is a dcpo whose elements are coherent sequences $(x_0, x_1, \ldots)$ with $\pi_n(x_{n+1}) = x_n$. The isomorphism $D_\infty \cong [D_\infty \to D_\infty]$ follows from the continuity of the functor $[\cdot \to \cdot]$ on projective limits. ☐

Intuition

Scott's model solves a fundamental problem: in naive set theory, there is no set D such that $D \cong D^D$ (by a cardinality argument, unless D is a singleton). By passing to dcpos and continuous functions, one escapes this constraint because $[D \to D]$ contains only the continuous functions, not all functions.

Definition 9.9 (Denotational interpretation)**.** In a Scott model D_∞, each closed term M of the lambda calculus receives an interpretation $[\![M]\!] \in D_\infty$:

- $[\![x]\!]_\rho = \rho(x)$ where ρ is an environment;
- $[\![\lambda x.\, M]\!]_\rho = d \mapsto [\![M]\!]_{\rho[x \mapsto d]}$;
- $[\![MN]\!]_\rho = [\![M]\!]_\rho([\![N]\!]_\rho)$.

9.3 Continuous Lattices and Programming Language Theory

Definition 9.10 (Continuous lattice)**.** A complete lattice L is *continuous* if it is a continuous domain, i.e., if for every $x \in L$, $x = \bigvee\{a \in L : a \ll x\}$.

Theorem 9.11 (Lawson's characterization theorem)**.** *A complete lattice L is continuous if and only if the function $x \mapsto \Downarrow x$ is an isomorphism from L onto a sublattice of the lattice of ideals of the basis of L.*

Proposition 9.12 (Continuous lattices and types)**.** In denotational semantics:

- Base types (integers, Booleans) are interpreted by flat algebraic domains (domains with a bottom element $\bot$ and incomparable maximal elements).
- The function type $A \to B$ is interpreted by $[D_A \to D_B]$.
- Recursive types $T = F(T)$ are interpreted by solutions of domain equations, obtained as projective limits.

9.4 Powerdomains

Definition 9.13 (Powerdomains)**.** Let D be a domain. The *powerdomains* extend D to model nondeterminism. There are three variants, corresponding to three different observations:

Definition 9.14 (Hoare (lower) powerdomain)**.** The *Hoare powerdomain* $\mathcal{H}(D)$ is the dcpo of nonempty closed sets of D (in the Scott topology), ordered by:

$$A \leq_H B \iff A \subseteq \downarrow B.$$

Intuitively, $A \leq_H B$ means that every possible result of A is approximated by a result of B. This captures "partial correctness" (safety).

Definition 9.15 (Smyth (upper) powerdomain)**.** The *Smyth powerdomain* $\mathcal{S}(D)$ is the set of nonempty saturated compact sets of D, ordered by reverse inclusion:

$$Q_1 \leq_S Q_2 \iff Q_2 \subseteq Q_1.$$

This captures "termination" (liveness): $Q_1 \leq_S Q_2$ means that the set of possible behaviors shrinks.

Definition 9.16 (Plotkin (convex) powerdomain)**.** The *Plotkin powerdomain* $\mathcal{P}\ell(D)$ combines both orderings:

$$A \leq_P B \iff A \leq_H B \text{ and } A \leq_S B.$$

This is the Egli-Milner order. It captures both partial correctness and termination.

Theorem 9.17 (Properties of powerdomains)**.** *Let D be a continuous domain.*

1. *$\mathcal{H}(D)$ is a dcpo (under mild hypotheses).*
2. *$\mathcal{S}(D)$ is a dcpo if D is coherent (the intersection of saturated compact sets is a saturated compact set).*
3. *If D is a Scott domain (continuous domain with a countable basis), all three powerdomains are continuous domains.*

Example 9.18. Consider the flat Boolean domain $\mathbb{B}_\perp = \{\perp, \texttt{true}, \texttt{false}\}$. Then:

- $\mathcal{H}(\mathbb{B}_\perp)$ contains the closed sets: $\{\perp\}$, $\{\perp, \texttt{true}\}$, $\{\perp, \texttt{false}\}$, $\{\perp, \texttt{true}, \texttt{false}\}$.
- $\mathcal{S}(\mathbb{B}_\perp)$ contains the saturated compact sets: $\mathbb{B}_\perp$, $\{\texttt{true}\}$, $\{\texttt{false}\}$, $\{\texttt{true}, \texttt{false}\}$.

9.5 Topological Models of Computation

Definition 9.19 (Effectively presented space)**.** A domain D is *effectively presented* if it has a countable basis $(b_n)_{n\in\mathbb{N}}$ such that the relations $b_n \leq b_m$ and $b_n \ll b_m$ are decidable (semi-decidable for $\ll$).

Theorem 9.20 (Computability–continuity equivalence)**.** *On an effectively presented domain D, the computable functions $D \to D$ (in the sense of Type-2 computability theory) coincide with the effectively given Scott-continuous functions.*

Proposition 9.21 (Name spaces and domains)**.** The Böhm domain $\mathcal{B}$ of Böhm trees of the lambda calculus is a continuous domain such that:

- Finite elements are finite Böhm trees (with $\perp$ at the leaves).

- The order is information refinement: $t \leq s$ if s is obtained by replacing some $\perp$'s in t with subtrees.
- The Scott topology captures exactly observation by head contexts.

Theorem 9.22 (Adequacy and full abstraction)**.** *For the pure lambda calculus, the denotational semantics in a continuous Scott domain is:*

1. Adequate*: if $[\![M]\!] \leq [\![N]\!]$, then M is approximated by N in the operational sense.*
2. Fully abstract *for filter models: the denotational order coincides with the operational preorder.*

Topology–computer science correspondence

Topological concept	Computer science concept
Scott-open set	Semi-decidable property
Scott-closed set	Limit-verifiable property
Saturated compact set	Co-semi-decidable property
Continuous function	Program (computable function)
T_0 topology	Observational distinguishability
Bottom element $\perp$	Divergence / nontermination
Directed supremum	Limit of finite approximations

9.6 Exercises

Exercise 9.1. Show that the Scott topology on $(\mathbb{N}, \leq)$ is the cofinite topology on $\mathbb{N}$.

Exercise 9.2. Let $D = \mathbb{N} \cup \{\infty\}$ equipped with the usual order. Determine the Scott topology, verify that it is sober and locally compact, and compute the saturated compact sets.

Exercise 9.3. Show that the space $[D \to D]$ of Scott-continuous functions from a dcpo D to itself, equipped with the pointwise order, is a dcpo. Deduce that the category **DCPO** is Cartesian closed.

Exercise 9.4. Let D be the flat natural numbers domain $\mathbb{N}_\perp$. Describe explicitly the elements of the Hoare powerdomain $\mathcal{H}(D)$ and the Smyth powerdomain $\mathcal{S}(D)$.

Exercise 9.5. Let $f : D \to D$ be a Scott-continuous function on a pointed dcpo D (with $\perp$). Using Kleene's theorem (Chapter 10), show that the semantics of a `while` loop can be defined as the least fixed point of a continuous operator.

Exercise 9.6 (Powerdomain and monad)**.** Show that the Hoare powerdomain $\mathcal{H}$ defines a functor $\mathcal{H} : \mathbf{DCPO} \to \mathbf{DCPO}$. Verify that the unit $\eta_D : D \to \mathcal{H}(D)$ defined by $\eta_D(d) = {\downarrow}d$ and the multiplication $\mu_D : \mathcal{H}(\mathcal{H}(D)) \to \mathcal{H}(D)$ defined by $\mu_D(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$ make $\mathcal{H}$ into a monad.

Chapter 10

Fixed Point Theory in Advanced Topology

Intuition

Fixed point theorems play a central role in theoretical computer science and domain theory. Unlike the classical theorems of Banach or Brouwer, which rely on metric or classical topological structures, the theorems presented here exploit order and directed completeness. Tarski's theorem guarantees the existence of fixed points in complete lattices, Kleene's theorem constructs them effectively by iteration, and Pataraia's theorem extends these results to dcpos without any countability hypothesis.

10.1 Tarski's Theorem for Complete Lattices

Definition 10.1 (Complete lattice)**.** A partially ordered set $(L, \leq)$ is a *complete lattice* if every subset $S \subseteq L$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$. In particular, L has a least element $\bot = \bigvee \emptyset$ and a greatest element $\top = \bigwedge \emptyset$.

Theorem 10.2 (Knaster–Tarski)**.** *Let $(L, \leq)$ be a complete lattice and $f\colon L \to L$ a monotone function. Then:*

1. *f has a least fixed point* $\mathrm{lfp}(f) = \bigwedge\{x \in L : f(x) \leq x\}$.
2. *f has a greatest fixed point* $\mathrm{gfp}(f) = \bigvee\{x \in L : x \leq f(x)\}$.
3. *The set of fixed points* $\mathrm{Fix}(f)$ *is a complete lattice.*

Proof. (1) Let $P = \{x \in L : f(x) \leq x\}$ and $a = \bigwedge P$. For every $x \in P$, $a \leq x$, so $f(a) \leq f(x) \leq x$ by monotonicity. Thus $f(a)$ is a lower bound of P, hence $f(a) \leq a$. This shows $a \in P$.

By monotonicity, $f(f(a)) \leq f(a)$, so $f(a) \in P$, hence $a \leq f(a)$. Combined with $f(a) \leq a$, we get $f(a) = a$.

If b is another fixed point, then $b \in P$ so $a \leq b$: a is indeed the least fixed point.

(2) Dual of (1).

(3) Let $S \subseteq \mathrm{Fix}(f)$. The set $\{x \in L : x \geq s \ \forall s \in S,\ f(x) \leq x\}$ is nonempty (it contains $\top$). Its infimum is a fixed point of f that is the supremum of S in $\mathrm{Fix}(f)$. □

Remark 10.3. The Knaster–Tarski theorem requires no continuity hypothesis: monotonicity alone suffices. However, it does not provide a constructive method for computing the fixed point.

Example 10.4 (Application to language equations)**.** Let Σ be an alphabet and $\mathcal{P}(\Sigma^*)$ the lattice of languages over Σ. A system of language equations $X_i = f_i(X_1, \dots, X_n)$ where the f_i are monotone has a least solution, by Tarski's theorem applied to the lattice $\mathcal{P}(\Sigma^*)^n$.

10.2 Kleene's Fixed Point Theorem

Definition 10.5 (Pointed dcpo)**.** A *pointed dcpo* is a dcpo $(D, \leq)$ that has a least element $\bot$.

Theorem 10.6 (Kleene)**.** *Let $(D, \leq)$ be a pointed dcpo and $f\colon D \to D$ a Scott-continuous function. Then f has a least fixed point, given by:*

$$\mathrm{lfp}(f) = \bigvee_{n \geq 0} f^n(\bot).$$

Proof. First we show that the sequence $(f^n(\bot))_{n\geq 0}$ is increasing. We have $\bot \leq f(\bot)$ since $\bot$ is the least element. By induction, if $f^n(\bot) \leq f^{n+1}(\bot)$, then by monotonicity, $f^{n+1}(\bot) \leq f^{n+2}(\bot)$.

The set $\{f^n(\bot) : n \geq 0\}$ is directed (it is a chain), so its supremum $a = \bigvee_n f^n(\bot)$ exists in D.

By Scott-continuity:

$$f(a) = f\left(\bigvee_{n\geq 0} f^n(\bot)\right) = \bigvee_{n \geq 0} f^{n+1}(\bot) = a.$$

Hence a is a fixed point.

If b is another fixed point, then $\bot \leq b$ and by induction $f^n(\bot) \leq b$ for all n, whence $a = \bigvee_n f^n(\bot) \leq b$. □

Attention

Kleene's theorem requires Scott-continuity (not merely monotonicity). A monotone function on a pointed dcpo may not have a least fixed point obtained by iteration from $\bot$.

Corollary 10.7 (Transfinite iteration)**.** *If $f\colon L \to L$ is a monotone function on a complete lattice, the least fixed point is reached by transfinite iteration:*

$$f^\alpha(\bot) = \begin{cases} f(f^{\alpha-1}(\bot)) & \text{if } \alpha \text{ is a successor ordinal,} \\ \bigvee_{\beta<\alpha} f^\beta(\bot) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

There exists an ordinal α_0 such that $f^{\alpha_0}(\bot) = \mathrm{lfp}(f)$.

Example 10.8 (Semantics of loops)**.** The semantics of a `while b do c` loop is the least fixed point of the operator $\Phi\colon [S \to S_\perp] \to [S \to S_\perp]$ defined by:

$$\Phi(g)(s) = \begin{cases} g([\![c]\!](s)) & \text{if } [\![b]\!](s) = \texttt{true}, \\ s & \text{if } [\![b]\!](s) = \texttt{false}, \\ \perp & \text{otherwise.} \end{cases}$$

By Kleene's theorem, $\mathrm{lfp}(\Phi) = \bigvee_n \Phi^n(\perp_{S \to S_\perp})$.

10.3 Pataraia's Theorem

Theorem 10.9 (Pataraia)**.** *Let $(D, \leq)$ be a pointed dcpo and $f\colon D \to D$ a monotone function (not necessarily continuous) such that $f(x) \geq x$ for all x (*inflationary*). Then f has a fixed point.*

Proof. Define the collection $\mathcal{I}$ of all subsets S of D such that:

1. $\perp \in S$;
2. S is closed under f;
3. S is closed under directed suprema (in D).

Let $C = \bigcap \mathcal{I}$ be the smallest such set. One shows that C is a chain (by verifying that the set of elements comparable to every element of C satisfies the three conditions). The supremum $c = \bigvee C$ exists in D and satisfies $f(c) \leq c$ (since $f(c) \in C$ by closure, whence $f(c) \leq c$). Since f is inflationary, $c \leq f(c)$, hence $f(c) = c$. □

Remark 10.10. Pataraia's theorem generalizes the Bourbaki–Witt theorem (which assumes that every chain has a supremum). The advantage of Pataraia's theorem is that it requires only directed completeness, not chain completeness.

Corollary 10.11. *Every pointed dcpo in which every monotone inflationary function has a fixed point is a complete lattice.*

10.4 Least Fixed Points in Domain Theory

Theorem 10.12 (Existence of the least fixed point)**.** *Let D be a pointed continuous domain and $f\colon D \to D$ a Scott-continuous function. Then:*

1. *The least fixed point* $\mathrm{lfp}(f)$ *exists and is given by Kleene's formula* $\mathrm{lfp}(f) = \bigvee_n f^n(\perp)$.
2. *If moreover D is an effectively presented algebraic domain and f is effectively given, then* $\mathrm{lfp}(f)$ *is computable.*
3. *The operator* $\mathrm{lfp}\colon [D \to D] \to D$ *is itself Scott-continuous.*

Proof. (1) This is Kleene's theorem (Theorem 10.6).

(2) The finite approximations $f^n(\perp)$ are computable, and the compact elements below $\mathrm{lfp}(f)$ are enumerable.

(3) Let $(f_i)_{i\in I}$ be a directed family in $[D \to D]$ and $g = \bigvee_i f_i$. Then:

$$\mathrm{lfp}(g) = \bigvee_n g^n(\bot) = \bigvee_n \left(\bigvee_i f_i\right)^n (\bot) = \bigvee_i \bigvee_n f_i^n(\bot) = \bigvee_i \mathrm{lfp}(f_i).$$

The interchange of suprema is justified by the Scott-continuity of composition and application. □

10.5 Applications to Recursive Definitions

Proposition 10.13 (Recursive function definitions)**.** Let D be a pointed dcpo. A recursive definition of the form

$$f(x) = \Phi(f, x)$$

where $\Phi\colon [D \to D] \times D \to D$ is Scott-continuous, has a least solution $f^* \in [D \to D]$, given by $f^* = \mathrm{lfp}(\lambda g.\, \lambda x.\, \Phi(g, x))$.

Example 10.14 (Factorial)**.** The recursive definition

$$\texttt{fact}(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot \texttt{fact}(n-1) & \text{otherwise} \end{cases}$$

corresponds to $\texttt{fact} = \mathrm{lfp}(\Phi)$ where $\Phi(g)(n) = \text{if } n = 0 \text{ then } 1 \text{ else } n \cdot g(n-1)$. The operator Φ is Scott-continuous on $[\mathbb{N}_\bot \to \mathbb{N}_\bot]$.

Theorem 10.15 (Recursive domain equations)**.** *If F is a locally continuous functor on the category of continuous domains with retraction pairs, then the domain equation*

$$D \cong F(D)$$

has a solution, obtained as a projective limit (and also inductive colimit) of the sequence $D_0 \leftarrow D_1 \leftarrow \cdots$ where $D_{n+1} = F(D_n)$.

Summary of fixed point theorems

Theorem	**Structure**	**Hypothesis on f**	**Result**
Knaster–Tarski	Complete lattice	Monotone	$\mathrm{Fix}(f)$ complete lattice
Kleene	Pointed dcpo	Scott-continuous	$\mathrm{lfp}(f) = \bigvee f^n(\bot)$
Pataraia	Pointed dcpo	Monotone inflat.	Existence of fixed point
Bourbaki–Witt	Chain-complete	Inflationary	Existence of fixed point

10.6 Exercises

Exercise 10.1. Let $L = \mathcal{P}(\mathbb{N})$ be the lattice of subsets of $\mathbb{N}$. Determine the least and greatest fixed points of $f(A) = \mathbb{N} \setminus A$.

Exercise 10.2. Show that in Kleene's theorem, the sequence $(f^n(\bot))$ can stabilize after finitely many steps if and only if the least fixed point is a compact element of the dcpo.

Exercise 10.3. Give an example of a pointed dcpo D and a monotone function $f\colon D \to D$ such that the sequence $(f^n(\bot))$ does not converge to the least fixed point of f (i.e., such that $\bigvee_n f^n(\bot) < \mathrm{lfp}(f)$).

Exercise 10.4. Let D be a pointed dcpo and $f, g\colon D \to D$ two Scott-continuous functions with $f \leq g$ pointwise. Show that $\mathrm{lfp}(f) \leq \mathrm{lfp}(g)$.

Exercise 10.5 (Bekic's theorem)**.** Let D, E be pointed dcpos and $f\colon D \times E \to D$, $g\colon D \times E \to E$ Scott-continuous functions. Show that the least fixed point of $(f, g)\colon D \times E \to D \times E$ is given by $(\mathrm{lfp}(\hat{f}), \mathrm{lfp}(\hat{g}))$ where $\hat{f}$ and $\hat{g}$ are defined by mutual substitution.

Exercise 10.6. Let $\Sigma = \{a, b\}$ and $\mathcal{P}(\Sigma^*)$ the lattice of languages. The operator $\Phi(X) = \{\varepsilon\} \cup \{a\} \cdot X \cdot \{b\}$ is monotone. Compute $\mathrm{lfp}(\Phi)$ using Kleene's theorem.

Exercise 10.7. Show that Pataraia's theorem implies Zorn's lemma: every inductive partially ordered set (every chain is bounded) has a maximal element. *Hint: consider a pointed dcpo and a choice function.*

Bibliography

[1] Künzi, H.-P.A., *Nonsymmetric Distances and Their Associated Topologies*, 2001.

[2] Kelly, J.C., *Bitopological Spaces*, Proc. London Math. Soc., 1963.

[3] Gierz, G. et al., *Continuous Lattices and Domains*, Cambridge, 2003.

[4] Goubault-Larrecq, J., *Non-Hausdorff Topology and Domain Theory*, Cambridge, 2013.

[5] Johnstone, P.T., *Stone Spaces*, Cambridge, 1982.

[6] Smyth, M.B., *Topology*, in Handbook of Logic in CS, Oxford, 1992.