

General Topology

Lecture Notes

L2–L3 — 2025–2026

Yaë Ulrich Gaba

*“Geometry is the art of reasoning correctly
from badly drawn figures.” — Henri Poincaré*

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Contents

Preface	vii
Notation	ix
1 Topological Spaces — Definitions and Axioms	1
Motivation: from metric spaces to topological spaces	1
1.1 Definition of a topological space	2
1.2 First examples	2
1.3 Comparing topologies	3
1.4 Neighborhoods	4
1.5 Why finite intersections? A counterexample	5
1.6 Worked examples	5
1.7 Exercises	6
Chapter summary	7
2 Open Sets, Closed Sets, Interior, Closure, Boundary	9
Motivation: beyond ε -balls	9
2.1 Closed sets	9
2.2 Interior of a set	10
2.3 Closure of a set	10
2.4 Boundary	11
2.5 Derived set, limit points, isolated points	12
2.6 Dense sets, nowhere dense sets, and meager sets	13
2.7 Kuratowski closure axioms	13
2.8 Kuratowski's 14-set theorem	14
2.9 Interior, closure, and boundary in \mathbb{R}^2 : visualizations	15
2.10 Counterexamples and subtleties	16
2.11 Worked examples	16
2.12 Exercises	17
Chapter summary	18
3 Bases, Sub-bases, and Generated Topologies	19
3.1 Bases for a topology	19
3.2 Sub-bases and generated topologies	20
3.3 Countability and separability axioms	21
3.4 Standard examples of bases	21
3.5 Visualizing base elements	22
3.6 Counterexample: the Sorgenfrey line	22
3.7 Worked examples	22
3.8 Exercises	23
Chapter summary	23

4	Continuity, Homeomorphisms, and Topological Invariants	25
4.1	Continuous maps	25
4.2	Homeomorphisms and topological invariants	26
4.3	Open, closed, and continuous maps	26
4.4	A compact-to-Hausdorff theorem	27
4.5	Topological embeddings	27
4.6	The pasting lemma	27
4.7	Key examples and constructions	28
4.8	Visualizations	28
4.9	Counterexample: a continuous bijection that is not a homeomorphism	29
4.10	Invariance of domain	29
4.11	Worked examples	29
4.12	Exercises	30
	Chapter summary	30
5	Separation Axioms	33
5.1	The T_0 Axiom (Kolmogorov Spaces)	33
5.2	The T_1 Axiom	33
5.3	The T_2 Axiom (Hausdorff Spaces)	34
5.4	Regular Spaces and the T_3 Axiom	34
5.5	Normal Spaces and the T_4 Axiom	35
5.6	The Hierarchy of Separation Axioms	35
5.7	Compact Hausdorff Spaces Are Normal	35
5.8	Every Metric Space Is Normal	36
5.9	Hierarchy Diagram	36
5.10	Counterexamples	36
5.11	Worked Examples	37
5.12	Exercises	37
5.13	Chapter Summary	38
6	Compactness and Tychonoff's Theorem	39
6.1	Definition and First Properties	39
6.2	The Finite Intersection Property	39
6.3	Fundamental Theorems on Compactness	40
6.4	Compactness in \mathbb{R}^n	40
6.5	Sequential and Limit Point Compactness	41
6.6	Alexander's Sub-base Theorem	42
6.7	Tychonoff's Theorem	42
6.8	The Alexandroff One-Point Compactification	43
6.9	The Stone–Čech Compactification	44
6.10	Visualisation: Open Covers	44
6.11	Counterexamples	44
6.12	Worked Examples	45
6.13	Exercises	45
6.14	Chapter Summary	46
7	Connectedness and Path-Connectedness	47
7.1	Connected Spaces	47
7.2	Fundamental Theorems on Connectedness	48
7.3	Path-Connectedness	49
7.4	Counterexample: The Topologist's Sine Curve	49
7.5	Local Connectedness and Local Path-Connectedness	50

7.6	Totally Disconnected Spaces	50
7.7	Product of Connected Spaces	51
7.8	Illustrations	51
7.9	Worked Examples	52
7.10	Exercises	52
8	Product Spaces and Quotient Topology	55
8.1	Finite Products	55
8.2	Infinite Products: Product vs. Box Topology	55
8.3	Universal Property and Projections	56
8.4	The Tube Lemma	57
8.5	Products and Separation	57
8.6	Quotient Topology	57
8.7	Quotient Examples with Identification Diagrams	58
8.8	Disjoint Union (Coproduct) Topology	59
8.9	Box Topology Pathologies	60
8.10	Worked Examples	61
8.11	Exercises	61
9	Urysohn’s Lemma and the Tietze Extension Theorem	63
9.1	Completely Regular Spaces	63
9.2	Urysohn’s Lemma	64
9.3	The Tietze Extension Theorem	65
9.4	The Urysohn Metrization Theorem	66
9.5	Applications: Partitions of Unity	67
9.6	Counterexample: The Sorgenfrey Plane	67
9.7	Worked Examples	68
9.8	Exercises	68
10	A Preview of Algebraic Topology	71
10.1	Homotopy	71
10.2	The Fundamental Group	72
10.3	The Fundamental Group of the Circle	73
10.4	Covering Spaces	73
10.5	The Brouwer Fixed Point Theorem	74
10.6	The Borsuk–Ulam Theorem	75
10.7	Higher Invariants: A Brief Glimpse	75
10.8	Historical Notes	75
10.9	Exercises	76
A	Catalogue of Counterexamples	79
A.1	The Sorgenfrey Line	79
A.2	The Sorgenfrey Plane	80
A.3	Cofinite Topology on an Infinite Set	80
A.4	Cocountable Topology on an Uncountable Set	81
A.5	The Sierpiński Space	81
A.6	The Cantor Set	81
A.7	The Topologist’s Sine Curve	82
A.8	The Long Line	82
A.9	The Double Origin Line (Bug-Eyed Line)	83
A.10	The Particular Point Topology	83
A.11	Fort Space	84

A.12 The Moore Plane (Niemytzki Plane)	84
A.13 The Tychonoff Plank	85

Preface

The concept of *nearness* lies at the heart of mathematical analysis. In a first course on real analysis one learns to manipulate ε - δ arguments: a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at a when, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. In a course on metric spaces one discovers that the absolute value may be replaced by any metric d , and the same ε - δ game still works. But a natural question arises:

How far can we go? What is the minimal structure on a set X that allows us to speak meaningfully of “open sets,” “convergence,” and “continuity”?

The answer is the notion of a *topological space*, and the study of its properties is called **general topology** (or *point-set topology*). Rather than relying on a distance function, one axiomatizes directly the collection of open sets. The resulting framework is astonishingly flexible: it encompasses metric spaces, function spaces with the topology of pointwise convergence, algebraic varieties with the Zariski topology, and many structures that resist metrization.

Historical context. The foundations of general topology were laid in the early twentieth century. Felix Hausdorff’s *Grundzüge der Mengenlehre* (1914) introduced the axiomatic approach to topological spaces and formulated the separation axiom that today bears his name. Kazimierz Kuratowski (1920s) showed that a topology can equivalently be specified by a closure operator satisfying four elegant axioms, an approach that highlights the algebraic flavour of the subject. In the mid-twentieth century the Bourbaki group undertook a sweeping reorganization of topology within their treatise *Topologie générale*, establishing the vocabulary and conventions that remain standard in the French-speaking mathematical tradition and far beyond.

Goals of this course. These notes cover the core material of a one-semester course at the L2–L3 level. Starting from the axioms for a topological space, we develop the notions of open and closed sets, interior, closure, and boundary; bases and subbases; continuity and homeomorphisms; subspaces, products, and quotients; connectedness and compactness; separation and countability axioms; and an introduction to completeness and metrization. Throughout, we emphasize *examples* and *counterexamples*: it is by seeing where theorems fail that one truly understands their hypotheses.

Prerequisites. The reader should be comfortable with naive set theory (unions, intersections, complements, Cartesian products, functions, equivalence relations) and with the basic theory of metric spaces (\mathbb{R}^n with the Euclidean metric is sufficient). No prior knowledge of abstract algebra or measure theory is assumed.

Conventions. We follow the notation of Munkres’ *Topology* and, where appropriate, Bourbaki’s conventions. Proofs end with the symbol \square . Exercises are graded by difficulty: ★ basic, ★★ intermediate, ★★★ challenging.

Good luck, and enjoy the journey.

Notation

Throughout these notes we use the following conventions.

Symbol	Meaning
X, Y, Z	Sets (usually underlying sets of topological spaces)
$\mathcal{T}, \mathcal{T}'$	Topologies on a set X (collections of open sets)
$\mathcal{P}(X)$	Power set of X
(X, \mathcal{T})	Topological space
U, V, W	Open sets
F, C	Closed sets
$\overset{\circ}{A}$ or A° or $\text{Int}(A)$	Interior of a set A
\overline{A} or $\text{Cl}(A)$	Closure of a set A
∂A or $\text{Bd}(A)$	Boundary of a set A
A'	Derived set (set of limit points of A)
$\mathcal{V}(x)$ or $\mathcal{N}(x)$	Neighborhood system (filter) of a point x
$B(x, r)$ or $B_d(x, r)$	Open ball of center x and radius r (in a metric space)
$\overline{B}(x, r)$	Closed ball of center x and radius r
$\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}, \mathbb{C}$	Real, rational, integer, natural, complex numbers
\mathbb{R}^n	Euclidean n -space with the standard topology
$f: X \rightarrow Y$	A function from X to Y
$f^{-1}(B)$	Preimage of $B \subseteq Y$ under f
id_X	Identity function on X
\subseteq, \subsetneq	Subset, proper subset
A^c or $X \setminus A$	Complement of A in X
\cup, \cap	Union, intersection (possibly over arbitrary index sets)
\square or \square	End of proof

Chapter 1

Topological Spaces — Definitions and Axioms

Motivation: from metric spaces to topological spaces

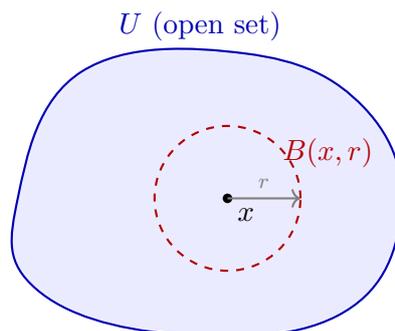
In a metric space (X, d) the notion of *open set* rests on open balls. Recall that the **open ball** of center x and radius $r > 0$ is

$$B(x, r) = \{ y \in X : d(x, y) < r \}.$$

A subset $U \subseteq X$ is declared *open* if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$. From this definition one proves three fundamental properties of the collection \mathcal{T}_d of all open subsets of (X, d) :

- (i) \emptyset and X belong to \mathcal{T}_d .
- (ii) Any union of sets in \mathcal{T}_d belongs to \mathcal{T}_d .
- (iii) Any *finite* intersection of sets in \mathcal{T}_d belongs to \mathcal{T}_d .

These three properties do *not* depend on the particular values produced by the metric; they depend only on the *collection* \mathcal{T}_d . The key insight of general topology is to take these three properties as *axioms*.



1.1 Definition of a topological space

Definition 1.1.1 (Topological space). Let X be a non-empty set. A **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying

(T1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

(T2) If $(U_i)_{i \in I}$ is any family of elements of \mathcal{T} (with I an arbitrary index set), then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

(T3) If $U_1, U_2, \dots, U_n \in \mathcal{T}$ (finitely many), then $\bigcap_{k=1}^n U_k \in \mathcal{T}$.

The pair (X, \mathcal{T}) is called a **topological space**. The elements of \mathcal{T} are called **open sets**.

Remark 1.1.1. Axiom (T2) allows the empty union ($I = \emptyset$), which gives $\emptyset \in \mathcal{T}$, and axiom (T3) allows the empty intersection, which gives $X \in \mathcal{T}$. Hence axiom (T1) is in fact redundant; it is included for clarity.

1.2 First examples

Definition 1.2.1 (Discrete topology). The **discrete topology** on a set X is $\mathcal{T}_{\text{disc}} = \mathcal{P}(X)$, the collection of *all* subsets of X .

Definition 1.2.2 (Indiscrete (trivial) topology). The **indiscrete topology** on a set X is $\mathcal{T}_{\text{ind}} = \{\emptyset, X\}$.

Remark 1.2.1. The discrete topology is the finest possible topology on X , and the indiscrete topology is the coarsest. In the discrete topology *every* subset is open (and closed); in the indiscrete topology the only open sets are \emptyset and X .

Definition 1.2.3 (Cofinite topology). The **cofinite topology** on a set X is

$$\mathcal{T}_{\text{cof}} = \{U \subseteq X : X \setminus U \text{ is finite}\} \cup \{\emptyset\}.$$

Proposition 1.2.2. *The cofinite topology is indeed a topology.*

Proof. We verify the three axioms.

(i) $\emptyset \in \mathcal{T}_{\text{cof}}$ by definition. $X \in \mathcal{T}_{\text{cof}}$ because $X \setminus X = \emptyset$ is finite.

(ii) Let $(U_i)_{i \in I}$ be a family in \mathcal{T}_{cof} . If every $U_i = \emptyset$ the union is $\emptyset \in \mathcal{T}_{\text{cof}}$. Otherwise, pick some $j \in I$ with $U_j \neq \emptyset$. Then

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i) \subseteq X \setminus U_j,$$

which is finite. So the union is in \mathcal{T}_{cof} .

(iii) Let $U_1, \dots, U_n \in \mathcal{T}_{\text{cof}}$. If some $U_k = \emptyset$ the intersection is $\emptyset \in \mathcal{T}_{\text{cof}}$. Otherwise,

$$X \setminus \bigcap_{k=1}^n U_k = \bigcup_{k=1}^n (X \setminus U_k),$$

which is a finite union of finite sets, hence finite. □

Definition 1.2.4 (Cocountable topology). The **cocountable topology** on a set X is

$$\mathcal{T}_{\text{coc}} = \{U \subseteq X : X \setminus U \text{ is countable}\} \cup \{\emptyset\}.$$

Remark 1.2.3. The verification that \mathcal{T}_{coc} is a topology proceeds exactly as for the cofinite topology, replacing “finite” by “countable” throughout.

Example 1.2.4 (The metric topology). Let (X, d) be a metric space. The collection

$$\mathcal{T}_d = \{U \subseteq X : \forall x \in U, \exists r > 0, B(x, r) \subseteq U\}$$

is a topology on X , called the **metric topology** induced by d . Classical examples include

- \mathbb{R} with the Euclidean metric $d(x, y) = |x - y|$, giving the **standard topology** on \mathbb{R} .
- \mathbb{R}^n with $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2$.
- Any normed vector space $(V, \|\cdot\|)$ with $d(x, y) = \|x - y\|$.

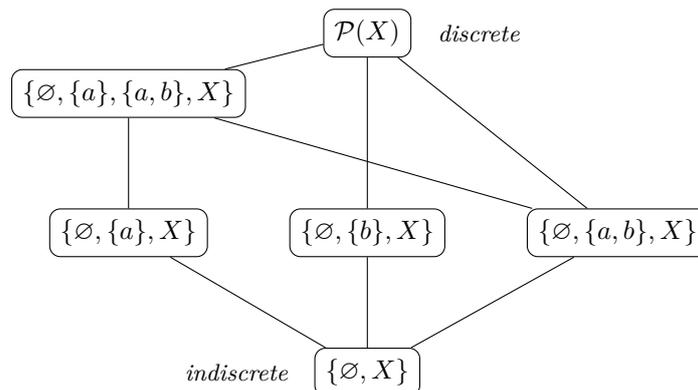
Example 1.2.5 (The Zariski topology on \mathbb{R} — a brief preview). Let $X = \mathbb{R}$ (or more generally an algebraically closed field k). Declare a subset $C \subseteq \mathbb{R}$ to be *closed* if it is the zero set of some collection of polynomials. Since a non-zero polynomial in one variable has only finitely many roots, the closed sets in this topology on \mathbb{R} are precisely the finite sets and \mathbb{R} itself. In other words the Zariski topology on \mathbb{R} coincides with the cofinite topology. In higher dimensions (on k^n or on algebraic varieties) the Zariski topology is far richer and plays a central role in algebraic geometry.

1.3 Comparing topologies

Definition 1.3.1 (Finer and coarser topologies). Let \mathcal{T}_1 and \mathcal{T}_2 be two topologies on the same set X . We say that \mathcal{T}_2 is **finer** (or **stronger**) than \mathcal{T}_1 , and \mathcal{T}_1 is **coarser** (or **weaker**) than \mathcal{T}_2 , if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Remark 1.3.1. The relation $\mathcal{T}_1 \subseteq \mathcal{T}_2$ is a partial order on the set of all topologies on X . In this partial order the indiscrete topology $\{\emptyset, X\}$ is the minimum and the discrete topology $\mathcal{P}(X)$ is the maximum.

Example 1.3.2 (Lattice of topologies on a three-element set). Let $X = \{a, b, c\}$. The following diagram shows several topologies on X , ordered by inclusion (lines go upward toward finer topologies):



(Only a selection of the 29 topologies on a 3-element set is shown.)

Proposition 1.3.3 (Intersection of topologies). *Let $(\mathcal{T}_i)_{i \in I}$ be a family of topologies on a set X . Then*

$$\mathcal{T} = \bigcap_{i \in I} \mathcal{T}_i$$

is a topology on X .

Proof. We check the three axioms.

- (i) Since $\emptyset \in \mathcal{T}_i$ and $X \in \mathcal{T}_i$ for every $i \in I$, we have $\emptyset, X \in \mathcal{T}$.
- (ii) Let $(U_j)_{j \in J}$ be a family of elements of \mathcal{T} . For each $j \in J$ and each $i \in I$ we have $U_j \in \mathcal{T}_i$. Because each \mathcal{T}_i is a topology, $\bigcup_{j \in J} U_j \in \mathcal{T}_i$. This holds for every i , so $\bigcup_{j \in J} U_j \in \mathcal{T}$.
- (iii) Similarly, if $U_1, \dots, U_n \in \mathcal{T}$ then $\bigcap_{k=1}^n U_k \in \mathcal{T}_i$ for every $i \in I$, hence $\bigcap_{k=1}^n U_k \in \mathcal{T}$. □

Remark 1.3.4. The union of two topologies need not be a topology. For instance on $X = \{a, b, c\}$ let $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, X\}$, which is not closed under unions since $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

1.4 Neighborhoods

Definition 1.4.1 (Neighborhood). Let (X, \mathcal{T}) be a topological space and $x \in X$. A subset $V \subseteq X$ is a **neighborhood** of x if there exists an open set $U \in \mathcal{T}$ with $x \in U \subseteq V$. The collection of all neighborhoods of x is denoted $\mathcal{V}(x)$ and is called the **neighborhood system** (or **neighborhood filter**) of x .

Remark 1.4.1. Note that a neighborhood of x is *not* required to be open; it merely needs to contain an open set around x . An open set U is a neighborhood of each of its points.

Proposition 1.4.2. *Let (X, \mathcal{T}) be a topological space and $x \in X$. The neighborhood system $\mathcal{V}(x)$ satisfies*

- (N1) $\mathcal{V}(x) \neq \emptyset$, and every $V \in \mathcal{V}(x)$ contains x .
- (N2) If $V \in \mathcal{V}(x)$ and $V \subseteq W \subseteq X$, then $W \in \mathcal{V}(x)$.
- (N3) If $V_1, V_2 \in \mathcal{V}(x)$ then $V_1 \cap V_2 \in \mathcal{V}(x)$.
- (N4) For every $V \in \mathcal{V}(x)$ there exists $W \in \mathcal{V}(x)$ with $W \subseteq V$ such that $V \in \mathcal{V}(y)$ for all $y \in W$.

Proof. (N1): X is open and $x \in X$, so $X \in \mathcal{V}(x)$. By definition every neighborhood contains x .

(N2): There exists an open U with $x \in U \subseteq V \subseteq W$, so W is a neighborhood of x .

(N3): There exist open sets U_1, U_2 with $x \in U_1 \subseteq V_1$ and $x \in U_2 \subseteq V_2$. Then $U_1 \cap U_2$ is open (finite intersection) and $x \in U_1 \cap U_2 \subseteq V_1 \cap V_2$.

(N4): Let $V \in \mathcal{V}(x)$ and pick an open U with $x \in U \subseteq V$. Set $W = U$. For every $y \in W = U$, we have $y \in U \subseteq V$ with U open, so $V \in \mathcal{V}(y)$. □

Proposition 1.4.3 (Characterization of open sets via neighborhoods). *Let (X, \mathcal{T}) be a topological space and $U \subseteq X$. Then U is open if and only if U is a neighborhood of each of its points; that is,*

$$U \in \mathcal{T} \iff \forall x \in U, U \in \mathcal{V}(x).$$

Proof. (\Rightarrow) If U is open and $x \in U$, then $x \in U \subseteq U$ with U open, so $U \in \mathcal{V}(x)$.

(\Leftarrow) Suppose that for every $x \in U$ there exists an open set U_x with $x \in U_x \subseteq U$. Then

$$U = \bigcup_{x \in U} U_x.$$

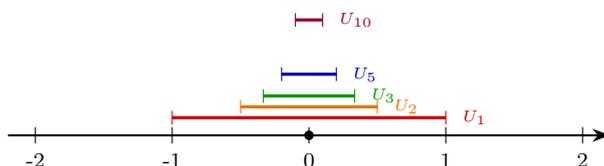
As an arbitrary union of open sets, U is open. □

1.5 Why finite intersections? A counterexample

Example 1.5.1 (Arbitrary intersections of open sets need not be open). In \mathbb{R} with the standard topology, consider the open intervals $U_n = (-1/n, 1/n)$ for $n \geq 1$. Each U_n is open. However,

$$\bigcap_{n=1}^{\infty} U_n = \{0\},$$

which is *not* open in \mathbb{R} (no open interval is contained in a single point). This is why axiom (T3) requires only *finite* intersections.



1.6 Worked examples

Example 1.6.1 (Worked Example 1: verifying a topology). Let $X = \{1, 2, 3, 4\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}$. Is (X, \mathcal{T}) a topological space?

Solution. We check the axioms.

- (T1): $\emptyset, X \in \mathcal{T}$. ✓
- (T2, unions): List all possible unions of elements of \mathcal{T} . Since \mathcal{T} is a chain $\emptyset \subseteq \{1\} \subseteq \{1, 2\} \subseteq \{1, 2, 3\} \subseteq X$, any union of elements is the largest element in the union, which is again in \mathcal{T} . ✓
- (T3, finite intersections): Similarly, any finite intersection is the smallest element, which is in \mathcal{T} . ✓

Hence (X, \mathcal{T}) is a topological space.

Example 1.6.2 (Worked Example 2: the cofinite topology on \mathbb{Z}). In $(\mathbb{Z}, \mathcal{T}_{\text{cof}})$, determine whether $A = \{0, 1, 2\}$ is open, closed, both, or neither.

Solution.

- *Open?* $\mathbb{Z} \setminus A = \mathbb{Z} \setminus \{0, 1, 2\}$ is infinite (it contains all integers outside $\{0, 1, 2\}$). So A is **not open**.
- *Closed?* A set is closed if its complement is open. $A^c = \mathbb{Z} \setminus \{0, 1, 2\}$ is not open (since $\mathbb{Z} \setminus A^c = A$ is finite). Wait — let us recheck: $\mathbb{Z} \setminus (\mathbb{Z} \setminus A) = A$, which *is* finite. So $\mathbb{Z} \setminus A$ *is* open in the cofinite topology, meaning A is **closed**.

Therefore $A = \{0, 1, 2\}$ is closed but not open in $(\mathbb{Z}, \mathcal{T}_{\text{cof}})$. This illustrates that in the cofinite topology on an infinite set, the finite non-empty subsets are closed but not open.

Example 1.6.3 (Worked Example 3: neighborhood systems). Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$. Determine $\mathcal{V}(a)$, $\mathcal{V}(b)$, and $\mathcal{V}(c)$.

Solution. A set V is a neighborhood of x if there exists $U \in \mathcal{T}$ with $x \in U \subseteq V$.

- $\mathcal{V}(a)$: The open sets containing a are $\{a\}$, $\{a, b\}$, and X . Every superset of $\{a\}$ is a neighborhood of a . Hence

$$\mathcal{V}(a) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}.$$

- $\mathcal{V}(b)$: The open sets containing b are $\{a, b\}$ and X . Every superset of $\{a, b\}$ is a neighborhood of b . Hence

$$\mathcal{V}(b) = \{\{a, b\}, \{a, b, c\}\} = \{\{a, b\}, X\}.$$

- $\mathcal{V}(c)$: The only open set containing c is X . Hence $\mathcal{V}(c) = \{X\}$.

Notice that the point c is “barely visible” to the topology: the only way to “surround” it with an open set is to take the entire space.

1.7 Exercises

Exercise 1.7.1 (★). Let $X = \{1, 2, 3\}$. List *all* topologies on X that contain the set $\{1, 2\}$.

Exercise 1.7.2 (★). Show that on any set X , the discrete topology is the only topology in which every singleton $\{x\}$ is open.

Exercise 1.7.3 (★). Let (X, d) be a metric space. Prove directly (using the triangle inequality) that every open ball $B(x, r)$ is an open set in the metric topology \mathcal{T}_d .

Exercise 1.7.4 (★★). On \mathbb{R} , consider the collection

$$\mathcal{T}_{[-\infty)} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

Prove that $\mathcal{T}_{[-\infty)}$ is a topology on \mathbb{R} . Is it finer or coarser than the standard topology?

Exercise 1.7.5 (★★). Let X be a set and $p \in X$ a fixed point. Define

$$\mathcal{T}_p = \{\emptyset\} \cup \{U \subseteq X : p \in U\}.$$

- Prove that \mathcal{T}_p is a topology on X (the *particular point topology*).
- Determine the neighborhood system $\mathcal{V}(p)$ and $\mathcal{V}(x)$ for $x \neq p$.
- Is (X, \mathcal{T}_p) metrizable when $|X| \geq 2$? (Hint: consider the T_1 axiom.)

Exercise 1.7.6 (★★). Let $f: X \rightarrow Y$ be a function and let \mathcal{T}_Y be a topology on Y . Show that

$$\mathcal{T}_f = \{f^{-1}(V) : V \in \mathcal{T}_Y\}$$

is a topology on X (the *initial topology* or *pullback topology* induced by f).

Exercise 1.7.7 (★★★). Prove that the set of all topologies on a set X , ordered by inclusion, forms a **complete lattice**: every non-empty family of topologies has a greatest lower bound (infimum) and a least upper bound (supremum).

- Show that the infimum of a family $(\mathcal{T}_i)_{i \in I}$ is $\bigcap_{i \in I} \mathcal{T}_i$.
- Show that the supremum is the coarsest topology containing $\bigcup_{i \in I} \mathcal{T}_i$ (describe it explicitly).

Exercise 1.7.8 (★★★). Let X be an infinite set. Prove that the cofinite topology \mathcal{T}_{cof} on X is **not** metrizable. *Hint:* Suppose $\mathcal{T}_{\text{cof}} = \mathcal{T}_d$ for some metric d . Show that distinct points can be separated by disjoint open sets in \mathcal{T}_d (Hausdorff property), then derive a contradiction with the cofinite topology.

Exercise 1.7.9 (★). In \mathbb{R} with the standard topology, give an example of a collection of open sets whose intersection is:

- (a) open,
- (b) closed but not open,
- (c) neither open nor closed.

Chapter summary

- A **topology** on a set X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ that is closed under arbitrary unions and finite intersections and contains both \emptyset and X .
- Key examples include the **discrete**, **indiscrete**, **cofinite**, **countable**, and **metric** topologies.
- Topologies on X are partially ordered by inclusion (**finer** vs. **coarser**). The intersection of any family of topologies is again a topology; the union need not be.
- A **neighborhood** of x is any set containing an open set around x . A set is open if and only if it is a neighborhood of each of its points.
- Arbitrary intersections of open sets need not be open: the finiteness condition in axiom (T3) is essential.

Chapter 2

Open Sets, Closed Sets, Interior, Closure, Boundary

Motivation: beyond ε -balls

In a metric space (X, d) , one says that a point x is in the *interior* of a set A if some ε -ball around x lies entirely within A , and that x belongs to the *closure* of A if every ε -ball around x meets A . The *boundary* consists of those points whose every ε -ball meets both A and its complement.

These notions extend verbatim to topological spaces: one simply replaces “ ε -ball” by “open neighborhood.” This chapter develops the theory of interior, closure, boundary, and the related notions of dense and nowhere dense sets. A highlight is Kuratowski’s elegant axiomatization of topology via a closure operator, and the remarkable “14-set theorem.”

2.1 Closed sets

Definition 2.1.1 (Closed set). Let (X, \mathcal{T}) be a topological space. A subset $F \subseteq X$ is **closed** if its complement $X \setminus F$ is open, i.e., $X \setminus F \in \mathcal{T}$.

Proposition 2.1.1. *The collection of closed subsets of a topological space (X, \mathcal{T}) satisfies*

- (C1) \emptyset and X are closed.
- (C2) Any intersection of closed sets is closed.
- (C3) Any finite union of closed sets is closed.

Proof. These follow from the axioms for open sets via De Morgan’s laws.

- (i) $X \setminus \emptyset = X \in \mathcal{T}$ and $X \setminus X = \emptyset \in \mathcal{T}$.
- (ii) If $(F_i)_{i \in I}$ are closed, then each $X \setminus F_i$ is open, and $X \setminus \bigcap_{i \in I} F_i = \bigcup_{i \in I} (X \setminus F_i) \in \mathcal{T}$.
- (iii) If F_1, \dots, F_n are closed, then $X \setminus \bigcup_{k=1}^n F_k = \bigcap_{k=1}^n (X \setminus F_k) \in \mathcal{T}$. □

Remark 2.1.2. “Open” and “closed” are *not* mutually exclusive. The empty set and the whole space X are always both open and closed (**clopen**). In the discrete topology every set is clopen. A set may also be neither open nor closed; for instance $[0, 1)$ in \mathbb{R} with the standard topology.

2.2 Interior of a set

Definition 2.2.1 (Interior). Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. The **interior** of A , denoted $\overset{\circ}{A}$, A° , or $\text{Int}(A)$, is

$$\text{Int}(A) = \bigcup \{U \in \mathcal{T} : U \subseteq A\}.$$

Proposition 2.2.1 (Characterizations and properties of the interior). Let (X, \mathcal{T}) be a topological space and $A, B \subseteq X$.

- (i) $\text{Int}(A)$ is the **largest open set** contained in A .
- (ii) A is open if and only if $A = \text{Int}(A)$.
- (iii) $x \in \text{Int}(A)$ if and only if A is a neighborhood of x (i.e., $A \in \mathcal{V}(x)$).
- (iv) If $A \subseteq B$ then $\text{Int}(A) \subseteq \text{Int}(B)$.
- (v) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$.
- (vi) $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$.
- (vii) $\text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B)$, and equality need not hold.

Proof. (i): By definition $\text{Int}(A)$ is a union of open subsets of A , so it is open and contained in A . If V is any open set with $V \subseteq A$, then V is one of the sets in the union defining $\text{Int}(A)$, so $V \subseteq \text{Int}(A)$.

(ii): If A is open then A appears in the union $\text{Int}(A) = \bigcup \{U \in \mathcal{T} : U \subseteq A\}$, so $A \subseteq \text{Int}(A)$; the reverse inclusion is clear. Conversely, if $A = \text{Int}(A)$ then A is a union of open sets, hence open.

(iii): $x \in \text{Int}(A)$ iff there exists $U \in \mathcal{T}$ with $x \in U \subseteq A$ iff $A \in \mathcal{V}(x)$.

(iv): Every open subset of A is an open subset of B .

(v): $\text{Int}(A)$ is open, so $\text{Int}(\text{Int}(A)) = \text{Int}(A)$ by (ii).

(vi): “ \subseteq ”: $\text{Int}(A) \cap \text{Int}(B)$ is open and contained in $A \cap B$, hence $\text{Int}(A) \cap \text{Int}(B) \subseteq \text{Int}(A \cap B)$.

“ \supseteq ”: $\text{Int}(A \cap B) \subseteq A \cap B \subseteq A$ and $\text{Int}(A \cap B)$ is open, so $\text{Int}(A \cap B) \subseteq \text{Int}(A)$. Similarly $\text{Int}(A \cap B) \subseteq \text{Int}(B)$.

(vii): $\text{Int}(A) \subseteq A \subseteq A \cup B$ and $\text{Int}(A)$ is open, so $\text{Int}(A) \subseteq \text{Int}(A \cup B)$; similarly for $\text{Int}(B)$. For a counter-example to equality, take $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ in \mathbb{R} : $\text{Int}(A) = \text{Int}(B) = \emptyset$ but $\text{Int}(A \cup B) = \text{Int}(\mathbb{R}) = \mathbb{R}$. \square

2.3 Closure of a set

Definition 2.3.1 (Closure). Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. The **closure** of A , denoted \bar{A} or $\text{Cl}(A)$, is

$$\text{Cl}(A) = \bigcap \{F \subseteq X : F \text{ is closed and } A \subseteq F\}.$$

Proposition 2.3.1 (Characterizations and properties of the closure). Let (X, \mathcal{T}) be a topological space and $A, B \subseteq X$.

- (i) $\text{Cl}(A)$ is the **smallest closed set** containing A .
- (ii) A is closed if and only if $A = \text{Cl}(A)$.

- (iii) If $A \subseteq B$ then $\text{Cl}(A) \subseteq \text{Cl}(B)$.
- (iv) $\text{Cl}(\text{Cl}(A)) = \text{Cl}(A)$.
- (v) $\text{Cl}(A \cup B) = \text{Cl}(A) \cup \text{Cl}(B)$.
- (vi) $\text{Cl}(A \cap B) \subseteq \text{Cl}(A) \cap \text{Cl}(B)$, and equality need not hold.
- (vii) $\text{Cl}(A) = X \setminus \text{Int}(X \setminus A)$ and $\text{Int}(A) = X \setminus \text{Cl}(X \setminus A)$.

Proof. The proofs are dual to those of Proposition 2.2.1 via complements. We prove the key identity (vii) and leave the others as exercises.

(vii): We have

$$\begin{aligned} X \setminus \text{Int}(X \setminus A) &= X \setminus \bigcup \{U \in \mathcal{T} : U \subseteq X \setminus A\} \\ &= \bigcap \{X \setminus U : U \in \mathcal{T}, U \subseteq X \setminus A\} \\ &= \bigcap \{F : F \text{ closed}, A \subseteq F\} = \text{Cl}(A), \end{aligned}$$

where in the third equality we set $F = X \setminus U$ and note that U ranges over all open subsets of $X \setminus A$ iff $F = X \setminus U$ ranges over all closed supersets of A . The second identity follows by replacing A with $X \setminus A$. \square

Theorem 2.3.2 (Characterization of closure via neighborhoods). *Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. Then*

$$x \in \text{Cl}(A) \iff \text{every open set containing } x \text{ meets } A,$$

i.e., $x \in \text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in \mathcal{T}$ with $x \in U$.

Proof. We prove both directions by contraposition.

(\Rightarrow) Suppose there exists an open set U with $x \in U$ and $U \cap A = \emptyset$. Then $A \subseteq X \setminus U$, and $X \setminus U$ is closed. Hence

$$\text{Cl}(A) \subseteq X \setminus U,$$

and since $x \notin X \setminus U$ we get $x \notin \text{Cl}(A)$.

(\Leftarrow) Suppose $x \notin \text{Cl}(A)$. Since $\text{Cl}(A)$ is closed, $U = X \setminus \text{Cl}(A)$ is open. We have $x \in U$ and $U \cap A \subseteq U \cap \text{Cl}(A) = \emptyset$, so U is an open neighborhood of x that does not meet A . \square

2.4 Boundary

Definition 2.4.1 (Boundary). Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. The **boundary** (or **frontier**) of A is

$$\partial A = \text{Cl}(A) \setminus \text{Int}(A) = \text{Cl}(A) \cap \text{Cl}(X \setminus A).$$

Proposition 2.4.1. *Let (X, \mathcal{T}) be a topological space and $A \subseteq X$.*

- (i) ∂A is closed.
- (ii) $\partial A = \partial(X \setminus A)$.
- (iii) $X = \text{Int}(A) \sqcup \partial A \sqcup \text{Int}(X \setminus A)$ (disjoint union).
- (iv) A is open if and only if $A \cap \partial A = \emptyset$.

(v) A is closed if and only if $\partial A \subseteq A$.

(vi) A is clopen if and only if $\partial A = \emptyset$.

Proof. (i) $\partial A = \text{Cl}(A) \cap \text{Cl}(X \setminus A)$ is the intersection of two closed sets.

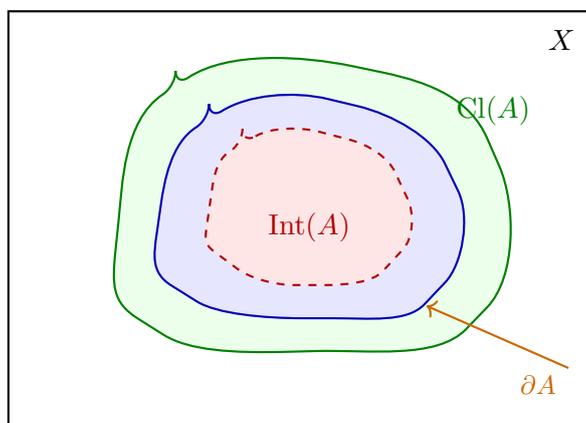
(ii) $\partial(X \setminus A) = \text{Cl}(X \setminus A) \cap \text{Cl}(A) = \partial A$.

(iii) Every $x \in X$ belongs to exactly one of: $\text{Int}(A)$ (an open neighborhood inside A), $\text{Int}(X \setminus A)$ (an open neighborhood inside $X \setminus A$), or neither (every open neighborhood meets both A and $X \setminus A$, i.e., $x \in \partial A$).

(iv) A is open iff $A = \text{Int}(A)$ iff A contains no boundary points of A iff $A \cap \partial A = \emptyset$.

(v) A is closed iff $\text{Cl}(A) = A$ iff $\partial A = \text{Cl}(A) \setminus \text{Int}(A) \subseteq \text{Cl}(A) = A$.

(vi) Combine (iv) and (v). □



2.5 Derived set, limit points, isolated points

Definition 2.5.1 (Limit point and derived set). Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. A point $x \in X$ is a **limit point** (or **accumulation point**) of A if every open set containing x meets $A \setminus \{x\}$:

$$\forall U \in \mathcal{T}, \quad x \in U \implies U \cap (A \setminus \{x\}) \neq \emptyset.$$

The set of all limit points of A is called the **derived set** of A and is denoted A' .

Definition 2.5.2 (Isolated point). A point $a \in A$ is an **isolated point** of A if there exists an open set U such that $U \cap A = \{a\}$.

Proposition 2.5.1. For any subset A of a topological space (X, \mathcal{T}) ,

$$\text{Cl}(A) = A \cup A'.$$

Proof. By Theorem 2.3.2, $x \in \text{Cl}(A)$ iff every open neighborhood of x meets A . If $x \in A$ then $x \in A \cup A'$ trivially. If $x \notin A$, then every open neighborhood of x meeting A in fact meets $A \setminus \{x\} = A$, so $x \in A'$. Hence $\text{Cl}(A) \subseteq A \cup A'$.

Conversely, if $x \in A$ then certainly every open neighborhood meets A (it contains x), so $x \in \text{Cl}(A)$. If $x \in A'$ then every open neighborhood meets $A \setminus \{x\} \subseteq A$, so $x \in \text{Cl}(A)$. □

2.6 Dense sets, nowhere dense sets, and meager sets

Definition 2.6.1 (Dense set). A subset A of a topological space (X, \mathcal{T}) is **dense** in X if $\text{Cl}(A) = X$, equivalently, if every non-empty open set meets A .

Example 2.6.1. \mathbb{Q} is dense in \mathbb{R} with the standard topology, because every non-empty open interval (a, b) contains a rational number (by the Archimedean property). Similarly, $\mathbb{R} \setminus \mathbb{Q}$ is dense in \mathbb{R} .

Definition 2.6.2 (Nowhere dense set). A subset A of (X, \mathcal{T}) is **nowhere dense** if $\text{Int}(\text{Cl}(A)) = \emptyset$; equivalently, $\text{Cl}(A)$ has empty interior.

Example 2.6.2. \mathbb{Z} is nowhere dense in \mathbb{R} : $\text{Cl}(\mathbb{Z}) = \mathbb{Z}$ (since \mathbb{Z} is closed in \mathbb{R}), and $\text{Int}(\mathbb{Z}) = \emptyset$ (no open interval is contained in \mathbb{Z}).

Definition 2.6.3 (Meager set (set of first category)). A subset M of (X, \mathcal{T}) is **meager** (or of **first category**) if it can be written as a countable union of nowhere dense sets:

$$M = \bigcup_{n=1}^{\infty} A_n, \quad \text{each } A_n \text{ nowhere dense.}$$

A set that is not meager is said to be of **second category**.

Remark 2.6.3 (Preview of Baire's theorem). The **Baire category theorem** asserts that in a complete metric space (and more generally in a locally compact Hausdorff space), the whole space is of second category: it cannot be written as a countable union of nowhere dense sets. We shall prove this theorem in a later chapter.

2.7 Kuratowski closure axioms

Kuratowski observed that a topology on X can be recovered from its closure operator. This gives an equivalent axiomatization of topological spaces.

Theorem 2.7.1 (Kuratowski closure axioms). *Let X be a set and $c: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ an operator satisfying*

(K1) $c(\emptyset) = \emptyset$.

(K2) $A \subseteq c(A)$ for every $A \subseteq X$.

(K3) $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \subseteq X$.

(K4) $c(c(A)) = c(A)$ for every $A \subseteq X$ (idempotence).

Then the collection

$$\mathcal{T}_c = \{U \subseteq X : c(X \setminus U) = X \setminus U\}$$

is a topology on X , and the closure of any set A in (X, \mathcal{T}_c) is precisely $c(A)$.

Conversely, in any topological space (X, \mathcal{T}) , the closure operator $\text{Cl}: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfies (K1)–(K4).

Proof. Step 1: \mathcal{T}_c is a topology.

Observe that $U \in \mathcal{T}_c$ iff $F = X \setminus U$ satisfies $c(F) = F$; call such sets *c-closed*.

- (i) $c(X) = c(X \setminus \emptyset) \supseteq X$ by (K2), so $c(X) = X$; thus X is c -closed and $\emptyset \in \mathcal{T}_c$. By (K1), \emptyset is c -closed, so $X \in \mathcal{T}_c$.
- (ii) Let $(F_i)_{i \in I}$ be c -closed sets. We show $F = \bigcap_{i \in I} F_i$ is c -closed. By (K2), $F \subseteq c(F)$. For each i , $F \subseteq F_i$ so by monotonicity of c (which follows from (K2) and (K3): if $A \subseteq B$ then $c(B) = c(A \cup B) = c(A) \cup c(B) \supseteq c(A)$), $c(F) \subseteq c(F_i) = F_i$. Hence $c(F) \subseteq \bigcap_i F_i = F$, giving $c(F) = F$. Thus arbitrary intersections of c -closed sets are c -closed, equivalently, arbitrary unions of sets in \mathcal{T}_c are in \mathcal{T}_c .
- (iii) Let F_1, F_2 be c -closed. Then $c(F_1 \cup F_2) = c(F_1) \cup c(F_2) = F_1 \cup F_2$ by (K3), so $F_1 \cup F_2$ is c -closed. By induction, finite unions of c -closed sets are c -closed, equivalently, finite intersections of sets in \mathcal{T}_c are in \mathcal{T}_c .

Step 2: The closure in (X, \mathcal{T}_c) equals c .

Let $\text{Cl}_c(A)$ denote the closure of A in (X, \mathcal{T}_c) . By definition, $\text{Cl}_c(A) = \bigcap \{F : F \text{ } c\text{-closed, } A \subseteq F\}$.

Since $A \subseteq c(A)$ by (K2) and $c(c(A)) = c(A)$ by (K4), the set $c(A)$ is c -closed and contains A . Hence $\text{Cl}_c(A) \subseteq c(A)$.

Conversely, if F is c -closed and $A \subseteq F$, then $c(A) \subseteq c(F) = F$. So $c(A) \subseteq \bigcap \{F : \dots\} = \text{Cl}_c(A)$.

Therefore $\text{Cl}_c(A) = c(A)$.

Step 3: The converse.

In any topological space (X, \mathcal{T}) , the closure operator Cl satisfies (K1)–(K4) by Proposition 2.3.1 (items concerning \emptyset , monotonicity, $\text{Cl}(A \cup B) = \text{Cl}(A) \cup \text{Cl}(B)$, and idempotence). \square

2.8 Kuratowski's 14-set theorem

Theorem 2.8.1 (Kuratowski's 14-set theorem). *Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. By repeatedly applying the operations of closure (Cl) and complement $((\cdot)^c)$ to A in any order, one can obtain **at most 14** distinct sets. Moreover, there exists a set in \mathbb{R} (with the standard topology) for which all 14 are distinct.*

Proof sketch. Denote closure by k and complementation by c . Each of these is a function $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$. The key algebraic identities are:

$$c \circ c = \text{id}, \tag{2.1}$$

$$k \circ k = k, \tag{2.2}$$

$$k \circ c \circ k \circ c \circ k \circ c \circ k = k \circ c \circ k. \tag{2.3}$$

Identities (2.1) and (2.2) are immediate. The crucial identity (2.3) follows from the fact that $k \circ c \circ k \circ c$ is idempotent (i.e., $(k \circ c \circ k \circ c)^2 = k \circ c \circ k \circ c$), which can be proved using the identity $\text{Int}(\text{Cl}(\text{Int}(\text{Cl}(A)))) = \text{Int}(\text{Cl}(A))$ that holds in any topological space.

Starting from A , one generates sets by applying k and c alternately. Using the three identities above, one can show that the distinct sets obtainable are at most the following 14:

1. A
2. kA
3. ckA
4. $kckA$
5. $ckckA$
6. $kckckA$
7. $ckckckA$
8. cA
9. kcA
10. $ckcA$
11. $kckcA$
12. $ckckcA$
13. $kckckcA$
14. $ckckckcA$

Any further application of k or c yields one of these 14, by virtue of (2.1)–(2.3).

Sharpness. The set

$$A = (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, 5] \cap \mathbb{Q})$$

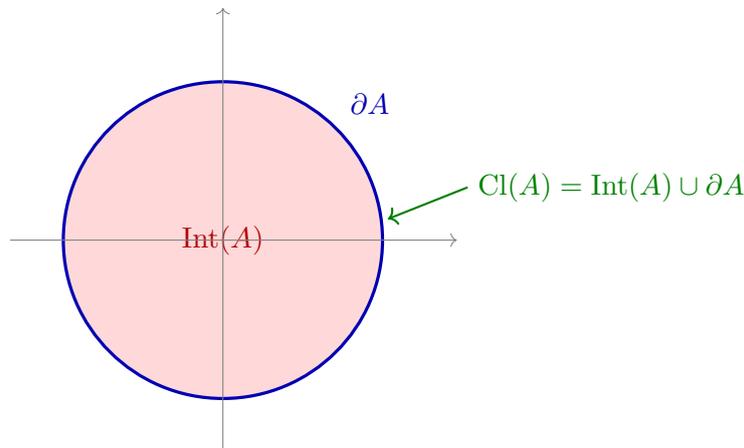
in \mathbb{R} with the standard topology attains all 14 distinct sets. One verifies by direct computation:

- A has an isolated point (3), a “thick” part $([4, 5] \cap \mathbb{Q})$ whose closure is $[4, 5]$ but whose interior is empty), and two intervals separated by a point.
- Successive applications of closure and complement produce 14 genuinely distinct subsets of \mathbb{R} .

The verification is elementary but lengthy; we refer the reader to Kuratowski’s original paper (1922) or to Munkres’ *Topology*, Problem 2.17.9, for the full computation. \square

2.9 Interior, closure, and boundary in \mathbb{R}^2 : visualizations

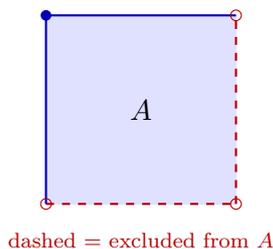
Example 2.9.1 (The open disk in \mathbb{R}^2). Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open unit disk. Then $\text{Int}(A) = A$ (it is already open), $\text{Cl}(A) = \{(x, y) : x^2 + y^2 \leq 1\}$ (the closed disk), and $\partial A = \{(x, y) : x^2 + y^2 = 1\}$ (the unit circle).



Example 2.9.2 (A set with non-trivial interior and closure). In \mathbb{R}^2 consider $A = [0, 1) \times (0, 1]$. Then

- $\text{Int}(A) = (0, 1) \times (0, 1)$,

- $\text{Cl}(A) = [0, 1] \times [0, 1]$,
- $\partial A = \text{Cl}(A) \setminus \text{Int}(A)$, which consists of the four edges of the unit square.



2.10 Counterexamples and subtleties

Example 2.10.1 (Sequential closure \neq closure in general). In a metric space (or more generally a first-countable space), the closure of a set A coincides with the *sequential closure*:

$$\text{Cl}(A) = \{x \in X : \exists (x_n)_{n \geq 1} \text{ in } A \text{ with } x_n \rightarrow x\}.$$

This fails in general topological spaces. Consider the ordinal space $X = [0, \omega_1]$ with the order topology, and let $A = [0, \omega_1)$. Then $\omega_1 \in \text{Cl}(A)$ (every open neighborhood of ω_1 meets A), but no *sequence* in A converges to ω_1 (since ω_1 has uncountable cofinality). Thus the sequential closure of A is A itself, which is strictly smaller than $\text{Cl}(A) = X$.

Example 2.10.2 (\mathbb{Q} is dense in \mathbb{R} but not in the Sorgenfrey line). The **Sorgenfrey line** (or **lower limit topology**) on \mathbb{R} has as a basis the half-open intervals $[a, b)$ with $a < b$. In this topology \mathbb{Q} is still dense: every non-empty basic open set $[a, b)$ contains a rational. However, the Sorgenfrey line is *not* second-countable (it has no countable basis), and many properties that hold for the standard topology fail: for instance, the Sorgenfrey line is separable but not second-countable, showing these two properties are independent outside the metric setting.

More strikingly, consider the set $A = \mathbb{R} \setminus \mathbb{Q}$ of irrationals. In the standard topology, $\text{Cl}(A) = \mathbb{R}$ and $\text{Int}(A) = \emptyset$ (the irrationals are dense with empty interior). In the Sorgenfrey line the same conclusions hold: $\text{Cl}(A) = \mathbb{R}$ (every $[a, b)$ contains an irrational) and $\text{Int}(A) = \emptyset$ (no $[a, b)$ is contained in $\mathbb{R} \setminus \mathbb{Q}$, since every such interval contains rationals).

2.11 Worked examples

Example 2.11.1 (Worked Example 1: interior and closure in the cofinite topology). Let $X = \mathbb{R}$ with the cofinite topology, and let $A = (0, 1)$.

Solution. In the cofinite topology, the closed sets are \mathbb{R} , \emptyset , and all finite subsets of \mathbb{R} .

- **Closure:** $\text{Cl}(A)$ is the smallest closed set containing A . Since $A = (0, 1)$ is infinite, the only closed set containing A is \mathbb{R} itself. Hence $\text{Cl}(A) = \mathbb{R}$.
- **Interior:** $\text{Int}(A)$ is the largest open set contained in A . A set U is open in the cofinite topology iff $\mathbb{R} \setminus U$ is finite (or $U = \emptyset$). If $U \subseteq (0, 1)$ is open and non-empty, then $\mathbb{R} \setminus U$ is finite, but $\mathbb{R} \setminus U \supseteq \mathbb{R} \setminus (0, 1) = (-\infty, 0] \cup [1, \infty)$, which is infinite. Contradiction. Hence $\text{Int}(A) = \emptyset$.
- **Boundary:** $\partial A = \text{Cl}(A) \setminus \text{Int}(A) = \mathbb{R} \setminus \emptyset = \mathbb{R}$.

Example 2.11.2 (Worked Example 2: derived set computation). In \mathbb{R} with the standard topology, let $A = \{1/n : n \in \mathbb{N}, n \geq 1\}$. Find A' , $\text{Cl}(A)$, $\text{Int}(A)$, and ∂A .

Solution.

- **Limit points (A'):** A point x is a limit point of A if every open neighborhood of x contains a point of A distinct from x .

Claim: $A' = \{0\}$. Indeed, every open interval around 0 contains $1/n$ for n sufficiently large, so $0 \in A'$. For $x \neq 0$, if $x \notin A$ pick $\varepsilon > 0$ smaller than the distance from x to the nearest point of $A \cup \{0\}$; then $(x - \varepsilon, x + \varepsilon) \cap A = \emptyset$. If $x = 1/n \in A$, pick ε smaller than $\min(1/n - 1/(n+1), 1/(n-1) - 1/n)$ (with the obvious convention for $n = 1$); then $(x - \varepsilon, x + \varepsilon) \cap (A \setminus \{x\}) = \emptyset$.

- **Closure:** $\text{Cl}(A) = A \cup A' = \{0\} \cup \{1/n : n \geq 1\}$.
- **Interior:** $\text{Int}(A) = \emptyset$ (no open interval is contained in A , since A is countable).
- **Boundary:** $\partial A = \text{Cl}(A) \setminus \text{Int}(A) = \text{Cl}(A) = \{0, 1, 1/2, 1/3, \dots\}$.

Example 2.11.3 (Worked Example 3: the closure of $\mathbb{Q} \times \mathbb{Q}$ in \mathbb{R}^2). In \mathbb{R}^2 with the standard (product) topology, let $A = \mathbb{Q} \times \mathbb{Q}$. Find $\text{Cl}(A)$, $\text{Int}(A)$, and ∂A .

Solution.

- **Closure:** Let $(x, y) \in \mathbb{R}^2$ and let U be any open set containing (x, y) . Then U contains a product of open intervals $(a, b) \times (c, d)$ with $x \in (a, b)$ and $y \in (c, d)$. Since \mathbb{Q} is dense in \mathbb{R} , there exist $q_1 \in (a, b) \cap \mathbb{Q}$ and $q_2 \in (c, d) \cap \mathbb{Q}$, so $(q_1, q_2) \in U \cap A$. By Theorem 2.3.2, $(x, y) \in \text{Cl}(A)$. Therefore $\text{Cl}(A) = \mathbb{R}^2$.
- **Interior:** If $U \subseteq \mathbb{Q} \times \mathbb{Q}$ were a non-empty open set, it would contain a product of open intervals, which contains uncountably many points — but $\mathbb{Q} \times \mathbb{Q}$ is countable. Contradiction. Hence $\text{Int}(A) = \emptyset$.
- **Boundary:** $\partial A = \text{Cl}(A) \setminus \text{Int}(A) = \mathbb{R}^2$.

2.12 Exercises

Exercise 2.12.1 (★). In \mathbb{R} with the standard topology, find $\text{Int}(A)$, $\text{Cl}(A)$, and ∂A for each of the following sets:

- $A = [0, 1)$,
- $A = \mathbb{Q}$,
- $A = \mathbb{Z}$,
- $A = \{1/n : n \in \mathbb{N}, n \geq 1\}$.

Exercise 2.12.2 (★). Prove that a set A is closed if and only if $A' \subseteq A$ (i.e., A contains all its limit points).

Exercise 2.12.3 (★). Let (X, \mathcal{T}) be a topological space. Prove that $\partial A = \emptyset$ if and only if A is clopen.

Exercise 2.12.4 (★★). Let A and B be subsets of a topological space. Prove or disprove:

- $\partial(A \cup B) \subseteq \partial A \cup \partial B$.
- $\partial(A \cap B) \subseteq \partial A \cup \partial B$.

Exercise 2.12.5 (★★). Let A be a dense subset of (X, \mathcal{T}) and let U be an open set. Prove that $\text{Cl}(A \cap U) = \text{Cl}(U)$.

Exercise 2.12.6 (★★). Verify the Kuratowski closure axioms (K1)–(K4) for the closure operator of the cofinite topology on an infinite set X .

Exercise 2.12.7 (★★★). (Kuratowski’s 14-set theorem — explicit computation.) Let $A = (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, 5] \cap \mathbb{Q})$ in \mathbb{R} with the standard topology. Compute all 14 distinct sets obtained by repeatedly applying closure and complement to A .

Exercise 2.12.8 (★★★). Prove that in a topological space (X, \mathcal{T}) , the following are equivalent:

- (i) Every subset of X is either open or closed (or both).
- (ii) The topology \mathcal{T} is the discrete topology.

Hint: Show that (i) implies every singleton is closed, then that every singleton is open.

Exercise 2.12.9 (★★). Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. Prove that $\text{Int}(A)$, ∂A , and $\text{Int}(X \setminus A)$ form a partition of X (i.e., they are pairwise disjoint and their union is X).

Exercise 2.12.10 (★★★). A set A in a topological space is called **regular open** if $A = \text{Int}(\text{Cl}(A))$, and **regular closed** if $A = \text{Cl}(\text{Int}(A))$.

- (a) Give an example of an open set in \mathbb{R} that is not regular open.
- (b) Prove that for any set A , $\text{Int}(\text{Cl}(A))$ is regular open.
- (c) Prove that the collection of regular open sets of a topological space forms a Boolean algebra under the operations $A \wedge B = A \cap B$ and $A \vee B = \text{Int}(\text{Cl}(A \cup B))$.

Chapter summary

- A set is **closed** if its complement is open. Closed sets are stable under arbitrary intersections and finite unions.
- The **interior** $\text{Int}(A)$ is the largest open subset of A ; the **closure** $\text{Cl}(A)$ is the smallest closed superset of A . They are related by complementation: $\text{Cl}(A) = X \setminus \text{Int}(X \setminus A)$.
- $x \in \text{Cl}(A)$ if and only if every open neighborhood of x meets A (Theorem 2.3.2).
- The **boundary** $\partial A = \text{Cl}(A) \setminus \text{Int}(A)$ consists of points every neighborhood of which meets both A and $X \setminus A$.
- The **derived set** A' consists of all limit points; one has $\text{Cl}(A) = A \cup A'$.
- A set is **dense** if its closure is the whole space; **nowhere dense** if the interior of its closure is empty.
- The **Kuratowski closure axioms** (K1)–(K4) provide an equivalent axiomatization of topological spaces via the closure operator.
- **Kuratowski’s 14-set theorem:** starting from any set, repeated application of closure and complement yields at most 14 distinct sets.
- Caution: in non-first-countable spaces, the sequential closure of a set may be strictly smaller than the topological closure.

Chapter 3

Bases, Sub-bases, and Generated Topologies

In the study of metric spaces one quickly observes that every open set can be written as a union of open balls. This single observation carries enormous organizational power: instead of describing a topology by listing *all* of its open sets, it suffices to specify a much smaller collection—a *base*—from which the full topology can be recovered by taking unions. The notion of a base, and the still more economical notion of a *sub-base*, are the central themes of this chapter. They provide systematic machinery for constructing topologies and for comparing different topological spaces.

3.1 Bases for a topology

Definition 3.1.1 (Base for a topology). Let (X, \mathcal{T}) be a topological space. A collection $\mathcal{B} \subseteq \mathcal{T}$ is called a **base** (or **basis**) for \mathcal{T} if every open set $U \in \mathcal{T}$ can be written as a union of members of \mathcal{B} ; equivalently, for every $U \in \mathcal{T}$ and every $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Remark 3.1.1. If \mathcal{B} is a base for \mathcal{T} , then \mathcal{T} equals the collection of all unions of subfamilies of \mathcal{B} (including the empty union, which gives \emptyset). We say that \mathcal{B} *generates* \mathcal{T} .

The following criterion tells us exactly when a collection of subsets of a set X is a base for *some* topology on X .

Theorem 3.1.2 (Base criterion). *Let X be a set and let \mathcal{B} be a collection of subsets of X . Then \mathcal{B} is a base for some topology on X if and only if the following two conditions hold:*

(B1) \mathcal{B} covers X : for every $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.

(B2) For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

When these conditions hold, the topology $\mathcal{T}_{\mathcal{B}}$ generated by \mathcal{B} is

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B} : x \in B \subseteq U\}.$$

Proof. (\Rightarrow) Suppose \mathcal{B} is a base for a topology \mathcal{T} on X . Since $X \in \mathcal{T}$, every point of X lies in some member of \mathcal{B} , giving (B1). For (B2), let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Because $\mathcal{B} \subseteq \mathcal{T}$ and \mathcal{T} is closed under finite intersections, $B_1 \cap B_2 \in \mathcal{T}$. Since \mathcal{B} is a base, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

(\Leftarrow) Assume (B1) and (B2). Define

$$\mathcal{T} = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B} : x \in B \subseteq U\}.$$

We verify the axioms of a topology.

- \emptyset and X belong to \mathcal{T} . The empty set satisfies the condition vacuously. By (B1), every point of X lies in some $B \in \mathcal{B} \subseteq \mathcal{P}(X)$, hence $X \in \mathcal{T}$.
- *Arbitrary unions.* Let $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ and set $U = \bigcup_\alpha U_\alpha$. If $x \in U$, then $x \in U_\alpha$ for some α , so there exists $B \in \mathcal{B}$ with $x \in B \subseteq U_\alpha \subseteq U$. Thus $U \in \mathcal{T}$.
- *Finite intersections.* It suffices to treat the intersection of two members. Let $U_1, U_2 \in \mathcal{T}$ and $x \in U_1 \cap U_2$. Choose $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. By (B2), there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Hence $U_1 \cap U_2 \in \mathcal{T}$.

Finally, each $B \in \mathcal{B}$ belongs to \mathcal{T} (take $B_3 = B$ for every $x \in B$), and by construction every member of \mathcal{T} is a union of members of \mathcal{B} . Therefore \mathcal{B} is a base for \mathcal{T} . \square

3.2 Sub-bases and generated topologies

Definition 3.2.1 (Sub-base). Let X be a set. A collection \mathcal{S} of subsets of X is called a **sub-base** for a topology on X if \mathcal{S} covers X , i.e. $\bigcup_{S \in \mathcal{S}} S = X$. The topology *generated* by \mathcal{S} is the collection $\mathcal{T}_\mathcal{S}$ of all unions of finite intersections of members of \mathcal{S} :

$$\mathcal{T}_\mathcal{S} = \left\{ \bigcup_{\alpha \in A} (S_{\alpha,1} \cap \cdots \cap S_{\alpha,n_\alpha}) \mid S_{\alpha,j} \in \mathcal{S} \right\}.$$

Equivalently, the collection $\mathcal{B}_\mathcal{S} = \{S_1 \cap \cdots \cap S_n \mid S_i \in \mathcal{S}, n \geq 1\}$ is a base for $\mathcal{T}_\mathcal{S}$.

Proposition 3.2.1. Let \mathcal{S} be a sub-base on a set X . Then the collection $\mathcal{B}_\mathcal{S}$ of all finite intersections of members of \mathcal{S} satisfies conditions (B1) and (B2) of Theorem 3.1.2 and hence is a base for a topology on X .

Proof. Since \mathcal{S} covers X , so does $\mathcal{B}_\mathcal{S}$, giving (B1). For (B2), if $B_1 = \bigcap_{i=1}^m S_i$ and $B_2 = \bigcap_{j=1}^n T_j$ with $S_i, T_j \in \mathcal{S}$, then $B_1 \cap B_2 = \bigcap_{i=1}^m S_i \cap \bigcap_{j=1}^n T_j$ is itself a finite intersection of members of \mathcal{S} , hence an element of $\mathcal{B}_\mathcal{S}$. Taking $B_3 = B_1 \cap B_2$ satisfies (B2). \square

Definition 3.2.2 (Generated topology — intersection characterization). Let \mathcal{A} be any collection of subsets of a set X . The **topology generated by \mathcal{A}** is

$$\mathcal{T}(\mathcal{A}) = \bigcap \{ \mathcal{T} \mid \mathcal{T} \text{ is a topology on } X \text{ and } \mathcal{A} \subseteq \mathcal{T} \}.$$

This intersection is nonempty (since the discrete topology contains \mathcal{A}) and is itself a topology on X — the *smallest* topology containing every member of \mathcal{A} .

Proposition 3.2.2. If \mathcal{S} covers X , then $\mathcal{T}(\mathcal{S}) = \mathcal{T}_\mathcal{S}$; that is, the topology generated by \mathcal{S} in the sense of Definition 3.2.2 coincides with the sub-base topology of Definition 3.2.1.

Proof. Every topology containing \mathcal{S} must contain all finite intersections of members of \mathcal{S} and all unions thereof, hence contains $\mathcal{T}_\mathcal{S}$. Thus $\mathcal{T}(\mathcal{S}) \supseteq \mathcal{T}_\mathcal{S}$. Conversely, $\mathcal{T}_\mathcal{S}$ is a topology containing \mathcal{S} (by Proposition 3.2.1), so it is one of the topologies in the intersection, giving $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}_\mathcal{S}$. \square

3.3 Countability and separability axioms

Definition 3.3.1 (First and second countability). A topological space X is said to be

- **first countable** if every point $x \in X$ has a countable neighborhood base, i.e. a countable collection $\{B_n\}_{n \in \mathbb{N}}$ of open neighborhoods of x such that for every open set U containing x there exists n with $B_n \subseteq U$;
- **second countable** if X has a countable base for its topology.

Definition 3.3.2 (Separable space). A topological space X is **separable** if it contains a countable dense subset, i.e. a countable set $D \subseteq X$ with $\overline{D} = X$.

Theorem 3.3.1 (Second countable implies separable). *Every second countable space is separable.*

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable base for X . For each n with $B_n \neq \emptyset$, choose a point $x_n \in B_n$ (using the axiom of countable choice). Set $D = \{x_n\}$. Then D is countable. To see that D is dense, let U be a nonempty open set. Since \mathcal{B} is a base, there exists $B_n \subseteq U$ with $B_n \neq \emptyset$; then $x_n \in D \cap U$. Hence every nonempty open set meets D , so $\overline{D} = X$. \square

Definition 3.3.3 (Lindelöf space). A topological space X is called a **Lindelöf space** if every open cover of X admits a countable subcover.

Theorem 3.3.2 (Second countable implies Lindelöf). *Every second countable space is a Lindelöf space.*

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable base for X , and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X . For each $x \in X$ there exist $\alpha_x \in A$ and $B_{n_x} \in \mathcal{B}$ with $x \in B_{n_x} \subseteq U_{\alpha_x}$. The collection $\{B_{n_x}\}_{x \in X}$ is a sub-collection of the countable family \mathcal{B} , so the set of indices $I = \{n_x : x \in X\} \subseteq \mathbb{N}$ is countable. For each $n \in I$, choose one $\alpha_n \in A$ such that $B_n \subseteq U_{\alpha_n}$. Then $\{U_{\alpha_n}\}_{n \in I}$ is a countable sub-collection of the original cover, and

$$X = \bigcup_{x \in X} B_{n_x} \subseteq \bigcup_{n \in I} B_n \subseteq \bigcup_{n \in I} U_{\alpha_n} \subseteq X.$$

Hence $\{U_{\alpha_n}\}_{n \in I}$ is a countable subcover. \square

3.4 Standard examples of bases

Example 3.4.1 (Standard base for \mathbb{R}). *The collection of all open intervals (a, b) with $a < b$ is a base for the standard (Euclidean) topology on \mathbb{R} . More economically, the sub-collection $\{(p, q) : p, q \in \mathbb{Q}, p < q\}$ is a countable base, showing that \mathbb{R} is second countable.*

Example 3.4.2 (Standard base for \mathbb{R}^n). *The collection of all open balls $B_d(\mathbf{x}, \varepsilon)$ with $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ is a base for the Euclidean topology. A countable base is obtained by restricting to rational centers $\mathbf{x} \in \mathbb{Q}^n$ and rational radii $\varepsilon \in \mathbb{Q}_{>0}$. Hence \mathbb{R}^n is second countable.*

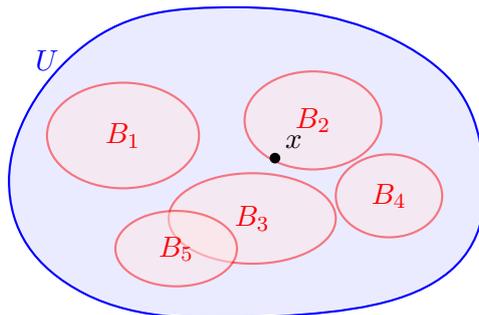
Example 3.4.3 (Sorgenfrey line). *The **Sorgenfrey line** \mathbb{R}_ℓ is \mathbb{R} equipped with the topology generated by the base $\mathcal{B} = \{[a, b) : a < b, a, b \in \mathbb{R}\}$. One readily checks that \mathcal{B} satisfies (B1) and (B2). The resulting topology is strictly finer than the standard topology.*

Example 3.4.4 (Order topology). Let (X, \leq) be a linearly ordered set with more than one element. The **order topology** on X has as a base all sets of the following forms:

1. $(a, b) = \{x \in X : a < x < b\}$ for $a < b$ in X ,
2. $[a_0, b) = \{x \in X : a_0 \leq x < b\}$ if X has a smallest element a_0 ,
3. $(a, b_0] = \{x \in X : a < x \leq b_0\}$ if X has a largest element b_0 .

3.5 Visualizing base elements

The following diagram illustrates how an open set U is expressed as a union of base elements.



$$U = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup \dots \quad (\text{each } B_i \in \mathcal{B})$$

3.6 Counterexample: the Sorgenfrey line

Proposition 3.6.1. *The Sorgenfrey line \mathbb{R}_ℓ is first countable.*

Proof. For each $x \in \mathbb{R}$, the collection $\{[x, x + 1/n) : n \in \mathbb{N}, n \geq 1\}$ is a countable neighborhood base at x . Indeed, if U is open in \mathbb{R}_ℓ and $x \in U$, there exist $a \leq x < b$ with $[a, b) \subseteq U$; then for n large enough, $[x, x + 1/n) \subseteq [a, b) \subseteq U$. \square

Proposition 3.6.2. *The Sorgenfrey line \mathbb{R}_ℓ is separable.*

Proof. The set \mathbb{Q} is dense in \mathbb{R}_ℓ : every nonempty basic open set $[a, b)$ with $a < b$ contains a rational number. \square

Proposition 3.6.3. *The Sorgenfrey line \mathbb{R}_ℓ is not second countable.*

Proof. Suppose for contradiction that $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ is a countable base for \mathbb{R}_ℓ . For each $x \in \mathbb{R}$, the set $[x, x + 1)$ is open, so there exists $B_{n_x} \in \mathcal{B}$ with $x \in B_{n_x} \subseteq [x, x + 1)$. Since x is the infimum of $[x, x + 1)$ and $B_{n_x} \subseteq [x, x + 1)$ with $x \in B_{n_x}$, we see that $\inf B_{n_x} = x$. Therefore the map $x \mapsto n_x$ is injective (distinct reals yield distinct indices), giving an injection $\mathbb{R} \hookrightarrow \mathbb{N}$, which contradicts the uncountability of \mathbb{R} . \square

3.7 Worked examples

Example 3.7.1 (Comparing bases). Let $\mathcal{B}_1 = \{(a, b) : a < b\}$ and $\mathcal{B}_2 = \{[a, b) : a < b\}$ be bases on \mathbb{R} . We show $\mathcal{T}_{\mathcal{B}_2}$ is strictly finer than $\mathcal{T}_{\mathcal{B}_1}$.

$\mathcal{T}_{\mathcal{B}_1} \subseteq \mathcal{T}_{\mathcal{B}_2}$: Given (a, b) and $x \in (a, b)$, we have $[x, b) \subseteq (a, b)$, so each standard open set is a union of Sorgenfrey-open sets.

Strict containment: The set $[0, 1)$ is open in $\mathcal{T}_{\mathcal{B}_2}$ but not in $\mathcal{T}_{\mathcal{B}_1}$, since no open interval (a, b) contains 0 and is contained in $[0, \infty)$ with $b \leq 1$ while also having $a < 0$; more precisely, 0 has no standard neighborhood contained in $[0, 1)$.

Example 3.7.2 (Product topology sub-base). Let X_1, X_2 be topological spaces with topologies $\mathcal{T}_1, \mathcal{T}_2$. The collection

$$\mathcal{S} = \{\pi_1^{-1}(U_1) : U_1 \in \mathcal{T}_1\} \cup \{\pi_2^{-1}(U_2) : U_2 \in \mathcal{T}_2\}$$

is a sub-base for the product topology on $X_1 \times X_2$. A finite intersection of sub-base elements has the form $\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) = U_1 \times U_2$, recovering the standard base of open rectangles.

Example 3.7.3 (A topology from a sub-base on a finite set). Let $X = \{1, 2, 3\}$ and $\mathcal{S} = \{\{1, 2\}, \{2, 3\}\}$. Then \mathcal{S} covers X since $\{1, 2\} \cup \{2, 3\} = X$. The finite intersections of members of \mathcal{S} are $\{1, 2\}$, $\{2, 3\}$, and $\{1, 2\} \cap \{2, 3\} = \{2\}$. Taking all unions, the generated topology is

$$\mathcal{T} = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Note that $\{1\}$ and $\{3\}$ are not open.

Example 3.7.4 (Countable base for a metric space). Let (X, d) be a separable metric space with countable dense subset D . Then $\mathcal{B} = \{B_d(q, 1/n) : q \in D, n \in \mathbb{N}_{\geq 1}\}$ is a countable base, so (X, d) is second countable. Thus, for metric spaces, separability and second countability are equivalent.

3.8 Exercises

Exercise 3.8.1 (*). Show that the collection of singletons $\{\{x\} : x \in X\}$ is a base for the discrete topology on X .

Exercise 3.8.2 (*). Let X be an infinite set. Does the cofinite topology on X have a countable base? Justify your answer.

Exercise 3.8.3 (*). Let $X = \mathbb{R}$ and $\mathcal{S} = \{\mathbb{R} \setminus \{x\} : x \in \mathbb{R}\}$. Show that \mathcal{S} is a sub-base for the cofinite topology on \mathbb{R} .

Exercise 3.8.4 (**). Let \mathcal{B} and \mathcal{B}' be bases for topologies \mathcal{T} and \mathcal{T}' on X . Show that $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for every $B \in \mathcal{B}$ and $x \in B$ there exists $B' \in \mathcal{B}'$ with $x \in B' \subseteq B$.

Exercise 3.8.5 (**). Show that every subspace of a second countable space is second countable.

Exercise 3.8.6 (**). Give an example of a topological space that is separable but not second countable. (*Hint*: consider the Sorgenfrey line, or an uncountable set with a suitable topology.)

Exercise 3.8.7 (**). Let X be a metrizable Lindelöf space. Show that X is second countable. (*Hint*: for each n , cover X by balls of radius $1/n$.)

Exercise 3.8.8 (***)). Show that the Sorgenfrey plane $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is separable but not Lindelöf. (*Hint*: consider the anti-diagonal $\{(x, -x) : x \in \mathbb{R}\}$ and cover it by basic open sets.)

Exercise 3.8.9 (***)). Let \mathcal{A} and \mathcal{B} be collections of subsets of X . Prove that $\mathcal{T}(\mathcal{A} \cup \mathcal{B})$ is the smallest topology containing both $\mathcal{T}(\mathcal{A})$ and $\mathcal{T}(\mathcal{B})$. Is $\mathcal{T}(\mathcal{A} \cup \mathcal{B}) = \mathcal{T}(\mathcal{A}) \cup \mathcal{T}(\mathcal{B})$ in general? Give a proof or counterexample.

Chapter summary

- A **base** for a topology is a sub-collection such that every open set is a union of base elements. The base criterion (Theorem 3.1.2) provides a direct test.
- A **sub-base** generates a topology by first taking all finite intersections (to produce a base) and then all unions.

- The **generated topology** $\mathcal{T}(\mathcal{A})$ is the intersection of all topologies containing \mathcal{A} .
- **Second countable** \Rightarrow **separable** \Rightarrow (in metrizable spaces) second countable.
- **Second countable** \Rightarrow **Lindelöf**.
- The Sorgenfrey line is first countable and separable, but not second countable, demonstrating that the converses fail.

Chapter 4

Continuity, Homeomorphisms, and Topological Invariants

The notion of a continuous function lies at the very heart of topology. In real analysis, continuity is defined via the ε - δ formalism: $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. A standard exercise shows that this is equivalent to requiring that the preimage of every open set be open. It is precisely this preimage formulation that generalizes to arbitrary topological spaces, freeing the concept of continuity from any dependence on a metric. We then single out the most important class of continuous maps—the *homeomorphisms*—and introduce the fundamental idea of a topological invariant.

4.1 Continuous maps

Definition 4.1.1 (Continuous map). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A function $f: X \rightarrow Y$ is **continuous** if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V)$ belongs to \mathcal{T}_X .

Theorem 4.1.1 (Equivalent characterizations of continuity). *Let $f: X \rightarrow Y$ be a function between topological spaces. The following are equivalent:*

1. f is continuous (preimages of open sets are open).
2. For every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X .
3. For every subset $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.
4. For every subset $B \subseteq Y$, $f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B))$.
5. For every $x \in X$ and every neighborhood V of $f(x)$ in Y , the preimage $f^{-1}(V)$ is a neighborhood of x in X .

Proof. **(1) \Leftrightarrow (2):** A set C is closed in Y if and only if $Y \setminus C$ is open. Since $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$, we see that $f^{-1}(C)$ is closed in X if and only if $f^{-1}(Y \setminus C)$ is open in X . Hence the equivalence.

(2) \Rightarrow (3): Let $A \subseteq X$. The set $\overline{f(A)}$ is closed in Y , so by (2), $f^{-1}(\overline{f(A)})$ is closed in X . Since $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$ and the latter is closed, $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Applying f , $f(\overline{A}) \subseteq \overline{f(A)}$.

(3) \Rightarrow (2): Let $C \subseteq Y$ be closed. Set $A = f^{-1}(C)$. By (3), $f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{C} = C$. Hence $\overline{A} \subseteq f^{-1}(C) = A$, so A is closed.

(1)⇒(4): Let $B \subseteq Y$. Since $\text{Int}(B)$ is open in Y , $f^{-1}(\text{Int}(B))$ is open in X by (1). Moreover, $f^{-1}(\text{Int}(B)) \subseteq f^{-1}(B)$. Since $f^{-1}(\text{Int}(B))$ is an open subset of $f^{-1}(B)$, it is contained in $\text{Int}(f^{-1}(B))$.

(4)⇒(1): Let V be open in Y . Then $\text{Int}(V) = V$, so $f^{-1}(V) = f^{-1}(\text{Int}(V)) \subseteq \text{Int}(f^{-1}(V)) \subseteq f^{-1}(V)$. Equality holds throughout, so $f^{-1}(V) = \text{Int}(f^{-1}(V))$, meaning $f^{-1}(V)$ is open.

(1)⇔(5): If f is continuous and V is a neighborhood of $f(x)$, there exists an open set U with $f(x) \in U \subseteq V$. Then $x \in f^{-1}(U) \subseteq f^{-1}(V)$ and $f^{-1}(U)$ is open, so $f^{-1}(V)$ is a neighborhood of x . Conversely, if (5) holds and V is open in Y , then for every $x \in f^{-1}(V)$, V is a neighborhood of $f(x)$, so $f^{-1}(V)$ is a neighborhood of x ; hence $f^{-1}(V)$ is a neighborhood of each of its points, so it is open. \square

Theorem 4.1.2 (Composition of continuous maps). *If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous, then $g \circ f: X \rightarrow Z$ is continuous.*

Proof. Let W be open in Z . Since g is continuous, $g^{-1}(W)$ is open in Y . Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X . But $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$, so $(g \circ f)^{-1}(W)$ is open in X . \square

4.2 Homeomorphisms and topological invariants

Definition 4.2.1 (Homeomorphism). A function $f: X \rightarrow Y$ is a **homeomorphism** if f is a bijection and both f and f^{-1} are continuous. When such a map exists, X and Y are said to be **homeomorphic**, written $X \cong Y$.

Definition 4.2.2 (Topological property / invariant). A property of a topological space is a **topological invariant** (or **topological property**) if it is preserved by homeomorphisms; that is, if X has the property and $X \cong Y$, then Y has the property as well.

Remark 4.2.1. Examples of topological invariants include: being Hausdorff, being compact, being connected, being second countable, cardinality of the space, and the fundamental group $\pi_1(X, x_0)$.

4.3 Open, closed, and continuous maps

Definition 4.3.1 (Open and closed maps). A function $f: X \rightarrow Y$ is called

- an **open map** if for every open set $U \subseteq X$, the image $f(U)$ is open in Y ;
- a **closed map** if for every closed set $C \subseteq X$, the image $f(C)$ is closed in Y .

Remark 4.3.1. These notions are logically independent of continuity. A continuous map need not be open or closed, and an open (or closed) map need not be continuous.

Example 4.3.2 (Continuous but not open). *The constant function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 0$, is continuous. However, $f(\mathbb{R}) = \{0\}$, which is not open in \mathbb{R} . So f is continuous but not an open map.*

Example 4.3.3 (Open but not continuous). *Let $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the standard topology. Consider $f: Y \rightarrow X$ defined by $f(x) = x$. Every subset of X is open, so the image of any set under f is open in X ; hence $f: Y \rightarrow X$ is an open map. But f is*

not continuous: the singleton $\{0\}$ is open in X (discrete topology) while $f^{-1}(\{0\}) = \{0\}$ is not open in Y (standard topology).

Example 4.3.4 (Closed but not continuous). Let $X = \mathbb{R}$ with the standard topology and $Y = \mathbb{R}$ with the indiscrete topology. The identity $f: X \rightarrow Y$, $f(x) = x$, maps every set to itself. In Y the only closed sets are \emptyset and \mathbb{R} , so f is vacuously closed (the only closed images we need to check collapse to \emptyset or \mathbb{R}). But f is not continuous: $Y \setminus \emptyset = \mathbb{R}$ is the only nonempty open set in Y , yet many open sets in Y (there are only two) pull back to open sets in X ; more precisely, f^{-1} from the indiscrete topology gives only \emptyset and \mathbb{R} as open, but in X open sets like $(0, 1)$ have $f^{-1}((0, 1)) = (0, 1)$ — wait, f goes $X \rightarrow Y$, and $(0, 1)$ is not open in Y . So f is trivially continuous. To correct: let $g: Y \rightarrow X$ (from indiscrete to standard). Then g is closed because the closed sets in Y are \emptyset and \mathbb{R} , and $g(\emptyset) = \emptyset$, $g(\mathbb{R}) = \mathbb{R}$, both closed in X . But g is not continuous: $(0, 1)$ is open in X while $g^{-1}((0, 1)) = (0, 1)$ is not open in the indiscrete topology on Y .

Proposition 4.3.5. A function $f: X \rightarrow Y$ is a homeomorphism if and only if f is a continuous open bijection; equivalently, if and only if f is a continuous closed bijection.

Proof. If f is a homeomorphism, then f is a continuous bijection and f^{-1} is continuous. For any open $U \subseteq X$, $f(U) = (f^{-1})^{-1}(U)$ is open in Y since f^{-1} is continuous; so f is open. Conversely, if f is a continuous open bijection, then for any open $U \subseteq X$, $(f^{-1})^{-1}(U) = f(U)$ is open, so f^{-1} is continuous. The argument for “closed” is analogous, replacing “open” by “closed” throughout. \square

4.4 A compact-to-Hausdorff theorem

Theorem 4.4.1. Let $f: X \rightarrow Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. By Proposition 4.3.5, it suffices to show that f is a closed map. Let $C \subseteq X$ be closed. Since X is compact and C is a closed subset of a compact space, C is compact. The continuous image of a compact set is compact, so $f(C)$ is compact in Y . Since Y is Hausdorff, every compact subset of Y is closed. Therefore $f(C)$ is closed in Y . Hence f is a closed map, and since it is also a continuous bijection, it is a homeomorphism. \square

4.5 Topological embeddings

Definition 4.5.1 (Topological embedding). A function $f: X \rightarrow Y$ is a **topological embedding** (or simply an **embedding**) if f is an injective continuous map such that $f: X \rightarrow f(X)$ is a homeomorphism, where $f(X)$ carries the subspace topology inherited from Y .

Remark 4.5.1. Equivalently, $f: X \rightarrow Y$ is an embedding if and only if f is injective, continuous, and for every open $U \subseteq X$ there exists an open $V \subseteq Y$ with $f(U) = V \cap f(X)$.

Example 4.5.2. The inclusion $\iota: (0, 1) \hookrightarrow \mathbb{R}$, $\iota(x) = x$, is a topological embedding. More generally, the inclusion of any subspace into the ambient space is always an embedding.

4.6 The pasting lemma

Theorem 4.6.1 (Pasting lemma). Let $X = A \cup B$ where A and B are closed subsets of X . Let $f: A \rightarrow Y$ and $g: B \rightarrow Y$ be continuous functions that agree on the overlap: $f(x) = g(x)$ for all

$x \in A \cap B$. Then the function $h: X \rightarrow Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B, \end{cases}$$

is continuous.

Proof. We use the characterization of continuity via closed sets (Theorem 4.1.1, (2)). Let $C \subseteq Y$ be closed. Then

$$h^{-1}(C) = \{x \in X : h(x) \in C\} = \{x \in A : f(x) \in C\} \cup \{x \in B : g(x) \in C\} = f^{-1}(C) \cup g^{-1}(C).$$

Since $f: A \rightarrow Y$ is continuous and C is closed in Y , the set $f^{-1}(C)$ is closed in the subspace A . Because A is closed in X , a set closed in A is closed in X (a closed subset of a closed set is closed). Hence $f^{-1}(C)$ is closed in X . By the same reasoning, $g^{-1}(C)$ is closed in X . A finite union of closed sets is closed, so $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed in X . Therefore h is continuous. \square

Remark 4.6.2. An analogous statement holds when A and B are both *open* (and the proof is even simpler, working directly with open sets). The lemma generalizes to finite covers by closed sets $X = A_1 \cup \dots \cup A_n$, but *not* to arbitrary covers by closed sets.

4.7 Key examples and constructions

Example 4.7.1 ($(0, 1) \cong \mathbb{R}$). Define $f: (0, 1) \rightarrow \mathbb{R}$ by

$$f(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right).$$

This is a composition of continuous functions: $x \mapsto \pi(x - \frac{1}{2})$ (affine, hence continuous) followed by \tan (continuous on $(-\pi/2, \pi/2)$). The map f is a bijection from $(0, 1)$ onto \mathbb{R} with continuous inverse $f^{-1}(y) = \frac{1}{\pi} \arctan(y) + \frac{1}{2}$. Therefore f is a homeomorphism. In particular, $(0, 1)$ and \mathbb{R} are homeomorphic.

Example 4.7.2 (Stereographic projection). Let $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$ denote the n -sphere and let $N = (0, \dots, 0, 1)$ be its north pole. The stereographic projection $\sigma: S^n \setminus \{N\} \rightarrow \mathbb{R}^n$ is defined by

$$\sigma(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n).$$

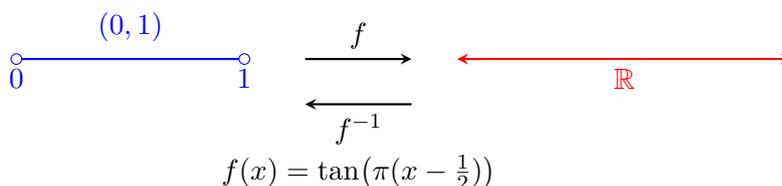
One verifies that σ is a bijection with inverse

$$\sigma^{-1}(u_1, \dots, u_n) = \frac{1}{|\mathbf{u}|^2 + 1}(2u_1, \dots, 2u_n, |\mathbf{u}|^2 - 1),$$

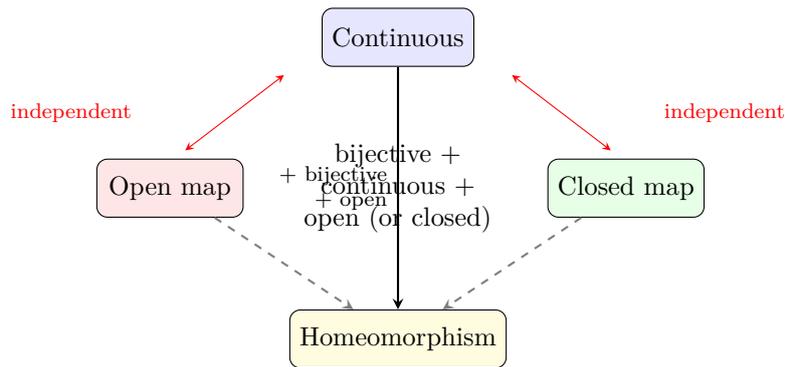
and both maps are continuous (being compositions of rational functions with nonvanishing denominators). Hence $S^n \setminus \{N\} \cong \mathbb{R}^n$.

4.8 Visualizations

Homeomorphism between $(0, 1)$ and \mathbb{R} .



Open, closed, and continuous maps.



4.9 Counterexample: a continuous bijection that is not a homeomorphism

Example 4.9.1. Define $f: [0, 2\pi) \rightarrow S^1$ by $f(t) = (\cos t, \sin t)$, where $[0, 2\pi)$ carries the subspace topology from \mathbb{R} and S^1 carries the subspace topology from \mathbb{R}^2 . Then f is a continuous bijection.

However, f is not a homeomorphism. To see this, consider the set $U = [0, 1)$, which is open in $[0, 2\pi)$ (as $[0, 1) = [0, 2\pi) \cap (-1, 1)$). Its image $f(U)$ is the arc $\{(\cos t, \sin t) : 0 \leq t < 1\}$. This arc contains the point $(1, 0) = f(0)$, but every open neighborhood of $(1, 0)$ in S^1 contains points $(\cos t, \sin t)$ with t slightly less than 2π , which lie outside $f(U)$. Hence $f(U)$ is not open in S^1 , so f is not an open map, and therefore f^{-1} is not continuous.

Note that $[0, 2\pi)$ is not compact (it is not closed in \mathbb{R} , and in particular it is not a closed bounded subset of \mathbb{R}), so Theorem 4.4.1 does not apply.

4.10 Invariance of domain

Theorem 4.10.1 (Invariance of domain — Brouwer, 1912). Let $U \subseteq \mathbb{R}^n$ be open and let $f: U \rightarrow \mathbb{R}^n$ be a continuous injective map. Then $f(U)$ is open in \mathbb{R}^n , and $f: U \rightarrow f(U)$ is a homeomorphism.

Remark 4.10.2. The proof of invariance of domain requires tools from algebraic topology (specifically, homology theory) and is beyond the scope of this course. Nevertheless, the result has profound consequences:

1. It implies that $\mathbb{R}^m \cong \mathbb{R}^n$ only if $m = n$, establishing that *topological dimension* is a well-defined invariant of Euclidean spaces.
2. It guarantees that any continuous injection from an open subset of \mathbb{R}^n into \mathbb{R}^n is automatically an open map (and hence an embedding), a fact that fails in general topological spaces.
3. It shows that the notion of “dimension” in \mathbb{R}^n is a topological invariant, not merely an algebraic artifact.

4.11 Worked examples

Example 4.11.1 (Continuity via sub-base). Let $f: X \rightarrow Y$ be a function and let \mathcal{S} be a sub-base for the topology on Y . Then f is continuous if and only if $f^{-1}(S)$ is open in X for every $S \in \mathcal{S}$.

Proof. The “only if” direction is immediate. For the converse, suppose $f^{-1}(S)$ is open for all $S \in \mathcal{S}$. Every basic open set is a finite intersection $S_1 \cap \cdots \cap S_n$, and $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$, a finite intersection of open sets, hence open. Every open set is a union of basic open sets, and f^{-1} commutes with unions, so the preimage of every open set is open.

Example 4.11.2 (The projection map is continuous and open but not closed). Let $\pi_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection $\pi_1(x, y) = x$. The preimage of an open set $(a, b) \subseteq \mathbb{R}$ is $(a, b) \times \mathbb{R}$, which is open in \mathbb{R}^2 , so π_1 is continuous. For any basic open set $U \times V \subseteq \mathbb{R}^2$ (with U, V open in \mathbb{R}), $\pi_1(U \times V) = U$ (provided $V \neq \emptyset$), which is open; hence π_1 is an open map. However, π_1 is not closed: the set $C = \{(x, y) : xy = 1\}$ is closed in \mathbb{R}^2 (it is the preimage of $\{1\}$ under the continuous map $(x, y) \mapsto xy$), but $\pi_1(C) = \mathbb{R} \setminus \{0\}$, which is not closed in \mathbb{R} .

Example 4.11.3 (Homeomorphism via the compact-Hausdorff theorem). Consider the map $f: [0, 1]/(0 \sim 1) \rightarrow S^1$ induced by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. The quotient space $[0, 1]/(0 \sim 1)$ is compact (it is a quotient of a compact space) and S^1 is Hausdorff (as a subspace of \mathbb{R}^2). The map f is a continuous bijection (it descends from the continuous surjection $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ and becomes injective after the identification $0 \sim 1$). By Theorem 4.4.1, f is a homeomorphism: $[0, 1]/(0 \sim 1) \cong S^1$.

4.12 Exercises

Exercise 4.12.1 (*). Prove that every constant function $f: X \rightarrow Y$ is continuous.

Exercise 4.12.2 (*). Let $\mathcal{T}_1 \subseteq \mathcal{T}_2$ be two topologies on a set X . Show that the identity map $\text{id}: (X, \mathcal{T}_2) \rightarrow (X, \mathcal{T}_1)$ is continuous but that the reverse identity $\text{id}: (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2)$ need not be.

Exercise 4.12.3 (*). Prove that if $f: X \rightarrow Y$ is continuous and $A \subseteq X$ carries the subspace topology, then $f|_A: A \rightarrow Y$ is continuous.

Exercise 4.12.4 (**). Let $f: X \rightarrow Y$ be continuous with Y Hausdorff. Show that the graph $\Gamma_f = \{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$.

Exercise 4.12.5 (**). Prove that any two nonempty open intervals in \mathbb{R} are homeomorphic. (*Hint:* use affine maps.)

Exercise 4.12.6 (**). State and prove the pasting lemma for three closed sets: if $X = A \cup B \cup C$ with A, B, C closed, and $f: A \rightarrow Y, g: B \rightarrow Y, h: C \rightarrow Y$ are continuous and agree on overlaps, then the combined function is continuous.

Exercise 4.12.7 (**). Show that “having a countable dense subset” (separability) is a topological invariant.

Exercise 4.12.8 (***). Show that there is no continuous bijection from \mathbb{R} to \mathbb{R}^2 . (*Hint:* consider removing a point from each and examine connectedness.)

Exercise 4.12.9 (***). Let $f: X \rightarrow Y$ be injective and continuous. Prove that f is an embedding if and only if for every open set $U \subseteq X$ and every $x \in U$, there exists an open set $V \subseteq Y$ such that $f(x) \in V$ and $f^{-1}(V) \subseteq U$.

Chapter summary

- A map $f: X \rightarrow Y$ is **continuous** if preimages of open sets are open. Equivalent formulations involve closed sets, closures, interiors, or neighborhoods (Theorem 4.1.1).

- The composition of continuous maps is continuous (Theorem 4.1.2).
- A **homeomorphism** is a continuous bijection with continuous inverse. Properties preserved under homeomorphisms are **topological invariants**.
- Open maps, closed maps, and continuous maps are logically independent notions. A homeomorphism is precisely a continuous open (or closed) bijection.
- A continuous bijection from a **compact** space to a **Hausdorff** space is automatically a homeomorphism (Theorem 4.4.1).
- The **pasting lemma** (Theorem 4.6.1) allows one to build continuous maps from compatible pieces defined on closed (or open) subsets.
- **Invariance of domain** guarantees that continuous injections from open subsets of \mathbb{R}^n into \mathbb{R}^n are open maps.

Chapter 5

Separation Axioms

In a metric space (X, d) , distinct points $x \neq y$ can always be separated by disjoint open balls: take $r = d(x, y)/2$ and observe that $B(x, r) \cap B(y, r) = \emptyset$. Closed sets can likewise be separated from points outside them, and even pairs of disjoint closed sets can be enclosed in disjoint open neighbourhoods. These properties, however, need not hold in an arbitrary topological space. The *separation axioms* form a hierarchy of increasingly restrictive conditions that a topology may or may not satisfy, each capturing a different degree of “separating power.” This chapter develops the axioms T_0 through T_4 , proves the strict chain of implications among them, and illustrates each level with examples and counterexamples.

5.1 The T_0 Axiom (Kolmogorov Spaces)

Definition 5.1.1 (T_0 space). A topological space (X, τ) is T_0 (or a **Kolmogorov space**) if for every pair of distinct points $x, y \in X$, there exists an open set $U \in \tau$ that contains exactly one of them: either $x \in U, y \notin U$ or $y \in U, x \notin U$.

Example 5.1.1 (The Sierpiński space). Let $X = \{0, 1\}$ with topology $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$. For the pair $0, 1$ the open set $\{1\}$ contains 1 but not 0. Hence (X, τ) is T_0 . Note, however, that no open set contains 0 without also containing 1, so the space is not T_1 .

Example 5.1.2 (Indiscrete space is not T_0). If $|X| \geq 2$ and $\tau = \{\emptyset, X\}$, then the only non-empty open set is X itself, which contains every point. No pair of distinct points can be topologically distinguished; thus (X, τ) is not T_0 .

5.2 The T_1 Axiom

Definition 5.2.1 (T_1 space). A topological space (X, τ) is T_1 if for every pair of distinct points $x, y \in X$, there exists an open set U with $x \in U$ and $y \notin U$.

The key difference from T_0 is that the requirement is *symmetric*: for each ordered pair (x, y) with $x \neq y$, a separating open set for x must exist.

Theorem 5.2.1 (Characterisation of T_1 spaces). A topological space (X, τ) is T_1 if and only if every singleton $\{x\}$ is closed.

Proof. (\Rightarrow) Assume (X, τ) is T_1 and fix $x \in X$. For every $y \neq x$ there is an open set U_y with $y \in U_y$ and $x \notin U_y$. Then

$$X \setminus \{x\} = \bigcup_{y \neq x} U_y,$$

which is a union of open sets and hence open. Therefore $\{x\}$ is closed.

(\Leftarrow) Suppose every singleton is closed. Given distinct x, y , the set $U = X \setminus \{y\}$ is open, contains x , and does not contain y . Hence (X, τ) is T_1 . \square

Example 5.2.2 (Cofinite topology). Let X be any infinite set and let τ_{cof} consist of \emptyset together with all subsets whose complement is finite. Every singleton $\{x\}$ has complement $X \setminus \{x\}$, which is cofinite and hence open. By Theorem 5.2.1, (X, τ_{cof}) is T_1 . We shall see shortly that it is not T_2 .

5.3 The T_2 Axiom (Hausdorff Spaces)

Definition 5.3.1 (Hausdorff space). A topological space (X, τ) is T_2 (or **Hausdorff**) if for every pair of distinct points $x, y \in X$, there exist open sets U, V with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Theorem 5.3.1 (Uniqueness of limits in Hausdorff spaces). In a Hausdorff space, a net can converge to at most one point.

Proof. Let $(x_\alpha)_{\alpha \in A}$ be a net in a Hausdorff space X and suppose $x_\alpha \rightarrow x$ and $x_\alpha \rightarrow y$ with $x \neq y$. Choose disjoint open sets $U \ni x$ and $V \ni y$. By convergence to x , there exists α_0 such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. By convergence to y , there exists α_1 such that $x_\alpha \in V$ for all $\alpha \geq \alpha_1$. Since A is directed, there exists $\beta \geq \alpha_0, \alpha_1$. Then $x_\beta \in U \cap V = \emptyset$, a contradiction. \square

Theorem 5.3.2 (Compact subsets of Hausdorff spaces are closed). Let X be a Hausdorff space and $K \subseteq X$ compact. Then K is closed in X .

Proof. We show $X \setminus K$ is open. Fix $x \in X \setminus K$. For each $y \in K$, the Hausdorff condition gives open sets $U_y \ni x$ and $V_y \ni y$ with $U_y \cap V_y = \emptyset$. The collection $\{V_y\}_{y \in K}$ is an open cover of K . By compactness, there exist $y_1, \dots, y_n \in K$ with $K \subseteq V_{y_1} \cup \dots \cup V_{y_n}$. Set $U = U_{y_1} \cap \dots \cap U_{y_n}$. Then U is a finite intersection of open sets (hence open), $x \in U$, and

$$U \cap K \subseteq U \cap (V_{y_1} \cup \dots \cup V_{y_n}) = \bigcup_{i=1}^n (U \cap V_{y_i}) \subseteq \bigcup_{i=1}^n (U_{y_i} \cap V_{y_i}) = \emptyset.$$

Therefore $U \subseteq X \setminus K$. Since each $x \in X \setminus K$ has such a neighbourhood, $X \setminus K$ is open, so K is closed. \square

5.4 Regular Spaces and the T_3 Axiom

Definition 5.4.1 (Regular space). A topological space X is **regular** if for every closed set F and every point $x \notin F$, there exist disjoint open sets $U \ni x$ and $V \supseteq F$.

Definition 5.4.2 (T_3 space). A space is T_3 if it is both T_1 and regular.

Remark 5.4.1. The naming convention varies in the literature. We follow the convention where T_3 requires both T_1 and regularity. Some authors use the terms in the opposite sense; care is needed when consulting different sources.

Example 5.4.2. Every metric space is regular. Indeed, if F is closed and $x \notin F$, then $d(x, F) = \inf_{y \in F} d(x, y) > 0$. Set $r = d(x, F)/2$ and take $U = B(x, r)$ and $V = \bigcup_{y \in F} B(y, r)$. These are open and disjoint.

5.5 Normal Spaces and the T_4 Axiom

Definition 5.5.1 (Normal space). A topological space X is **normal** if for every pair of disjoint closed sets F_1, F_2 , there exist disjoint open sets $U_1 \supseteq F_1$ and $U_2 \supseteq F_2$.

Definition 5.5.2 (T_4 space). A space is T_4 if it is both T_1 and normal.

5.6 The Hierarchy of Separation Axioms

Theorem 5.6.1 (The separation chain). *For any topological space,*

$$T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0.$$

Proof. $T_4 \implies T_3$: Suppose X is T_4 (hence T_1 and normal). Let F be closed and $x \notin F$. Since X is T_1 , the singleton $\{x\}$ is closed. The sets $\{x\}$ and F are disjoint closed sets, so normality provides disjoint open sets $U \supseteq \{x\}$ and $V \supseteq F$. Hence X is regular. Combined with T_1 , X is T_3 .

$T_3 \implies T_2$: Suppose X is T_3 (hence T_1 and regular). Let $x \neq y$. Since X is T_1 , $\{y\}$ is closed. Since $x \notin \{y\}$, regularity gives disjoint open sets $U \ni x$ and $V \ni y$. Hence X is T_2 .

$T_2 \implies T_1$: If $x \neq y$, the Hausdorff condition gives disjoint open $U \ni x$ and $V \ni y$. In particular, U is open, contains x , and misses y . Hence X is T_1 .

$T_1 \implies T_0$: Immediate from the definitions: T_1 requires a separating open set for every *ordered* pair, which a fortiori gives one for every *unordered* pair. \square

None of these implications reverses:

Proposition 5.6.2 (No implication reverses).

1. The Sierpiński space (Example 5.1.1) is T_0 but not T_1 .
2. The cofinite topology on an infinite set (Example 5.2.2) is T_1 but not T_2 .
3. There exist T_2 spaces that are not T_3 (see Example 5.10.3).
4. The Sorgenfrey plane (Example 5.10.4) shows that normality is not preserved by products; in particular one can produce T_3 spaces that are not T_4 .

Proof. (1) and (2) follow from the examples already discussed.

(3) See Example 5.10.3 below.

(4) The Sorgenfrey line \mathbb{R}_ℓ is T_4 , but the product $\mathbb{R}_\ell \times \mathbb{R}_\ell$ (the Sorgenfrey plane) is T_3 (as a product of regular T_1 spaces) but not normal. The proof that the Sorgenfrey plane is not normal is given in Example 5.10.4. \square

5.7 Compact Hausdorff Spaces Are Normal

Theorem 5.7.1. *Every compact Hausdorff space is normal (and hence T_4).*

Proof. Let X be compact and Hausdorff.

Step 1: X is regular. Let F be closed and $x \notin F$. Since X is compact and F is a closed subset of a compact space, F is compact. By the proof of Theorem 5.3.2, for each $y \in F$ we obtain open sets $U_y \ni x$ and $V_y \ni y$ with $U_y \cap V_y = \emptyset$. Covering F by finitely many V_{y_1}, \dots, V_{y_n} and setting $U = \bigcap_{i=1}^n U_{y_i}$, $V = \bigcup_{i=1}^n V_{y_i}$, we get disjoint open sets with $x \in U$ and $F \subseteq V$.

Step 2: X is normal. Let F_1, F_2 be disjoint closed subsets of X . Both are compact. By Step 1, for each $x \in F_1$ there exist disjoint open sets $U_x \ni x$ and $V_x \supseteq F_2$. The open cover $\{U_x\}_{x \in F_1}$ of the compact set F_1 admits a finite subcover U_{x_1}, \dots, U_{x_m} . Set

$$U = U_{x_1} \cup \dots \cup U_{x_m}, \quad V = V_{x_1} \cap \dots \cap V_{x_m}.$$

Then U and V are open, $F_1 \subseteq U$, $F_2 \subseteq V$, and $U \cap V = \emptyset$.

Since X is Hausdorff (hence T_1) and normal, X is T_4 . □

5.8 Every Metric Space Is Normal

Theorem 5.8.1. *Every metrizable space is normal.*

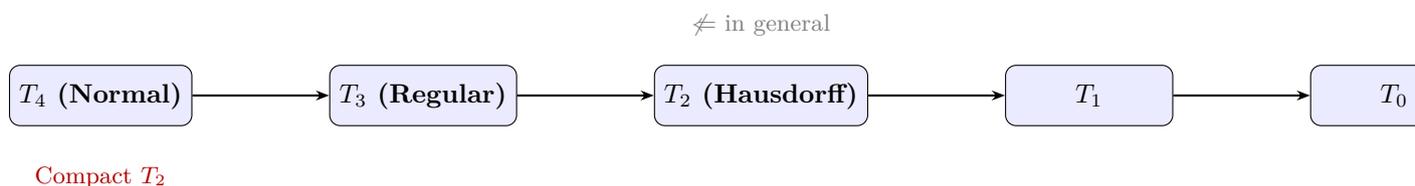
Proof. Let (X, d) be a metric space and let F_1, F_2 be disjoint closed sets. For each $x \in X$, define $d(x, F_i) = \inf_{y \in F_i} d(x, y)$. Since F_i is closed, $d(x, F_i) = 0$ if and only if $x \in F_i$. Define

$$U_1 = \{x \in X : d(x, F_1) < d(x, F_2)\}, \quad U_2 = \{x \in X : d(x, F_2) < d(x, F_1)\}.$$

The function $x \mapsto d(x, F_1) - d(x, F_2)$ is continuous (as a difference of continuous functions), so U_1 and U_2 are open. They are clearly disjoint. If $x \in F_1$, then $d(x, F_1) = 0$ and $d(x, F_2) > 0$ (since $F_1 \cap F_2 = \emptyset$), so $x \in U_1$. Similarly $F_2 \subseteq U_2$. □

5.9 Hierarchy Diagram

The following diagram summarises the relationships among the separation axioms.



5.10 Counterexamples

Example 5.10.1 (Particular point topology: T_0 not T_1). Let X be a set with $|X| \geq 2$ and fix $p \in X$. Define $\tau = \{U \subseteq X : p \in U\} \cup \{\emptyset\}$. For any $x \neq p$, the open set $\{p\}$ contains p but not x , so the space is T_0 . However, if $x \neq p$, every open set containing x must also contain p , so the space is not T_1 .

Example 5.10.2 (Cofinite topology on an infinite set: T_1 not T_2). On an infinite set X with the cofinite topology, any two non-empty open sets U and V satisfy $|X \setminus U| < \infty$ and $|X \setminus V| < \infty$, so $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is finite. Since X is infinite, $U \cap V \neq \emptyset$. Hence no two distinct points can be separated by disjoint open sets, and the space is not T_2 .

Example 5.10.3 (T_2 but not T_3). Let $X = \mathbb{R}$ with the topology generated by the usual open sets together with the set $S = \{1/n : n \in \mathbb{N}^*\}$. Formally, a basis is $\{U \setminus A : U \text{ usual open, } A \subseteq S\}$. This is the **K -topology** on \mathbb{R} . It is Hausdorff (finer than the usual topology), but the closed set $F = \{1/n : n \geq 1\}$ (which is closed because its complement $\mathbb{R} \setminus S$ is open by design) and the point $0 \notin F$ cannot be separated by disjoint open sets, so the space is not regular.

Example 5.10.4 (The Sorgenfrey plane: T_3 not T_4). The **Sorgenfrey line** \mathbb{R}_ℓ has basis $\{[a, b) : a < b\}$. It is T_4 . The product $\mathbb{R}_\ell \times \mathbb{R}_\ell$, the **Sorgenfrey plane**, is T_3 (products of regular T_1 spaces are regular and T_1) but not normal. Indeed, the anti-diagonal $D = \{(x, -x) : x \in \mathbb{R}\}$ is closed and discrete, and one can show (using a cardinality argument) that the two disjoint closed subsets $D_1 = \{(x, -x) : x \in \mathbb{Q}\}$ and $D_2 = \{(x, -x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ cannot be separated by disjoint open sets. (The full proof uses a Baire category argument on the irrationals.)

5.11 Worked Examples

Example 5.11.1 (Subspaces of Hausdorff spaces). **Claim:** Every subspace of a Hausdorff space is Hausdorff.

Solution. Let $Y \subseteq X$ with X Hausdorff, and let $y_1, y_2 \in Y$ be distinct. In X , there are disjoint open sets $U_1 \ni y_1$ and $U_2 \ni y_2$. Then $U_1 \cap Y$ and $U_2 \cap Y$ are disjoint open sets in Y separating y_1 and y_2 .

Example 5.11.2 (Products of Hausdorff spaces). **Claim:** An arbitrary product $\prod_{\alpha \in A} X_\alpha$ (with the product topology) is Hausdorff if and only if each X_α is Hausdorff.

Solution. (\Rightarrow) Each X_α is a continuous image of the product (via the projection π_α), and the Hausdorff property passes to subspaces but not to arbitrary continuous images. However, each π_α is an open map with a right inverse (fixing a point in every other factor), so X_α is homeomorphic to a subspace of the product.

(\Leftarrow) Let (x_α) and (y_α) be distinct points. There exists β with $x_\beta \neq y_\beta$. Since X_β is Hausdorff, choose disjoint open $U \ni x_\beta, V \ni y_\beta$ in X_β . Then $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in the product separating the two points.

Example 5.11.3 (The diagonal characterisation of Hausdorff). **Claim:** X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Solution. X is Hausdorff iff for every $(x, y) \notin \Delta$ (i.e. $x \neq y$), there exist open $U \ni x, V \ni y$ with $U \cap V = \emptyset$. Then $U \times V$ is an open neighbourhood of (x, y) in $X \times X$ disjoint from Δ . This holds for all $(x, y) \notin \Delta$ iff $(X \times X) \setminus \Delta$ is open, i.e. Δ is closed.

Example 5.11.4 (A normal space whose subspace is not normal). **Claim:** Normality is not hereditary.

Solution. The ordinal space $[0, \omega_1]$ with the order topology is compact Hausdorff, hence normal by Theorem 5.7.1. However, the subspace $[0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$ of $[0, \omega_1] \times [0, \omega]$ can be shown to be Hausdorff but not normal (this is the Tychonoff plank).

5.12 Exercises

Exercise 5.12.1 (★). Show that every finite T_1 space carries the discrete topology. Conclude that a finite topological space is T_1 if and only if it is T_2 .

Exercise 5.12.2 (★). Let X be Hausdorff and let (x_n) be a sequence converging to x . Show that $\{x\} = \bigcap_{n=1}^{\infty} \overline{\{x_n, x_{n+1}, \dots\}}$.

Exercise 5.12.3 (★). Let $f: X \rightarrow Y$ be a continuous, closed, surjective map with X normal. Prove that Y is normal.

Exercise 5.12.4 (★★). Prove that every regular Lindelöf space is normal. *Hint:* mimic the proof of Theorem 5.7.1, replacing “finite subcover” by “countable subcover”, then use a countable shrinking argument.

Exercise 5.12.5 (★★). Let X be uncountable and let τ be the *cocountable topology* (open sets are \emptyset and those with countable complement). Show that X is T_1 but not T_2 .

Exercise 5.12.6 (★★). Let $f: X \rightarrow Y$ be a function with Y Hausdorff. Prove that the graph $\Gamma_f = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$ if and only if for every $x \in X$ and every net (x_α) converging to x , if $(f(x_\alpha))$ converges to y then $y = f(x)$.

Exercise 5.12.7 (★★). Let X be a second-countable T_3 space. Using Urysohn's lemma (which applies because X is regular Lindelöf, hence normal by Exercise 5.12.4), outline how to embed X into $[0, 1]^\mathbb{N}$, thereby showing X is metrizable.

Exercise 5.12.8 (★★★). (**Jones' Lemma.**) Let X be a normal space containing a closed discrete subset D and a dense subset S . Prove that $2^{|D|} \leq 2^{|S|}$. *Hint:* use Urysohn's lemma to assign to each $A \subseteq D$ a continuous function $f_A: X \rightarrow [0, 1]$, and show that distinct subsets give distinct restrictions to S .

5.13 Chapter Summary

- The separation axioms T_0 – T_4 form a strictly increasing chain of topological properties: $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$, with no implication reversing.
- T_1 is equivalent to every singleton being closed.
- In Hausdorff (T_2) spaces, limits of nets are unique and compact subsets are closed.
- Every compact Hausdorff space is normal (T_4); every metrizable space is normal.
- Normality is *not* hereditary and *not* productive: the Sorgenfrey line is normal, but the Sorgenfrey plane is not.
- The K -topology on \mathbb{R} provides a Hausdorff space that fails regularity.

Chapter 6

Compactness and Tychonoff's Theorem

Compactness is one of the most powerful and far-reaching concepts in topology. Its origins lie in analysis: the Heine–Borel theorem asserts that a subset of \mathbb{R}^n is compact if and only if it is closed and bounded, and the Bolzano–Weierstrass theorem guarantees that every bounded sequence has a convergent subsequence. These results underpin the extreme value theorem, uniform continuity, and much of the rigorous foundation of calculus. In general topology, compactness plays an analogous role: compact spaces behave, in many ways, like finite sets. This chapter develops the theory of compactness from the definition through to Tychonoff's theorem on arbitrary products, one of the most celebrated results in all of mathematics.

6.1 Definition and First Properties

Definition 6.1.1 (Compact space). A topological space X is **compact** if every open cover of X has a finite subcover. That is, whenever $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets with $X = \bigcup_{\alpha \in A} U_\alpha$, there exist $\alpha_1, \dots, \alpha_n \in A$ with $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. A subset K of a topological space X is **compact** if K is compact as a subspace (equivalently, every cover of K by open sets of X has a finite subcover).

6.2 The Finite Intersection Property

Definition 6.2.1 (Finite intersection property). A collection \mathcal{F} of subsets of X has the **finite intersection property** (FIP) if every finite subcollection has non-empty intersection:

$$F_1, \dots, F_n \in \mathcal{F} \implies F_1 \cap \dots \cap F_n \neq \emptyset.$$

Theorem 6.2.1 (FIP characterisation of compactness). *A space X is compact if and only if every collection of closed subsets of X with the finite intersection property has non-empty total intersection.*

Proof. This is the contrapositive of the definition, obtained by taking complements.

(\implies) Let $\{F_\alpha\}_{\alpha \in A}$ be a collection of closed sets with $\bigcap_{\alpha} F_\alpha = \emptyset$. Setting $U_\alpha = X \setminus F_\alpha$, we have $\bigcup_{\alpha} U_\alpha = X$, so $\{U_\alpha\}$ is an open cover. By compactness, there is a finite subcover: $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. Taking complements, $F_{\alpha_1} \cap \dots \cap F_{\alpha_n} = \emptyset$. Hence the collection $\{F_\alpha\}$ does not have the FIP. Contrapositively, every collection with the FIP has non-empty intersection.

(\Leftarrow) Given an open cover $\{U_\alpha\}$ with no finite subcover, the closed sets $F_\alpha = X \setminus U_\alpha$ satisfy $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} \neq \emptyset$ for every finite subcollection (otherwise the corresponding U_{α_i} would cover X). So $\{F_\alpha\}$ has the FIP, hence $\bigcap_\alpha F_\alpha \neq \emptyset$, which means $\bigcup_\alpha U_\alpha \neq X$, contradicting the assumption that $\{U_\alpha\}$ covers X . \square

6.3 Fundamental Theorems on Compactness

Theorem 6.3.1 (Closed subsets of compact spaces). *A closed subset of a compact space is compact.*

Proof. Let X be compact, $F \subseteq X$ closed, and let $\{U_\alpha\}$ be an open cover of F (by open sets of X). Then $\{U_\alpha\} \cup \{X \setminus F\}$ is an open cover of X . By compactness of X , a finite subcollection covers X :

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup (X \setminus F).$$

Then $F \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. \square

Theorem 6.3.2 (Compact subsets of Hausdorff spaces are closed). *If X is Hausdorff and $K \subseteq X$ is compact, then K is closed.*

Proof. This was proved as Theorem 5.3.2 in Chapter 5. \square

Theorem 6.3.3 (Continuous image of a compact space). *If $f: X \rightarrow Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$. Then $\{f^{-1}(V_\alpha)\}$ is an open cover of X (by continuity). Compactness of X yields a finite subcover $X = f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})$. Applying f ,

$$f(X) \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}. \quad \square$$

Corollary 6.3.4 (Extreme value theorem). *If X is compact and $f: X \rightarrow \mathbb{R}$ is continuous, then f attains its maximum and minimum on X .*

Proof. By Theorem 6.3.3, $f(X)$ is a compact subset of \mathbb{R} . By the Heine–Borel theorem (Theorem 6.4.2 below, or its classical form), $f(X)$ is closed and bounded. A closed bounded subset of \mathbb{R} contains its supremum and infimum. \square

6.4 Compactness in \mathbb{R}^n

Theorem 6.4.1 (Compactness of $[a, b]$). *Every closed bounded interval $[a, b] \subseteq \mathbb{R}$ is compact.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of $[a, b]$. Define

$$S = \{s \in [a, b] : [a, s] \text{ is covered by finitely many members of } \mathcal{U}\}.$$

The set S is non-empty (a belongs to some U_α , and for s sufficiently close to a we have $[a, s] \subseteq U_\alpha$, so $s \in S$). Since $S \subseteq [a, b]$, S is bounded above and we set $c = \sup S$.

Claim: $c \in S$. Choose $U_\beta \ni c$. Since U_β is open in $[a, b]$, there exists $\varepsilon > 0$ with $(c - \varepsilon, c] \cap [a, b] \subseteq U_\beta$. By the definition of supremum, there exists $s \in S$ with $s > c - \varepsilon$. Then $[a, s]$ is covered by finitely many members of \mathcal{U} , and $[s, c] \subseteq U_\beta$, so $[a, c]$ is covered by finitely many members of \mathcal{U} . Hence $c \in S$.

Claim: $c = b$. Suppose $c < b$. The open set $U_\beta \ni c$ contains an interval $[c, c + \delta)$ for some $\delta > 0$ (taking δ small enough that $c + \delta \leq b$). Then $[a, c + \delta/2]$ is covered by the finite subcover of $[a, c]$ together with U_β , contradicting $c = \sup S$.

Therefore $c = b \in S$, and $[a, b]$ has a finite subcover. \square

Theorem 6.4.2 (Heine–Borel). *A subset of \mathbb{R}^n is compact if and only if it is closed and bounded.*

Proof. (\Leftarrow) A closed bounded subset $K \subseteq \mathbb{R}^n$ is contained in some box $[-M, M]^n$. By Theorem 6.4.1, $[-M, M]$ is compact. By Tychonoff’s theorem (Theorem 6.7.1 below, or by an elementary induction on n), the finite product $[-M, M]^n$ is compact. Since K is a closed subset of a compact space, K is compact by Theorem 6.3.1.

(\Rightarrow) A compact subset of a Hausdorff space is closed (Theorem 6.3.2). To see it is bounded, the open cover $\{B(0, n)\}_{n=1}^\infty$ of K has a finite subcover, so $K \subseteq B(0, N)$ for some N . \square

6.5 Sequential and Limit Point Compactness

Definition 6.5.1. A space X is **sequentially compact** if every sequence in X has a convergent subsequence.

Definition 6.5.2. A space X is **limit point compact** (or **Bolzano–Weierstrass**) if every infinite subset of X has a limit point (accumulation point) in X .

Proposition 6.5.1. *Compact \Rightarrow limit point compact.*

Proof. Suppose X is compact and $A \subseteq X$ is infinite with no limit point. Then A is closed (its complement is open since no point outside A is a limit point). For each $a \in A$, there is an open set U_a with $U_a \cap A = \{a\}$. Then $\{U_a\}_{a \in A} \cup \{X \setminus A\}$ is an open cover of X with no finite subcover (since A is infinite), contradicting compactness. \square

Theorem 6.5.2 (Equivalence in metric spaces). *For a metric space (X, d) , the following are equivalent:*

1. X is compact.
2. X is sequentially compact.
3. X is complete and totally bounded.

Proof. (1) \Rightarrow (2): Compact implies limit point compact (Proposition 6.5.1). In a metric space (which is first countable), limit point compactness implies sequential compactness. Indeed, if (x_n) is a sequence and $A = \{x_n : n \in \mathbb{N}\}$ is finite, then some value repeats infinitely often, giving a (constant, hence convergent) subsequence. If A is infinite, it has a limit point p ; since X is first countable, we can extract a subsequence converging to p .

(2) \Rightarrow (3): *Completeness:* Every Cauchy sequence in a sequentially compact space has a convergent subsequence, and a Cauchy sequence with a convergent subsequence converges.

Total boundedness: Suppose X is not totally bounded. Then there exists $\varepsilon > 0$ such that no finite set of ε -balls covers X . Inductively choose $x_1 \in X$, and given x_1, \dots, x_n , choose $x_{n+1} \notin \bigcup_{i=1}^n B(x_i, \varepsilon)$. The sequence (x_n) satisfies $d(x_m, x_n) \geq \varepsilon$ for all $m \neq n$, so it has no convergent subsequence, contradicting sequential compactness.

(3) \Rightarrow (1): Let $\{U_\alpha\}$ be an open cover with no finite subcover. Since X is totally bounded, cover X by finitely many balls of radius 1. At least one, say B_1 , cannot be covered by finitely many U_α . The closure $\overline{B_1}$ is totally bounded; cover it by finitely many balls of radius $1/2$. At least one, say B_2 , cannot be finitely covered. Continue inductively with radii $1/n$ to obtain a nested sequence of sets $\overline{B_n}$ with $\text{diam}(\overline{B_n}) \leq 2/n$. Pick $x_n \in \overline{B_n}$. Then (x_n) is Cauchy; by completeness, $x_n \rightarrow x$ for some $x \in X$. Choose $U_\beta \ni x$. For large n , $\overline{B_n} \subseteq U_\beta$ (since $\text{diam}(\overline{B_n}) \rightarrow 0$ and $x_n \rightarrow x$). But then $\overline{B_n}$ is covered by a single U_β , contradicting the choice of B_n . \square

6.6 Alexander's Sub-base Theorem

Theorem 6.6.1 (Alexander's sub-base theorem). *Let \mathcal{S} be a sub-base for the topology on X . If every cover of X by members of \mathcal{S} has a finite subcover, then X is compact.*

Proof. We prove the contrapositive. Suppose X is not compact. We must find a cover by sub-basic open sets with no finite subcover.

Consider the collection Σ of all open covers of X that have no finite subcover. By assumption, $\Sigma \neq \emptyset$. Order Σ by inclusion: $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{U} \subseteq \mathcal{V}$. We apply Zorn's lemma.

Chain condition: Let $\{\mathcal{U}_i\}_{i \in I}$ be a chain in Σ . Set $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$. This is a collection of open sets covering X (since each \mathcal{U}_i covers X). If \mathcal{U} had a finite subcover $\{V_1, \dots, V_k\}$, then each $V_j \in \mathcal{U}_{i_j}$ for some i_j . Since the chain is totally ordered, there exists i_0 with $\mathcal{U}_{i_j} \subseteq \mathcal{U}_{i_0}$ for all j , so $\{V_1, \dots, V_k\} \subseteq \mathcal{U}_{i_0}$, contradicting $\mathcal{U}_{i_0} \in \Sigma$. Hence $\mathcal{U} \in \Sigma$, and \mathcal{U} is an upper bound for the chain.

By Zorn's lemma, Σ has a maximal element \mathcal{M} . Thus \mathcal{M} is an open cover of X with no finite subcover, and it is maximal with respect to this property (adding any open set not in \mathcal{M} creates a cover admitting a finite subcover).

Claim: $\mathcal{M} \cap \mathcal{S}$ already covers X .

Suppose not. Then there exists $x \in X$ not covered by any member of $\mathcal{M} \cap \mathcal{S}$. Since \mathcal{M} covers X , there exists $U \in \mathcal{M}$ with $x \in U$. Since \mathcal{S} is a sub-base, U contains a basic open set $S_1 \cap \dots \cap S_m$ with $x \in S_1 \cap \dots \cap S_m$ and each $S_j \in \mathcal{S}$. Since x is not covered by $\mathcal{M} \cap \mathcal{S}$, for each j we have $S_j \notin \mathcal{M}$.

By the maximality of \mathcal{M} , for each j the cover $\mathcal{M} \cup \{S_j\}$ admits a finite subcover. Hence there exist $M_{j,1}, \dots, M_{j,k_j} \in \mathcal{M}$ such that

$$X = S_j \cup M_{j,1} \cup \dots \cup M_{j,k_j}.$$

Now consider $x \in S_1 \cap \dots \cap S_m \subseteq U$. We have

$$X = \bigcap_{j=1}^m (S_j \cup M_{j,1} \cup \dots \cup M_{j,k_j}).$$

For any point $p \in X$: if $p \in S_j$ for all j , then $p \in S_1 \cap \dots \cap S_m \subseteq U \in \mathcal{M}$. Otherwise $p \notin S_j$ for some j , so $p \in M_{j,1} \cup \dots \cup M_{j,k_j}$. In either case, p is covered by $\{U\} \cup \bigcup_{j=1}^m \{M_{j,1}, \dots, M_{j,k_j}\}$, which is a finite subcollection of \mathcal{M} . This contradicts $\mathcal{M} \in \Sigma$.

Hence $\mathcal{M} \cap \mathcal{S}$ covers X , and it has no finite subcover (since \mathcal{M} does not). Therefore we have found a sub-basic cover with no finite subcover. \square

6.7 Tychonoff's Theorem

Theorem 6.7.1 (Tychonoff). *Let $\{X_\alpha\}_{\alpha \in A}$ be a (possibly infinite) collection of non-empty topological spaces. Then the product $X = \prod_{\alpha \in A} X_\alpha$ (with the product topology) is compact if and only if each X_α is compact.*

Proof. (\Rightarrow) Each projection $\pi_\alpha: X \rightarrow X_\alpha$ is continuous and surjective. By Theorem 6.3.3, if X is compact then each $X_\alpha = \pi_\alpha(X)$ is compact.

(\Leftarrow) Assume each X_α is compact. We use Alexander's sub-base theorem (Theorem 6.6.1).

Recall that the product topology has a sub-base

$$\mathcal{S} = \{ \pi_\alpha^{-1}(U_\alpha) : \alpha \in A, U_\alpha \subseteq X_\alpha \text{ open} \}.$$

By Alexander's theorem, it suffices to show that every cover of X by members of \mathcal{S} has a finite subcover.

Let $\mathcal{C} \subseteq \mathcal{S}$ be a cover of X with no finite subcover. For each $\alpha \in A$, define

$$\mathcal{C}_\alpha = \{U_\alpha \subseteq X_\alpha : \pi_\alpha^{-1}(U_\alpha) \in \mathcal{C}\}.$$

Claim: for each α , \mathcal{C}_α does not cover X_α .

Suppose \mathcal{C}_α covers X_α for some α . Since X_α is compact, there exist $U_{\alpha,1}, \dots, U_{\alpha,n} \in \mathcal{C}_\alpha$ with $X_\alpha = U_{\alpha,1} \cup \dots \cup U_{\alpha,n}$. Then

$$X = \pi_\alpha^{-1}(X_\alpha) = \pi_\alpha^{-1}(U_{\alpha,1}) \cup \dots \cup \pi_\alpha^{-1}(U_{\alpha,n}),$$

which is a finite subcover of \mathcal{C} , contradicting our assumption.

So for each $\alpha \in A$, there exists $x_\alpha \in X_\alpha$ with $x_\alpha \notin \bigcup \mathcal{C}_\alpha$. Define $x = (x_\alpha)_{\alpha \in A} \in X$.

Since \mathcal{C} covers X , there exists $\pi_\beta^{-1}(U_\beta) \in \mathcal{C}$ with $x \in \pi_\beta^{-1}(U_\beta)$. This means $x_\beta = \pi_\beta(x) \in U_\beta$, so $U_\beta \in \mathcal{C}_\beta$, contradicting the choice $x_\beta \notin \bigcup \mathcal{C}_\beta$.

The contradiction shows that no such cover \mathcal{C} exists, so every sub-basic cover has a finite subcover. By Alexander's theorem, X is compact. \square

Remark 6.7.2. The proof of Tychonoff's theorem uses the Axiom of Choice (via Zorn's lemma in Alexander's sub-base theorem, and also in selecting the points x_α). In fact, Tychonoff's theorem is *equivalent* to the Axiom of Choice in ZF, as proved by Kelley (1950).

6.8 The Alexandroff One-Point Compactification

Definition 6.8.1 (One-point compactification). Let X be a non-compact, locally compact, Hausdorff topological space. The **one-point (Alexandroff) compactification** of X is the space $X^* = X \cup \{\infty\}$ (where $\infty \notin X$) with the topology

$$\tau^* = \tau \cup \{(X \setminus K) \cup \{\infty\} : K \subseteq X \text{ compact}\}.$$

Proposition 6.8.1. *The space X^* is compact and Hausdorff, X embeds as an open dense subspace, and the topology τ^* is the unique compact Hausdorff topology on $X \cup \{\infty\}$ that restricts to τ on X .*

Proof. τ^* is a topology. The empty set and X^* are in τ^* ($\emptyset \in \tau$ and $X^* = (X \setminus \emptyset) \cup \{\infty\}$ with \emptyset compact). Unions and finite intersections of sets in τ^* remain in τ^* : for sets not containing ∞ this follows from τ being a topology; for sets containing ∞ , it follows from the fact that finite unions of compact sets are compact and arbitrary intersections of compact (closed) sets in a Hausdorff space are compact (closed).

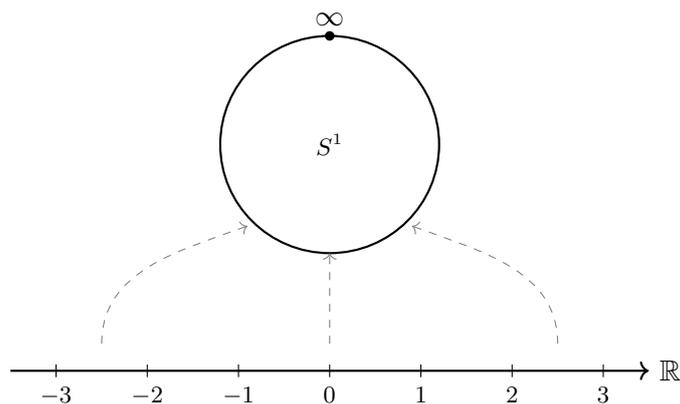
Compactness. Let $\{V_\beta\}$ be an open cover of X^* . Some V_{β_0} contains ∞ , so $V_{\beta_0} = (X \setminus K) \cup \{\infty\}$ for some compact $K \subseteq X$. The remaining sets $\{V_\beta\}_{\beta \neq \beta_0}$ (intersected with X) cover K . By compactness of K , finitely many suffice. Adding V_{β_0} gives a finite subcover of X^* .

Hausdorff. Two points in X are separated because X is Hausdorff. For $x \in X$ and ∞ : by local compactness, there is an open $U \ni x$ with \bar{U} compact. Then U and $(X \setminus \bar{U}) \cup \{\infty\}$ are disjoint open sets separating x and ∞ . \square

Example 6.8.2 ($\mathbb{R} \cup \{\infty\} \cong S^1$). *The one-point compactification of \mathbb{R} is homeomorphic to the circle S^1 . A homeomorphism is given by stereographic projection from the "north pole" of S^1 : the map sends each $t \in \mathbb{R}$ to the corresponding point on $S^1 \setminus \{N\}$ and ∞ to N .*

Example 6.8.3 ($\mathbb{C} \cup \{\infty\} \cong S^2$). *Similarly, the one-point compactification of $\mathbb{C} \cong \mathbb{R}^2$ is the Riemann sphere S^2 , via stereographic projection.*

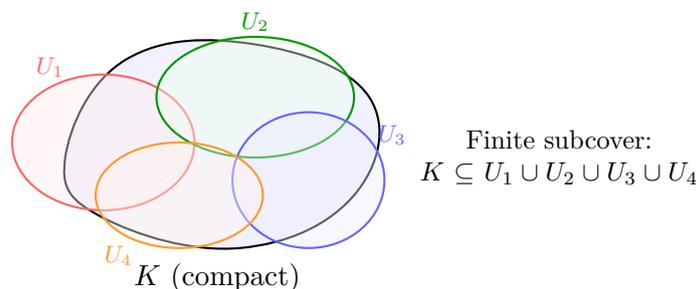
The following TikZ diagram illustrates the one-point compactification of the real line.



6.9 The Stone–Čech Compactification

While the Alexandroff compactification adds a single point, the **Stone–Čech compactification** βX is in a precise sense the “largest” compactification of a completely regular ($T_{3\frac{1}{2}}$) space X . It is characterised by the universal property: every continuous function $f: X \rightarrow K$ into a compact Hausdorff space K extends uniquely to a continuous function $\tilde{f}: \beta X \rightarrow K$. The construction can be carried out by embedding X into a product of copies of $[0, 1]$ (one for each bounded continuous function $X \rightarrow [0, 1]$) and taking the closure; Tychonoff’s theorem guarantees compactness of the ambient product. A detailed treatment lies beyond our scope, but the interested reader is referred to Engelking [9] or Willard [3].

6.10 Visualisation: Open Covers



6.11 Counterexamples

Example 6.11.1 ($(0, 1)$ is not compact). *The open interval $(0, 1)$ is not compact. The open cover $\{(1/n, 1) : n \geq 2\}$ has no finite subcover: any finite subcollection misses points near 0.*

Example 6.11.2 ($\mathbb{Q} \cap [0, 1]$ is not compact). *The space $\mathbb{Q} \cap [0, 1]$ (with the subspace topology from \mathbb{R}) is not compact. Choose an irrational $r \in (0, 1)$ (say $r = 1/\sqrt{2}$) and consider the open cover $\{[0, r - 1/n) \cup (r + 1/n, 1] : n \geq 1\} \cap \mathbb{Q}$; no finite subcover exists. Alternatively, $\mathbb{Q} \cap [0, 1]$ is not closed in \mathbb{R} , yet is bounded, so by the Heine–Borel theorem it cannot be compact (as a subset of \mathbb{R}).*

Example 6.11.3 (Sequential compactness \neq compactness in general). *The product $\{0, 1\}^{[0, 1]}$ (with the product topology, where $\{0, 1\}$ is discrete) is compact by Tychonoff’s theorem. However, the subspace $\omega_1 = [0, \omega_1)$ with the order topology is sequentially compact (every sequence of countable ordinals is bounded by a countable ordinal and hence has a convergent subsequence) but not compact (the open cover $\{[0, \alpha) : \alpha < \omega_1\}$ has no finite subcover).*

Conversely, the product $\{0, 1\}^{\mathbb{R}}$ is compact (Tychonoff) but not sequentially compact. These examples show that compactness and sequential compactness are genuinely distinct in general topological spaces.

6.12 Worked Examples

Example 6.12.1 (Continuous bijection from compact to Hausdorff is a homeomorphism). **Claim:** If $f: X \rightarrow Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.

Solution. It suffices to show f is a closed map. Let $F \subseteq X$ be closed. Since X is compact and F is closed, F is compact (Theorem 6.3.1). Then $f(F)$ is compact (Theorem 6.3.3). Since Y is Hausdorff, $f(F)$ is closed (Theorem 6.3.2). Hence f is closed, and a continuous closed bijection is a homeomorphism.

Example 6.12.2 (Compactness of the Cantor set). **Claim:** The Cantor set $C \subseteq [0, 1]$ is compact.

Solution. The Cantor set is constructed as $C = \bigcap_{n=0}^{\infty} C_n$ where each C_n is a finite union of closed intervals (hence closed). Thus C is an intersection of closed sets in $[0, 1]$, hence closed in $[0, 1]$. Since $[0, 1]$ is compact (Theorem 6.4.1) and C is a closed subset, C is compact by Theorem 6.3.1.

Alternatively, C is homeomorphic to $\{0, 1\}^{\mathbb{N}}$ (via the ternary expansion), which is compact by Tychonoff's theorem.

Example 6.12.3 (The product of $[0, 1]$ with itself uncountably many times). **Claim:** For any index set A , the product $[0, 1]^A$ is compact.

Solution. Each factor $[0, 1]$ is compact (being a closed bounded interval in \mathbb{R}). By Tychonoff's theorem (Theorem 6.7.1), the arbitrary product $[0, 1]^A$ is compact. When $A = \mathbb{N}$, this gives the **Hilbert cube** $[0, 1]^{\mathbb{N}}$, a fundamental object in infinite-dimensional topology.

Example 6.12.4 (Tube lemma application). **Claim (Tube Lemma):** If X is compact and Y is any space, $x_0 \in X$, and N is an open set in $X \times Y$ containing the "slice" $X \times \{y_0\}$, then there exists an open set $V \ni y_0$ in Y such that $X \times V \subseteq N$.

Solution. For each $x \in X$, the point $(x, y_0) \in N$, so there exist open sets $U_x \ni x$ and $V_x \ni y_0$ in X and Y respectively with $U_x \times V_x \subseteq N$. The sets $\{U_x\}_{x \in X}$ cover the compact space X , so finitely many suffice: $X = U_{x_1} \cup \dots \cup U_{x_n}$. Set $V = V_{x_1} \cap \dots \cap V_{x_n}$. Then V is open, $y_0 \in V$, and $X \times V \subseteq N$.

6.13 Exercises

Exercise 6.13.1 (★). Prove that a discrete topological space is compact if and only if it is finite.

Exercise 6.13.2 (★). Let X be a compact space and let $\{F_n\}_{n \geq 1}$ be a decreasing sequence of non-empty closed subsets ($F_1 \supseteq F_2 \supseteq \dots$). Prove that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Exercise 6.13.3 (★). Show that every compact metric space is bounded, and that every compact metric space is complete.

Exercise 6.13.4 (★★). (**Lebesgue number lemma.**) Let (X, d) be a compact metric space and $\{U_\alpha\}$ an open cover. Prove there exists $\delta > 0$ (the *Lebesgue number*) such that every subset of X with diameter less than δ is contained in some U_α .

Exercise 6.13.5 (★★). Let X be compact, Y Hausdorff, and $f: X \rightarrow Y$ a continuous bijection. Prove f is a homeomorphism. (This generalises Example 6.12.1; give a complete proof.)

Exercise 6.13.6 (★★). Without using Tychonoff's theorem, prove directly that if X and Y are compact, then $X \times Y$ is compact. *Hint:* use the tube lemma (Example 6.12.4).

Exercise 6.13.7 (★★). Let X be a locally compact Hausdorff space. Show that the one-point compactification X^* is (up to homeomorphism) the unique compact Hausdorff space containing X as an open dense subspace.

Exercise 6.13.8 (★★★). Show that Tychonoff's theorem implies the Axiom of Choice. *Hint:* Given a family $\{X_\alpha\}_{\alpha \in A}$ of non-empty sets, consider $Y_\alpha = X_\alpha \cup \{\star\}$ with the topology $\{\emptyset, X_\alpha, Y_\alpha\}$, show each Y_α is compact, and use Tychonoff to find a point in $\prod X_\alpha$.

Exercise 6.13.9 (★★★). (**Baire category theorem for compact Hausdorff spaces.**) Prove that in a compact Hausdorff space, the intersection of countably many dense open sets is dense. *Hint:* given a non-empty open W , inductively construct a nested sequence of open sets $U_1 \supseteq \overline{U_2} \supseteq U_2 \supseteq \cdots$ with $\overline{U_n}$ compact and $\overline{U_n} \subseteq G_n \cap W$, then use the FIP.

6.14 Chapter Summary

- A space is compact if every open cover has a finite subcover; equivalently, every collection of closed sets with the FIP has non-empty intersection.
- Closed subsets of compact spaces are compact; compact subsets of Hausdorff spaces are closed; continuous images of compact spaces are compact.
- The extreme value theorem is a corollary: continuous real-valued functions on compact spaces attain their bounds.
- $[a, b]$ is compact (direct proof via the least upper bound property), and the Heine–Borel theorem characterises compact subsets of \mathbb{R}^n as the closed and bounded sets.
- In metric spaces, compactness is equivalent to sequential compactness and to completeness plus total boundedness. In general spaces, these notions diverge.
- Alexander's sub-base theorem reduces checking compactness to sub-basic covers; this is the key tool in proving Tychonoff's theorem.
- **Tychonoff's theorem:** an arbitrary product of compact spaces is compact (in the product topology). This result is equivalent to the Axiom of Choice.
- The Alexandroff one-point compactification adds a single point to a locally compact Hausdorff space; the Stone–Čech compactification provides the “maximal” compactification.

Chapter 7

Connectedness and Path-Connectedness

One of the most intuitive topological properties is that of being “in one piece.” In analysis on the real line, this idea is captured by the classical result that the connected subsets of \mathbb{R} are precisely the intervals (bounded or unbounded, open, closed, or half-open). The goal of this chapter is to formalize and extend this notion to arbitrary topological spaces. We shall meet two flavors of the concept—*connectedness* and *path-connectedness*—study their interplay, and exhibit spaces that separate the two notions.

7.1 Connected Spaces

Definition 7.1.1 (Separation). Let X be a topological space. A **separation** of X is a pair (U, V) of non-empty open subsets $U, V \subseteq X$ such that

$$U \cap V = \emptyset \quad \text{and} \quad U \cup V = X.$$

The space X is called **disconnected** if it admits a separation, and **connected** if it does not.

Proposition 7.1.1 (Clopen characterization). *A topological space X is connected if and only if the only subsets of X that are both open and closed (clopen) are \emptyset and X itself.*

Proof. Suppose X is connected and let $A \subseteq X$ be clopen. If $A \neq \emptyset$ and $A \neq X$, then $(A, X \setminus A)$ is a separation of X , since both A and $X \setminus A$ are non-empty and open. This contradicts connectedness, so $A = \emptyset$ or $A = X$.

Conversely, suppose the only clopen subsets are \emptyset and X . If (U, V) were a separation, then U would be open and $U = X \setminus V$ would be closed, hence clopen. Since $U \neq \emptyset$, we would need $U = X$, forcing $V = \emptyset$, a contradiction. \square

Definition 7.1.2 (Connected subset). A subset A of a topological space X is **connected** if the subspace A (with the subspace topology inherited from X) is a connected space.

Definition 7.1.3 (Connected component). Let X be a topological space and $x \in X$. The **connected component** of x in X is the union of all connected subsets of X containing x :

$$C(x) = \bigcup \{ A \subseteq X : A \text{ is connected and } x \in A \}.$$

Proposition 7.1.2. *Let X be a topological space.*

1. *Each connected component $C(x)$ is connected and closed in X .*
2. *The connected components of X form a partition of X .*
3. *$C(x)$ is the largest connected subset of X containing x .*

Proof. (1) That $C(x)$ is connected follows from Theorem 7.2.4 below (all the sets in the union share the point x). Since the closure of a connected set is connected (Theorem 7.2.3), and $C(x) \subseteq \overline{C(x)}$ with $\overline{C(x)}$ connected and containing x , maximality gives $\overline{C(x)} \subseteq C(x)$, so $C(x)$ is closed.

(2) Every point x lies in $C(x)$ (since $\{x\}$ is connected), so the components cover X . If $C(x) \cap C(y) \neq \emptyset$, then $C(x) \cup C(y)$ is connected (by Theorem 7.2.4), so by maximality $C(x) \cup C(y) \subseteq C(x)$ and similarly $C(x) \cup C(y) \subseteq C(y)$; hence $C(x) = C(y)$.

(3) This is immediate from the definition. \square

7.2 Fundamental Theorems on Connectedness

Theorem 7.2.1 (Continuous image of a connected space). *Let $f: X \rightarrow Y$ be a continuous surjection. If X is connected, then Y is connected. More generally, if $A \subseteq X$ is connected, then $f(A)$ is connected.*

Proof. It suffices to prove the “more generally” statement. Equip $f(A)$ with the subspace topology from Y and suppose for contradiction that (U, V) is a separation of $f(A)$. Then $f^{-1}(U) \cap A$ and $f^{-1}(V) \cap A$ are non-empty open subsets of A (open because f is continuous), their union is A (since $U \cup V = f(A)$), and their intersection is empty (since $U \cap V = \emptyset$). This gives a separation of A , contradicting the connectedness of A . \square

Corollary 7.2.2 (Intermediate Value Theorem). *Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous and suppose $f(a) < c < f(b)$ (or $f(b) < c < f(a)$). Then there exists $x \in (a, b)$ such that $f(x) = c$.*

Proof. The interval $[a, b]$ is connected (a classical fact proved in Example 7.9.1 below). By Theorem 7.2.1, $f([a, b])$ is a connected subset of \mathbb{R} , hence an interval. Since $f(a)$ and $f(b)$ belong to this interval and c lies between them, $c \in f([a, b])$. \square

Theorem 7.2.3 (Closure preserves connectedness). *Let A be a connected subset of a topological space X . If $A \subseteq B \subseteq \overline{A}$, then B is connected. In particular, \overline{A} is connected.*

Proof. Suppose (U, V) is a separation of B . Then $A \subseteq B = U \cup V$, and the sets $U \cap A, V \cap A$ are open in A . Since A is connected, one of them must be empty; say $A \cap V = \emptyset$, so $A \subseteq U$. Since V is open in B and $V \subseteq B \subseteq \overline{A}$, every point of V is a limit point of A or belongs to A . But $A \subseteq U$ and $U \cap V = \emptyset$, so no point of V belongs to A ; hence every point $v \in V$ is a limit point of A . Then every open neighborhood of v in B meets $A \subseteq U$, so in particular V meets U —a contradiction since $U \cap V = \emptyset$ and $V \neq \emptyset$ forces the existence of such a point. \square

Theorem 7.2.4 (Union with a common point). *Let $\{A_i\}_{i \in I}$ be a family of connected subsets of X with $\bigcap_{i \in I} A_i \neq \emptyset$. Then $A = \bigcup_{i \in I} A_i$ is connected.*

Proof. Fix a point $p \in \bigcap_{i \in I} A_i$. Suppose $f: A \rightarrow \{0, 1\}$ is continuous, where $\{0, 1\}$ carries the discrete topology. For each $i \in I$, the restriction $f|_{A_i}: A_i \rightarrow \{0, 1\}$ is continuous and A_i is connected, so $f|_{A_i}$ is constant. Since $p \in A_i$ for every i , we have $f|_{A_i} \equiv f(p)$ for all i . Hence $f \equiv f(p)$ on A , and A is connected. \square

7.3 Path-Connectedness

Definition 7.3.1 (Path). Let X be a topological space. A **path** in X from x to y is a continuous map $\gamma: [0, 1] \rightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$. The space X is **path-connected** if for every $x, y \in X$ there exists a path from x to y .

Theorem 7.3.1. *Every path-connected space is connected.*

Proof. Let X be path-connected and fix $x_0 \in X$. For each $x \in X$, let $\gamma_x: [0, 1] \rightarrow X$ be a path from x_0 to x . Then $\gamma_x([0, 1])$ is a connected subset of X containing both x_0 and x (it is the continuous image of the connected set $[0, 1]$). Therefore

$$X = \bigcup_{x \in X} \gamma_x([0, 1]),$$

and all these connected sets share the point x_0 . By Theorem 7.2.4, X is connected. \square

Remark 7.3.2. The converse of Theorem 7.3.1 is false in general. The classical counterexample is the *topologist's sine curve*, presented in Section 7.4.

7.4 Counterexample: The Topologist's Sine Curve

Definition 7.4.1. The **topologist's sine curve** is the subspace $T \subseteq \mathbb{R}^2$ defined by

$$T = \bar{S}, \quad S = \{(x, \sin(1/x)) : x \in (0, 1]\}.$$

Equivalently, $T = S \cup J$ where $J = \{0\} \times [-1, 1]$ is the segment on the y -axis.

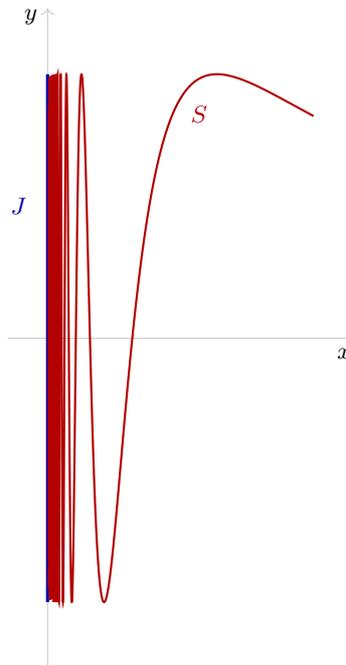


Figure 7.1: The topologist's sine curve $T = S \cup J$. The curve $S = \{(x, \sin(1/x)) : 0 < x \leq 1\}$ oscillates faster and faster as $x \rightarrow 0^+$, and its closure picks up the entire segment J .

Theorem 7.4.1. *The topologist's sine curve T is connected but not path-connected.*

Proof. Connected. The set S is the image of the continuous map $(0, 1] \rightarrow \mathbb{R}^2$, $x \mapsto (x, \sin(1/x))$, and $(0, 1]$ is connected (it is an interval), so S is connected. Since $S \subseteq T = \bar{S}$, Theorem 7.2.3 implies T is connected.

Not path-connected. We show there is no path in T from the origin $(0, 0) \in J$ to the point $(1, \sin 1) \in S$. Suppose for contradiction that $\gamma: [0, 1] \rightarrow T$ is continuous with $\gamma(0) = (0, 0)$ and $\gamma(1) = (1, \sin 1)$. Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$.

Define $t_0 = \sup\{t \in [0, 1] : \gamma_1(t) = 0\}$. Since $\gamma_1^{-1}(\{0\})$ is closed, $\gamma_1(t_0) = 0$. Moreover $t_0 < 1$ because $\gamma_1(1) = 1 > 0$. For $t > t_0$, we have $\gamma_1(t) > 0$ and $\gamma(t) \in S$, so $\gamma_2(t) = \sin(1/\gamma_1(t))$.

Since γ_1 is continuous at t_0 with $\gamma_1(t_0) = 0$, the values $\gamma_1(t) \rightarrow 0^+$ as $t \rightarrow t_0^+$. Hence $1/\gamma_1(t) \rightarrow +\infty$, and $\sin(1/\gamma_1(t))$ oscillates between -1 and $+1$ without converging. But γ_2 is continuous at t_0 , so $\gamma_2(t) \rightarrow \gamma_2(t_0)$ as $t \rightarrow t_0^+$. This is a contradiction: $\gamma_2(t)$ cannot converge and oscillate simultaneously. \square

7.5 Local Connectedness and Local Path-Connectedness

Definition 7.5.1. A topological space X is **locally connected** if for every $x \in X$ and every open neighborhood U of x , there exists a connected open neighborhood V of x with $V \subseteq U$.

Similarly, X is **locally path-connected** if every point has arbitrarily small path-connected open neighborhoods.

Remark 7.5.1. Local (path-)connectedness and global (path-)connectedness are logically independent. For instance, the disjoint union $\mathbb{R} \sqcup \mathbb{R}$ is locally path-connected but not connected. The topologist's sine curve is connected but not locally connected at points of J .

Theorem 7.5.2. *If X is connected and locally path-connected, then X is path-connected.*

Proof. Fix $x_0 \in X$ and let P be the set of all points in X that can be joined to x_0 by a path. We show P is clopen; since X is connected and $P \neq \emptyset$ ($x_0 \in P$), we conclude $P = X$.

P is open. Let $x \in P$. Since X is locally path-connected, there exists a path-connected open neighborhood V of x . For every $y \in V$, we can concatenate a path from x_0 to x with a path from x to y in V , showing $y \in P$. Hence $V \subseteq P$.

P is closed. We show $X \setminus P$ is open. Let $x \in X \setminus P$ and choose a path-connected open neighborhood V of x . If some $y \in V$ belonged to P , then a path from x_0 to y followed by a path from y to x in V would give a path from x_0 to x , contradicting $x \notin P$. So $V \subseteq X \setminus P$. \square

7.6 Totally Disconnected Spaces

Definition 7.6.1. A topological space X is **totally disconnected** if the only connected subsets of X are singletons. Equivalently, for every $x \in X$, the connected component of x is $\{x\}$.

Example 7.6.1 (The rationals). *The space \mathbb{Q} (with the subspace topology from \mathbb{R}) is totally disconnected. Indeed, if $A \subseteq \mathbb{Q}$ contains two distinct points $p < q$, choose an irrational $r \in (p, q)$. Then $A = (A \cap (-\infty, r)) \cup (A \cap (r, +\infty))$ is a separation of A in \mathbb{Q} , so A is not connected.*

Example 7.6.2 (The Cantor set). *The Cantor set $C \subseteq [0, 1]$ is totally disconnected. Each stage of its construction removes an open middle third, and for any two distinct points $x, y \in C$ one can find a removed interval between them, providing a separation of any subset containing both x and y .*

7.7 Product of Connected Spaces

Theorem 7.7.1. *Let $\{X_i\}_{i \in I}$ be a family of non-empty topological spaces. The product $\prod_{i \in I} X_i$ (with the product topology) is connected if and only if each X_i is connected.*

Proof. “Only if.” Each projection $\pi_j: \prod_{i \in I} X_i \rightarrow X_j$ is continuous and surjective, so by Theorem 7.2.1, X_j is connected.

“If.” We prove the result in two steps.

Step 1: Finite products. It suffices to prove $X \times Y$ is connected whenever X and Y are connected, and then iterate. Fix a point $(a, b) \in X \times Y$. For each $x \in X$, the “horizontal slice” $\{x\} \times Y \cong Y$ is connected, and the “vertical slice” $X \times \{b\} \cong X$ is connected. The set

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$$

is the union of two connected sets sharing the point (x, b) , so T_x is connected by Theorem 7.2.4. Moreover, each T_x contains (a, b) (because $(a, b) \in X \times \{b\} \subseteq T_x$), so

$$X \times Y = \bigcup_{x \in X} T_x$$

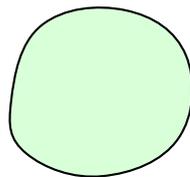
is connected by Theorem 7.2.4 again.

Step 2: Arbitrary products. Fix a point $\mathbf{a} = (a_i)_{i \in I} \in \prod X_i$. For each finite subset $F \subseteq I$, let

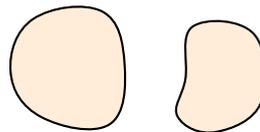
$$Y_F = \left\{ \mathbf{x} \in \prod X_i : x_i = a_i \text{ for all } i \notin F \right\}.$$

Then $Y_F \cong \prod_{i \in F} X_i$, which is connected by Step 1. Moreover $\mathbf{a} \in Y_F$ for all F , so $Z = \bigcup_{F \subseteq I, |F| < \infty} Y_F$ is connected. We claim Z is dense in $\prod X_i$. A basic open set in the product topology restricts only finitely many coordinates, say those in a finite set F , and agrees with $\prod X_i$ on the remaining coordinates. Hence any basic open set meeting $\prod X_i$ also meets $Y_F \subseteq Z$. Therefore $\overline{Z} = \prod X_i$, and by Theorem 7.2.3, $\prod X_i$ is connected. \square

7.8 Illustrations



Connected



Disconnected

Figure 7.2: A connected set (left) versus a disconnected set consisting of two components (right).

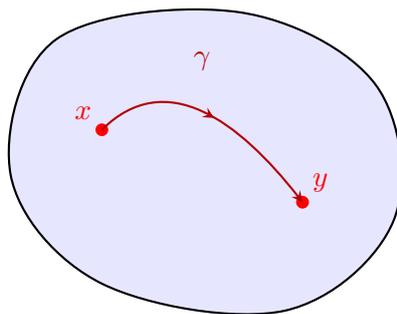
Path from x to y

Figure 7.3: A path $\gamma: [0, 1] \rightarrow X$ joining x to y in a path-connected space.

7.9 Worked Examples

Example 7.9.1 (Intervals are connected). *Every interval $I \subseteq \mathbb{R}$ is connected. Suppose (U, V) is a separation of I and pick $a \in U$, $b \in V$ with $a < b$. Let $c = \sup(U \cap [a, b])$. Since U is closed in I (it is the complement of the open set V), we have $c \in U$. Since U is also open in I , there is an $\varepsilon > 0$ with $(c - \varepsilon, c + \varepsilon) \cap I \subseteq U$. Because $c < b$ (as $b \in V$ and $U \cap V = \emptyset$), there exist points of $U \cap [a, b]$ greater than c , contradicting the definition of c . Hence no separation exists.*

Example 7.9.2 (\mathbb{R}^n is connected). *For any two points $x, y \in \mathbb{R}^n$, the map $\gamma(t) = (1 - t)x + ty$ is a path from x to y . Hence \mathbb{R}^n is path-connected, therefore connected.*

Example 7.9.3 (S^n is connected for $n \geq 1$). *The sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ for $n \geq 1$ is path-connected: given $p, q \in S^n$, if $p \neq -q$ the great-circle arc gives a path; if $p = -q$ we pass through any third point. Hence S^n is connected. (For $n = 0$, $S^0 = \{-1, 1\}$ is disconnected.)*

Example 7.9.4 ($\text{GL}_n(\mathbb{R})$ is disconnected). *The determinant $\det: \text{GL}_n(\mathbb{R}) \rightarrow \mathbb{R} \setminus \{0\}$ is continuous. The image $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is disconnected, and the preimages $\text{GL}_n^+(\mathbb{R})$ and $\text{GL}_n^-(\mathbb{R})$ are both non-empty, open, and disjoint. Hence $\text{GL}_n(\mathbb{R})$ is disconnected (it has exactly two connected components).*

7.10 Exercises

Exercise 7.10.1 (★). Show that $\mathbb{R} \setminus \{0\}$ is disconnected, and determine its connected components.

Exercise 7.10.2 (★). Prove that the connected subsets of \mathbb{R} are precisely the intervals (including the empty set, singletons, rays, and \mathbb{R} itself).

Exercise 7.10.3 (★). Use the Intermediate Value Theorem to prove that every continuous map $f: [0, 1] \rightarrow [0, 1]$ has a fixed point.

Exercise 7.10.4 (★★). Show that $\mathbb{R}^n \setminus \{p\}$ is connected for $n \geq 2$ and any $p \in \mathbb{R}^n$. *Hint:* show it is path-connected.

Exercise 7.10.5 (★★). Let $A \subseteq \mathbb{R}^n$ be open and connected. Is \bar{A} necessarily connected? Is the interior of a closed connected set necessarily connected? Justify or give counterexamples.

Exercise 7.10.6 (★★). Let X be a connected space and \sim an equivalence relation on X . Show that the quotient space X/\sim is connected.

Exercise 7.10.7 (★★). Let X be a topological space and let X/\sim be the quotient space obtained by collapsing each connected component to a point. Show that X/\sim is totally disconnected.

Exercise 7.10.8 (★★★). The *Warsaw circle* (or *Polish circle*) is obtained by connecting the endpoints $(0, -1)$ and $(1, \sin 1)$ of the topologist's sine curve T by an arc in \mathbb{R}^2 that does not otherwise intersect T . Show that the Warsaw circle is connected but not path-connected.

Exercise 7.10.9 (★★★). Prove that the Cantor set \mathcal{C} is totally disconnected, compact, metrizable, and has no isolated points. (In fact, up to homeomorphism, \mathcal{C} is the unique such space.)

Chapter Summary

- A space is **connected** if it has no separation. Equivalently, its only clopen subsets are \emptyset and the whole space.
- Continuous images of connected spaces are connected, yielding the Intermediate Value Theorem.
- If A is connected and $A \subseteq B \subseteq \overline{A}$, then B is connected. Unions of connected sets sharing a common point are connected.
- **Path-connected** \implies **connected**, but the converse fails (topologist's sine curve).
- Connected + locally path-connected \implies path-connected.
- **Totally disconnected** means every connected component is a singleton (e.g. \mathbb{Q} , the Cantor set).
- Arbitrary products of connected spaces are connected (product topology).

Chapter 8

Product Spaces and Quotient Topology

Topology provides systematic machinery for building new spaces from old ones. Two of the most fundamental constructions are the *product*, which assembles spaces “side by side,” and the *quotient*, which identifies points according to an equivalence relation, effectively “gluing” parts of a space together. These constructions underpin nearly every geometric object studied in algebraic topology and differential geometry: tori, projective spaces, and fiber bundles all arise through products and quotients.

8.1 Finite Products

Definition 8.1.1 (Product topology — finite case). Let X_1, \dots, X_n be topological spaces. The **product topology** on $X_1 \times \cdots \times X_n$ is the topology generated by the basis

$$\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \text{ open in } X_i, i = 1, \dots, n\}.$$

The projections $\pi_j : X_1 \times \cdots \times X_n \rightarrow X_j$ are defined by $\pi_j(x_1, \dots, x_n) = x_j$.

Remark 8.1.1. In the finite case, the set \mathcal{B} is itself a basis (it is closed under finite intersections), and the product topology coincides with the topology generated by all sets of the form $\pi_j^{-1}(U_j) = X_1 \times \cdots \times U_j \times \cdots \times X_n$.

8.2 Infinite Products: Product vs. Box Topology

Definition 8.2.1. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and let $X = \prod_{i \in I} X_i$.

1. The **box topology** on X is the topology generated by the basis

$$\mathcal{B}_{\text{box}} = \left\{ \prod_{i \in I} U_i : U_i \text{ open in } X_i \text{ for all } i \in I \right\}.$$

2. The **product topology** on X is the topology generated by the subbasis

$$\mathcal{S} = \{ \pi_i^{-1}(U_i) : i \in I, U_i \text{ open in } X_i \}.$$

A basis for the product topology consists of sets of the form $\prod_{i \in I} U_i$ where $U_i = X_i$ for all but finitely many i .

Remark 8.2.1. For finite index sets, the two topologies coincide. For infinite index sets, the product topology is strictly coarser than the box topology. The product topology is the “correct” topology for most purposes: it makes the product a categorical product (see Theorem 8.3.2 and Section 8.3).

8.3 Universal Property and Projections

Theorem 8.3.1 (Coarsest topology for projections). *The product topology on $X = \prod_{i \in I} X_i$ is the coarsest topology on X such that every projection $\pi_j: X \rightarrow X_j$ is continuous.*

Proof. In the product topology, each π_j is continuous because $\pi_j^{-1}(U_j) \in \mathcal{S}$ for every open $U_j \subseteq X_j$.

Conversely, let τ be any topology on X making all projections continuous. Then $\pi_j^{-1}(U_j) \in \tau$ for every $j \in I$ and every open $U_j \subseteq X_j$. Since the product topology is generated by exactly these sets, we have $\tau_{\text{prod}} \subseteq \tau$. \square

Theorem 8.3.2 (Universal property of the product). *Let $X = \prod_{i \in I} X_i$ carry the product topology. A map $f: Y \rightarrow X$ is continuous if and only if $\pi_i \circ f$ is continuous for every $i \in I$.*

Proof. “Only if.” If f is continuous, then $\pi_i \circ f$ is a composition of continuous maps, hence continuous.

“If.” Suppose $\pi_i \circ f$ is continuous for all i . For each subbasis element $\pi_i^{-1}(U_i)$,

$$f^{-1}(\pi_i^{-1}(U_i)) = (\pi_i \circ f)^{-1}(U_i)$$

is open in Y . Since the product topology is generated by the subbasis $\mathcal{S} = \{\pi_i^{-1}(U_i)\}$, and the preimage of any finite intersection or arbitrary union of such sets is likewise open, f is continuous. \square

Proposition 8.3.3 (Projections are open maps). *Each projection $\pi_j: \prod_{i \in I} X_i \rightarrow X_j$ is an open map.*

Proof. It suffices to show that π_j maps basis elements to open sets. A basic open set in the product topology is $B = \prod_{i \in I} U_i$ with $U_i = X_i$ for all but finitely many i and U_i open in X_i for all i . Then $\pi_j(B) = U_j$, which is open in X_j . \square

Remark 8.3.4. Projections are not necessarily closed maps. For example, the set $C = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ is closed in \mathbb{R}^2 , but $\pi_1(C) = \mathbb{R} \setminus \{0\}$ is not closed in \mathbb{R} .

8.4 The Tube Lemma

Lemma 8.4.1 (Tube Lemma). *Let X and Y be topological spaces with Y compact. Let $x_0 \in X$ and let N be an open subset of $X \times Y$ containing the “slice” $\{x_0\} \times Y$. Then there exists an open set $W \subseteq X$ containing x_0 such that $\{x_0\} \times Y \subseteq W \times Y \subseteq N$.*

Proof. For each $y \in Y$, the point $(x_0, y) \in N$, so there exist open sets $U_y \subseteq X$ and $V_y \subseteq Y$ with $(x_0, y) \in U_y \times V_y \subseteq N$. The collection $\{V_y\}_{y \in Y}$ is an open cover of Y . Since Y is compact, there is a finite subcover V_{y_1}, \dots, V_{y_m} . Set

$$W = U_{y_1} \cap \dots \cap U_{y_m}.$$

Then W is open, $x_0 \in W$, and for any $(w, y) \in W \times Y$, we have $y \in V_{y_k}$ for some k , so $(w, y) \in U_{y_k} \times V_{y_k} \subseteq N$. Hence $W \times Y \subseteq N$. \square

8.5 Products and Separation

Theorem 8.5.1. *The product $\prod_{i \in I} X_i$ (with the product topology) is Hausdorff if and only if each X_i is Hausdorff.*

Proof. “Only if.” Each X_j is the continuous image of the projection, and a subspace of a Hausdorff space is Hausdorff. More precisely, if (x_i) and (y_i) differ in coordinate j with $x_j \neq y_j$, and we can separate them in the product, then projecting gives a separation in X_j .

“If.” Let $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ be distinct points in $\prod X_i$. There exists $j \in I$ with $x_j \neq y_j$. Since X_j is Hausdorff, choose disjoint open sets $U_j, V_j \subseteq X_j$ with $x_j \in U_j$ and $y_j \in V_j$. Then $\pi_j^{-1}(U_j)$ and $\pi_j^{-1}(V_j)$ are disjoint open sets in $\prod X_i$ separating \mathbf{x} and \mathbf{y} . \square

8.6 Quotient Topology

Definition 8.6.1 (Quotient map and quotient topology). Let X be a topological space, Y a set, and $q: X \rightarrow Y$ a surjection. The **quotient topology** on Y (induced by q) is

$$\tau_Y = \{V \subseteq Y : q^{-1}(V) \text{ is open in } X\}.$$

With this topology, q is called a **quotient map**.

If \sim is an equivalence relation on X , the **quotient space** X/\sim is the set of equivalence classes equipped with the quotient topology induced by the canonical projection $q: X \rightarrow X/\sim$, $q(x) = [x]$.

Definition 8.6.2 (Saturated set). A subset $A \subseteq X$ is **saturated** with respect to a surjection $q: X \rightarrow Y$ if $A = q^{-1}(q(A))$; that is, A is a union of fibers $q^{-1}(\{y\})$.

Remark 8.6.1. A subset $V \subseteq Y$ is open in the quotient topology if and only if $q^{-1}(V)$ is a saturated open subset of X . Thus the open sets of X/\sim correspond bijectively to the saturated open subsets of X .

Theorem 8.6.2 (Universal property of the quotient). *Let $q: X \rightarrow Y$ be a quotient map and $f: X \rightarrow Z$ a continuous map that is constant on the fibers of q (i.e., $q(x) = q(x')$ implies $f(x) = f(x')$). Then there exists a unique continuous map $\bar{f}: Y \rightarrow Z$ such that $f = \bar{f} \circ q$.*

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ & \searrow f & \vdots \bar{f} \\ & & Z \end{array}$$

Proof. Existence and uniqueness. Define $\bar{f}(y) = f(x)$ for any $x \in q^{-1}(\{y\})$. This is well-defined because f is constant on fibers of q . By construction $f = \bar{f} \circ q$, and uniqueness is immediate since q is surjective.

Continuity. Let $W \subseteq Z$ be open. Then $q^{-1}(\bar{f}^{-1}(W)) = (\bar{f} \circ q)^{-1}(W) = f^{-1}(W)$, which is open in X because f is continuous. By definition of the quotient topology, $\bar{f}^{-1}(W)$ is open in Y . \square

Theorem 8.6.3 (Continuity via quotient maps). *Let $q: X \rightarrow Y$ be a quotient map. A map $g: Y \rightarrow Z$ is continuous if and only if $g \circ q: X \rightarrow Z$ is continuous.*

Proof. "Only if." A composition of continuous maps is continuous.

"If." Let $W \subseteq Z$ be open. Then $q^{-1}(g^{-1}(W)) = (g \circ q)^{-1}(W)$ is open in X . By definition of the quotient topology, $g^{-1}(W)$ is open in Y . \square

8.7 Quotient Examples with Identification Diagrams

Example 8.7.1 (The circle S^1). *Define an equivalence relation on $[0, 1]$ by $0 \sim 1$ (and $x \sim x$ for all x). The quotient $[0, 1]/(0 \sim 1)$ is homeomorphic to S^1 .*

The quotient map is $q: [0, 1] \rightarrow S^1$, $q(t) = e^{2\pi it}$. This map is continuous, surjective, and since $[0, 1]$ is compact and S^1 is Hausdorff, q is a quotient map. It identifies only 0 and 1, so it descends to a continuous bijection $[0, 1]/(0 \sim 1) \rightarrow S^1$, which is a homeomorphism (continuous bijection from a compact space to a Hausdorff space).

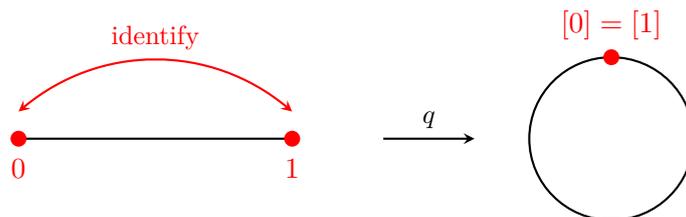


Figure 8.1: The circle S^1 as the quotient $[0, 1]/(0 \sim 1)$.

Example 8.7.2 (The torus T^2). *Define an equivalence relation on $[0, 1]^2$ by identifying opposite edges:*

$$(0, t) \sim (1, t) \quad \text{and} \quad (s, 0) \sim (s, 1) \quad \text{for all } s, t \in [0, 1].$$

The quotient $[0, 1]^2/\sim$ is homeomorphic to the torus $T^2 = S^1 \times S^1$. Indeed, the map $q(s, t) = (e^{2\pi is}, e^{2\pi it})$ is a quotient map realizing this identification.

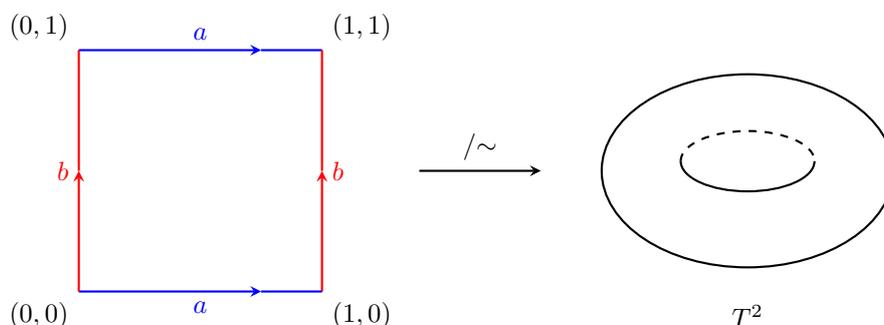


Figure 8.2: The torus T^2 obtained by identifying opposite edges of the unit square with matching orientations. Edges labeled a are identified, as are edges labeled b .

Example 8.7.3 (The Möbius band). On $[0, 1]^2$, identify $(0, t) \sim (1, 1 - t)$ for all $t \in [0, 1]$. This reverses the orientation of one pair of opposite edges, yielding the Möbius band.

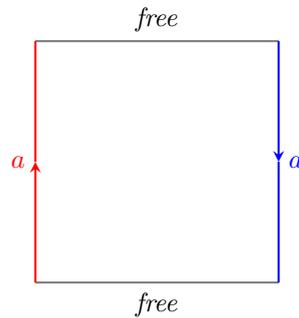


Figure 8.3: The Möbius band: the edges labeled a are identified with reversed orientations (note the opposing arrows).

Example 8.7.4 (The Klein bottle). On $[0, 1]^2$, identify $(0, t) \sim (1, 1 - t)$ and $(s, 0) \sim (s, 1)$. The first identification reverses one pair of edges (as for the Möbius band), and the second identifies the other pair with matching orientations. The resulting quotient is the Klein bottle, which cannot be embedded in \mathbb{R}^3 without self-intersection.

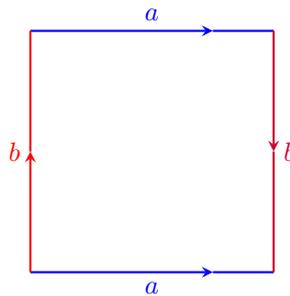


Figure 8.4: The Klein bottle: edges a are identified with the same orientation, but edges b are identified with reversed orientation.

Example 8.7.5 (The real projective plane $\mathbb{R}P^2$). There are two common descriptions.

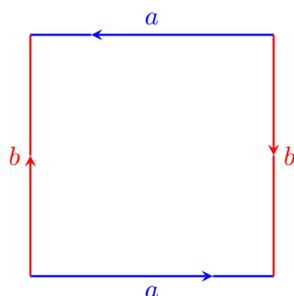
1. As a quotient of S^2 . On the 2-sphere S^2 , identify antipodal points: $x \sim -x$. The quotient S^2/\sim is $\mathbb{R}P^2$.
2. As a quotient of a square. On $[0, 1]^2$, identify $(0, t) \sim (1, 1 - t)$ and $(s, 0) \sim (1 - s, 1)$. Both pairs of opposite edges are identified with reversed orientations.

The resulting space is non-orientable and cannot be embedded in \mathbb{R}^3 .

8.8 Disjoint Union (Coproduct) Topology

Definition 8.8.1. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. The **disjoint union** (or **coproduct**) is the set

$$\coprod_{i \in I} X_i = \bigcup_{i \in I} \{i\} \times X_i,$$



Both pairs reversed

Figure 8.5: The real projective plane $\mathbb{R}P^2$: both pairs of opposite edges are identified with reversed orientations.

equipped with the topology in which $U \subseteq \coprod X_i$ is open if and only if $U \cap (\{i\} \times X_i)$ is open in X_i for every $i \in I$. Equivalently, it is the finest topology making each canonical injection $\iota_j: X_j \rightarrow \coprod X_i$ continuous.

Proposition 8.8.1. *A map $f: \coprod_{i \in I} X_i \rightarrow Y$ is continuous if and only if $f \circ \iota_i: X_i \rightarrow Y$ is continuous for every $i \in I$.*

Proof. “Only if.” Compositions of continuous maps are continuous.

“If.” Let $V \subseteq Y$ be open. Then $f^{-1}(V) \cap (\{i\} \times X_i) = \iota_i^{-1}(f^{-1}(V)) = (f \circ \iota_i)^{-1}(V)$, which is open in X_i by hypothesis. By definition of the disjoint union topology, $f^{-1}(V)$ is open. \square

8.9 Box Topology Pathologies

The box topology, while seemingly natural, has severe pathological properties for infinite products that make it poorly suited to general theory.

Theorem 8.9.1. *Let $\mathbb{R}^\omega = \prod_{n=1}^\infty \mathbb{R}$ equipped with the box topology. Then:*

1. \mathbb{R}^ω with the box topology is **not connected**.
2. \mathbb{R}^ω with the box topology is **not second countable**.
3. \mathbb{R}^ω with the box topology is **not metrizable**.

Proof. (1) Let B denote the set of bounded sequences and U the set of unbounded sequences. We claim both are open in the box topology. For any bounded sequence (x_n) with $|x_n| \leq M$ for all n , the box-open set $\prod_{n=1}^\infty (-M - 1, M + 1)$ is a neighborhood contained in B . For any unbounded sequence (x_n) , choose a subsequence $|x_{n_k}| \rightarrow \infty$ and for each n_k take an interval $(x_{n_k} - 1, x_{n_k} + 1)$, and for other coordinates take all of \mathbb{R} ; the resulting box-open set lies within U . Since B and U are non-empty, open, disjoint, and cover \mathbb{R}^ω , it is disconnected.

(2) For each function $f: \mathbb{N} \rightarrow \mathbb{R}$, the box-open set $\prod_{n=1}^\infty (f(n) - 1, f(n) + 1)$ contains the point $(f(n))_{n \geq 1}$ but does not contain any point $(g(n))$ with $|g(k) - f(k)| \geq 1$ for some k . These form an uncountable collection of open sets with pairwise distinct “centers,” and any basis must contain at least one basis element inside each, yielding uncountably many basis elements. Hence no countable basis exists.

(3) We show that \mathbb{R}^ω with the box topology is not first countable, hence not metrizable. Consider the origin $\mathbf{0} = (0, 0, 0, \dots)$ and suppose $\{B_k\}_{k \geq 1}$ is a countable neighborhood basis at $\mathbf{0}$. For

each k , since B_k is open, there exists $\varepsilon_k^{(n)} > 0$ for each n such that $\prod_n(-\varepsilon_k^{(n)}, \varepsilon_k^{(n)}) \subseteq B_k$. Define $\delta_n = \varepsilon_n^{(n)}/2 > 0$ and consider the box-open set $W = \prod_n(-\delta_n, \delta_n)$. For each k , the point with n -th coordinate $\frac{2}{3}\varepsilon_k^{(k)}$ (choosing $n = k$) lies in B_k but not in W (when $\frac{2}{3}\varepsilon_k^{(k)} \geq \delta_k = \varepsilon_k^{(k)}/2$, which holds since $\frac{2}{3} > \frac{1}{2}$). Hence $B_k \not\subseteq W$ for any k , contradicting the assumption that $\{B_k\}$ is a neighborhood basis. \square

8.10 Worked Examples

Example 8.10.1 (Convergence in the product topology). *A sequence $(x^{(k)})_{k \geq 1}$ in $\prod_{i \in I} X_i$ converges to $x = (x_i)$ in the product topology if and only if $\pi_i(x^{(k)}) \rightarrow x_i$ in X_i for every $i \in I$. That is, convergence in the product topology is coordinate-wise convergence.*

Indeed, the “only if” direction follows from continuity of projections. For the “if” direction, let $U = \bigcap_{j=1}^n \pi_{i_j}^{-1}(V_{i_j})$ be a basic open neighborhood of x . For each j , there exists N_j such that $\pi_{i_j}(x^{(k)}) \in V_{i_j}$ for $k \geq N_j$. Taking $N = \max(N_1, \dots, N_n)$, we get $x^{(k)} \in U$ for $k \geq N$.

Example 8.10.2 (Quotient that is not Hausdorff). *Even if X is Hausdorff, X/\sim may fail to be Hausdorff. Consider $X = \mathbb{R}$ with the equivalence relation $x \sim y$ if $x - y \in \mathbb{Q}$. Then \mathbb{R}/\sim has uncountably many points (one for each coset of \mathbb{Q}), but the quotient topology is the indiscrete topology $\{\emptyset, \mathbb{R}/\mathbb{Q}\}$.*

To see this, suppose $U \subseteq \mathbb{R}/\mathbb{Q}$ is open and non-empty. Then $q^{-1}(U)$ is a non-empty open subset of \mathbb{R} that is saturated, meaning it is a union of cosets of \mathbb{Q} . But every non-empty open set in \mathbb{R} contains representatives of every coset of \mathbb{Q} (since \mathbb{Q} is dense), so $q^{-1}(U) = \mathbb{R}$, hence $U = \mathbb{R}/\mathbb{Q}$.

Example 8.10.3 (The cone on a space). *Let X be a topological space. The **cone** on X is the quotient*

$$CX = (X \times [0, 1]) / (X \times \{1\}),$$

where all points of the form $(x, 1)$ are identified to a single “apex” point. If $X = S^{n-1}$, then $CX \cong D^n$ (the closed n -disk). The cone is always contractible (path-connected to the apex), hence connected.

Example 8.10.4 (The suspension of a space). *The **suspension** of X is*

$$\Sigma X = (X \times [-1, 1]) / (X \times \{-1\}, X \times \{1\}),$$

collapsing the top and bottom to two distinct points. For instance, $\Sigma S^n \cong S^{n+1}$.

8.11 Exercises

Exercise 8.11.1 (★). Show that the product of two discrete spaces is discrete. Is the product of infinitely many discrete spaces (each with at least two points) discrete in the product topology?

Exercise 8.11.2 (★). Find an explicit closed set $C \subseteq \mathbb{R}^2$ whose projection $\pi_1(C)$ onto the first coordinate is not closed in \mathbb{R} .

Exercise 8.11.3 (★). Let $A_i \subseteq X_i$ for each $i \in I$. Show that the subspace topology on $\prod A_i$ (as a subset of $\prod X_i$ with the product topology) coincides with the product of the subspace topologies.

Exercise 8.11.4 (★★). Use the Tube Lemma to give a direct proof that the product of two compact spaces is compact (without invoking Tychonoff’s theorem).

Exercise 8.11.5 (★★). Let $q: X \rightarrow Y$ be a quotient map. Show that q is an open map if and only if for every open set $U \subseteq X$, the saturated set $q^{-1}(q(U))$ is open in X .

Exercise 8.11.6 (★★). Let X be a compact Hausdorff space and \sim an equivalence relation on X . Show that X/\sim is Hausdorff if and only if the set $R = \{(x, y) \in X \times X : x \sim y\}$ is closed in $X \times X$.

Exercise 8.11.7 (★★). Show that the product $\prod_{i \in I} X_i$ (with the product topology) is first countable if and only if each X_i is first countable and all but countably many of the X_i are indiscrete.

Exercise 8.11.8 (★★★). Let X and Y be topological spaces, $A \subseteq X$ closed, and $f: A \rightarrow Y$ continuous. The **adjunction space** $Y \cup_f X$ is the quotient of $X \sqcup Y$ by the relation generated by $a \sim f(a)$ for all $a \in A$. Show that the natural inclusion $Y \hookrightarrow Y \cup_f X$ is an embedding (homeomorphism onto its image) and that Y is closed in $Y \cup_f X$.

Exercise 8.11.9 (★★★). Show that in \mathbb{R}^ω with the box topology, the sequence $x^{(k)} = (1/k, 1/k, 1/k, \dots)$ does **not** converge to the origin, even though each coordinate converges to 0. Conclude that the box topology does not satisfy the universal property of the product.

Chapter Summary

- The **product topology** on $\prod X_i$ is generated by the subbasis of preimages $\pi_i^{-1}(U_i)$. It is the coarsest topology making all projections continuous and satisfies a universal property: $f: Y \rightarrow \prod X_i$ is continuous iff each $\pi_i \circ f$ is.
- The **box topology** agrees with the product topology for finite products but is strictly finer for infinite products. It leads to pathologies: \mathbb{R}^ω with the box topology is not connected, not second countable, and not metrizable.
- Projections are continuous, open, and surjective, but not necessarily closed. The **Tube Lemma** compensates in the compact case.
- Products preserve the Hausdorff property, connectedness (Chapter 7), and compactness (Tychonoff).
- The **quotient topology** on $Y = X/\sim$ is the finest topology making the projection q continuous: $V \subseteq Y$ is open iff $q^{-1}(V)$ is open in X .
- The universal property of quotients: continuous maps constant on fibers descend uniquely to continuous maps on the quotient.
- Fundamental geometric objects— S^1 , T^2 , the Möbius band, the Klein bottle, $\mathbb{R}P^2$ —all arise naturally as quotient spaces.
- The **disjoint union** (coproduct) is the dual construction: it carries the finest topology making all injections continuous.

Chapter 9

Urysohn's Lemma and the Tietze Extension Theorem

Motivation

A central question in topology is the following: given two disjoint closed subsets of a topological space, can they be separated by a continuous real-valued function? In a metric space the answer is immediate—the distance function provides such a separation—but for a general topological space the situation is far more subtle. This chapter is devoted to the celebrated *Urysohn lemma*, which answers the question affirmatively for *normal* spaces, and to its powerful corollary, the *Tietze extension theorem*. Together, these results form the cornerstone of the theory of normal spaces and have far-reaching applications: metrization theorems, embedding theorems, and the construction of partitions of unity.

9.1 Completely Regular Spaces

Definition 9.1.1 (Completely regular space). A topological space X is called **completely regular** (or **Tychonoff**, or a $T_{3\frac{1}{2}}$ **space**) if it is T_1 and for every closed set $F \subseteq X$ and every point $x \in X \setminus F$ there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f|_F \equiv 1$.

Remark 9.1.1. The separation axioms studied so far satisfy the strict implications

$$T_4 \implies T_{3\frac{1}{2}} \implies T_3 \implies T_2 \implies T_1 \implies T_0.$$

The implication $T_4 \implies T_{3\frac{1}{2}}$ is precisely the content of Urysohn's lemma (Theorem 9.2.1 below). None of the reverse implications holds in general.

Example 9.1.2. *Every metrizable space (X, d) is completely regular. Indeed, given a closed set F and a point $x \notin F$, the function*

$$f(y) = \frac{d(y, F)}{d(y, F) + d(y, x)}$$

is continuous, satisfies $f(x) = 0$, and $f|_F \equiv 1$.

Example 9.1.3. *An arbitrary product of completely regular spaces is completely regular (in the product topology). This follows directly from the definition since projections are continuous and continuous functions on factors pull back to continuous functions on the product.*

9.2 Urysohn's Lemma

Theorem 9.2.1 (Urysohn's Lemma). *Let X be a normal topological space, and let A and B be disjoint closed subsets of X . Then there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f(a) = 0$ for every $a \in A$ and $f(b) = 1$ for every $b \in B$.*

Proof. The proof proceeds in three steps: we first assign an open set U_r to each dyadic rational $r \in [0, 1]$, then define f in terms of these sets, and finally prove that f is continuous.

Step 1. Construction of the sets U_r . Let D denote the set of *dyadic rationals* in $[0, 1]$:

$$D = \left\{ \frac{k}{2^n} : n \in \mathbb{N}_0, 0 \leq k \leq 2^n \right\}.$$

We shall construct, for each $r \in D$, an open set $U_r \subseteq X$ such that

$$A \subseteq U_r, \quad U_r \cap B = \emptyset, \quad \text{and} \quad r < s \implies \overline{U_r} \subseteq U_s. \quad (9.1)$$

We proceed by induction on the “level” of the dyadic decomposition.

Base case. Since A and B are disjoint closed sets in the normal space X , there exists an open set U_1 with $A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$ (using normality: for the disjoint closed sets A and B find open sets separating them, then take the closure). More precisely, set $U_1 = X \setminus B$; this is open and satisfies $A \subseteq U_1$. Now apply normality to the disjoint closed sets A and $X \setminus U_1 = B$: there exist disjoint open sets V, W with $A \subseteq V$ and $B \subseteq W$. Set $U_0 = V$. Then $A \subseteq U_0 \subseteq \overline{U_0} \subseteq X \setminus W \subseteq X \setminus B = U_1$.

Inductive step. Suppose we have already constructed open sets U_r for all dyadic rationals r of the form $k/2^n$, $0 \leq k \leq 2^n$, satisfying (9.1). Consider a new dyadic rational of the form $r = (2j+1)/2^{n+1}$ with $0 \leq j < 2^n$. Set $r^- = j/2^n$ and $r^+ = (j+1)/2^n$; these are consecutive dyadics at the previous level, with $r^- < r < r^+$. By induction, $\overline{U_{r^-}} \subseteq U_{r^+}$. Hence $\overline{U_{r^-}}$ and $X \setminus U_{r^+}$ are disjoint closed sets. By normality, there exists an open set U_r with

$$\overline{U_{r^-}} \subseteq U_r \subseteq \overline{U_r} \subseteq U_{r^+}.$$

This completes the induction. By construction, the family $(U_r)_{r \in D}$ satisfies (9.1).

Step 2. Definition of f . For each $x \in X$, define

$$f(x) = \inf \{ r \in D : x \in U_r \},$$

with the convention $\inf \emptyset = 1$.

Claim. $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Indeed, $A \subseteq U_r$ for every $r \in D$ with $r \geq 0$, so $f(a) = \inf D = 0$ for every $a \in A$. For $b \in B$, note that $U_r \cap B = \emptyset$ for every $r \in D \cap [0, 1)$ (since $\overline{U_r} \subseteq U_1 = X \setminus B$), hence $b \notin U_r$ for $r < 1$; the only dyadic rational r with $b \in U_r$ is $r = 1$ (since $U_1 = X \setminus B$ may or may not contain b —but actually $b \in B$ so $b \notin U_1 = X \setminus B$). Hence the infimum is taken over the empty set, giving $f(b) = 1$.

Step 3. Continuity of f . It suffices to show that $f^{-1}((-\infty, \alpha))$ and $f^{-1}((\beta, +\infty))$ are open for all $\alpha, \beta \in \mathbb{R}$, since intervals of this form constitute a subbasis for the topology on \mathbb{R} .

Preimage of $(-\infty, \alpha)$. We have $f(x) < \alpha$ if and only if there exists $r \in D$ with $r < \alpha$ and $x \in U_r$. Hence

$$f^{-1}((-\infty, \alpha)) = \bigcup_{\substack{r \in D \\ r < \alpha}} U_r,$$

which is a union of open sets and therefore open.

Preimage of $(\beta, +\infty)$. We claim that $f(x) > \beta$ if and only if there exists $s \in D$ with $s > \beta$ and $x \notin \overline{U_s}$. Indeed, if $f(x) > \beta$, pick $s \in D$ with $\beta < s < f(x)$; then $x \notin U_s$, hence $x \notin \overline{U_s}$ for

some r with $\beta < r < s$ (choosing r dyadic with $r < s$ and noting $\overline{U_r} \subseteq U_s$, so $x \notin U_s$ implies we can find an appropriate r). Conversely, if $x \notin \overline{U_s}$ then $x \notin U_r$ for all $r \leq s$ (since $U_r \subseteq \overline{U_r} \subseteq U_s$ for $r < s$ —but we need $x \notin U_r$ for $r \leq s$, which follows from $x \notin \overline{U_s}$ only when $\overline{U_r} \subseteq U_s$ for $r < s$; indeed, for any $r < s$ with r, s dyadic we have $\overline{U_r} \subseteq U_s$, so $x \notin U_s \supseteq \overline{U_r} \supseteq U_r$, i.e., $x \notin U_r$). Thus $f(x) \geq s > \beta$.

More precisely, we obtain

$$f^{-1}((\beta, +\infty)) = \bigcup_{\substack{s \in D \\ s > \beta}} (X \setminus \overline{U_s}),$$

which is again a union of open sets. Hence f is continuous. □

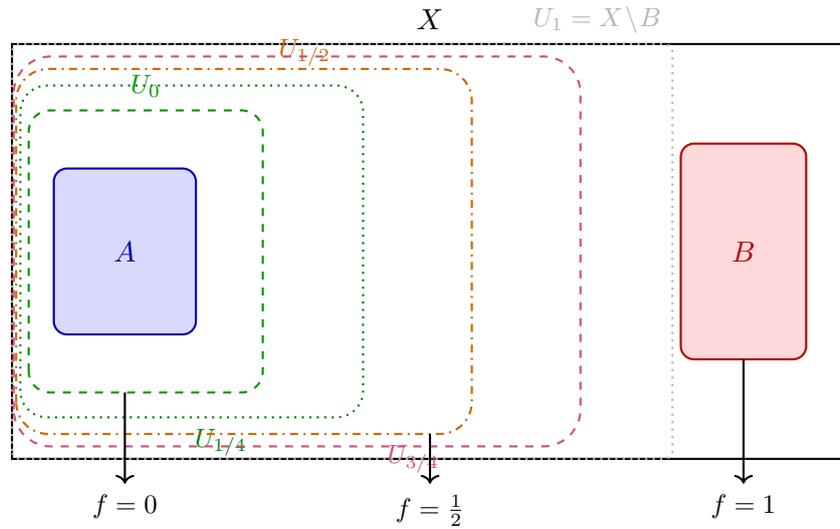


Figure 9.1: The Urysohn function construction. The nested open sets $U_0 \subseteq U_{1/4} \subseteq U_{1/2} \subseteq U_{3/4} \subseteq U_1$ satisfy $\overline{U_r} \subseteq U_s$ for $r < s$. The function f takes value 0 on A and value 1 on B , varying continuously through the nested layers.

Corollary 9.2.2. *Every normal (T_4) space is completely regular ($T_{3\frac{1}{2}}$).*

Proof. Let X be normal, $F \subseteq X$ closed, and $x \in X \setminus F$. Since X is T_1 , the singleton $\{x\}$ is closed. By normality, $\{x\}$ and F are disjoint closed sets, so Urysohn’s lemma provides a continuous $f: X \rightarrow [0, 1]$ with $f(x) = 0$ and $f|_F \equiv 1$. □

9.3 The Tietze Extension Theorem

Theorem 9.3.1 (Tietze Extension Theorem). *Let X be a normal space and $A \subseteq X$ a closed subspace. If $f: A \rightarrow [a, b]$ is continuous, then there exists a continuous function $F: X \rightarrow [a, b]$ such that $F|_A = f$.*

Proof. By an affine change of variables it suffices to treat the case $[a, b] = [-1, 1]$. We construct a sequence of continuous functions $g_n: X \rightarrow \mathbb{R}$ and set $F = \sum_{n=0}^{\infty} g_n$.

Step 1. Iterative approximation. Given a continuous function $h: A \rightarrow [-c, c]$ (for some $c > 0$) and the normal space X , we show there exists a continuous function $g: X \rightarrow [-c/3, c/3]$ such that

$$|h(a) - g(a)| \leq \frac{2c}{3} \quad \text{for all } a \in A. \tag{9.2}$$

Define

$$A_1 = h^{-1}([-c, -c/3]), \quad A_2 = h^{-1}([c/3, c]).$$

Since h is continuous and A carries the subspace topology, A_1 and A_2 are closed in A . Since A is closed in X , both A_1 and A_2 are closed in X . Moreover, $A_1 \cap A_2 = \emptyset$. By Urysohn's lemma (Theorem 9.2.1) there exists a continuous function $g: X \rightarrow [-c/3, c/3]$ with $g|_{A_1} \equiv -c/3$ and $g|_{A_2} \equiv c/3$. (Explicitly, apply Urysohn's lemma to obtain $\varphi: X \rightarrow [0, 1]$ with $\varphi|_{A_1} \equiv 0$ and $\varphi|_{A_2} \equiv 1$, then set $g = \frac{2c}{3}\varphi - \frac{c}{3}$.)

We verify (9.2). For $a \in A_1$: $h(a) \in [-c, -c/3]$ and $g(a) = -c/3$, so $|h(a) - g(a)| \leq 2c/3$. For $a \in A_2$: $h(a) \in [c/3, c]$ and $g(a) = c/3$, so $|h(a) - g(a)| \leq 2c/3$. For $a \in A \setminus (A_1 \cup A_2)$: $h(a) \in (-c/3, c/3)$ and $|g(a)| \leq c/3$, so $|h(a) - g(a)| < 2c/3$.

Step 2. Inductive construction. Set $f_0 = f$ (so $f_0: A \rightarrow [-1, 1]$) and $c_0 = 1$. Applying Step 1 with $h = f_0$ and $c = c_0 = 1$, we obtain $g_0: X \rightarrow [-1/3, 1/3]$ such that $|f_0(a) - g_0(a)| \leq 2/3$ for all $a \in A$.

Define $f_1 = f_0 - g_0|_A$; then $f_1: A \rightarrow [-2/3, 2/3]$. Apply Step 1 with $h = f_1$ and $c_1 = 2/3$ to get $g_1: X \rightarrow [-(2/3)/3, (2/3)/3] = [-2/9, 2/9]$ with $|f_1(a) - g_1(a)| \leq (2/3)^2$ for all $a \in A$.

Inductively, at stage n we have $f_n: A \rightarrow [-(2/3)^n, (2/3)^n]$, and we obtain $g_n: X \rightarrow [-(2/3)^n/3, (2/3)^n/3]$ with $|f_n(a) - g_n(a)| \leq (2/3)^{n+1}$. Set $f_{n+1} = f_n - g_n|_A$.

Step 3. Convergence and properties of F . Define

$$F(x) = \sum_{n=0}^{\infty} g_n(x).$$

We have $\|g_n\|_{\infty} \leq \frac{1}{3}(\frac{2}{3})^n$, so $\sum_{n=0}^{\infty} \|g_n\|_{\infty} = \frac{1}{3} \cdot \frac{1}{1-2/3} = 1 < \infty$. By the Weierstrass M -test, the series converges uniformly on X , so F is continuous.

Moreover, $|F(x)| \leq \sum_{n=0}^{\infty} \|g_n\|_{\infty} = 1$, so $F: X \rightarrow [-1, 1]$.

Finally, for $a \in A$:

$$F(a) = \sum_{n=0}^{\infty} g_n(a) = \lim_{N \rightarrow \infty} \sum_{n=0}^N g_n(a).$$

By construction, $f(a) - \sum_{n=0}^N g_n(a) = f_{N+1}(a)$, and $|f_{N+1}(a)| \leq (2/3)^{N+1} \rightarrow 0$. Hence $F(a) = f(a)$ for all $a \in A$. \square

Remark 9.3.2. The Tietze extension theorem also holds for $f: A \rightarrow \mathbb{R}$ (unbounded case). One may reduce to the bounded case via the homeomorphism $\mathbb{R} \cong (-1, 1)$ given by $t \mapsto t/(1+|t|)$, or alternatively extend f to a bounded function first and then post-compose. The resulting extension $F: X \rightarrow \mathbb{R}$ is continuous.

Remark 9.3.3. The converse also holds: if every continuous real-valued function on a closed subspace of X extends to X , then X is normal. Hence the Tietze extension property characterizes normality (among T_1 spaces).

9.4 The Urysohn Metrization Theorem

Theorem 9.4.1 (Urysohn Metrization Theorem). *Every regular (T_3), second countable space is metrizable.*

Proof. Let X be a regular, second countable space with countable basis $\mathcal{B} = \{B_n\}_{n \geq 1}$. Recall that a regular, second countable space is normal (a classical result).

Step 1. Constructing a countable separating family of functions. For each pair (m, n) with $\overline{B_m} \subseteq B_n$, apply Urysohn's lemma to the disjoint closed sets $\overline{B_m}$ and $X \setminus B_n$: there exists a continuous function $f_{m,n}: X \rightarrow [0, 1]$ with $f_{m,n}|_{\overline{B_m}} \equiv 1$ and $f_{m,n}|_{X \setminus B_n} \equiv 0$. The collection $\{f_{m,n}\}$ is countable.

This family separates points from closed sets. Given a closed set F and $x \notin F$, choose (by regularity) a basis element B_n with $x \in B_n \subseteq X \setminus F$. By regularity again (and second countability), find B_m with $x \in B_m \subseteq \overline{B_m} \subseteq B_n$. Then $f_{m,n}(x) = 1$ and $f_{m,n}|_F \equiv 0$.

Step 2. Embedding in \mathbb{R}^ω . Enumerate the family of functions as $\{f_k\}_{k \geq 1}$. Define $\Phi: X \rightarrow \mathbb{R}^\omega$ by

$$\Phi(x) = (f_1(x), f_2(x), f_3(x), \dots).$$

Since each f_k is continuous and \mathbb{R}^ω carries the product topology, Φ is continuous. The separating property ensures that Φ is injective and that Φ is an embedding (i.e., a homeomorphism onto its image): if $U \subseteq X$ is open, $x \in U$, then there exists f_k with $f_k(x) > 0$ and $f_k|_{X \setminus U} \equiv 0$, so $\Phi(U)$ is open in $\Phi(X)$.

Step 3. Metrizable. The space \mathbb{R}^ω (with the product topology) is metrizable; one compatible metric is

$$d((x_k), (y_k)) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}.$$

Since Φ is an embedding, X is homeomorphic to a subspace of a metrizable space, hence metrizable. □

Corollary 9.4.2. *Every compact Hausdorff second countable space is metrizable.*

Proof. A compact Hausdorff space is normal, hence regular. Apply Theorem 9.4.1. □

9.5 Applications: Partitions of Unity

Definition 9.5.1 (Partition of unity). Let X be a topological space and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X . A **partition of unity subordinate to $\{U_\alpha\}$** is a family $\{\varphi_\alpha\}_{\alpha \in I}$ of continuous functions $\varphi_\alpha: X \rightarrow [0, 1]$ such that:

1. $\text{supp}(\varphi_\alpha) \subseteq U_\alpha$ for each $\alpha \in I$;
2. the family $\{\text{supp}(\varphi_\alpha)\}$ is locally finite;
3. $\sum_{\alpha \in I} \varphi_\alpha(x) = 1$ for every $x \in X$.

Theorem 9.5.1. *A Hausdorff space X is paracompact if and only if every open cover of X admits a partition of unity subordinate to it.*

Proof sketch. The “if” direction is straightforward: the supports give a locally finite refinement. For the “only if” direction, one uses the shrinking lemma for locally finite covers and Urysohn’s lemma on each pair (shrunk closed set, complement of original open set) to construct the functions, then normalizes. □

Remark 9.5.2. Partitions of unity are indispensable in differential geometry and the theory of manifolds. They allow one to pass from local constructions (defined on coordinate charts) to global objects on the manifold—for instance, Riemannian metrics, integration of differential forms, and extension of sections of vector bundles.

9.6 Counterexample: The Sorgenfrey Plane

Example 9.6.1. *The **Sorgenfrey plane** $S = \mathbb{R}_\ell \times \mathbb{R}_\ell$ (the product of two copies of the Sorgenfrey line, see Appendix A.1) is not normal.*

Consider the anti-diagonal $\Delta' = \{(x, -x) : x \in \mathbb{R}\}$ and its subsets $A = \{(x, -x) : x \in \mathbb{Q}\}$ and $B = \{(x, -x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$. Both A and B are closed in S (they are closed discrete subsets of the closed subspace Δ'), and $A \cap B = \emptyset$. However, one can show (using a Baire category argument or a cardinality argument on the real line) that there do not exist disjoint open sets in S separating A and B .

Since S is not normal, Urysohn's lemma cannot be applied: there is no continuous function $f: S \rightarrow [0, 1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.

9.7 Worked Examples

Example 9.7.1 (Extending a continuous function). Let $X = \mathbb{R}$ (which is normal) and $A = [0, 1] \cup [2, 3]$. Define $f: A \rightarrow [0, 1]$ by $f(x) = x$ for $x \in [0, 1]$ and $f(x) = 3 - x$ for $x \in [2, 3]$. The Tietze extension theorem guarantees a continuous extension $F: \mathbb{R} \rightarrow [0, 1]$. In fact, one explicit extension is

$$F(x) = \begin{cases} x & \text{if } x \in [0, 1], \\ 1 & \text{if } x \in [1, 2], \\ 3 - x & \text{if } x \in [2, 3], \\ 0 & \text{if } x \notin [0, 3]. \end{cases}$$

Example 9.7.2 (Urysohn function in a metric space). In \mathbb{R}^2 with the Euclidean metric, let $A = \overline{B}(0, 1)$ (closed unit disk) and $B = \{x \in \mathbb{R}^2 : \|x\| \geq 2\}$. An explicit Urysohn function is

$$f(x) = \begin{cases} 0 & \text{if } \|x\| \leq 1, \\ \|x\| - 1 & \text{if } 1 \leq \|x\| \leq 2, \\ 1 & \text{if } \|x\| \geq 2. \end{cases}$$

This is continuous, takes value 0 on A and 1 on B .

Example 9.7.3 (A non-extendable function on a non-normal space). Let $S = \mathbb{R}_\ell \times \mathbb{R}_\ell$ be the Sorgenfrey plane and $\Delta' = \{(x, -x) : x \in \mathbb{R}\}$ the anti-diagonal. The subspace Δ' is discrete in S , so every function $f: \Delta' \rightarrow \mathbb{R}$ is continuous. However, not every such function extends continuously to S . For instance, the characteristic function χ_A of $A = \{(q, -q) : q \in \mathbb{Q}\}$ on Δ' does not extend to a continuous function $S \rightarrow \mathbb{R}$, since S is not normal and the Tietze extension theorem fails.

9.8 Exercises

Exercise 9.8.1 (★). Let X be a normal space and let $A, B \subseteq X$ be disjoint closed sets. Show that there exists a continuous function $f: X \rightarrow [0, 1]$ with $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$ if and only if A and B are G_δ sets.

Exercise 9.8.2 (★). Prove that every metrizable space is completely regular directly (without using Urysohn's lemma), using the distance function.

Exercise 9.8.3 (★). Let X be a compact Hausdorff space. Show that X is metrizable if and only if X is second countable.

Exercise 9.8.4 (★★). Let X be a normal space, $A \subseteq X$ closed, and $f: A \rightarrow \mathbb{R}$ continuous (unbounded). Prove that f extends to a continuous function $F: X \rightarrow \mathbb{R}$. *Hint:* compose with a homeomorphism $\mathbb{R} \cong (-1, 1)$.

Exercise 9.8.5 (★★). Let X be a second countable, regular space. Show that X embeds in the Hilbert cube $[0, 1]^\mathbb{N}$.

Exercise 9.8.6 (★★). Show that a compact Hausdorff space X is normal. Deduce that Urysohn's lemma and the Tietze extension theorem apply to every compact Hausdorff space.

Exercise 9.8.7 (★★★). Prove in detail that the Sorgenfrey plane is not normal. *Hint:* use the Jones lemma: if X is normal and has a closed discrete subspace D and a dense subspace S , then $2^{|D|} \leq 2^{|S|}$.

Exercise 9.8.8 (★★★). Let X be a normal space and $\{U_1, U_2\}$ a finite open cover. Construct an explicit partition of unity $\{\varphi_1, \varphi_2\}$ subordinate to this cover using Urysohn's lemma. Generalize to a finite open cover $\{U_1, \dots, U_n\}$.

Chapter Summary

- A space is **completely regular** ($T_{3\frac{1}{2}}$) if points can be separated from closed sets by continuous functions into $[0, 1]$.
- **Urysohn's lemma:** in a normal space, disjoint closed sets can be separated by a continuous function $f: X \rightarrow [0, 1]$. The proof constructs nested open sets indexed by dyadic rationals.
- **Tietze extension theorem:** in a normal space, every continuous function $f: A \rightarrow [a, b]$ on a closed subspace A extends to a continuous function $F: X \rightarrow [a, b]$. The proof uses an iterative approximation based on Urysohn's lemma.
- The **Urysohn metrization theorem** states that every regular, second countable space is metrizable, via embedding in \mathbb{R}^ω .
- **Partitions of unity** subordinate to open covers exist for paracompact Hausdorff spaces and are essential tools in analysis and geometry.
- The Sorgenfrey plane provides a key counterexample: it is separable and first countable, but not normal.

Chapter 10

A Preview of Algebraic Topology

Motivation

Throughout this course, we have classified topological spaces using properties such as compactness, connectedness, and separation axioms. Yet these properties are often insufficient to distinguish between spaces that are intuitively very different. For example, both \mathbb{R}^2 and $\mathbb{R}^2 \setminus \{0\}$ are connected, locally compact, Hausdorff, and metrizable—yet they are not homeomorphic. How can we prove this?

The key insight, due to Poincaré, is to assign *algebraic invariants*—groups, rings, modules—to topological spaces in such a way that homeomorphic spaces receive isomorphic invariants. This is the domain of **algebraic topology**, and it transforms topological questions into algebraic computations. In this chapter, we give a brief, self-contained preview of the main ideas: homotopy, the fundamental group, covering spaces, and some spectacular applications.

10.1 Homotopy

Definition 10.1.1 (Homotopy of maps). Let X and Y be topological spaces and let $f, g: X \rightarrow Y$ be continuous maps. A **homotopy** from f to g is a continuous map $H: X \times [0, 1] \rightarrow Y$ such that

$$H(x, 0) = f(x) \quad \text{and} \quad H(x, 1) = g(x) \quad \text{for all } x \in X.$$

If such an H exists, we say f and g are **homotopic** and write $f \simeq g$.

Proposition 10.1.1. *Homotopy is an equivalence relation on the set of continuous maps from X to Y .*

Proof. Reflexivity: $H(x, t) = f(x)$ for all t . *Symmetry:* if H is a homotopy from f to g , then $\bar{H}(x, t) = H(x, 1 - t)$ is a homotopy from g to f . *Transitivity:* if $H_1: f \simeq g$ and $H_2: g \simeq h$, define

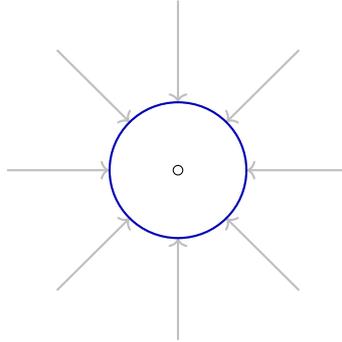
$$H(x, t) = \begin{cases} H_1(x, 2t) & \text{if } t \in [0, \frac{1}{2}], \\ H_2(x, 2t - 1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

By the pasting lemma, H is continuous. □

Definition 10.1.2 (Homotopy equivalence). Spaces X and Y are **homotopy equivalent** (or have the same **homotopy type**), written $X \simeq Y$, if there exist continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 10.1.3 (Deformation retract). A subspace $A \subseteq X$ is a **deformation retract** of X if there exists a continuous map $H: X \times [0, 1] \rightarrow X$ such that $H(x, 0) = x$, $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$. In this case, $X \simeq A$.

- Example 10.1.2.**
1. $\mathbb{R}^n \setminus \{0\}$ deformation-retracts onto S^{n-1} via $H(x, t) = (1-t)x + t \cdot \frac{x}{\|x\|}$.
 2. The Möbius band deformation-retracts onto its central circle.
 3. Any convex subset of \mathbb{R}^n is contractible (homotopy equivalent to a point).



$\mathbb{R}^2 \setminus \{0\}$ retracts onto S^1

Figure 10.1: The deformation retract of $\mathbb{R}^2 \setminus \{0\}$ onto S^1 : each point moves radially towards the unit circle.

10.2 The Fundamental Group

Definition 10.2.1 (Loop and fundamental group). Let X be a topological space and $x_0 \in X$ a **basepoint**. A **loop** based at x_0 is a continuous map $\gamma: [0, 1] \rightarrow X$ with $\gamma(0) = \gamma(1) = x_0$. Two loops γ and δ are **path-homotopic** if there exists a continuous map $H: [0, 1] \times [0, 1] \rightarrow X$ with $H(s, 0) = \gamma(s)$, $H(s, 1) = \delta(s)$, and $H(0, t) = H(1, t) = x_0$ for all s, t . The **fundamental group** of X at x_0 is the set of path-homotopy classes of loops based at x_0 :

$$\pi_1(X, x_0) = \{[\gamma] : \gamma \text{ is a loop at } x_0\},$$

equipped with the group operation given by concatenation of loops:

$$([\gamma] \cdot [\delta])(s) = \begin{cases} \gamma(2s) & \text{if } s \in [0, \frac{1}{2}], \\ \delta(2s - 1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

The identity element is the class of the constant loop at x_0 , and the inverse of $[\gamma]$ is $[\bar{\gamma}]$ where $\bar{\gamma}(s) = \gamma(1 - s)$.

Theorem 10.2.1. $\pi_1(X, x_0)$ is a group under the concatenation operation.

Proof sketch. One must verify that concatenation is well-defined on homotopy classes (if $\gamma \simeq \gamma'$ and $\delta \simeq \delta'$, then $\gamma * \delta \simeq \gamma' * \delta'$), associative up to homotopy, and that the constant loop and reversal of loops provide identity and inverses. Each verification amounts to constructing an explicit homotopy by reparametrizing the interval $[0, 1]$. \square

Theorem 10.2.2. *If $f: X \rightarrow Y$ is a homeomorphism with $f(x_0) = y_0$, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$. More generally, if $X \simeq Y$ (homotopy equivalence), then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ (for appropriate basepoints). Hence the fundamental group is a **topological invariant**, and more generally, a **homotopy invariant**.*

Proof sketch. A homeomorphism f induces a group isomorphism $f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ by $f_*([\gamma]) = [f \circ \gamma]$. For homotopy equivalences, one shows that the induced map f_* has an inverse ($g \circ f \simeq \text{id}$ implies $g_* \circ f_* = \text{id}$ up to a change-of-basepoint isomorphism). \square

Definition 10.2.2 (Simply connected space). A path-connected space X is **simply connected** if $\pi_1(X, x_0)$ is trivial (i.e., $\pi_1(X, x_0) \cong \{e\}$) for some (hence every) basepoint x_0 .

Example 10.2.3. 1. \mathbb{R}^n is simply connected for all $n \geq 1$: every loop is homotopic to a constant via the straight-line homotopy $H(s, t) = (1 - t)\gamma(s) + t \cdot x_0$.

2. S^n is simply connected for $n \geq 2$. (This requires a more careful argument using the Lebesgue number lemma.)

3. The torus $T^2 = S^1 \times S^1$ is not simply connected: $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$.

10.3 The Fundamental Group of the Circle

Theorem 10.3.1. $\pi_1(S^1, 1) \cong \mathbb{Z}$.

The integer associated to a loop γ is its **winding number**: the number of times γ winds around the circle, counted with sign.

Proof sketch via covering spaces. Consider the covering map $p: \mathbb{R} \rightarrow S^1$ defined by $p(t) = e^{2\pi it}$ (or equivalently, $p(t) = (\cos 2\pi t, \sin 2\pi t)$). The key facts are:

1. **Path lifting:** every loop $\gamma: [0, 1] \rightarrow S^1$ with $\gamma(0) = 1$ lifts uniquely to a path $\tilde{\gamma}: [0, 1] \rightarrow \mathbb{R}$ with $\tilde{\gamma}(0) = 0$. Then $\tilde{\gamma}(1) \in \mathbb{Z}$ (since $p(\tilde{\gamma}(1)) = \gamma(1) = 1$).
2. **Homotopy lifting:** if $\gamma \simeq \delta$ (as loops at 1), then their lifts have the same endpoint: $\tilde{\gamma}(1) = \tilde{\delta}(1)$.

The map $\Phi: \pi_1(S^1, 1) \rightarrow \mathbb{Z}$ defined by $\Phi([\gamma]) = \tilde{\gamma}(1)$ is therefore well-defined. One verifies that Φ is a group homomorphism (lifting of a concatenation corresponds to addition of endpoints) and a bijection (the loop $\gamma_n(s) = e^{2\pi i ns}$ has winding number n , so Φ is surjective; and a loop with winding number 0 lifts to a closed path in \mathbb{R} , which is null-homotopic since \mathbb{R} is simply connected, so Φ is injective). \square

10.4 Covering Spaces

Definition 10.4.1 (Covering space). A **covering space** of a topological space X is a space \tilde{X} together with a surjective continuous map $p: \tilde{X} \rightarrow X$ (the **covering map**) such that for every $x \in X$ there exists an open neighbourhood U of x with the property that $p^{-1}(U)$ is a disjoint union of open sets in \tilde{X} , each of which is mapped homeomorphically onto U by p . Such a U is called **evenly covered**.

Example 10.4.1. 1. $p: \mathbb{R} \rightarrow S^1, p(t) = e^{2\pi it}$, is the universal covering of the circle.

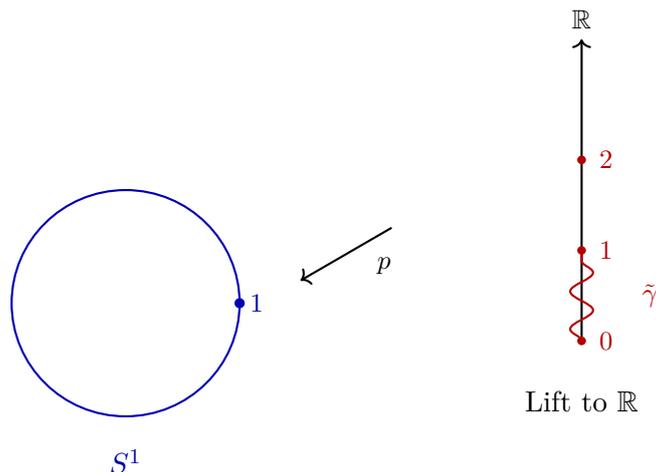


Figure 10.2: The covering space $p: \mathbb{R} \rightarrow S^1$. A loop on S^1 lifts to a path in \mathbb{R} ; the endpoint of the lift is the winding number.

2. $p: S^n \rightarrow \mathbb{R}P^n$ (the quotient by the antipodal map) is a double covering for $n \geq 1$.

3. $p: \mathbb{R}^2 \rightarrow T^2$ given by the quotient $\mathbb{R}^2/\mathbb{Z}^2$ is the universal covering of the torus.

Theorem 10.4.2 (Path Lifting Lemma). *Let $p: \tilde{X} \rightarrow X$ be a covering map, $\gamma: [0, 1] \rightarrow X$ a path, and $\tilde{x}_0 \in p^{-1}(\gamma(0))$. Then there exists a unique path $\tilde{\gamma}: [0, 1] \rightarrow \tilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ and $p \circ \tilde{\gamma} = \gamma$.*

Theorem 10.4.3 (Homotopy Lifting Lemma). *Let $p: \tilde{X} \rightarrow X$ be a covering map, $H: [0, 1] \times [0, 1] \rightarrow X$ a homotopy with $H(0, t) = x_0$ and $H(1, t) = x_1$ for all t , and $\tilde{x}_0 \in p^{-1}(x_0)$. Then H lifts to a unique homotopy $\tilde{H}: [0, 1] \times [0, 1] \rightarrow \tilde{X}$ with $\tilde{H}(0, 0) = \tilde{x}_0$ and $p \circ \tilde{H} = H$. In particular, $\tilde{H}(1, t)$ is independent of t .*

10.5 The Brouwer Fixed Point Theorem

Theorem 10.5.1 (Brouwer Fixed Point Theorem, dimension 2). *Every continuous map $f: D^2 \rightarrow D^2$ has a fixed point, where $D^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ is the closed unit disk.*

Proof. We proceed by contradiction. Suppose $f: D^2 \rightarrow D^2$ is continuous with no fixed point, i.e., $f(x) \neq x$ for all $x \in D^2$.

Step 1. Constructing a retraction. For each $x \in D^2$, consider the ray starting at $f(x)$ and passing through x . Since $f(x) \neq x$, this ray is well-defined. Let $r(x)$ be the point where this ray intersects the boundary circle $S^1 = \partial D^2$. Explicitly, write $r(x) = x + t(x - f(x))$ for the unique $t \geq 0$ such that $\|r(x)\| = 1$. Standard arguments show that $r: D^2 \rightarrow S^1$ is continuous.

Moreover, if $x \in S^1$ already, then since $\|x\| = 1$, we have $r(x) = x$ (one can verify this directly, or note that the ray from $f(x) \in D^2$ through $x \in S^1$ hits S^1 first at x itself when $t = 0$). Hence r is a **retraction** of D^2 onto S^1 : $r|_{S^1} = \text{id}_{S^1}$.

Step 2. Deriving a contradiction via π_1 . Let $\iota: S^1 \hookrightarrow D^2$ be the inclusion. Then $r \circ \iota = \text{id}_{S^1}$, so the induced homomorphisms satisfy

$$r_* \circ \iota_* = (\text{id}_{S^1})_* = \text{id}_{\pi_1(S^1)}.$$

In particular, $\iota_*: \pi_1(S^1) \rightarrow \pi_1(D^2)$ is injective. But $\pi_1(S^1) \cong \mathbb{Z}$ (Theorem 10.3.1) and $\pi_1(D^2) \cong \{0\}$ (since D^2 is convex, hence contractible). An injective homomorphism $\mathbb{Z} \hookrightarrow \{0\}$ is impossible. Contradiction. \square

Remark 10.5.2. The Brouwer fixed point theorem holds in all dimensions: every continuous map $f: D^n \rightarrow D^n$ has a fixed point. For $n \geq 3$, the proof requires either singular homology or higher homotopy groups, as $\pi_1(S^n)$ is trivial for $n \geq 2$.

10.6 The Borsuk–Ulam Theorem

Theorem 10.6.1 (Borsuk–Ulam, dimension 2). *For every continuous map $f: S^2 \rightarrow \mathbb{R}^2$, there exists a point $x \in S^2$ such that $f(x) = f(-x)$.*

Remark 10.6.2. Informally, at any given moment there exist two antipodal points on the surface of the Earth that have exactly the same temperature and the same atmospheric pressure. This striking result generalizes to S^n and \mathbb{R}^n for all n , and has applications in combinatorics (the ham sandwich theorem) and in geometry.

10.7 Higher Invariants: A Brief Glimpse

The fundamental group is only the first of a family of algebraic invariants:

- **Higher homotopy groups** $\pi_n(X, x_0)$, $n \geq 2$, are defined using maps $S^n \rightarrow X$. They are always abelian for $n \geq 2$.
- **Singular homology groups** $H_n(X; \mathbb{Z})$ measure “ n -dimensional holes” in X . For instance, $H_1(S^1) \cong \mathbb{Z}$ reflects the single hole enclosed by the circle, while $H_n(S^n) \cong \mathbb{Z}$ for all n .
- The **Euler characteristic** $\chi(X) = \sum_n (-1)^n \text{rank } H_n(X)$ is a numerical invariant. For a closed orientable surface of genus g , $\chi = 2 - 2g$.

The algebraization of homology—passing from Betti numbers to actual groups—was accomplished by Emmy Noether in the 1920s, a breakthrough that shaped modern algebraic topology.

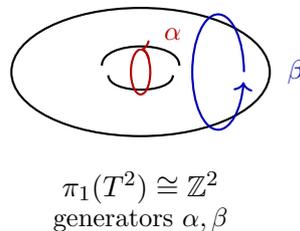


Figure 10.3: Two generating loops on the torus $T^2 = S^1 \times S^1$. The fundamental group is $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$, generated by the loops α (around the hole) and β (around the tube).

10.8 Historical Notes

- **Henri Poincaré** (1895) introduced the fundamental group and homology in his landmark paper *Analysis Situs*, founding algebraic topology.
- **L. E. J. Brouwer** (1911) proved his fixed point theorem and the invariance of domain, using simplicial methods.
- **Emmy Noether** (1925–1930) recognized that homology should be formulated in terms of groups and homomorphisms rather than mere numerical invariants (Betti numbers and torsion coefficients), thereby laying the algebraic foundations of the subject.
- The modern axiomatic approach to homology was developed by **Eilenberg** and **Steenrod** (1945), culminating in their book *Foundations of Algebraic Topology* (1952).

10.9 Exercises

Exercise 10.9.1 (★). Show that homotopy equivalence is an equivalence relation on topological spaces.

Exercise 10.9.2 (★). A space X is **contractible** if it is homotopy equivalent to a point. Show that every convex subset of \mathbb{R}^n is contractible.

Exercise 10.9.3 (★★). Let X be path-connected and let $x_0, x_1 \in X$. Show that $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. *Hint:* use a path from x_0 to x_1 to construct the isomorphism.

Exercise 10.9.4 (★★). Compute $\pi_1(S^1 \vee S^1)$, where $S^1 \vee S^1$ is the wedge (one-point union) of two circles. *Answer:* the free group on two generators, F_2 .

Exercise 10.9.5 (★★). Use the Brouwer fixed point theorem to prove the following: for every continuous function $f: [0, 1] \rightarrow [0, 1]$, there exists $x \in [0, 1]$ with $f(x) = x$. (Give a direct proof from the intermediate value theorem as well.)

Exercise 10.9.6 (★★★). Use the fundamental group to prove the **fundamental theorem of algebra**: every non-constant polynomial $p(z) \in \mathbb{C}[z]$ has a root in \mathbb{C} . *Hint:* consider the map $\gamma_R(t) = p(Re^{2\pi it})/|p(Re^{2\pi it})|$ from S^1 to S^1 for large R , and show its winding number equals $\deg p$.

Chapter Summary

- **Homotopy** formalizes the idea of continuous deformation of maps. **Homotopy equivalence** is a coarser relation than homeomorphism but preserves algebraic invariants.
- The **fundamental group** $\pi_1(X, x_0)$ consists of homotopy classes of loops at a basepoint, and is a topological (and homotopy) invariant.
- $\pi_1(S^1) \cong \mathbb{Z}$, proved via the covering space $\mathbb{R} \rightarrow S^1$. A **simply connected** space has trivial fundamental group.
- The **Brouwer fixed point theorem** (dimension 2) states that every continuous self-map of the closed disk has a fixed point; its proof uses π_1 .
- **Covering spaces** provide the key technical tool: path lifting and homotopy lifting.
- Algebraic topology offers a wealth of further invariants: higher homotopy groups, homology, cohomology, and the Euler characteristic.

Concluding Remarks: Where to Go from Here

This course has taken us from the bare axioms of a topological space through compactness, connectedness, separation, and metrization, all the way to the threshold of algebraic topology. We close with a brief guide to further study.

- **Algebraic topology.** A full treatment of the fundamental group, covering space theory, singular homology and cohomology, and their applications. The standard reference is Hatcher [8].
- **Differential topology.** The study of smooth manifolds, tangent bundles, transversality, and degree theory. Milnor's *Topology from the Differentiable Viewpoint* is an excellent starting point.

- **Geometric topology.** Classification of surfaces, 3-manifold theory (including Thurston's geometrization and the Poincaré conjecture, proved by Perelman in 2003), knot theory, and 4-manifold theory.
- **Functional analysis and descriptive set theory.** Many results from general topology—Tychonoff's theorem, the Baire category theorem, Urysohn's lemma—find deep applications in the study of Banach spaces, operator algebras, and the structure of Borel and analytic sets.

Appendix A

Catalogue of Counterexamples

The following spaces are standard sources of counterexamples in general topology. For each, we give the definition, the key properties, and the theorems it illuminates.

A.1 The Sorgenfrey Line

Definition. Let \mathbb{R}_ℓ denote the real line equipped with the *lower limit topology* (or *half-open interval topology*): the topology generated by the basis $\{[a, b) : a < b, a, b \in \mathbb{R}\}$.

Key properties.

- T_1 : yes. T_2 (Hausdorff): yes. T_3 (regular): yes. T_4 (normal): yes.
- Completely regular ($T_{3\frac{1}{2}}$): yes.
- Separable: yes (\mathbb{Q} is dense).
- First countable: yes. Second countable: **no**.
- Lindelöf: yes.
- Compact: **no**. Locally compact: **no**.
- Connected: **no** (totally disconnected).
- Metrizable: **no** (separable but not second countable).
- Paracompact: yes.

Counterexamples provided.

1. A separable space need not be second countable.
2. A normal, separable space need not be metrizable (refutes the converse of the Urysohn metrization theorem without the second countability hypothesis).
3. A Lindelöf space need not be second countable.

A.2 The Sorgenfrey Plane

Definition. $S = \mathbb{R}_\ell \times \mathbb{R}_\ell$, the product of two copies of the Sorgenfrey line, with the product topology.

Key properties.

- T_1 : yes. T_2 : yes. T_3 : yes. $T_{3\frac{1}{2}}$: yes.
- T_4 (normal): **no**.
- Separable: yes (\mathbb{Q}^2 is dense). First countable: yes. Second countable: **no**.
- Lindelöf: **no**. Compact: **no**.
- Metrizable: **no**. Paracompact: **no**.

Counterexamples provided.

1. The product of two normal spaces need not be normal.
2. The product of two Lindelöf spaces need not be Lindelöf.
3. The product of two paracompact spaces need not be paracompact.
4. A separable, first countable, completely regular space need not be normal.
5. Urysohn's lemma and the Tietze extension theorem fail in non-normal spaces (see Example 9.6.1).

A.3 Cofinite Topology on an Infinite Set

Definition. Let X be an infinite set. The *cofinite topology* (or *finite complement topology*) on X consists of \emptyset and all subsets whose complement is finite.

Key properties.

- T_0 : yes. T_1 : yes. T_2 (Hausdorff): **no** (any two non-empty open sets intersect).
- Compact: yes (every open cover has a finite subcover, since removing one open set leaves only finitely many points to cover).
- Connected: yes (no proper non-empty clopen set exists).
- Second countable: yes if X is countable; no if X is uncountable.
- Metrizable: **no** (not Hausdorff).

Counterexamples provided.

1. A compact, T_1 space need not be Hausdorff.
2. A compact space need not be metrizable.
3. Limits of sequences are not unique in non-Hausdorff spaces.

A.4 Cocountable Topology on an Uncountable Set

Definition. Let X be an uncountable set. The *cocountable topology* on X consists of \emptyset and all subsets whose complement is countable.

Key properties.

- T_0 : yes. T_1 : yes. T_2 : **no**.
- Compact: **no** (if X is uncountable; consider a partition into uncountably many singletons—though singletons are not open. More carefully: X is Lindelöf but not compact when $|X| > \aleph_1$; the situation depends on the cardinality).
- Connected: yes.
- Second countable: **no**. First countable: **no** (when $|X| > \aleph_0$).
- Metrizable: **no**.

Counterexamples provided.

1. A T_1 connected space that is not Hausdorff.
2. A space in which every convergent sequence is eventually constant, yet the space is not discrete.

A.5 The Sierpiński Space

Definition. $S = \{0, 1\}$ with the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Key properties.

- T_0 : yes. T_1 : **no** ($\{0\}$ is closed but not open, $\{1\}$ is open but not closed).
- Compact: yes. Connected: yes.
- The continuous maps $X \rightarrow S$ correspond bijectively to the open sets of X (via $f^{-1}(\{1\})$).

Counterexamples provided.

1. The simplest T_0 space that is not T_1 .
2. Illustrates that T_0 is strictly weaker than T_1 .

A.6 The Cantor Set

Definition. The *middle-thirds Cantor set* $C \subseteq [0, 1]$ is defined by $C = \bigcap_{n=0}^{\infty} C_n$, where $C_0 = [0, 1]$ and C_{n+1} is obtained from C_n by removing the open middle third of each component interval. Equivalently,

$$C = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k} : a_k \in \{0, 2\} \text{ for all } k \right\}.$$

Key properties.

- Compact: yes (closed and bounded in \mathbb{R}).
- Hausdorff: yes. Metrizable: yes.

- Perfect (no isolated points): yes.
- Totally disconnected: yes.
- Uncountable: yes (in bijection with $\{0, 1\}^{\mathbb{N}}$).
- Lebesgue measure zero.
- Homeomorphic to $\{0, 1\}^{\mathbb{N}}$ (with the product topology).

Counterexamples provided.

1. An uncountable set of measure zero.
2. A compact, perfect, totally disconnected, metrizable space (the unique such space, up to homeomorphism, that is non-empty and has no isolated points: this is Brouwer's theorem).
3. A closed subset of $[0, 1]$ with empty interior that is uncountable.

A.7 The Topologist's Sine Curve

Definition.

$$T = \{(x, \sin(1/x)) : x \in (0, 1]\} \cup (\{0\} \times [-1, 1]) \subseteq \mathbb{R}^2.$$

Key properties.

- Compact: yes (if one takes $x \in (0, 1]$; the closure in \mathbb{R}^2 is compact). Hausdorff: yes.
Metrizable: yes.
- Connected: yes.
- Path-connected: **no** (no path from $(0, 0)$ reaches the oscillating part).
- Locally connected: **no** (at points on the segment $\{0\} \times [-1, 1]$).

Counterexamples provided.

1. A connected space that is not path-connected.
2. A connected space that is not locally connected.

A.8 The Long Line

Definition. Let ω_1 denote the first uncountable ordinal. The *long ray* is $L^+ = \omega_1 \times [0, 1)$ with the order topology (using the lexicographic order), and the *long line* L is obtained by gluing two copies of L^+ at their endpoints (or equivalently by considering $(-\omega_1, \omega_1) \times [0, 1)$ with appropriate identifications).

Key properties.

- T_1 : yes. T_2 : yes. T_3 : yes. T_4 (normal): yes.
- Locally compact: yes. Compact: **no**.
- Connected: yes. Path-connected: yes. Locally path-connected: yes.
- Second countable: **no**. First countable: yes.

- Paracompact: **no**. Lindelöf: **no**.
- Metrizable: **no**.
- Sequentially compact: yes (every sequence is eventually contained in a compact interval).

Counterexamples provided.

1. A sequentially compact space that is not compact (showing that sequential compactness $\not\Rightarrow$ compactness without second countability).
2. A locally compact, normal space that is not metrizable and not paracompact.
3. A connected, locally compact space that is not σ -compact.

A.9 The Double Origin Line (Bug-Eyed Line)

Definition. Start with the real line \mathbb{R} and add a second copy $0'$ of the origin. A basis for the topology is:

- all open intervals (a, b) with $0 \notin (a, b)$;
- sets of the form $(a, 0) \cup \{0\} \cup (0, b)$ for $a < 0 < b$ (neighbourhoods of 0);
- sets of the form $(a, 0) \cup \{0'\} \cup (0, b)$ for $a < 0 < b$ (neighbourhoods of $0'$).

Key properties.

- T_0 : yes. T_1 : yes. T_2 (Hausdorff): **no** (0 and $0'$ cannot be separated by disjoint open sets).
- Locally Euclidean (a non-Hausdorff 1-manifold).
- Connected: yes. Path-connected: yes.
- Second countable: yes.

Counterexamples provided.

1. A locally Euclidean, second countable space that is not Hausdorff, showing why the Hausdorff condition is needed in the definition of a manifold.
2. Two points that share every neighbourhood filter base element (except for the points themselves), yet are distinct.

A.10 The Particular Point Topology

Definition. Let X be any set with $|X| \geq 2$ and fix a point $p \in X$. The *particular point topology* on X consists of \emptyset and all subsets of X containing p .

Key properties.

- T_0 : yes. T_1 : **no** (the closure of $\{p\}$ is X). T_2 : **no**.
- Compact: if X is finite, yes; if X is infinite, **no** (the cover $\{\{p, x\} : x \in X\}$ has no finite subcover when $|X|$ is infinite—actually each $\{p, x\}$ is open and their union is X ; a finite subcover covers only finitely many points). More precisely: compact iff X is finite.

- Connected: yes (p belongs to every non-empty open set, so there is no non-trivial clopen partition).
- Metrizable: **no**.

Counterexamples provided.

1. A connected, T_0 space that fails all higher separation axioms.
2. A hyperconnected space (every two non-empty open sets intersect).

A.11 Fort Space

Definition. Let X be an uncountable set and fix a point $p \in X$. The *Fort topology* on X consists of all subsets of $X \setminus \{p\}$ (which are declared open) together with all subsets $U \ni p$ whose complement $X \setminus U$ is finite.

Key properties.

- T_1 : yes. T_2 : yes. T_3 : yes. T_4 : yes.
- Compact: yes (every open cover must contain a set $U \ni p$ with $X \setminus U$ finite; finitely many additional sets cover the remainder).
- Second countable: **no** (uncountably many isolated points). First countable: **no** at p (if X is uncountable).
- Metrizable: **no** (compact T_2 but not second countable).
- Sequentially compact: depends on $|X|$; if $|X| = \aleph_1$, then yes.

Counterexamples provided.

1. A compact Hausdorff space that is not metrizable (hence not second countable, by the Urysohn metrization theorem).
2. A compact T_2 space that is not first countable.

A.12 The Moore Plane (Niemytzki Plane)

Definition. Let $\Gamma = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ (the closed upper half-plane). For a point $p = (a, b)$ with $b > 0$, basic open sets are ordinary Euclidean open disks centred at p contained in the open upper half-plane. For a point $p = (a, 0)$ on the x -axis, a basic open neighbourhood is $\{p\} \cup B_\varepsilon((a, \varepsilon))$ (the open disk of radius ε tangent to the x -axis at p , together with the point p itself).

Key properties.

- T_1 : yes. T_2 : yes. T_3 (regular): yes.
- $T_{3\frac{1}{2}}$ (completely regular): yes.
- T_4 (normal): **no**.
- Separable: yes ($\mathbb{Q}^2 \cap \Gamma$ is dense).
- First countable: yes. Second countable: **no**.

- Metrizable: **no**. Paracompact: **no**.
- Lindelöf: **no**.

Counterexamples provided.

1. A separable, completely regular space that is not normal (another example besides the Sorgenfrey plane).
2. A separable space with a closed discrete subspace of cardinality \mathfrak{c} (the x -axis), which by Jones' lemma prevents normality.
3. A first countable, separable space that is not second countable and not metrizable.

A.13 The Tychonoff Plank

Definition. Let ω_1 denote the first uncountable ordinal and ω_0 the first infinite ordinal. Set

$$T = [0, \omega_1] \times [0, \omega_0] \setminus \{(\omega_1, \omega_0)\},$$

equipped with the subspace topology from the product of order topologies. This is the *Tychonoff plank* (also called the *deleted Tychonoff plank*).

Key properties.

- T_1 : yes. T_2 : yes. T_3 : yes.
- $T_{3\frac{1}{2}}$: yes. T_4 (normal): **no**.
- Locally compact: yes.
- Compact: **no** (the full product $[0, \omega_1] \times [0, \omega_0]$ is compact, but removing the corner point destroys compactness).
- Countably compact: yes.
- Sequentially compact: yes.
- Pseudocompact: yes. Metrizable: **no**.

Counterexamples provided.

1. A completely regular space that is not normal: every continuous $f: T \rightarrow \mathbb{R}$ extends to the one-point compactification, yet the two “boundary edges” $\{\omega_1\} \times [0, \omega_0]$ and $[0, \omega_1] \times \{\omega_0\}$ cannot be separated by disjoint open sets.
2. A countably compact (hence pseudocompact), locally compact, completely regular space that is not compact and not normal.

Bibliography

- [1] J. R. Munkres, *Topology*, 2nd edition, Prentice Hall, Upper Saddle River, NJ, 2000.
- [2] N. Bourbaki, *General Topology* (Topologie Générale), Chapters 1–4, Springer-Verlag, Berlin, 1995. (Reprint of the 1989 English translation.)
- [3] S. Willard, *General Topology*, Dover Publications, Mineola, NY, 2004. (Unabridged republication of the 1970 Addison-Wesley edition.)
- [4] J. L. Kelley, *General Topology*, Graduate Texts in Mathematics, vol. 27, Springer-Verlag, New York, 1975. (Reprint of the 1955 Van Nostrand edition.)
- [5] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
- [6] J. Dixmier, *General Topology*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1984.
- [7] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in Topology*, Dover Publications, Mineola, NY, 1995. (Unabridged republication of the 1978 Springer-Verlag edition.)
- [8] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002. (Also freely available from the author's website.)
- [9] R. Engelking, *General Topology*, revised and completed edition, Heldermann Verlag, Berlin, 1989.

Index

- accumulation point, *see* limit point
- Alexander's sub-base theorem, 42
- Alexandroff compactification, 43
 - of \mathbb{C} , 43
 - of \mathbb{R} , 43
- algebraic topology, 71–77
- attaching spaces, 62

- Baire category, 13
- Baire category theorem, 13, 46
- base, 19–24
 - comparison, 22
 - criterion, 19
 - for \mathbb{R} , 21
 - for \mathbb{R}^n , 21
 - for a topology, 19
- basepoint, 72
- Borsuk–Ulam theorem, 75
- boundary, 11
- Bourbaki, Nicolas, 87
- box topology, 55–61
 - pathologies, 60
- Brouwer fixed point theorem, 74–75
- Brouwer, L. E. J., 75
- bug-eyed line, *see* double origin line

- Cantor set, 81–82
 - compactness, 45
 - totally disconnected, 51
- circle
 - as quotient, 58
- clopen set, 9, 47
- closed graph, 30
- closed map, 26
 - not continuous, 27
- closed set, 9
- closure, 10
 - characterization via neighborhoods, 11
- coarser topology, 3
- cocountable topology, 3, 81
- cofinite topology, 2, 34, 80
 - not Hausdorff, 36
- compact
 - continuous bijection to Hausdorff, 27
- compact Hausdorff space
 - normality, 35
- compact space, 39
- compactness, 39–46
 - closed subset, 40
 - continuous image, 40
 - counterexamples, 44
 - FIP characterisation, 39
 - in Hausdorff space, 40
 - in metric spaces, 41
 - of $[a, b]$, 40
- completely regular space, 63
 - definition, 63
- cone on a space, 61
- connected component, 47
- connected space, 47
- connectedness, 47–53
 - and closure, 48
 - and local path-connectedness, 50
 - preserved by continuous maps, 48
 - product, 51
 - union of connected sets, 48
- continuity, 25–31
 - composition, 26
 - constant map, 30
 - definition, 25
 - equivalent characterizations, 25
 - restriction, 30
 - via sub-base, 29
- contractible space, 76
- coproduct topology, 59
 - universal property, 60
- counterexamples, 79–85
- covering map, 73
- covering space, 73–74
 - definition, 73

- deformation retract, 72
- dense set, 13
- derived set, 12
- disconnected space, 47
- discrete topology, 2
- disjoint union topology, 59
- Dixmier, Jacques, 87

- double origin line, 83
- Dugundji, James, 87
- dyadic rationals, 64
- Eilenberg–Steenrod axioms, 75
- embedding
 - in \mathbb{R}^ω , 67
 - topological, 27
- Euclidean space
 - connected, 52
- Euler characteristic, 75
- evenly covered, 73
- extreme value theorem, 40
- finer topology, 3
- finite intersection property, 39
- first category, *see* meager set
- first countable, 21
- Fort space, 84
- fundamental group, 72
 - definition, 72
 - of S^1 , 73
 - topological invariant, 73
- fundamental theorem of algebra, 76
- general linear group
 - disconnected, 52
- generated topology, 20
- ham sandwich theorem, 75
- Hatcher, Allen, 87
- Hausdorff
 - product, 57
- Hausdorff space, *see* T_2 space
 - compact subsets closed, 34
 - diagonal characterisation, 37
 - not regular, 36
 - uniqueness of limits, 34
- Heine–Borel theorem, 41
- higher homotopy groups, 75
- Hilbert cube, 45
- homeomorphism, 25–31
 - $(0, 1) \cong \mathbb{R}$, 28
 - characterization, 27
 - compact to Hausdorff, 27, 45
 - continuous bijection that is not, 29
 - definition, 26
- homology
 - singular, 75
- homotopy, 71–72
 - equivalence, 71
 - of maps, 71
- homotopy group
 - higher, 75
- homotopy lifting, 73, 74
- indiscrete topology, 2, 33
- interior, 10
- Intermediate Value Theorem, 48
- intersection
 - infinite, of open sets, 5
- intersection of topologies, 4
- interval
 - connected, 52
- invariance of domain, 29
- isolated point, 12
- Jones’ lemma, 38
- K -topology, 36
- Kelley, John L., 87
- Klein bottle, 59
- Kolmogorov space, 33, *see* T_0 space
- Kuratowski closure axioms, 13
- Kuratowski’s 14-set theorem, 14
- Lebesgue number lemma, 45
- limit point, 12
- limit point compactness, 41
- Lindelöf space, 21
- locally connected, 50
- locally path-connected, 50
- long line, 82–83
- loop, 72
- lower limit topology, 21
- meager set, 13
- metric space
 - normality, 36
- metric topology, 3
- metrization
 - Urysohn theorem, 66
- Möbius band, 59
 - as quotient, 59
- Moore plane, 84–85
- Munkres, James, 87
- neighborhood, 4
- neighborhood system, 4
- Niemytzki plane, *see* Moore plane
- Noether, Emmy, 75
- normal space, 35, 64
 - is completely regular, 65
- nowhere dense set, 13
- one-point compactification, 43
- open ball, 1

- open map, 26
 - not continuous, 26
- open set, 2
 - characterization via neighborhoods, 4
 - in a metric space, 1
- order topology
 - base, 22
- paracompact space
 - partition of unity, 67
- particular point topology, 6, 36, 83–84
- partition of unity, 67
 - construction, 69
- pasting lemma, 27
- path, 49
- path homotopy, 72
- path lifting, 73, 74
- path-connected space, 49
- path-connectedness, 49–53
 - implies connectedness, 49
- Poincaré, Henri, 71, 75
- product topology, 55–62
 - coarsest for projections, 56
 - convergence, 61
 - finite, 55
 - infinite, 55
 - universal property, 56
- projection
 - continuous and open, 30
 - open map, 56
- quotient map, 57
- quotient topology, 55–62
 - non-Hausdorff example, 61
 - universal property, 57
- rationals
 - totally disconnected, 50
- real projective plane, 59
- regular open set, 18
- regular space, 34, 66
- retraction, 74
- Riemann sphere, 43
- saturated set, 57
- second countable, 21, 66
 - implies Lindelöf, 21
 - implies separable, 21
- Seebach, J. Arthur, 87
- separable, 21
- separation, 47
- separation axiom
 - $T_{3\frac{1}{2}}$, 63
- separation axioms, 33–38
 - hierarchy, 35
- sequential closure, 16
- sequential compactness, 41
 - vs compactness, 44
- Sierpiński space, 33, 81
- simply connected, 73
- singular homology, 75
- Sorgenfrey line, 16, 37, 79
 - base, 21
 - first countable, 22
 - not second countable, 22
 - separable, 22
- Sorgenfrey plane, 37, 67, 80
 - exercises, 69
 - non-normality, 67
- space-filling curve, 30
- sphere
 - connected, 52
- standard topology on \mathbb{R} , 3
- Steen, Lynn Arthur, 87
- stereographic projection, 28
- Stone–Čech compactification, 44
- sub-base, 19–24
 - definition, 20
 - generates a base, 20
 - product topology, 23
- suspension, 61
- T_0 space, 33
- T_1 space, 33
 - characterisation, 33
- T_2 space, 34
- T_3 space, 34
- T_4 space, 35
- Tietze extension theorem, 63–69
 - statement and proof, 65
- topological invariant, 26
- topological space, 2
- topologist’s sine curve, 49–50, 82
- topology, 2
 - cocountable, 3
 - cofinite, 2
 - discrete, 2
 - indiscrete, 2
 - metric, 3
- torus
 - as quotient, 58
- totally disconnected, 50–51
- tube lemma, 45, 57
- Tychonoff plank, 37, 85
- Tychonoff space, *see* completely regular space

Tychonoff's theorem, 42
 application, 45

Urysohn metrization theorem, 66

Urysohn's lemma, 63–69
 exercises, 68

 statement and proof, 64

Warsaw circle, 53

Willard, Stephen, 87

winding number, 73

Zariski topology, 3