Differential Topology

Notes de Cours / Lecture Notes

Graduate Level — 2025–2026

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Manifolds, embeddings, and Morse theory:
smooth structures at the heart of geometry.

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Contents

Table of Notation

Symbol	Meaning
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Natural numbers, integers, rationals, reals, complex numbers
$\mathbb{R}^n$	Euclidean n-space
S^n	Unit n-sphere in $\mathbb{R}^{n+1}$
$\mathbb{R}\mathrm{P}^n$	Real projective n-space
$\mathbb{C}\mathrm{P}^n$	Complex projective n-space
T^n	n-torus $S^1 \times \cdots \times S^1$
$\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$	General linear groups
$\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$	Special linear groups
$\mathrm{O}(n)$, $\mathrm{SO}(n)$	Orthogonal and special orthogonal groups
$\mathrm{U}(n)$, $\mathrm{SU}(n)$	Unitary and special unitary groups
$\mathrm{Gr}(k, n)$	Grassmannian of k-planes in $\mathbb{R}^n$
$\mathrm{Diff}(M)$	Diffeomorphism group of M
$C^\infty(M)$	Smooth real-valued functions on M
$C^\infty(M, N)$	Smooth maps from M to N
T_pM	Tangent space at p
TM	Tangent bundle of M
$\mathrm{d}f_p$, $Df(p)$	Differential of f at p
Id_M	Identity map on M
∂M	Boundary of the manifold M
$\mathrm{Int}(M)$	Interior of M
$\lvert\cdot\rvert$, $\lVert\cdot\rVert$	Absolute value, norm
$\sim$	Equivalence relation

Preface

Differential topology studies the properties of smooth manifolds and smooth maps that are invariant under diffeomorphisms. Unlike differential geometry, which equips manifolds with additional structure such as Riemannian metrics or connections, differential topology works with the bare smooth structure alone. The subject is therefore closer in spirit to algebraic topology, yet its methods are overwhelmingly analytic and geometric.

This text is addressed to graduate students who have completed a solid course in point-set topology and possess working knowledge of linear algebra, multivariable calculus, and basic group theory. Some familiarity with algebraic topology (fundamental group, covering spaces) is helpful but not strictly required for the first half of the book.

The first two chapters establish the language of smooth manifolds and smooth maps. We take care to present full proofs of the foundational results — the existence of partitions of unity, the inverse and implicit function theorems on manifolds, and the constant rank theorem — as they underpin everything that follows. The exposition is supplemented by numerous examples drawn from geometry and Lie group theory, as well as exercises that range from routine verifications to more demanding problems.

Prerequisites. Second-year graduate standing in mathematics; point-set topology (Hausdorff spaces, compactness, paracompactness, quotient topology); multivariable analysis (inverse function theorem in $\mathbb{R}^n$, implicit function theorem); linear algebra.

Chapter 1

Smooth Manifolds — Definitions and Examples

The notion of a *smooth manifold* lies at the very foundation of differential topology. Intuitively, a smooth manifold is a topological space that "locally looks like" Euclidean space and on which the tools of calculus can be applied in a consistent, coordinate-free manner. In this chapter we make this idea precise and develop the basic vocabulary that will accompany us throughout the course.

1.1 Topological Preliminaries

Before introducing manifolds, we recall the topological notions that appear in their definition.

Definition 1.1.1 (Second-countable space)**.** A topological space X is *second-countable* if its topology admits a countable base, i.e. there exists a countable collection $\mathcal{B} = \{B_i\}_{i\in\mathbb{N}}$ of open sets such that every open set $U \subseteq X$ can be written as a union of members of $\mathcal{B}$.

Definition 1.1.2 (Hausdorff space)**.** A topological space X is *Hausdorff* if for every pair of distinct points $p, q \in X$ there exist disjoint open sets $U \ni p$ and $V \ni q$.

Definition 1.1.3 (Paracompact space)**.** A topological space X is *paracompact* if every open cover of X admits a locally finite open refinement.

Proposition 1.1.4 (Second-countable implies paracompact)**.** *Every second-countable Hausdorff space is paracompact.*

Proof. Let $\{B_i\}_{i\in\mathbb{N}}$ be a countable base for the topology. Given an open cover $\{U_\alpha\}$, for each $x \in X$ choose some U_α containing x and then a basis element $B_{i(x)} \subseteq U_\alpha$ with $x \in B_{i(x)}$. The countable subcover $\{B_{i(x)}\}$ can be refined to a locally finite cover by a standard exhaustion argument using compact sets (which exist because a second-countable Hausdorff space is Lindelöf and every open cover has a countable subcover). □

Remark 1.1.5. We require manifolds to be Hausdorff and second-countable to exclude pathologies: the line with two origins is locally Euclidean but not Hausdorff, and the long line is Hausdorff and locally Euclidean but not second-countable. Both conditions together guarantee that manifolds are metrizable and paracompact, which is essential for the existence of partitions of unity.

1.2 Charts, Atlases, and Smooth Structures

Definition 1.2.1 (Topological manifold)**.** A *topological manifold of dimension n* is a topological space M that is:

(i) Hausdorff,

(ii) second-countable, and

(iii) locally Euclidean of dimension n: every point $p \in M$ possesses an open neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 1.2.2 (Chart)**.** A *chart* on a topological manifold M^n is a pair (U, φ) where $U \subseteq M$ is open and $\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. We call U the *chart domain* and φ the *coordinate map*. If $\varphi(p) = 0$ we say the chart is *centred* at p.

Definition 1.2.3 (Transition map)**.** Given two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) on M with $U_\alpha \cap U_\beta \neq \varnothing$, the *transition map* is

$$\varphi_{\beta\alpha} \;=\; \varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta).$$

This is a homeomorphism between open subsets of $\mathbb{R}^n$.

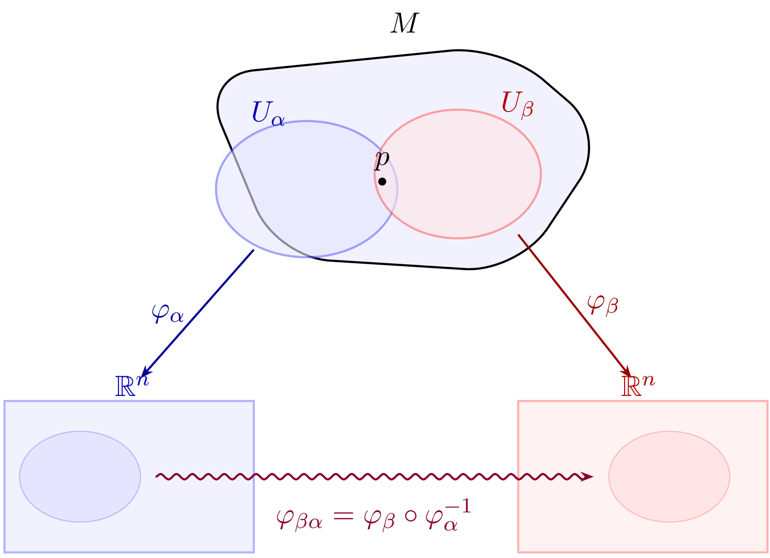


Figure 1.1: Two overlapping charts and their transition map.

Definition 1.2.4 (Smooth atlas)**.** A *smooth atlas* (or C^∞ atlas) on M is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of charts such that:

(i) $\{U_\alpha\}_{\alpha \in A}$ covers M;

(ii) for every $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, the transition map $\varphi_{\beta\alpha}$ is a smooth (C^∞) diffeomorphism between open subsets of $\mathbb{R}^n$.

Definition 1.2.5 (Compatible charts and maximal atlas)**.** Two smooth atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are *compatible* if $\mathcal{A} \cup \mathcal{A}'$ is again a smooth atlas. Compatibility is an equivalence relation. A *smooth structure* on M is a maximal smooth atlas $\overline{\mathcal{A}}$, i.e. the union of all atlases compatible with a given atlas $\mathcal{A}$. A *smooth manifold* is a pair $(M, \overline{\mathcal{A}})$.

Remark 1.2.6. In practice, we specify a smooth manifold by giving *any* smooth atlas; the maximal atlas is then determined. Two atlases define the same smooth structure if and only if they are compatible.

Proposition 1.2.7 (Dimension is well-defined)**.** *If M is a connected topological manifold that is locally Euclidean of dimension n and also of dimension m, then $n = m$.*

Proof. If $U \subseteq M$ lies in the domain of charts to both $\mathbb{R}^n$ and $\mathbb{R}^m$, then there is a homeomorphism between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$. By Brouwer's invariance of domain, this forces $n = m$. □

1.3 Examples of Smooth Manifolds

Example 1.3.1 (Euclidean space $\mathbb{R}^n$)**.** The space $\mathbb{R}^n$ with the single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$ is a smooth n-manifold. More generally, any open subset $U \subseteq \mathbb{R}^n$ inherits a smooth structure from the inclusion chart.

Example 1.3.2 (The n-sphere S^n)**.** Define $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. We equip S^n with a smooth structure via *stereographic projection*. Let $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$ be the north and south poles. Define

$$\varphi_N \colon S^n \setminus \{N\} \longrightarrow \mathbb{R}^n, \qquad \varphi_N(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n),$$
$$\varphi_S \colon S^n \setminus \{S\} \longrightarrow \mathbb{R}^n, \qquad \varphi_S(x_1, \dots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \dots, x_n).$$

The transition map on $S^n \setminus \{N, S\}$ is

$$\varphi_S \circ \varphi_N^{-1}(y) = \frac{y}{\|y\|^2},$$

which is smooth (indeed real-analytic) on $\mathbb{R}^n \setminus \{0\}$. Hence $\{(S^n \setminus \{N\}, \varphi_N),\ (S^n \setminus \{S\}, \varphi_S)\}$ is a smooth atlas on S^n.

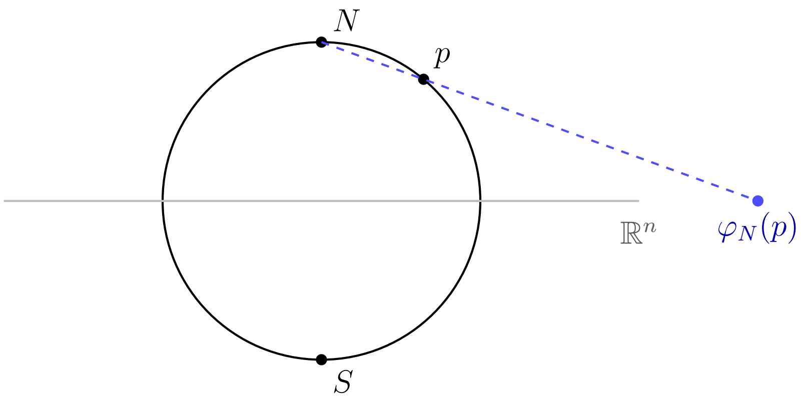


Figure 1.2: Stereographic projection from the north pole.

Example 1.3.3 (Real projective space $\mathbb{RP}^n$)**.** Define $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$ where $x \sim \lambda x$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Denote the equivalence class of $(x_0, \dots, x_n)$ by $[x_0 : \cdots : x_n]$. For $i = 0, \dots, n$ set

$$U_i = \{[x_0 : \cdots : x_n] : x_i \neq 0\}, \qquad \varphi_i([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right) \in \mathbb{R}^n,$$

where the hat denotes omission. The transition maps are rational functions with non-vanishing denominators, hence smooth. The resulting atlas gives $\mathbb{RP}^n$ the structure of a compact smooth n-manifold.

Example 1.3.4 (Complex projective space $\mathbb{CP}^n$)**.** Analogously, $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ with $z \sim \lambda z$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Charts $U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\}$ with $\varphi_i([z]) = (z_0/z_i, \dots, \widehat{z_i/z_i}, \dots, z_n/z_i) \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ yield a smooth atlas. Thus $\mathbb{CP}^n$ is a compact smooth manifold of (real) dimension $2n$.

Example 1.3.5 (The torus T^n)**.** The *n-torus* is $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$. As a product of smooth manifolds it inherits a product smooth structure. Equivalently, $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ via the quotient by the lattice, and the local sections of the projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ serve as charts.

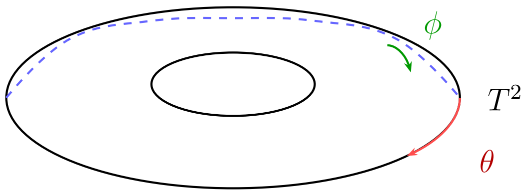


Figure 1.3: The 2-torus T^2 with its two angular coordinates (θ, ϕ).

Example 1.3.6 (General linear group $\mathrm{GL}(n, \mathbb{R})$)**.** The set $\mathrm{GL}(n, \mathbb{R})$ of invertible $n \times n$ real matrices is an open subset of $\mathbb{R}^{n^2}$ (as $\det\colon \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous and $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$). It is therefore a smooth manifold of dimension n^2. Matrix multiplication $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ and inversion $A \mapsto A^{-1}$ are smooth maps (the entries of A^{-1} are rational functions of the entries of A), so $\mathrm{GL}(n, \mathbb{R})$ is a

Lie group.

Example 1.3.7 (Special linear group $\mathrm{SL}(n,\mathbb{R})$)**.** The subgroup $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : \det A = 1\}$ is the preimage of the regular value 1 under $\det\colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$. By the preimage theorem (which we prove in Chapter 2), $\mathrm{SL}(n,\mathbb{R})$ is a smooth submanifold of dimension $n^2 - 1$, and is a Lie group.

Example 1.3.8 (Orthogonal group $\mathrm{O}(n)$ and $\mathrm{SO}(n)$)**.** The orthogonal group $\mathrm{O}(n) = \{A \in \mathrm{GL}(n,\mathbb{R}) : A^T A = I_n\}$ is the preimage of I_n under the smooth map $F(A) = A^T A$ from $\mathrm{GL}(n,\mathbb{R})$ to the space $\mathrm{Sym}(n)$ of symmetric matrices. Since I_n is a regular value of F, $\mathrm{O}(n)$ is a compact smooth manifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. Its identity component is $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\}$.

Example 1.3.9 (Unitary group $\mathrm{U}(n)$ and $\mathrm{SU}(n)$)**.** Similarly, $\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) : A^*A = I_n\}$ is a compact Lie group of (real) dimension n^2, and $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) : \det A = 1\}$ has dimension $n^2 - 1$. In particular $\mathrm{SU}(2)$ is diffeomorphic to S^3.

Example 1.3.10 (Grassmannian $\mathrm{Gr}(k,n)$)**.** The *Grassmannian* $\mathrm{Gr}(k,n)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$. For each subset $I \subseteq \{1,\dots,n\}$ of size k, let U_I consist of those k-planes V such that the projection $\mathbb{R}^n \to \mathbb{R}^I$ restricts to an isomorphism on V. Identifying each such V with the unique $k \times (n-k)$ matrix expressing the complementary coordinates in terms of the I-coordinates gives a chart $\varphi_I\colon U_I \xrightarrow{\sim} \mathbb{R}^{k(n-k)}$. The transition maps are rational, hence smooth, and $\mathrm{Gr}(k,n)$ is a compact smooth manifold of dimension $k(n-k)$. Note that $\mathrm{Gr}(1,n+1) \cong \mathbb{RP}^n$.

1.4 Smooth Functions on Manifolds

Before turning to manifolds with boundary, we single out the important special case of smooth *real-valued* functions.

Definition 1.4.1 (Smooth function)**.** Let M be a smooth manifold. A function $f\colon M \to \mathbb{R}$ is *smooth* if for every chart (U,φ) in the smooth structure of M, the composition $f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{R}$ is smooth in the ordinary sense of calculus. The set of all smooth functions on M is denoted $C^\infty(M)$.

Proposition 1.4.2. *The set $C^\infty(M)$ is a commutative $\mathbb{R}$-algebra under pointwise operations: for $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$,*

$$(f+g)(p) = f(p) + g(p), \qquad (\lambda f)(p) = \lambda \cdot f(p), \qquad (fg)(p) = f(p)\,g(p).$$

Proof. The sum, scalar multiple, and product of smooth functions on open subsets of $\mathbb{R}^n$ are smooth. The claim follows by checking in local coordinates. □

Remark 1.4.3. A fundamental fact (which we shall not prove here) is that the smooth structure on a manifold M is completely determined by the algebra $C^\infty(M)$. More precisely, if M and N are smooth manifolds and $\Phi\colon C^\infty(N) \to C^\infty(M)$ is an $\mathbb{R}$-algebra isomorphism, then there is a unique diffeomorphism $f\colon M \to N$ with $\Phi(g) = g \circ f$ for all $g \in C^\infty(N)$. This is the content of Milnor's exercise, and it foreshadows the sheaf-theoretic viewpoint on manifolds.

Example 1.4.4 (Height function on the sphere)**.** The function $h\colon S^n \to \mathbb{R}$ defined by $h(x_0, x_1, \ldots, x_n) = x_n$ is smooth. Indeed, in the stereographic chart φ_S from Example 1.3.2, one computes

$$h \circ \varphi_S^{-1}(y) = \frac{\|y\|^2 - 1}{\|y\|^2 + 1},$$

which is a smooth function on $\mathbb{R}^n$. The critical points of h are the north and south poles.

Example 1.4.5 (Smooth functions on $\mathbb{RP}^n$)**.** A function $f\colon \mathbb{RP}^n \to \mathbb{R}$ is smooth if and only if $f \circ \pi\colon S^n \to \mathbb{R}$ is smooth, where $\pi\colon S^n \to \mathbb{RP}^n$ is the quotient map. Moreover, $f \circ \pi$ must satisfy $(f \circ \pi)(-x) = (f \circ \pi)(x)$ for all $x \in S^n$. Hence $C^\infty(\mathbb{RP}^n)$ identifies with the even smooth functions on S^n.

1.5 Manifolds with Boundary

Notation 1.5.1. We denote the closed upper half-space by $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. Its boundary is $\partial\mathbb{H}^n = \{x_n = 0\} \cong \mathbb{R}^{n-1}$.

Definition 1.5.2 (Smooth manifold with boundary)**.** A *smooth n-manifold with boundary* is a second-countable Hausdorff space M equipped with a maximal smooth atlas whose charts map open subsets of M homeomorphically to open subsets of $\mathbb{H}^n$, with smooth transition maps.
The *boundary* ∂M consists of all points mapped to $\partial\mathbb{H}^n$ by some (hence every) chart. The *interior* is $\operatorname{Int}(M) = M \setminus \partial M$.

Proposition 1.5.3. *The boundary ∂M is well-defined: if p maps to $\partial\mathbb{H}^n$ under one chart, it does so under every chart containing p.*

Proof. Suppose charts (U, φ) and (V, ψ) both contain p, with $\varphi(p) \in \partial\mathbb{H}^n$ but $\psi(p) \in \operatorname{Int}(\mathbb{H}^n)$. Then the transition map $\psi \circ \varphi^{-1}$ sends a boundary point of $\mathbb{H}^n$ to an interior point, contradicting the smooth invariance of domain (a diffeomorphism between open subsets of $\mathbb{H}^n$ maps boundary to boundary). □

Proposition 1.5.4. *If M is a smooth n-manifold with boundary, then ∂M is a smooth $(n-1)$-manifold without boundary.*

Proof. For each boundary chart (U, φ) the restriction $\varphi|_{U \cap \partial M}$ maps $U \cap \partial M$ homeomorphically onto an open subset of $\mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$. The transition maps restrict to smooth diffeomorphisms between open subsets of $\mathbb{R}^{n-1}$. □

Example 1.5.5 (Closed disk)**.** The closed unit disk $\overline{D}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is a smooth n-manifold with boundary $\partial\overline{D}^n = S^{n-1}$.

Example 1.5.6 (Cylinder and Möbius band)**.** The cylinder $S^1 \times [0,1]$ is a 2-manifold with boundary $S^1 \sqcup S^1$. The Möbius band is a non-orientable 2-manifold with boundary S^1.

Example 1.5.7 (Half-spaces and quadrants)**.** The upper half-plane $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ is a 2-manifold with boundary $\partial\mathbb{H}^2 = \mathbb{R} \times \{0\}$. Note that the quadrant $Q = \{(x,y) : x \geq 0,\, y \geq 0\}$ is *not* a manifold with boundary (the origin has no neighbourhood homeomorphic to an open subset of $\mathbb{H}^2$); rather, Q is a *manifold with corners*, a notion we do not develop here.

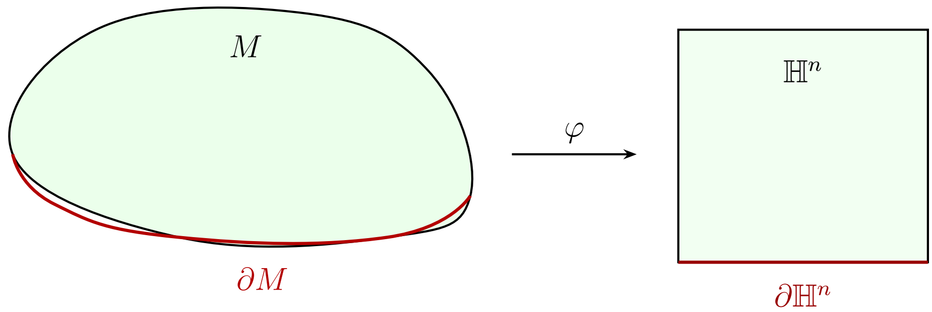


Figure 1.4: A boundary chart maps a neighbourhood of a boundary point to an open subset of the half-space $\mathbb{H}^n$.

1.6 Partitions of Unity

Partitions of unity are the single most important technical tool in the theory of smooth manifolds. They allow one to patch together local constructions into global ones.

Definition 1.6.1 (Support)**.** The *support* of a continuous function $f\colon M \to \mathbb{R}$ is $\operatorname{supp}(f) = \overline{\{p \in M : f(p) \neq 0\}}$.

Definition 1.6.2 (Partition of unity)**.** Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a smooth manifold M. A *smooth partition of unity subordinate to* $\{U_\alpha\}$ is a family $\{\rho_\alpha\}_{\alpha\in A}$ of smooth functions $\rho_\alpha\colon M \to [0,1]$ such that:

(i) $\operatorname{supp}(\rho_\alpha) \subseteq U_\alpha$ for every α;

(ii) the collection $\{\operatorname{supp}(\rho_\alpha)\}$ is locally finite;

(iii) $\sum_\alpha \rho_\alpha(p) = 1$ for every $p \in M$ (the sum is finite at each point by (ii)).

We need a bump function construction.

Lemma 1.6.3 (Smooth bump function on $\mathbb{R}^n$)**.** *For any $0 < r < R$ there exists a smooth function $\beta\colon \mathbb{R}^n \to [0,1]$ with $\beta \equiv 1$ on $\overline{B}_r(0)$ and* $\operatorname{supp}(\beta) \subseteq B_R(0)$.

Proof. Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} e^{-1/t} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then h is smooth. Set $g(t) = h(t)/(h(t) + h(1-t))$; this is a smooth non-decreasing function with $g(t) = 0$ for $t \leq 0$ and $g(t) = 1$ for $t \geq 1$. Now define

$$\beta(x) = g\left(\frac{R^2 - \|x\|^2}{R^2 - r^2}\right).$$

This satisfies the stated properties. □

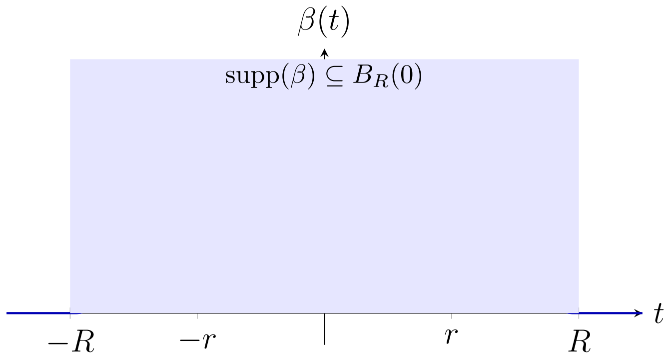


Figure 1.5: A smooth bump function: $\beta \equiv 1$ on $\overline{B}_r(0)$ and $\mathrm{supp}(\beta) \subseteq B_R(0)$.

Theorem 1.6.4 (Existence of partitions of unity)**.** *Let M be a smooth manifold (possibly with boundary) and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Then there exists a smooth partition of unity subordinate to $\{U_\alpha\}$.*

Proof. The proof proceeds in several steps.

Step 1: Exhaustion by compact sets. Since M is second-countable and locally compact Hausdorff, there exists an exhaustion $K_1 \subseteq K_2 \subseteq \cdots$ by compact sets with $K_j \subseteq \mathrm{Int}(K_{j+1})$ and $M = \bigcup_j K_j$. Set $K_0 = \varnothing$.

Step 2: Locally finite refinement. For each $p \in K_{j+1} \setminus \mathrm{Int}(K_j)$, choose $\alpha(p)$ such that $p \in U_{\alpha(p)}$, and a chart (V_p, φ_p) centred at p with

$$V_p \subseteq U_{\alpha(p)} \cap \Big(\mathrm{Int}(K_{j+2}) \setminus K_{j-1}\Big).$$

Choose $r_p > 0$ so that $\varphi_p^{-1}(\overline{B}_{r_p}(0)) \subseteq V_p$. By compactness of $K_{j+1} \setminus \mathrm{Int}(K_j)$, finitely many of these balls cover this annulus. Combining over all j gives a countable, locally finite refinement $\{W_i\}_{i \in \mathbb{N}}$ of the original cover, together with smooth bump functions $\psi_i\colon M \to [0,1]$ with $\mathrm{supp}(\psi_i) \subseteq W_i$ and $\psi_i > 0$ on a neighbourhood of the corresponding compact piece.

Step 3: Construction of the partition. For each i, associate W_i to some $U_{\alpha(i)}$ with $W_i \subseteq U_{\alpha(i)}$. For each $\alpha \in A$ define

$$\tilde{\rho}_\alpha = \sum_{\{i : \alpha(i) = \alpha\}} \psi_i.$$

Since $\{\operatorname{supp}(\psi_i)\}$ is locally finite, each $\tilde{\rho}_\alpha$ is smooth with $\operatorname{supp}(\tilde{\rho}_\alpha) \subseteq U_\alpha$. The sum $\Psi = \sum_\alpha \tilde{\rho}_\alpha = \sum_i \psi_i$ is everywhere positive (the bumps ψ_i cover M). Setting

$$\rho_\alpha = \frac{\tilde{\rho}_\alpha}{\Psi}$$

yields the desired partition of unity. □

Corollary 1.6.5. *Let $A \subseteq M$ be closed and $U \supseteq A$ open in a smooth manifold M. Then there exists $f \in C^\infty(M)$ with $0 \leq f \leq 1$, $f|_A = 1$, and $\operatorname{supp}(f) \subseteq U$.*

Proof. Apply Theorem 1.6.4 to the cover $\{U, M \setminus A\}$ and take $f = \rho_U$. □

Corollary 1.6.6. *Let $A \subseteq M$ be closed and $f\colon A \to \mathbb{R}$ a function that extends smoothly to a neighbourhood of each point in A. Then f extends to a smooth function $\tilde{f}\colon M \to \mathbb{R}$.*

Proof. Cover A by open sets $\{V_i\}$ on which smooth extensions f_i exist. Let $\{U_i\}$ be the cover $\{V_i\} \cup \{M \setminus A\}$ and $\{\rho_i\}$ a subordinate partition of unity. Define $\tilde{f} = \sum_i \rho_i f_i$, where we set $f_i = 0$ on $M \setminus A$. □

1.7 Exercises

Exercise 1.1. Construct a smooth atlas on S^1 using four charts (open arcs) and verify that the transition maps are smooth.

Exercise 1.2. Show that $\mathbb{RP}^1$ is diffeomorphic to S^1. *Hint:* exhibit an explicit smooth bijection with smooth inverse, or use the map $[x_0 : x_1] \mapsto e^{2i \arctan(x_1/x_0)}$.

Exercise 1.3. Prove that $\mathbb{CP}^1$ is diffeomorphic to S^2.

Exercise 1.4. Let M^m and N^n be smooth manifolds. Show that $M \times N$ has a natural smooth structure of dimension $m + n$, with charts $(U_\alpha \times V_\beta,\ \varphi_\alpha \times \psi_\beta)$.

Exercise 1.5. Verify explicitly that the atlas on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ obtained from local sections of the projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a smooth atlas, and that T^2 is diffeomorphic to $S^1 \times S^1$.

Exercise 1.6. Show that $\mathrm{GL}(n, \mathbb{R})$ has exactly two connected components: $\mathrm{GL}^+(n, \mathbb{R}) = \{A : \det A > 0\}$ and $\mathrm{GL}^-(n, \mathbb{R}) = \{A : \det A < 0\}$.

Exercise 1.7. Prove that $\mathrm{SU}(2)$ is diffeomorphic to S^3 by showing that every element of $\mathrm{SU}(2)$ can be written as $\left(\begin{smallmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{smallmatrix}\right)$ with $|\alpha|^2 + |\beta|^2 = 1$.

Exercise 1.8. Prove that $\mathrm{Gr}(k, n)$ is compact. *Hint:* exhibit it as a quotient of a compact group, e.g. $\mathrm{Gr}(k, n) \cong \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$.

Exercise 1.9 (Partition of unity application)**.** Use a partition of unity to show that every smooth manifold admits a Riemannian metric. (You need only show the existence of a smooth, positive-definite symmetric $(0, 2)$-tensor field.)

Exercise 1.10. Let M be a smooth manifold with boundary and N a smooth manifold without boundary. Show that $M \times N$ is a smooth manifold with boundary $\partial(M \times N) = (\partial M) \times N$.

Chapter 2

Smooth Maps and Diffeomorphisms

Having defined smooth manifolds, we now study the maps between them. The central idea is that smoothness of a map $f\colon M \to N$ can be tested in local coordinates; the compatibility of the smooth structures ensures that this notion is independent of the charts chosen. We develop the fundamental analytic tools — the inverse and implicit function theorems on manifolds, the constant rank theorem — and introduce the key notions of immersion, submersion, and diffeomorphism.

2.1 Smooth Maps Between Manifolds

Definition 2.1.1 (Smooth map)**.** Let M^m and N^n be smooth manifolds. A continuous map $f\colon M \to N$ is *smooth* (or C^∞) if for every $p \in M$ there exist charts (U, φ) on M around p and (V, ψ) on N around $f(p)$ with $f(U) \subseteq V$ such that the *coordinate representation*

$$\hat{f} = \psi \circ f \circ \varphi^{-1}\colon \varphi(U) \longrightarrow \psi(V)$$

is smooth as a map between open subsets of Euclidean spaces.
We write $C^\infty(M, N)$ for the set of all smooth maps from M to N, and $C^\infty(M) = C^\infty(M, \mathbb{R})$.

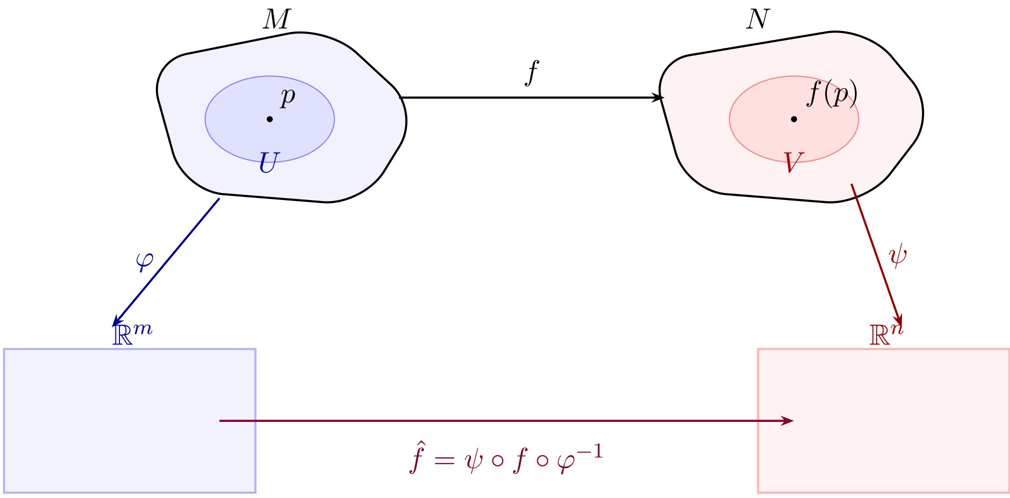


Figure 2.1: The coordinate representation of a smooth map f.

Remark 2.1.2. The definition is independent of the choice of charts: if (U', φ') and (V', ψ') are other charts, then $\psi' \circ f \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ \hat{f} \circ (\varphi \circ (\varphi')^{-1})$ is a composition of smooth maps, hence smooth.

Proposition 2.1.3 (Composition of smooth maps)**.** *If $f\colon M \to N$ and $g\colon N \to P$ are smooth, then $g \circ f\colon M \to P$ is smooth.*

Proof. In local coordinates, $\widehat{g \circ f} = \hat{g} \circ \hat{f}$, which is a composition of smooth maps between Euclidean open sets. □

Definition 2.1.4 (The smooth category)**.** Smooth manifolds and smooth maps form a category, denoted **Man** or **Diff**. The identity morphism on M is Id_M, and composition is associative by the previous proposition.

The following commutative diagram illustrates the "functorial" nature of the coordinate representation:

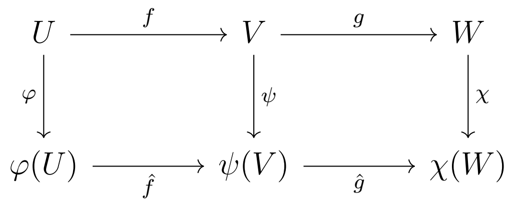


Figure 2.2: Composition in local coordinates.

2.2 Diffeomorphisms

Definition 2.2.1 (Diffeomorphism)**.** A smooth map $f\colon M \to N$ is a *diffeomorphism* if it is bijective and its inverse $f^{-1}\colon N \to M$ is also smooth. We say M and N are *diffeomorphic* and write $M \cong N$ (or $M \approx_{\text{diff}} N$ when we wish to distinguish from homeomorphism).

Remark 2.2.2. A smooth bijection need not be a diffeomorphism: $f\colon \mathbb{R} \to \mathbb{R}$, $f(t) = t^3$ is a smooth bijection but $f^{-1}(s) = s^{1/3}$ is not smooth at $s = 0$.

Proposition 2.2.3. *Diffeomorphism is an equivalence relation on the class of smooth manifolds.*

Proof. Reflexivity: Id_M is a diffeomorphism. Symmetry: if f is a diffeomorphism then so is f^{-1}. Transitivity: if $f\colon M \to N$ and $g\colon N \to P$ are diffeomorphisms, then $g \circ f$ is smooth with smooth inverse $f^{-1} \circ g^{-1}$. □

Definition 2.2.4 (Diffeomorphism group)**.** The group of all diffeomorphisms $f\colon M \to M$ under composition is denoted $\mathrm{Diff}(M)$.

2.3 The Inverse Function Theorem on Manifolds

We first recall the differential of a smooth map, which we shall define properly through the tangent space in a later chapter. For the present purposes, the following coordinate formulation suffices.

Definition 2.3.1 (Differential in coordinates)**.** Let $f\colon M^m \to N^n$ be smooth and $p \in M$. Choose charts (U, φ) around p and (V, ψ) around $f(p)$. The *differential of f at p* (in these coordinates) is the linear map

$$\mathrm{d}f_p\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n, \qquad \mathrm{d}f_p = D\hat{f}(\varphi(p)),$$

where $D\hat{f}$ denotes the usual (Euclidean) derivative of the coordinate representation $\hat{f} = \psi \circ f \circ \varphi^{-1}$. The rank of $\mathrm{d}f_p$ is independent of the charts chosen.

Theorem 2.3.2 (Inverse function theorem — manifold version)**.** *Let $f\colon M^n \to N^n$ be a smooth map between manifolds of the same dimension. If $\mathrm{d}f_p\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism at some point $p \in M$, then there exist open neighbourhoods U of p and V of $f(p)$ such that $f|_U\colon U \to V$ is a diffeomorphism.*

Proof. Choose charts (U_0, φ) centred at p and (V_0, ψ) centred at $f(p)$. The coordinate representation $\hat{f} = \psi \circ f \circ \varphi^{-1}$ satisfies $D\hat{f}(0) = \mathrm{d}f_p$, which is invertible by hypothesis. By the classical inverse function theorem in $\mathbb{R}^n$, there exist open sets $\hat{U} \ni 0$ and $\hat{V} \ni 0$ such that $\hat{f}|_{\hat{U}}\colon \hat{U} \to \hat{V}$ is a smooth diffeomorphism. Setting $U = \varphi^{-1}(\hat{U})$ and $V = \psi^{-1}(\hat{V})$, we get

$$f|_U = \psi^{-1} \circ \hat{f}|_{\hat{U}} \circ \varphi,$$

which is a diffeomorphism from U to V as a composition of diffeomorphisms. □

Corollary 2.3.3. *A local diffeomorphism (i.e. a smooth map whose differential is everywhere an isomorphism) is an open map.*

2.4 The Implicit Function Theorem on Manifolds

Theorem 2.4.1 (Implicit function theorem — manifold version)**.** *Let $f\colon M^m \to N^n$ be a smooth map with $m \geq n$. Suppose $q \in N$ is a* regular value *of f, meaning that $\mathrm{d}f_p$ is surjective for every $p \in f^{-1}(q)$. Then $f^{-1}(q)$ is a smooth submanifold of M of dimension $m - n$.*

Proof. Let $p \in f^{-1}(q)$. Choose charts (U, φ) centred at p and (V, ψ) centred at q with $f(U) \subseteq V$. The coordinate representation $\hat{f} = \psi \circ f \circ \varphi^{-1}$ satisfies $D\hat{f}(0)\colon \mathbb{R}^m \to \mathbb{R}^n$ surjective, hence of rank n.

By the classical implicit function theorem, after permuting coordinates if necessary, the set $\hat{f}^{-1}(0)$ is locally the graph of a smooth function $g\colon W \subseteq \mathbb{R}^{m-n} \to \mathbb{R}^n$. More precisely, there is an open neighbourhood $\hat{U}$ of 0 in $\mathbb{R}^m$ such that

$$\hat{f}^{-1}(0) \cap \hat{U} = \{(x', g(x')) : x' \in W\}.$$

The map $x' \mapsto \varphi^{-1}(x', g(x'))$ gives a chart for $f^{-1}(q)$ near p. Since transition maps between such charts are smooth, $f^{-1}(q)$ is a smooth submanifold of dimension $m - n$. □

Corollary 2.4.2 (Preimage theorem)**.** *If $f\colon M \to N$ is a smooth map and q is a regular value, then $f^{-1}(q)$ is either empty or a closed smooth submanifold of M of codimension $n = \dim N$.*

Example 2.4.3 (The sphere revisited)**.** Consider $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x) = \|x\|^2$. Then $Df(x) = 2x$, which is surjective for $x \neq 0$. Hence every $c > 0$ is a regular value, and $f^{-1}(1) = S^n$ is a smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$.

Example 2.4.4 ($\mathrm{O}(n)$ as a smooth manifold)**.** Define $F\colon M(n, \mathbb{R}) \to \mathrm{Sym}(n)$ by $F(A) = A^T A$. Then $\mathrm{O}(n) = F^{-1}(I_n)$. The derivative is $DF(A) \cdot H = H^T A + A^T H$. At $A \in \mathrm{O}(n)$ this becomes $DF(A) \cdot H = H^T A + A^T H$, and for any symmetric S the matrix $H = \frac{1}{2} AS$ satisfies $DF(A) \cdot H = S$. Hence I_n is a regular value, confirming that $\mathrm{O}(n)$ is a smooth submanifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

2.5 Immersions and Submersions

Definition 2.5.1 (Immersion, submersion)**.** Let $f\colon M^m \to N^n$ be smooth.

(i) f is an *immersion* at p if $\mathrm{d}f_p$ is injective (i.e. rank $\mathrm{d}f_p = m$).

(ii) f is a *submersion* at p if $\mathrm{d}f_p$ is surjective (i.e. rank $\mathrm{d}f_p = n$).

(iii) f is an *immersion* (resp. *submersion*) if it is one at every point.

Definition 2.5.2 (Local diffeomorphism)**.** A smooth map $f\colon M^n \to N^n$ is a *local diffeomorphism* if $\mathrm{d}f_p$ is an isomorphism for all $p \in M$ (equivalently, if f is both an immersion and a submersion).

Proposition 2.5.3 (Canonical form for immersions)**.** *If $f\colon M^m \to N^n$ is an immersion at p (so $m \leq n$), then there exist charts (U, φ) centred at p and (V, ψ) centred at $f(p)$ such that the coordinate representation takes the form*

$$\hat{f}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Proof. In coordinates, $D\hat{f}(0)$ has rank m. After reordering the coordinates of $\mathbb{R}^n$ we may assume the first m rows of the $n \times m$ Jacobian form an invertible $m \times m$ matrix. Define $F\colon \varphi(U) \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ by

$$F(x, y) = \hat{f}(x) + (0, y).$$

Then $DF(0,0)$ is invertible. By the inverse function theorem, F is a local diffeomorphism near $(0,0)$. Setting $\psi' = F^{-1} \circ \psi$ (restricted appropriately), we get $\psi' \circ f \circ \varphi^{-1}(x) = F^{-1}(\hat{f}(x)) = F^{-1}(F(x, 0)) = (x, 0)$. □

Proposition 2.5.4 (Canonical form for submersions). *If $f\colon M^m \to N^n$ is a submersion at p (so $m \geq n$), then there exist charts (U, φ) centred at p and (V, ψ) centred at $f(p)$ such that*

$$\hat{f}(x_1, \ldots, x_m) = (x_1, \ldots, x_n).$$

Proof. The Jacobian $D\hat{f}(0)$ has rank n. After reordering, assume the first n columns are linearly independent. Define $G\colon \varphi(U) \to \mathbb{R}^m$ by $G(x) = (\hat{f}(x), x_{n+1}, \ldots, x_m)$. Then $DG(0)$ is invertible, so G is a local diffeomorphism. Setting $\varphi' = G \circ \varphi$, the coordinate representation becomes $\psi \circ f \circ (\varphi')^{-1}(y_1, \ldots, y_m) = (y_1, \ldots, y_n)$. □

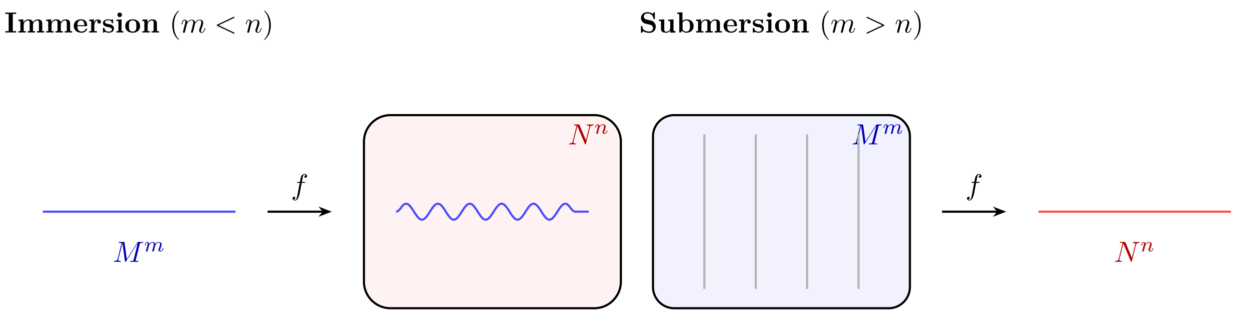


Figure 2.3: Schematic depiction of an immersion (left) and a submersion (right). The fibres of the submersion are shown as vertical lines.

2.6 The Constant Rank Theorem

The canonical forms for immersions and submersions are special cases of the following fundamental result.

Theorem 2.6.1 (Constant rank theorem). *Let $f\colon M^m \to N^n$ be a smooth map of constant rank r on an open set containing p. Then there exist charts (U, φ) centred at p and (V, ψ) centred at $f(p)$ in which f has the coordinate representation*

$$\hat{f}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

Proof. By the Euclidean constant rank theorem (see, e.g., Boothby), working in any pair of charts we may find smooth coordinate changes α on the domain side and β on the target side such that $\beta \circ \hat{f} \circ \alpha^{-1}$ has the stated form. Replacing φ by $\alpha \circ \varphi$ and ψ by $\beta \circ \psi$ gives charts with the desired property.

In more detail: after a linear change of coordinates we may assume $D\hat{f}(0)$ has its upper-left $r \times r$ block invertible. Write $\hat{f}(x) = (\hat{f}_1(x), \hat{f}_2(x))$ with $\hat{f}_1\colon \mathbb{R}^m \to \mathbb{R}^r$ and $\hat{f}_2\colon \mathbb{R}^m \to \mathbb{R}^{n-r}$. Define $\alpha(x) = (\hat{f}_1(x), x_{r+1}, \ldots, x_m)$; then $D\alpha(0)$ is invertible. In the new coordinates $y = \alpha(x)$, the map becomes $\hat{f} \circ \alpha^{-1}(y) = (y_1, \ldots, y_r, h(y))$ for some smooth h. The constant rank condition forces $\partial h/\partial y_j = 0$ for $j = r+1, \ldots, m$ (otherwise the rank would exceed r). Hence h depends only on $(y_1, \ldots, y_r)$. Define $\beta\colon \mathbb{R}^n \to \mathbb{R}^n$ by $\beta(z_1, \ldots, z_n) = (z_1, \ldots, z_r, z_{r+1} - h(z_1, \ldots, z_r), \ldots, z_n - h_{n-r}(z_1, \ldots, z_r))$. Then $\beta \circ \hat{f} \circ \alpha^{-1}(y) = (y_1, \ldots, y_r, 0, \ldots, 0)$. □

Corollary 2.6.2. *If $f\colon M \to N$ has constant rank r on all of M, then for every $q \in f(M)$ the level set $f^{-1}(q)$ is a closed smooth submanifold of M of dimension $m-r$.*

Proof. In the canonical coordinates of Theorem 4.4.1, $f^{-1}(q)$ locally coincides with $\{(0,\dots,0,x_{r+1},\dots,x_m)\}$, a smooth submanifold of dimension $m-r$. □

2.7 Examples

Example 2.7.1 (Projection maps)**.** Let $M \times N$ be a product manifold. The projections $\pi_M\colon M\times N \to M$ and $\pi_N\colon M \times N \to N$ are smooth submersions. In product charts $(\varphi\times\psi)$, $\hat{\pi}_M(x,y) = x$.

Example 2.7.2 (Inclusion of submanifolds)**.** If $S \subseteq M$ is a smooth submanifold, the inclusion $\iota\colon S \hookrightarrow M$ is a smooth immersion (indeed an embedding).

Example 2.7.3 (The Hopf map)**.** The *Hopf map* $h\colon S^3 \to S^2$ is defined as follows. View $S^3 \subseteq \mathbb{C}^2$ as $\{(z_1,z_2) : |z_1|^2 + |z_2|^2 = 1\}$ and $S^2 \cong \mathbb{C}\mathrm{P}^1$. Define

$$h(z_1,z_2) = [z_1 : z_2] \in \mathbb{C}\mathrm{P}^1 \cong S^2.$$

Explicitly, using the identification $S^2 \subseteq \mathbb{R}^3$,

$$h(z_1,z_2) = \Big(2\operatorname{Re}(z_1\bar{z}_2),\ 2\operatorname{Im}(z_1\bar{z}_2),\ |z_1|^2 - |z_2|^2\Big).$$

The Hopf map is a smooth submersion. Each fibre $h^{-1}(p)$ is a great circle $S^1 \subseteq S^3$, and h exhibits S^3 as an S^1-bundle over S^2.

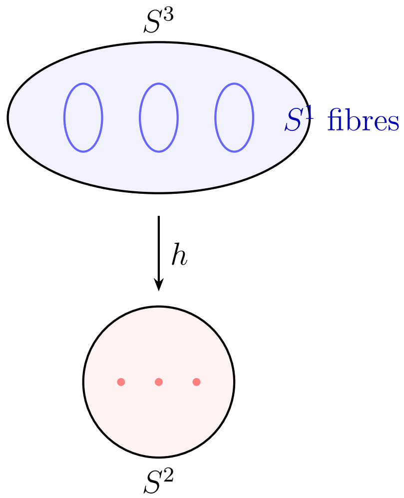


Figure 2.4: Schematic of the Hopf fibration $h\colon S^3 \to S^2$ with S^1 fibres.

Example 2.7.4 (The determinant map)**.** The map $\det\colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$ is a smooth submersion. Its derivative at A in the direction H is

$$D(\det)(A)\cdot H = \det(A)\,\operatorname{tr}(A^{-1}H).$$

For $A \in \mathrm{GL}(n, \mathbb{R})$ and any $c \in \mathbb{R}^*$, choosing $H = \frac{c}{\det(A) \cdot n} A$ shows surjectivity. The level set $\det^{-1}(1) = \mathrm{SL}(n, \mathbb{R})$.

Example 2.7.5 (Smooth maps between spheres)**.** The antipodal map $a\colon S^n \to S^n$, $a(x) = -x$, is a diffeomorphism. For n even, a is orientation-reversing; for n odd, it is orientation-preserving (as we shall see when we discuss orientations).

Example 2.7.6 (Covering maps as local diffeomorphisms)**.** The exponential map $e\colon \mathbb{R} \to S^1$, $e(t) = e^{2\pi i t}$, is a smooth surjective local diffeomorphism (and a covering map). More generally, the quotient map $\mathbb{R}^n \to T^n = \mathbb{R}^n/\mathbb{Z}^n$ is a smooth local diffeomorphism.

Example 2.7.7 (The Gauss map)**.** Let $S \subseteq \mathbb{R}^3$ be a smooth surface (a 2-dimensional submanifold). If S is orientable, there exists a smooth unit normal field $\nu\colon S \to S^2$, called the *Gauss map*. For instance, for the sphere $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ the outward normal is $\nu(x) = x$, so the Gauss map is the identity. For the torus $T^2 \subseteq \mathbb{R}^3$ parametrized by

$$\Phi(\theta, \phi) = \Big((R + r\cos\phi)\cos\theta,\ (R + r\cos\phi)\sin\theta,\ r\sin\phi\Big),$$

the Gauss map sends each point to its outward unit normal, and is a smooth map $T^2 \to S^2$ of degree zero.

Example 2.7.8 (Power maps on S^1)**.** For each integer $k \geq 1$, the map $p_k\colon S^1 \to S^1$ defined by $p_k(z) = z^k$ (viewing $S^1 \subseteq \mathbb{C}$) is smooth. It is a k-sheeted covering map for $k \geq 2$ and a diffeomorphism for $k = 1$. In real coordinates, writing $z = e^{i\theta}$, the map is $\theta \mapsto k\theta$, and the differential is multiplication by k, which is everywhere an isomorphism. Hence p_k is a local diffeomorphism for all $k \geq 1$.

2.8 Local Diffeomorphisms and Global Properties

Proposition 2.8.1. *A smooth map $f\colon M \to N$ that is both a local diffeomorphism and a bijection is a diffeomorphism.*

Proof. Since f is a local diffeomorphism, for each $q \in N$ there is a neighbourhood V_q of q on which $f^{-1}|_{V_q}$ is smooth. As f is bijective, these local inverses patch together to give the global inverse f^{-1}, which is therefore smooth. □

Proposition 2.8.2. *Let $f\colon M \to N$ be a local diffeomorphism with M compact and N connected. Then f is surjective and a (finite-sheeted) covering map. In particular, if N is simply connected, f is a diffeomorphism.*

Proof. Since f is a local diffeomorphism, it is an open map (Corollary 2.3.3). Since M is compact, $f(M)$ is compact, hence closed in N (as N is Hausdorff). As $f(M)$ is both open and closed in the connected space N, we get $f(M) = N$.

For each $q \in N$ the fibre $f^{-1}(q)$ is discrete (by the local diffeomorphism condition) and compact (as a closed subset of M), hence finite. A standard argument using the local inverse charts shows f is a covering map. If N is simply connected, covering theory gives that f is a homeomorphism, hence a diffeomorphism by Proposition 2.8.1. □

Remark 2.8.3. The converse fails: a local diffeomorphism from a non-compact manifold need not be a covering map. For instance, the inclusion of an open interval $(0,1) \hookrightarrow \mathbb{R}$ is a local diffeomorphism that is not surjective. A more interesting example: the restriction of $e\colon \mathbb{R} \to S^1$ to $(0,2)$ is a local diffeomorphism that is neither injective nor a covering.

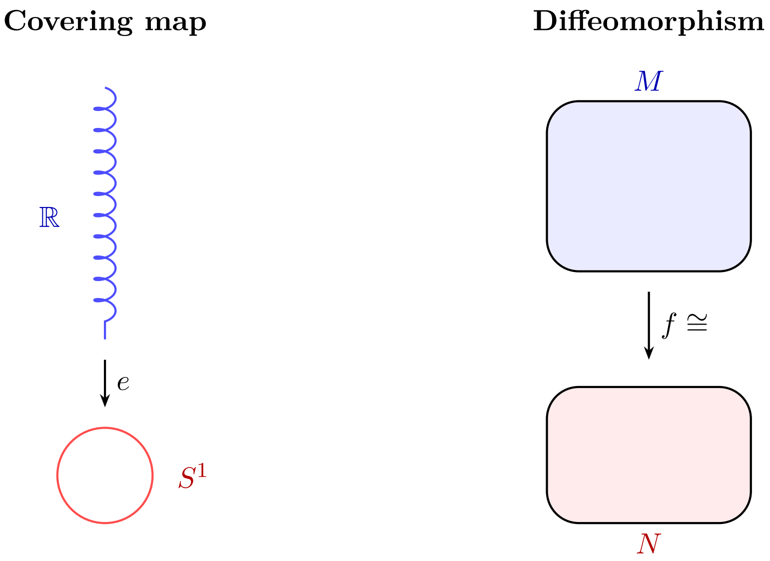


Figure 2.5: A covering map (local diffeomorphism, left) versus a global diffeomorphism (right).

2.9 Embeddings

Definition 2.9.1 (Smooth embedding)**.** A smooth map $f\colon M \to N$ is a *smooth embedding* if it is an injective immersion that is a homeomorphism onto its image $f(M)$ (with the subspace topology from N).

Remark 2.9.2. Every embedding is an injective immersion, but the converse fails. The standard example is an injective immersion of $\mathbb{R}$ into T^2 whose image is a dense subset of the torus (an irrational-slope line). This map is an injective immersion but not an embedding because the image, with the subspace topology, is not homeomorphic to $\mathbb{R}$.

Proposition 2.9.3 (Compact domain implies embedding)**.** *If M is compact and $f\colon M \to N$ is an injective immersion, then f is a smooth embedding.*

Proof. An injective continuous map from a compact space to a Hausdorff space is a homeomorphism onto its image (a closed subset). Since f is also an immersion, it is a smooth embedding. □

Proposition 2.9.4 (Image of an embedding is a submanifold)**.** *If $f\colon M^m \to N^n$ is a smooth embedding, then $f(M)$ is a smooth submanifold of N of dimension m, and $f\colon M \to f(M)$ is a diffeomorphism.*

Proof. Since f is an immersion, the canonical form (Proposition 2.5.3) provides, for each $p \in M$, charts in which f looks like the standard inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$. This shows that $f(M)$ is locally a "slice" of N, hence a submanifold. Since f is a homeomorphism onto $f(M)$, and the smooth structures agree via the canonical form, f is a diffeomorphism onto this submanifold. □

Figure 2.6: An embedding (left) versus a non-injective immersion producing a figure-eight (right).

Example 2.9.5 (Standard embeddings)**.** The following are smooth embeddings:

(a) The inclusion $\iota\colon S^n \hookrightarrow \mathbb{R}^{n+1}$.

(b) The map $f\colon \mathbb{R} \to \mathbb{R}^2$, $f(t) = (t, t^2)$ (a parabola).

(c) The Segre embedding $\sigma\colon \mathbb{CP}^m \times \mathbb{CP}^n \hookrightarrow \mathbb{CP}^{mn+m+n}$ defined by $\sigma([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) = [x_i y_j]_{0 \le i \le m, 0 \le j \le n}$.

(d) The Veronese embedding $v_d\colon \mathbb{CP}^n \hookrightarrow \mathbb{CP}^N$ ($N = \binom{n+d}{d} - 1$) sending $[x_0 : \cdots : x_n]$ to the list of all monomials of degree d.

2.10 Smooth Maps on Manifolds with Boundary

Definition 2.10.1. Let M and N be smooth manifolds, possibly with boundary. A map $f\colon M \to N$ is *smooth* if for every $p \in M$ there exist boundary-type charts (U, φ) and (V, ψ) with $f(U) \subseteq V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth in the sense that it extends to a smooth map on an open subset of $\mathbb{R}^m$.

Proposition 2.10.2. *If $f\colon M \to N$ is a diffeomorphism between manifolds with boundary, then $f(\partial M) = \partial N$ and $f|_{\partial M}\colon \partial M \to \partial N$ is a diffeomorphism.*

Proof. If $p \in \partial M$, then every chart around p sends it to $\partial \mathbb{H}^m$. If $f(p) \notin \partial N$, an interior chart around $f(p)$ composed with f and the inverse of a boundary chart would give a diffeomorphism mapping a boundary point to an interior point of $\mathbb{H}^n$, contradicting smooth invariance of domain. The same argument applied to f^{-1} shows $f^{-1}(\partial N) = \partial M$. The restriction $f|_{\partial M}$ is smooth with smooth inverse $f^{-1}|_{\partial N}$. □

2.11 Exercises

Exercise 2.1. Let $f\colon M \to \mathbb{R}^n$ be a map. Show that f is smooth if and only if each component function $f_i = \pi_i \circ f\colon M \to \mathbb{R}$ is smooth, where π_i is the i-th coordinate projection.

Exercise 2.2. Let $f\colon M \to N$ be smooth. Prove that its graph $\Gamma_f = \{(p, f(p)) : p \in M\} \subseteq M \times N$ is a smooth submanifold diffeomorphic to M.

Exercise 2.3. Give a detailed proof that $f\colon \mathbb{R} \to \mathbb{R}$, $f(t) = t^3$, is a smooth bijection that is *not* a diffeomorphism. Where does the inverse function theorem fail to apply?

Exercise 2.4. Verify that the Hopf map $h\colon S^3 \to S^2$ is a smooth submersion by computing its differential in suitable coordinates.

Exercise 2.5. Show that the quotient map $\pi\colon S^n \to \mathbb{RP}^n$, $\pi(x) = [x]$, is a smooth two-sheeted covering map and a local diffeomorphism.

Exercise 2.6. Prove that every submersion is an open map. *Hint:* use the canonical form for submersions (Proposition 2.5.4).

Exercise 2.7. Give an example of an immersion that is not injective, and an example of an injective immersion that is not an embedding (i.e. not a homeomorphism onto its image). *Hint for the second:* consider the figure-eight curve or a dense line on the torus.

Exercise 2.8. Let $f\colon M \to N$ and $g\colon N \to P$ be smooth maps. Show that $\operatorname{rank} \mathrm{d}(g \circ f)_p \leq \min(\operatorname{rank} \mathrm{d}f_p, \operatorname{rank} \mathrm{d}g_{f(p)})$. Give an example where strict inequality holds.

Exercise 2.9. Let $f\colon M \to N$ and $g\colon M' \to N'$ be submersions. Prove that $f \times g\colon M \times M' \to N \times N'$ is a submersion. State and prove the analogous result for immersions.

Exercise 2.10. Determine all critical points and critical values of the determinant map $\det\colon M(n, \mathbb{R}) \to \mathbb{R}$ (here $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ is the space of all $n \times n$ real matrices, not just invertible ones). *Answer:* the only critical value is 0; equivalently, $c \neq 0$ is always a regular value.

Exercise 2.11. Let $\gamma\colon (-\pi, \pi) \to \mathbb{R}^2$ be defined by $\gamma(t) = (\sin 2t, \sin t)$.

(a) Show that γ is an immersion.

(b) Show that γ is injective.

(c) Prove that γ is *not* an embedding. *Hint:* examine what happens as $t \to \pm\pi$.

Exercise 2.12. (a) Show that any open ball $B_r(0) \subseteq \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. *Hint:* use the map $x \mapsto x/(r - \|x\|)$ or a similar radial rescaling.

(b) Conclude that every connected smooth n-manifold has an open subset diffeomorphic to $\mathbb{R}^n$.

Exercise 2.13 (Ehresmann's lemma, special case)**.** Let $f\colon M \to \mathbb{R}$ be a smooth proper submersion (proper means preimages of compact sets are compact). Show that for any two regular values $a < b$ of f, the preimages $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic. *Hint:* this requires tools from Chapter 1 (partitions of unity) and the flow of a vector field; give a proof sketch assuming the existence of flows for compactly supported vector fields.

Exercise 2.14. Let M be a smooth manifold and $A \subseteq M$ a smooth submanifold. A *smooth retraction* onto A is a smooth map $r\colon M \to A$ with $r|_A = \mathrm{Id}_A$.

(a) Show that r is a submersion at every point of A.

(b) Conclude that if M is connected and $\dim A < \dim M$, then no smooth retraction $M \to A$ can exist when M is compact without boundary. *Hint:* what is $r^{-1}(q)$ for a regular value q?

Chapter 3

Tangent and Cotangent Bundles

Contents

The notion of tangent vector is fundamental in differential topology. On an open subset $U \subset \mathbb{R}^n$, tangent vectors are simply elements of $\mathbb{R}^n$ attached at a point. On a smooth manifold, one must define tangent vectors *intrinsically*, without reference to an ambient Euclidean space. We present two classical approaches—via derivations and via equivalence classes of curves—and prove their equivalence. We then assemble the tangent spaces into the *tangent bundle*, a smooth manifold in its own right, and dualise to obtain the cotangent bundle. Vector fields, the Lie bracket, and flows round out the chapter.

3.1 The tangent space

Throughout this chapter, M denotes a smooth manifold of dimension n and $p \in M$.

3.1.1 Derivations

Definition 3.1.1 (Tangent vector as derivation). Let $C_p^\infty(M)$ denote the $\mathbb{R}$-algebra of germs of smooth functions at p. A **derivation at** p is an $\mathbb{R}$-linear map $v\colon C_p^\infty(M) \to \mathbb{R}$ satisfying the *Leibniz rule*:

$$v(fg) \;=\; f(p)\,v(g) \;+\; g(p)\,v(f) \qquad \forall\, f, g \in C_p^\infty(M).$$

The set of all derivations at p is denoted T_pM and called the **tangent space of** M **at** p.

Lemma 3.1.2. *If $v \in T_pM$ and $c \in \mathbb{R}$ is a constant function, then $v(c) = 0$.*

Proof. Write $c = c \cdot 1$. Then $v(c) = v(c \cdot 1) = c\, v(1) + 1 \cdot v(c) = c\, v(1) + v(c)$, so $c\, v(1) = 0$. Applying this with $c = 1$ gives $v(1) = 0$, hence $v(c) = c\, v(1) = 0$ for all c. □

Proposition 3.1.3. *Let $(U, \varphi = (x^1, \dots, x^n))$ be a chart around p. Define, for $i = 1, \dots, n$,*

$$\frac{\partial}{\partial x^i}\bigg|_p : C_p^\infty(M) \to \mathbb{R}, \qquad f \longmapsto \frac{\partial(f \circ \varphi^{-1})}{\partial r^i}\bigg|_{\varphi(p)},$$

where $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$. Then $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis of T_pM. In particular, $\dim T_pM = n$.

Proof. Step 1: Spanning. By a standard lemma (Hadamard's lemma), every $f \in C_p^\infty(M)$ can be written in local coordinates as

$$f \circ \varphi^{-1}(r) = f(p) + \sum_{i=1}^n (r^i - r_0^i)\, g_i(r),$$

where $r_0 = \varphi(p)$ and g_i are smooth with $g_i(r_0) = \frac{\partial(f \circ \varphi^{-1})}{\partial r^i}(r_0)$. For any derivation v, Lemma 3.1.2 gives $v(f(p)) = 0$, so

$$v(f) = \sum_{i=1}^n v(x^i - x^i(p))\, g_i(\varphi(p)) = \sum_{i=1}^n v(x^i)\, \frac{\partial}{\partial x^i}\bigg|_p (f).$$

Hence $v = \sum_i v(x^i)\, \frac{\partial}{\partial x^i}\Big|_p$.

Step 2: Linear independence. If $\sum_i a^i \frac{\partial}{\partial x^i}\Big|_p = 0$, apply this to the coordinate function x^j to get $a^j = 0$. □

3.1.2 Curves approach

Definition 3.1.4 (Tangent vector as equivalence class of curves)**.** Two smooth curves $\gamma_1, \gamma_2 \colon (-\varepsilon, \varepsilon) \to M$ with $\gamma_1(0) = \gamma_2(0) = p$ are **equivalent** if for some (equivalently, every) chart (U, φ) around p,

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Denote the equivalence class of γ by $[\gamma]$ and the set of all such classes by $\widetilde{T}_pM$.

Remark 3.1.5. That the definition is chart-independent follows from the chain rule in $\mathbb{R}^n$: if (V, ψ) is another chart, then $(\psi \circ \gamma)'(0) = D(\psi \circ \varphi^{-1})_{\varphi(p)} \cdot (\varphi \circ \gamma)'(0)$, and the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

The set $\widetilde{T}_pM$ becomes a real vector space by defining, in a chart (U, φ),

$$[\gamma_1] + [\gamma_2] = \Big[\varphi^{-1}\Big(\varphi(p) + (\varphi \circ \gamma_1)'(0)\, t + (\varphi \circ \gamma_2)'(0)\, t\Big)\Big], \qquad \lambda\, [\gamma] = \Big[\varphi^{-1}\Big(\varphi(p) + \lambda\, (\varphi \circ \gamma)'(0)\, t\Big)\Big].$$

3.1.3 Equivalence of the two definitions

Theorem 3.1.6. *The map*

$$\Phi\colon \widetilde{T}_pM \longrightarrow T_pM, \qquad [\gamma] \longmapsto \Big(f \mapsto (f\circ\gamma)'(0)\Big)$$

is a well-defined linear isomorphism.

Proof. Well-definedness. If $[\gamma_1]=[\gamma_2]$, then for any smooth germ f at p,

$$(f\circ\gamma_1)'(0) = D(f\circ\varphi^{-1})_{\varphi(p)}\,(\varphi\circ\gamma_1)'(0) = D(f\circ\varphi^{-1})_{\varphi(p)}\,(\varphi\circ\gamma_2)'(0) = (f\circ\gamma_2)'(0),$$

so $\Phi([\gamma_1])=\Phi([\gamma_2])$. A direct computation shows that $\Phi([\gamma])$ satisfies the Leibniz rule, hence lies in T_pM.

Linearity. We verify $\Phi([\gamma_1]+[\gamma_2]) = \Phi([\gamma_1])+\Phi([\gamma_2])$ using the chain rule in coordinates; the argument for scalar multiplication is analogous.

Surjectivity. Given $v=\sum_i a^i \frac{\partial}{\partial x^i}\Big|_p$, let $\gamma(t)=\varphi^{-1}(\varphi(p)+t\,a)$, where $a=(a^1,\dots,a^n)$. Then $\Phi([\gamma])=v$.

Injectivity. If $\Phi([\gamma])=0$, then $(f\circ\gamma)'(0)=0$ for all f. Applying this to the coordinate functions x^i gives $(\varphi\circ\gamma)'(0)=0$, so $[\gamma]=0$. □

Henceforth, we identify $\widetilde{T}_pM$ with T_pM via Φ without further comment.

Example 3.1.7. For $M=\mathbb{R}^n$ with the identity chart, $T_p\mathbb{R}^n \cong \mathbb{R}^n$ via $\sum_i a^i \frac{\partial}{\partial r^i}\Big|_p \mapsto (a^1,\dots,a^n)$.

Example 3.1.8. Viewing $S^{n-1}\subset\mathbb{R}^n$ as a level set (see Chapter 4), one obtains $T_pS^{n-1}\cong\{v\in\mathbb{R}^n : \langle v,p\rangle = 0\}$, the orthogonal complement of p.

3.2 The differential

Definition 3.2.1 (Differential of a smooth map)**.** Let $f\colon M\to N$ be a smooth map between manifolds and $p\in M$. The **differential**[a] of f at p is the linear map

$$\mathrm{d}f_p\colon T_pM \longrightarrow T_{f(p)}N, \qquad (\mathrm{d}f_p\,v)(g) \;=\; v(g\circ f),$$

for $v\in T_pM$ and $g\in C^\infty_{f(p)}(N)$.

[a] Also called the *pushforward*, *tangent map*, or *derivative* of f at p.

In the curves picture, $\mathrm{d}f_p([\gamma]) = [f\circ\gamma]$.

Proposition 3.2.2 (Coordinate expression)**.** *Let* $(U,\varphi=(x^i))$ *and* $(V,\psi=(y^j))$ *be charts around* p *and* $f(p)$ *respectively. Write* $\hat f = \psi\circ f\circ\varphi^{-1}$*. Then*

$$\mathrm{d}f_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) \;=\; \sum_{j=1}^m \frac{\partial \hat f^j}{\partial r^i}(\varphi(p))\;\frac{\partial}{\partial y^j}\bigg|_{f(p)}.$$

Thus the matrix of $\mathrm{d}f_p$ *in these coordinate bases is the* ***Jacobian matrix*** $\left[\frac{\partial \hat{f}^j}{\partial r^i}(\varphi(p))\right]_{j,i}$.

Proof. For any smooth germ g at $f(p)$,

$$\mathrm{d}f_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right)(g) = \frac{\partial}{\partial x^i}\bigg|_p (g\circ f) = \frac{\partial(g\circ f\circ\varphi^{-1})}{\partial r^i}(\varphi(p)) = \sum_j \frac{\partial(g\circ\psi^{-1})}{\partial r^j}(\psi(f(p)))\,\frac{\partial \hat{f}^j}{\partial r^i}(\varphi(p)),$$

by the chain rule in $\mathbb{R}^n$. The right-hand side equals $\sum_j \frac{\partial \hat{f}^j}{\partial r^i}(\varphi(p))\,\frac{\partial}{\partial y^j}\Big|_{f(p)}(g)$. □

3.2.1 Chain rule and functoriality

Theorem 3.2.3 (Chain rule)**.** *Let* $f\colon M\to N$ *and* $g\colon N\to P$ *be smooth maps. Then for every* $p\in M$*,*

$$\mathrm{d}(g\circ f)_p \;=\; \mathrm{d}g_{f(p)}\circ \mathrm{d}f_p.$$

Proof. For $v\in T_pM$ and $h\in C^\infty_{g(f(p))}(P)$,

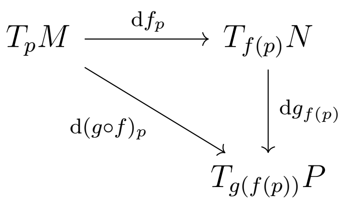
$$\mathrm{d}(g\circ f)_p(v)(h) = v(h\circ g\circ f) = \mathrm{d}f_p(v)(h\circ g) = \mathrm{d}g_{f(p)}(\,\mathrm{d}f_p(v))(h).$$ □

Corollary 3.2.4. *If* $f\colon M\to N$ *is a diffeomorphism, then* $\mathrm{d}f_p$ *is a linear isomorphism for all* $p\in M$*.*

Proof. Since $f^{-1}\circ f = \mathrm{Id}_M$, the chain rule gives $\mathrm{d}(f^{-1})_{f(p)}\circ \mathrm{d}f_p = \mathrm{d}(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$. Similarly, $\mathrm{d}f_p\circ \mathrm{d}(f^{-1})_{f(p)} = \mathrm{Id}_{T_{f(p)}N}$. □

Remark 3.2.5 (Functoriality). The assignments $M\mapsto TM$ and $f\mapsto \mathrm{d}f$ define a (covariant) functor T from the category **Man** of smooth manifolds and smooth maps to the category $\mathbf{Vect}_{\mathbb{R}}$ of real vector spaces and linear maps. The chain rule is precisely the statement that T preserves composition, and $\mathrm{d}(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$ says that T preserves identities.

The following commutative diagram summarises the chain rule:



3.3 The tangent bundle

Definition 3.3.1 (Tangent bundle)**.** The **tangent bundle** of M is the disjoint union

$$TM \;=\; \bigsqcup_{p\in M} T_pM \;=\; \{(p,v) : p\in M,\ v\in T_pM\}\,,$$

together with the natural projection $\pi\colon TM\to M$, $(p,v)\mapsto p$.

Theorem 3.3.2. *If M is a smooth n-manifold, then TM carries a natural smooth structure making it a smooth $2n$-manifold, with respect to which $\pi\colon TM \to M$ is a smooth surjective submersion.*

Proof. Step 1: Topology and charts. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas for M. For each α, define

$$\widetilde{\varphi}_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow \varphi_\alpha(U_\alpha) \times \mathbb{R}^n \;\subset\; \mathbb{R}^{2n},$$

by

$$\widetilde{\varphi}_\alpha(p, v) \;=\; \Big(\varphi_\alpha(p),\; v(x^1_\alpha),\, \ldots,\, v(x^n_\alpha)\Big),$$

where $x^i_\alpha = r^i \circ \varphi_\alpha$ are the coordinate functions. Concretely, if $v = \sum_i a^i \frac{\partial}{\partial x^i_\alpha}\Big|_p$, then $\widetilde{\varphi}_\alpha(p, v) = (\varphi_\alpha(p), a^1, \ldots, a^n)$. Since $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$, $\varphi_\alpha(U_\alpha) \times \mathbb{R}^n$ is open in $\mathbb{R}^{2n}$, and $\widetilde{\varphi}_\alpha$ is a bijection onto this open set. We give TM the topology generated by declaring a set $W \subset TM$ open if and only if $\widetilde{\varphi}_\alpha(W \cap \pi^{-1}(U_\alpha))$ is open in $\mathbb{R}^{2n}$ for every α.

Step 2: Transition maps. On $\pi^{-1}(U_\alpha \cap U_\beta)$, the transition map is

$$\begin{aligned} \widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n &\longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n, \\ (x, a) &\longmapsto \Big(\tau_{\beta\alpha}(x),\; D\tau_{\beta\alpha}(x)\, a\Big), \end{aligned}$$

where $\tau_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is the transition map of M. Since $\tau_{\beta\alpha}$ is a smooth diffeomorphism, the map $(x, a) \mapsto (\tau_{\beta\alpha}(x), D\tau_{\beta\alpha}(x)\, a)$ is smooth (in fact, the second component is linear in a and smooth in x).

Step 3: Hausdorff and second countability. Because M is Hausdorff and second-countable, and the fibres $\mathbb{R}^n$ have these properties, the topology on TM is Hausdorff and second-countable. (Alternatively, TM is covered by countably many of the sets $\pi^{-1}(U_\alpha)$, each homeomorphic to an open subset of $\mathbb{R}^{2n}$.)

Step 4: The projection π. In the chart $\widetilde{\varphi}_\alpha$, the map π is represented by the projection $(x, a) \mapsto x$, which is a smooth submersion. □

Example 3.3.3. $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ as a smooth manifold.

Example 3.3.4. $TS^1 \cong S^1 \times \mathbb{R}$, since S^1 is a Lie group and every Lie group has trivial tangent bundle. More generally, $T(S^1 \times \cdots \times S^1) \cong T^n \times \mathbb{R}^n$.

Example 3.3.5. The tangent bundle TS^2 is *non-trivial*: $TS^2 \not\cong S^2 \times \mathbb{R}^2$. This is a consequence of the **hairy ball theorem**, which states that there is no non-vanishing continuous vector field on S^2. If TS^2 were trivial, then the constant section $p \mapsto (p, e_1)$ (for a fixed $e_1 \in \mathbb{R}^2$) would give a non-vanishing vector field, a contradiction. More generally, TS^n is trivial if and only if $n \in \{1, 3, 7\}$ (Adams's theorem, 1962).

Remark 3.3.6 (The tangent functor on morphisms). If $f\colon M \to N$ is smooth, the **global differential** $\mathrm{d}f\colon TM \to TN$ defined by $\mathrm{d}f(p, v) = (f(p), \mathrm{d}f_p\, v)$ is a smooth map. In fact, if $\widetilde{\varphi}_\alpha$ and $\widetilde{\psi}_\beta$ are the induced charts on TM and TN, then

$$\widetilde{\psi}_\beta \circ \mathrm{d}f \circ \widetilde{\varphi}_\alpha^{-1}(x, a) = \Big(\hat{f}(x),\; D\hat{f}(x)\, a\Big),$$

which is smooth in (x, a). The diagram

$$\begin{array}{ccc} TM & \xrightarrow{\mathrm{d}f} & TN \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

commutes, expressing $\mathrm{d}f$ as a **bundle map** over f.

The following diagram illustrates the local trivialisation of TM:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\widetilde{\varphi}_\alpha} & \varphi_\alpha(U_\alpha) \times \mathbb{R}^n \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U_\alpha & \xrightarrow[\varphi_\alpha]{} & \varphi_\alpha(U_\alpha) \end{array}$$

3.4 The cotangent bundle

Definition 3.4.1 (Cotangent space and bundle)**.** The **cotangent space** at p is the dual vector space

$$T_p^*M \;=\; (T_pM)^* \;=\; \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R}).$$

Elements of T_p^*M are called **covectors** at p. The **cotangent bundle** is

$$T^*M \;=\; \bigsqcup_{p \in M} T_p^*M, \qquad \pi^* \colon T^*M \to M, \quad (p, \xi) \mapsto p.$$

It is a smooth $2n$-manifold by the same construction as for TM, with charts

$$\widetilde{\varphi}_\alpha^*(p, \xi) = \big(\varphi_\alpha(p),\ \xi_1, \dots, \xi_n\big), \qquad \text{where } \xi = \sum_i \xi_i \, \mathrm{d}x_\alpha^i\big|_p.$$

Definition 3.4.2 (Differential of a function)**.** For $f \in C^\infty(M)$, the **differential of f** is the smooth section $\mathrm{d}f \colon M \to T^*M$ defined by

$$\mathrm{d}f_p(v) = v(f), \qquad v \in T_pM.$$

In local coordinates $(x^1, \dots, x^n)$,

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i.$$

Proposition 3.4.3. *In a chart $(U, \varphi = (x^1, \dots, x^n))$, the differentials $\mathrm{d}x^1|_p, \dots, \mathrm{d}x^n|_p$*

form the basis of T_p^*M dual to $\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p$:

$$\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j.$$

Proof. By definition, $\mathrm{d}x^i|_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \frac{\partial(x^i\circ\varphi^{-1})}{\partial r^j}(\varphi(p)) = \frac{\partial r^i}{\partial r^j} = \delta^i_j$. □

Definition 3.4.4 (Pullback of a covector)**.** If $f\colon M \to N$ is smooth, the **pullback** is

$$f^*\colon T^*_{f(p)}N \to T^*_pM, \qquad (f^*\xi)(v) = \xi(\,\mathrm{d}f_p\, v).$$

Note that f^* goes in the opposite direction to $\mathrm{d}f$:

$$T_pM \xrightarrow{\ \mathrm{d}f_p\ } T_{f(p)}N$$

$$T_p^*M \xleftarrow{\ f^*\ } T^*_{f(p)}N$$

3.5 Vector fields

Definition 3.5.1 (Vector field)**.** A **(smooth) vector field** on M is a smooth section of the tangent bundle, i.e., a smooth map $X\colon M \to TM$ such that $\pi \circ X = \mathrm{Id}_M$. Equivalently, X assigns to each $p \in M$ a tangent vector $X_p \in T_pM$, smoothly in p. The set of all smooth vector fields on M is denoted $\mathfrak{X}(M)$.

In local coordinates $(x^1, \ldots, x^n)$, a vector field is written

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i},$$

where $X^i\colon U \to \mathbb{R}$ are smooth functions (the **component functions** of X).

Remark 3.5.2. The set $\mathfrak{X}(M)$ is a module over $C^\infty(M)$: for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, define $(fX)_p = f(p)\,X_p$. It is also a real vector space (taking f to be constant), but it is *not* finite-dimensional if $\dim M \geq 1$.

Remark 3.5.3. A vector field X acts as a **derivation of** $C^\infty(M)$, i.e., an $\mathbb{R}$-linear map $X\colon C^\infty(M) \to C^\infty(M)$ satisfying $X(fg) = fX(g) + gX(f)$. Conversely, every such derivation arises from a unique smooth vector field.

3.6 The Lie bracket

Definition 3.6.1 (Lie bracket). For $X, Y \in \mathfrak{X}(M)$, the **Lie bracket** $[X, Y]$ is the vector field defined by

$$[X, Y](f) \;=\; X(Y(f)) - Y(X(f))$$

for all $f \in C^\infty(M)$.

Proposition 3.6.2 (Coordinate formula). *In local coordinates $(x^1, \ldots, x^n)$, if $X = \sum_i X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_j Y^j \frac{\partial}{\partial x^j}$, then*

$$[X, Y] = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} X^i \frac{\partial Y^k}{\partial x^i} - Y^i \frac{\partial X^k}{\partial x^i} \right) \frac{\partial}{\partial x^k}.$$

Proof. Apply $[X, Y]$ to a coordinate function x^k:

$$\begin{aligned} [X, Y](x^k) &= X(Y(x^k)) - Y(X(x^k)) = X(Y^k) - Y(X^k) \\ &= \sum_i X^i \frac{\partial Y^k}{\partial x^i} - \sum_i Y^i \frac{\partial X^k}{\partial x^i}. \end{aligned}$$

Since $[X, Y] = \sum_k [X, Y](x^k) \frac{\partial}{\partial x^k}$ locally, the formula follows. □

Theorem 3.6.3 (Properties of the Lie bracket). *The Lie bracket satisfies, for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$:*

(i) $\mathbb{R}$***-bilinearity:*** $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ *for* $\alpha, \beta \in \mathbb{R}$.

(ii) ***Antisymmetry:*** $[X, Y] = -[Y, X]$.

(iii) ***Jacobi identity:*** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

(iv) $C^\infty(M)$***-Leibniz rule:*** $[fX, gY] = fg\,[X, Y] + f(Xg)\,Y - g(Yf)\,X$.

In particular, $(\mathfrak{X}(M), [\cdot, \cdot])$ *is a* ***Lie algebra*** *over* $\mathbb{R}$ *(but not over* $C^\infty(M)$*, by (iv)).*

Proof. Properties (i) and (ii) are immediate from the definition.

(iii) Jacobi identity. For $f \in C^\infty(M)$,

$$\begin{aligned} [X, [Y, Z]](f) &= X(Y(Z(f)) - Z(Y(f))) - (Y(Z(X(f))) - Z(Y(X(f)))) \\ &= X(Y(Z(f))) - X(Z(Y(f))) - Y(Z(X(f))) + Z(Y(X(f))). \end{aligned}$$

Summing over the three cyclic permutations of (X, Y, Z), all twelve terms cancel in pairs.

(iv) Leibniz rule. For $h \in C^\infty(M)$,

$$\begin{aligned} [fX, gY](h) &= fX(gY(h)) - gY(fX(h)) \\ &= fX(g)\,Y(h) + fg\,X(Y(h)) - gY(f)\,X(h) - gf\,Y(X(h)) \\ &= fg\,[X, Y](h) + f(Xg)\,Y(h) - g(Yf)\,X(h). \end{aligned}$$

□

Example 3.6.4. On $\mathbb{R}^2$ with coordinates (x, y), let $X = \frac{\partial}{\partial x}$ and $Y = x\,\frac{\partial}{\partial y}$. Then

$$[X,Y] = \frac{\partial}{\partial x}(x)\,\frac{\partial}{\partial y} - x\,\frac{\partial}{\partial x}(1)\,\frac{\partial}{\partial x} = \frac{\partial}{\partial y}.$$

Proposition 3.6.5 (f-relatedness)**.** *Let $f\colon M \to N$ be smooth. Vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are **f-related** (written $X \sim_f Y$) if $\mathrm{d}f_p(X_p) = Y_{f(p)}$ for all $p \in M$. If $X_1 \sim_f Y_1$ and $X_2 \sim_f Y_2$, then $[X_1, X_2] \sim_f [Y_1, Y_2]$.*

Proof. For $g \in C^\infty(N)$ and $p \in M$,

$$\begin{aligned}\mathrm{d}f_p([X_1,X_2]_p)(g) &= [X_1,X_2]_p(g\circ f) = X_{1,p}(X_2(g\circ f)) - X_{2,p}(X_1(g\circ f))\\ &= X_{1,p}((Y_2 g)\circ f) - X_{2,p}((Y_1 g)\circ f)\\ &= Y_{1,f(p)}(Y_2 g) - Y_{2,f(p)}(Y_1 g) = [Y_1,Y_2]_{f(p)}(g).\end{aligned}$$

□

3.7 Flows of vector fields

Definition 3.7.1 (Integral curve)**.** Let $X \in \mathfrak{X}(M)$. An **integral curve** of X through $p \in M$ is a smooth curve $\gamma\colon J \to M$, defined on an open interval $J \ni 0$, such that $\gamma(0) = p$ and $\gamma'(t) = X_{\gamma(t)}$ for all $t \in J$.

The existence and uniqueness of integral curves is a direct consequence of the classical Cauchy–Lipschitz theorem (Picard–Lindelöf theorem) applied in local coordinates.

Theorem 3.7.2 (Existence and uniqueness of integral curves)**.** *Let $X \in \mathfrak{X}(M)$ and $p \in M$. There exists a unique maximal integral curve $\gamma_p\colon J_p \to M$ of X with $\gamma_p(0) = p$, where $J_p \subset \mathbb{R}$ is the largest open interval on which the solution exists.*

Proof sketch. In a chart (U, φ), the equation $\gamma'(t) = X_{\gamma(t)}$ becomes the ODE system $\dot{y}(t) = F(y(t))$ in $\mathbb{R}^n$, where $F = (\varphi_* X) \circ \varphi^{-1}$ is smooth and hence locally Lipschitz. The Cauchy–Lipschitz theorem guarantees a unique local solution. By covering M with charts and using uniqueness on overlaps, one extends to a unique maximal solution. □

Definition 3.7.3 (Flow)**.** The **flow** of $X \in \mathfrak{X}(M)$ is the map

$$\theta\colon \mathcal{D} \to M, \qquad (t,p) \mapsto \theta_t(p) = \gamma_p(t),$$

where $\mathcal{D} = \{(t,p) \in \mathbb{R}\times M : t \in J_p\}$ is the **flow domain**.

Theorem 3.7.4 (Properties of the flow)**.** *Let θ be the flow of $X \in \mathfrak{X}(M)$.*

(i) *$\mathcal{D}$ is an open subset of $\mathbb{R}\times M$ containing $\{0\}\times M$.*

(ii) *$\theta\colon \mathcal{D} \to M$ is smooth.*

(iii) $\theta_0 = \mathrm{Id}_M$.

(iv) ***Group law:*** *$\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$ whenever both sides are defined.*

Definition 3.7.5 (Complete vector field)**.** A vector field X is **complete** if $J_p = \mathbb{R}$ for all $p \in M$, i.e., $\mathcal{D} = \mathbb{R} \times M$. In this case, $\theta_t : M \to M$ is a diffeomorphism for each t, and $t \mapsto \theta_t$ is a smooth group homomorphism from $(\mathbb{R}, +)$ to $\mathrm{Diff}(M)$, called the **one-parameter group of diffeomorphisms** generated by X.

Proposition 3.7.6. *Every smooth vector field on a compact manifold is complete.*

Proof. If M is compact, the maximal interval J_p must be all of $\mathbb{R}$ for every p: if $J_p = (a, b)$ with $b < \infty$, then $\gamma_p(t)$ leaves every compact set as $t \to b^-$ (by a standard escape lemma), contradicting the compactness of M. □

Example 3.7.7. On $S^1 \subset \mathbb{R}^2$, the vector field $X_{(x,y)} = (-y, x)$ (the unit tangent to the circle) has flow $\theta_t(x, y) = (x \cos t - y \sin t,\ x \sin t + y \cos t)$, which is rotation by angle t. This vector field is complete, and the one-parameter group $t \mapsto \theta_t$ is the group of rotations $\mathrm{SO}(2)$.

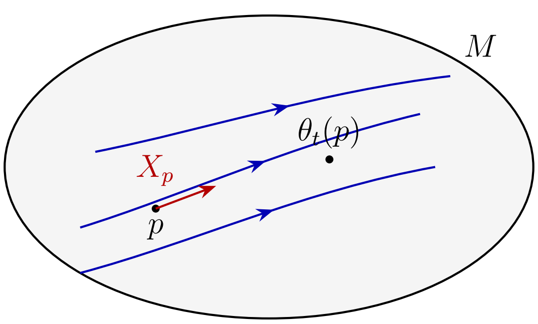
Example 3.7.8. On $M = \mathbb{R}$, the vector field $X = x^2 \frac{\mathrm{d}}{\mathrm{d}x}$ has integral curves $\gamma_p(t) = \frac{p}{1-pt}$ (for $p \neq 0$), defined on $(-\infty, 1/p)$ if $p > 0$ and $(1/p, +\infty)$ if $p < 0$. The flow is not complete: solutions blow up in finite time. This cannot happen on a compact manifold, confirming Proposition 3.7.6.

Remark 3.7.9 (Lie derivative of a function)*.* If $X \in \mathfrak{X}(M)$ with flow θ and $f \in C^\infty(M)$, the **Lie derivative** of f along X is

$$(\mathcal{L}_X f)(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\theta_t(p)) = X_p(f) = (Xf)(p).$$

Thus $\mathcal{L}_X f = Xf$: the Lie derivative of a function is simply the action of X as a derivation.

The flow is depicted schematically below.



3.8 Exercises

Exercise 3.1. Let M and N be smooth manifolds and $(p, q) \in M \times N$. Prove that

$$T_{(p,q)}(M \times N) \;\cong\; T_pM \oplus T_qN,$$

where the isomorphism is given by $v \mapsto (\,\mathrm{d}\mathrm{pr}_{1,(p,q)}\, v,\;\; \mathrm{d}\mathrm{pr}_{2,(p,q)}\, v)$.

Exercise 3.2. On $\mathbb{R}^3$ with coordinates (x, y, z), compute the Lie brackets $[X, Y]$, $[Y, Z]$, and $[X, Z]$, where

$$X = y\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial y}, \qquad Y = z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}, \qquad Z = x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}.$$

These are the infinitesimal generators of rotations. Verify the Jacobi identity.

Exercise 3.3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(u, v) = (u\cos v,\; u\sin v,\; v)$. Compute $\mathrm{d}f_{(u,v)}$ in the standard bases and find its rank.

Exercise 3.4. Prove that the tangent bundle TS^1 is trivial by exhibiting an explicit global trivialisation $TS^1 \xrightarrow{\sim} S^1 \times \mathbb{R}$.

Exercise 3.5. Verify directly from the definition that $[X, Y]$ satisfies the Leibniz rule, i.e., that $[X, Y]$ is a derivation of $C^\infty(M)$, not just an endomorphism.

Exercise 3.6. Let $A \in M_n(\mathbb{R})$ and define $X_A \in \mathfrak{X}(\mathbb{R}^n)$ by $(X_A)_x = Ax$ (identifying $T_x\mathbb{R}^n \cong \mathbb{R}^n$). Show that the flow of X_A is $\theta_t(x) = e^{tA}x$.

Exercise 3.7. Let $X, Y \in \mathfrak{X}(M)$ and let θ be the flow of X. Show that the Lie bracket can be expressed as

$$[X, Y]_p = \lim_{t\to 0} \frac{(\,\mathrm{d}\theta_{-t})_{\theta_t(p)}\, Y_{\theta_t(p)} - Y_p}{t} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\theta_t^* Y)_p.$$

This is the **Lie derivative** $\mathcal{L}_X Y = [X, Y]$.

Exercise 3.8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x^2 - y^2, 2xy)$. Compute $f^*(\,\mathrm{d}u)$ and $f^*(\,\mathrm{d}v)$, where (u, v) are coordinates on the codomain.

Exercise 3.9. Let $\pi\colon S^n \to \mathbb{RP}^n$ be the canonical double covering. Show that for each $p \in S^n$, the differential $\mathrm{d}\pi_p\colon T_pS^n \to T_{\pi(p)}\mathbb{RP}^n$ is an isomorphism. Use this to prove $\dim T_{[x]}\mathbb{RP}^n = n$.

Chapter 4

Submanifolds and the Regular Value Theorem

Contents

With the tangent space machinery in place, we can now study submanifolds and the key analytic result that produces them: the *regular value theorem*. This single theorem, combined with its generalisations, is responsible for establishing the smooth manifold structure of an enormous number of classical examples—spheres, orthogonal groups, special linear groups, and many more.

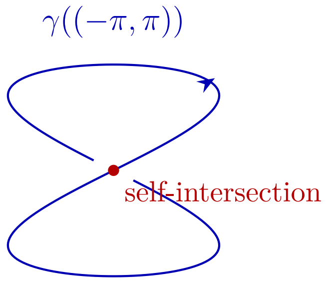
4.1 Immersions, embeddings, and submanifolds

Definition 4.1.1 (Immersion and submersion)**.** A smooth map $f\colon M \to N$ is:

- an **immersion** if $\mathrm{d}f_p$ is injective for every $p \in M$;
- a **submersion** if $\mathrm{d}f_p$ is surjective for every $p \in M$.

Definition 4.1.2 (Immersed submanifold)**.** An **immersed submanifold** of N is a subset $S \subset N$ together with a topology and smooth structure on S making the inclusion $\iota\colon S \hookrightarrow N$ a smooth immersion. The **codimension** is $\dim N - \dim S$.

Example 4.1.3. The figure-eight curve $\gamma\colon (-\pi, \pi) \to \mathbb{R}^2$, $\gamma(t) = (\sin 2t, \sin t)$, is an injective immersion whose image is not an embedded submanifold (the self-intersection at the origin has no manifold neighbourhood).



Definition 4.1.4 (Embedding)**.** A smooth map $f\colon M \to N$ is a **(smooth) embedding** if it is an injective immersion that is also a homeomorphism onto its image $f(M) \subset N$ (with the subspace topology).

Definition 4.1.5 (Embedded submanifold)**.** An **embedded submanifold** (or **regular submanifold**) of N is a subset $S \subset N$ such that for every $p \in S$ there exists a chart (U, φ) of N with $p \in U$ and

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^k \times \{0\}),$$

where $k = \dim S$ and we identify $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Such a chart is called a **slice chart** or **adapted chart**.

Proposition 4.1.6. *A subset $S \subset N$ is an embedded submanifold of dimension k if and only if the inclusion $\iota\colon S \hookrightarrow N$ is a smooth embedding, when S carries the subspace topology and the induced smooth structure.*

Remark 4.1.7. Every embedded submanifold is an immersed submanifold, but not conversely. The distinction lies in the topology: the topology of an immersed submanifold need not agree with the subspace topology inherited from the ambient manifold. A classical example is a line of irrational slope in the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$: it is an injective immersion of $\mathbb{R}$, dense in T^2, hence not embedded.

Proposition 4.1.8. *If M is compact and $f\colon M \to N$ is an injective immersion, then f is an embedding.*

Proof. An injective continuous map from a compact space to a Hausdorff space is a homeomorphism onto its image. Since f is also an immersion, it is a smooth embedding. □

4.2 Regular values and the preimage theorem

Definition 4.2.1 (Regular and critical values)**.** Let $f\colon M \to N$ be a smooth map. A point $p \in M$ is a **regular point** if $\mathrm{d}f_p$ is surjective; otherwise it is a **critical point**. A point $q \in N$ is a **regular value** if every $p \in f^{-1}(q)$ is a regular point (in particular, q is a regular value if $f^{-1}(q) = \varnothing$). Otherwise, q is a **critical value**.

The main tool is the following local result, which is a smooth version of the implicit function theorem.

Theorem 4.2.2 (Submersion theorem / local normal form)**.** *Let $f\colon M^m \to N^n$ be smooth with $\mathrm{d}f_p$ surjective ($m \geq n$). Then there exist charts (U, φ) around p and (V, ψ) around $f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n).$$

That is, in suitable coordinates, f is just the projection onto the first n factors.

Proof. By replacing M and N with coordinate neighbourhoods, we may assume $M \subset \mathbb{R}^m$ is open, $N \subset \mathbb{R}^n$ is open, and $p = 0$, $f(0) = 0$. Since $\mathrm{d}f_0\colon \mathbb{R}^m \to \mathbb{R}^n$ is surjective, after reordering coordinates we may assume the last n columns of the Jacobian $Df(0)$ form an invertible $n \times n$ matrix. Write $x = (x', x'') \in \mathbb{R}^{m-n} \times \mathbb{R}^n$ and define $\Phi\colon M \to \mathbb{R}^m$ by $\Phi(x', x'') = (x', f(x', x''))$. Then

$$D\Phi(0) = \begin{pmatrix} I_{m-n} & 0 \\ * & Df_0\big|_{\mathbb{R}^n} \end{pmatrix},$$

which is invertible. By the inverse function theorem, Φ is a local diffeomorphism near 0. Setting $\varphi = \Phi$ and $\psi = \mathrm{Id}$, one verifies

$$f \circ \varphi^{-1}(y', y'') = f \circ \Phi^{-1}(y', y'') = y'',$$

since $\Phi^{-1}(y', y'')$ maps to a point whose f-image is y''. Reindexing to place y'' as the first n coordinates gives the stated normal form. ☐

Theorem 4.2.3 (Regular value theorem / preimage theorem)**.** *Let $f\colon M^m \to N^n$ be a smooth map and $q \in N$ a regular value of f. Then $f^{-1}(q)$ is either empty or an embedded submanifold of M of dimension $m - n$.*

Proof. Assume $S = f^{-1}(q) \neq \varnothing$ and let $p \in S$. Since q is a regular value, $\mathrm{d}f_p$ is surjective. By the submersion theorem (Theorem 4.2.2), there exist charts (U, φ) around p with $\varphi(p) = 0$ and (V, ψ) around q with $\psi(q) = 0$ such that

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n).$$

Therefore

$$\begin{aligned} \varphi(U \cap S) = \varphi\Big(U \cap f^{-1}(q)\Big) &= \Big\{x \in \varphi(U) : \psi \circ f \circ \varphi^{-1}(x) = 0\Big\} \\ &= \Big\{x \in \varphi(U) : x^1 = \cdots = x^n = 0\Big\} = \varphi(U) \cap (\{0\} \times \mathbb{R}^{m-n}). \end{aligned}$$

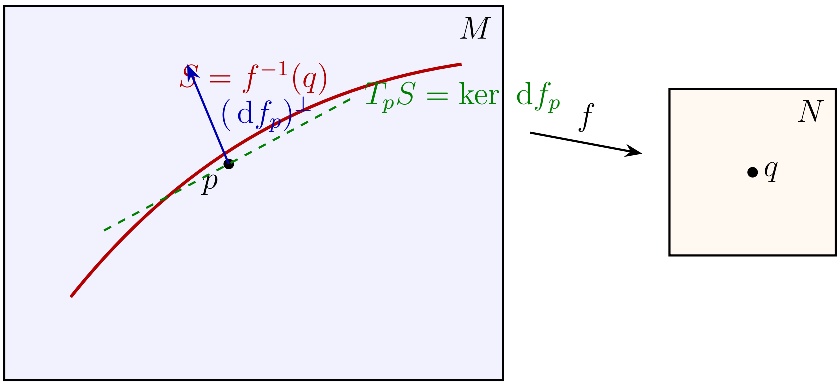
After reindexing, this shows (U, φ) is a slice chart for S of codimension n. Since $p \in S$ was arbitrary, S is an embedded submanifold of dimension $m - n$. ☐

Corollary 4.2.4 (Tangent space of a level set)**.** *Under the hypotheses of Theorem 4.2.3, if $S = f^{-1}(q)$ and $p \in S$, then*

$$T_pS \;=\; \ker \mathrm{d}f_p \;=\; \{v \in T_pM : \mathrm{d}f_p(v) = 0\}\,.$$

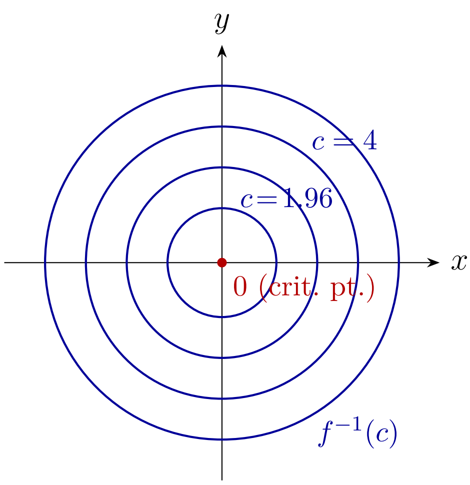
Proof. Since $f|_S$ is the constant map q, we have $\mathrm{d}(f \circ \iota)_p = \mathrm{d}f_p \circ \mathrm{d}\iota_p = 0$, where $\iota\colon S \hookrightarrow M$ is the inclusion. Hence $\mathrm{d}\iota_p(T_pS) \subset \ker \mathrm{d}f_p$. Since $\mathrm{d}\iota_p$ is injective (the inclusion is an embedding), we obtain $T_pS \subset \ker \mathrm{d}f_p$ (identifying T_pS with its image in T_pM). By dimension count, $\dim T_pS = m - n = m - \operatorname{rank}(\mathrm{d}f_p) = \dim \ker \mathrm{d}f_p$, giving equality. □

The situation is depicted in the following figure:



4.3 Applications of the regular value theorem

The following figure shows the level sets of $f(x, y) = x^2 + y^2$, illustrating that regular values ($c > 0$) give smooth manifolds (circles) while the critical value ($c = 0$) gives a single point.



We now apply Theorem 4.2.3 to establish the manifold structure of several classical spaces.

4.3.1 Spheres

Proposition 4.3.1. *The sphere* $S^{n-1} = \left\{x \in \mathbb{R}^n : |x|^2 = 1\right\}$ *is a smooth embedded submanifold of* $\mathbb{R}^n$ *of dimension* $n-1$.

Proof. Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = |x|^2 = \sum_{i=1}^n (x^i)^2$. Then $\mathrm{d}f_x = 2\sum_i x^i\, \mathrm{d}x^i$, so $\mathrm{d}f_x = 0$ only if $x = 0$. Since $0 \notin f^{-1}(1)$, every point of $S^{n-1} = f^{-1}(1)$ is a regular point. Hence 1 is a regular value, and $S^{n-1} = f^{-1}(1)$ is an embedded submanifold of dimension $n-1$. □

Corollary 4.3.2. *For* $p \in S^{n-1}$, $T_pS^{n-1} = \ker \mathrm{d}f_p = \{v \in \mathbb{R}^n : \sum_i p^i v^i = 0\} = p^\perp$.

4.3.2 The orthogonal group

Proposition 4.3.3. *The orthogonal group* $O(n) = \left\{A \in M_n(\mathbb{R}) : A^T A = I_n\right\}$ *is a smooth embedded submanifold of* $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ *of dimension* $\frac{n(n-1)}{2}$. *In particular,* $O(n)$ *is a Lie group.*

Proof. Let $\mathrm{Sym}_n(\mathbb{R})$ denote the space of $n\times n$ real symmetric matrices, which has dimension $\frac{n(n+1)}{2}$. Define $f\colon M_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ by $f(A) = A^T A$. Then $O(n) = f^{-1}(I_n)$.

We compute the differential. For $A \in M_n(\mathbb{R})$ and $B \in T_A M_n(\mathbb{R}) \cong M_n(\mathbb{R})$,

$$\mathrm{d}f_A(B) = \lim_{t\to 0} \frac{(A+tB)^T(A+tB) - A^T A}{t} = B^T A + A^T B.$$

We must show $\mathrm{d}f_A$ is surjective at every $A \in O(n)$. Let $S \in \mathrm{Sym}_n(\mathbb{R})$. Set $B = \frac{1}{2}AS$. Then

$$\mathrm{d}f_A(B) = \frac{1}{2}(AS)^T A + \frac{1}{2}A^T(AS) = \frac{1}{2}S^T \underbrace{A^T A}_{I_n} + \frac{1}{2}\underbrace{A^T A}_{I_n} S = \frac{1}{2}S + \frac{1}{2}S = S.$$

Hence $\mathrm{d}f_A$ is surjective for all $A \in O(n)$, so I_n is a regular value. The regular value theorem gives $O(n)$ as an embedded submanifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. □

Corollary 4.3.4. *The tangent space of* $O(n)$ *at* A *is*

$$T_A O(n) = \ker \mathrm{d}f_A = \left\{B \in M_n(\mathbb{R}) : B^T A + A^T B = 0\right\}.$$

At the identity, $T_I O(n) = \left\{B \in M_n(\mathbb{R}) : B^T + B = 0\right\} = \mathfrak{o}(n)$, *the space of skew-symmetric matrices.*

4.3.3 The special linear group

Proposition 4.3.5. *The special linear group* $\mathrm{SL}(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$ *is a smooth embedded submanifold of* $M_n(\mathbb{R})$ *of dimension* $n^2 - 1$.

Proof. Let $f = \det\colon M_n(\mathbb{R}) \to \mathbb{R}$. We use the well-known formula

$$\mathrm{d}(\det)_A(B) = \det(A)\,\mathrm{tr}(A^{-1}B)$$

for invertible A. If $A \in \mathrm{SL}(n,\mathbb{R})$, then $\det A = 1 \neq 0$, so $\mathrm{d}f_A(B) = \mathrm{tr}(A^{-1}B)$. This is surjective (onto $\mathbb{R}$): taking $B = A$ gives $\mathrm{d}f_A(A) = \mathrm{tr}(I_n) = n \neq 0$. Thus 1 is a regular value of det, and $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$ is an embedded submanifold of dimension $n^2 - 1$. □

Corollary 4.3.6. $T_I\,\mathrm{SL}(n,\mathbb{R}) = \ker \mathrm{d}(\det)_I = \{B \in M_n(\mathbb{R}) : \mathrm{tr}(B) = 0\} = \mathfrak{sl}(n,\mathbb{R})$.

4.3.4 Other classical groups

Example 4.3.7 ($\mathrm{SO}(n)$)**.** The special orthogonal group $\mathrm{SO}(n) = O(n) \cap \mathrm{SL}(n,\mathbb{R}) = \{A \in O(n) : \det A = 1\}$ is an open (hence embedded) submanifold of $O(n)$, since it is a connected component ($O(n)$ has exactly two connected components, distinguished by the sign of the determinant). In particular, $\dim \mathrm{SO}(n) = \frac{n(n-1)}{2}$, and $T_I\,\mathrm{SO}(n) = \mathfrak{o}(n)$.

Example 4.3.8 ($\mathrm{SU}(n)$)**.** A similar argument applies to the unitary group $U(n) = \{A \in M_n(\mathbb{C}) : A^*A = I_n\}$, using the map $f(A) = A^*A$ valued in Hermitian matrices, to show $U(n)$ is a smooth submanifold of $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$ of dimension n^2. The special unitary group $\mathrm{SU}(n) = \{A \in U(n) : \det A = 1\}$ is then a submanifold of dimension $n^2 - 1$.

4.4 The constant rank theorem

The submersion theorem generalises to maps of constant rank.

Theorem 4.4.1 (Constant rank theorem)**.** *Let $f\colon M^m \to N^n$ be a smooth map of constant rank r on an open set containing p. Then there exist charts (U,φ) around p and (V,ψ) around $f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1}(x^1,\dots,x^m) = (x^1,\dots,x^r,0,\dots,0).$$

Proof sketch. The proof proceeds in three steps. First, by the rank assumption, after permuting coordinates we may assume the upper-left $r \times r$ block of $Df(p)$ is invertible. Second, define φ by including $f^1,\dots,f^r$ among the coordinate functions and applying the inverse function theorem. Third, in the new coordinates the map has the form $(x',x'') \mapsto (x',g(x'))$ for some smooth g (since the rank is exactly r, the remaining components cannot depend on x''). A final coordinate change ψ on the target that subtracts $g(x')$ puts the map into the stated normal form. We omit the details and refer to [14, §4.12]. □

Corollary 4.4.2 (Immersion theorem / canonical form)**.** *If $f\colon M^m \to N^n$ is an immersion ($r = m \le n$), there exist charts (U,φ) around p and (V,ψ) around $f(p)$ in which f has the form*

$$(x^1,\dots,x^m) \longmapsto (x^1,\dots,x^m,0,\dots,0).$$

In particular, the image $f(U)$ is locally a "flat" m-dimensional slice in N.

Remark 4.4.3. Summarising the local normal forms for smooth maps of constant rank r:

Type	Rank	Normal form	
Submersion	$r = n$	$(x^1, \dots, x^m) \mapsto (x^1, \dots, x^n)$	$(m \geq n)$
Immersion	$r = m$	$(x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0)$	$(m \leq n)$
General, rank r	$r \leq \min(m, n)$	$(x^1, \dots, x^m) \mapsto (x^1, \dots, x^r, 0, \dots, 0)$	

All three are instances of the constant rank theorem.

Corollary 4.4.4. *If $f\colon M \to N$ has constant rank r on a neighbourhood of $f^{-1}(q)$, then $f^{-1}(q)$ (if non-empty) is an embedded submanifold of dimension $m - r$.*

4.5 Submanifolds with boundary

Definition 4.5.1 (Smooth manifold with boundary)**.** A **smooth n-manifold with boundary** is a second-countable Hausdorff space M locally modelled on the half-space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x^n \geq 0\}$, with smooth transition maps. The **boundary**

$$\partial M = \Big\{p \in M : \varphi(p) \in \partial\mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\}\Big\}$$

for any chart (U, φ), is a smooth $(n-1)$-manifold (without boundary). The **interior** is $\mathrm{Int}(M) = M \setminus \partial M$, an n-manifold without boundary.

Proposition 4.5.2. *If M is a smooth n-manifold with boundary, then ∂M is a smooth embedded submanifold of M of dimension $n - 1$.*

Definition 4.5.3 (Neat submanifold)**.** A submanifold S of a manifold with boundary M is **neat** if $\partial S = S \cap \partial M$ and S is transverse to ∂M.

Example 4.5.4. Consider the closed half-disk $D^n_+ = \{x \in \mathbb{R}^n : |x| \leq 1,\ x^n \geq 0\}$. Its boundary consists of two pieces: $\partial D^n_+ = S^{n-1}_+ \cup D^{n-1}$, where $S^{n-1}_+ = \{x \in S^{n-1} : x^n \geq 0\}$ and $D^{n-1} = \{x \in D^n : x^n = 0\}$. The equatorial disk D^{n-1} sits neatly in D^n_+.

4.6 Transversality

Definition 4.6.1 (Transversality)**.** Let $f\colon M \to N$ be smooth and $S \subset N$ an embedded submanifold. We say f is **transverse to** S, written $f \pitchfork S$, if for every $p \in f^{-1}(S)$,

$$\mathrm{im}(\,\mathrm{d}f_p) + T_{f(p)}S \;=\; T_{f(p)}N.$$

Two submanifolds $S_1, S_2 \subset N$ are **transverse** ($S_1 \pitchfork S_2$) if the inclusion $\iota_1\colon S_1 \hookrightarrow N$ is transverse to S_2.

Theorem 4.6.2 (Transverse preimage theorem). *If $f\colon M \to N$ is smooth and $f \pitchfork S$ for an embedded submanifold $S \subset N$ of codimension k, then $f^{-1}(S)$ is either empty or an embedded submanifold of M of codimension k. Moreover, $T_p(f^{-1}(S)) = (\mathrm{d}f_p)^{-1}(T_{f(p)}S)$ for $p \in f^{-1}(S)$.*

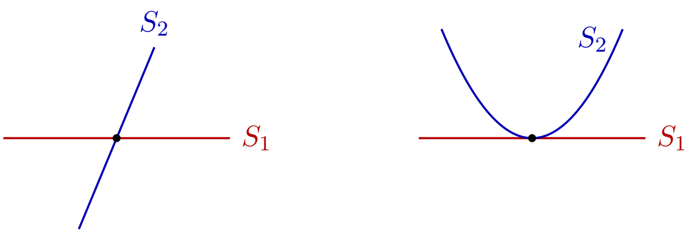
Proof. We reduce to the regular value theorem. Locally near $q = f(p) \in S$, choose a slice chart (V, ψ) for S in N so that $\psi(S \cap V) = \psi(V) \cap (\mathbb{R}^{n-k} \times \{0\})$. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the last k coordinates and set $g = \pi \circ \psi \circ f\colon f^{-1}(V) \to \mathbb{R}^k$. Then $f^{-1}(S) \cap f^{-1}(V) = g^{-1}(0)$. The transversality condition $\mathrm{im}(\mathrm{d}f_p) + T_qS = T_qN$ translates to $\mathrm{d}g_p$ being surjective. Hence 0 is a regular value of g and $g^{-1}(0)$ is an embedded submanifold of codimension k. □

Corollary 4.6.3. *If $S_1, S_2 \subset N$ are embedded submanifolds with $S_1 \pitchfork S_2$, then $S_1 \cap S_2$ is an embedded submanifold of N of dimension $\dim S_1 + \dim S_2 - \dim N$ (when non-empty).*

Example 4.6.4. In $\mathbb{R}^3$, the plane $S_1 = \{z = 0\}$ and the plane $S_2 = \{x = 0\}$ are transverse: at any point $p \in S_1 \cap S_2$, $T_pS_1 + T_pS_2 = \mathbb{R}^3$. Their intersection is the y-axis, a 1-dimensional submanifold, consistent with $\dim S_1 + \dim S_2 - \dim N = 2 + 2 - 3 = 1$.

Example 4.6.5. In $\mathbb{R}^2$, the x-axis $S_1 = \{y = 0\}$ and the curve $S_2 = \{y = x^2\}$ are *not* transverse at the origin, since $T_0S_1 = T_0S_2 = \mathbb{R} \times \{0\}$. The intersection $\{0\}$ is a point but this is "accidental"; a small perturbation of S_2 can create two intersection points, one, or none.

The following illustration contrasts transverse and non-transverse intersections:



transverse　　　　not transverse

4.7 Tubular neighbourhoods

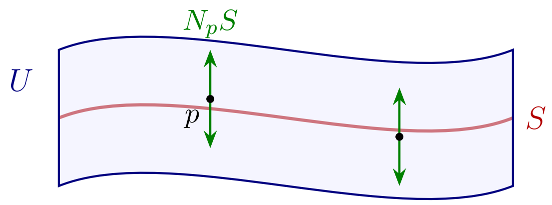
Definition 4.7.1 (Normal bundle). Let $S \subset M$ be an embedded submanifold. Choose a Riemannian metric g on M (which always exists by a partition of unity argument). The **normal bundle** of S in M is

$$NS = \bigsqcup_{p \in S} N_pS, \qquad N_pS = (T_pS)^{\perp} \subset T_pM,$$

where the orthogonal complement is taken with respect to g_p. It is a smooth vector bundle over S of rank $\dim M - \dim S$.

Theorem 4.7.2 (Tubular neighbourhood theorem)**.** *Let $S \subset M$ be a compact embedded submanifold. Then there exists an open neighbourhood $U \supset S$ in M and a diffeomorphism $\Phi\colon NS \supset V \xrightarrow{\sim} U$, where V is an open neighbourhood of the zero section in NS, such that $\Phi|_S = \mathrm{Id}_S$ (identifying S with the zero section).*
The image $U = \Phi(V)$ is called a ***tubular neighbourhood*** *of S in M.*

The proof uses the exponential map of the Riemannian metric restricted to normal vectors. We omit it here and refer to [14, §6.2].



Remark 4.7.3. The compactness assumption can be relaxed: the theorem holds for any closed (i.e., properly embedded) submanifold, though the proof requires more care with the width of the tubular neighbourhood.

4.8 Exercises

Exercise 4.1. The **Stiefel manifold** $V_k(\mathbb{R}^n)$ is the set of orthonormal k-frames in $\mathbb{R}^n$:

$$V_k(\mathbb{R}^n) = \left\{A \in M_{n\times k}(\mathbb{R}) : A^T A = I_k\right\}.$$

Show that $V_k(\mathbb{R}^n)$ is a smooth embedded submanifold of $M_{n\times k}(\mathbb{R})$ of dimension $nk - \frac{k(k+1)}{2}$. *Hint:* mimic the proof for $O(n)$ using the map $A \mapsto A^T A$.

Exercise 4.2. Let $f\colon M \to N$ be smooth. Prove that the graph $\Gamma_f = \{(p, f(p)) : p \in M\} \subset M \times N$ is an embedded submanifold diffeomorphic to M.

Exercise 4.3. Show that $S^m \times S^n \subset \mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$ is an embedded submanifold of $\mathbb{R}^{m+n+2}$, and determine its tangent space at (p, q).

Exercise 4.4. Consider the hyperboloid $H = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$.

(a) Show that H is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

(b) Compute $T_{(1,0,0)}H$.

(c) Is H compact? Connected?

Exercise 4.5. Let $C_1, C_2 \subset \mathbb{R}^2$ be smooth curves (embedded 1-dimensional submanifolds). Show that $C_1 \pitchfork C_2$ if and only if at every point $p \in C_1 \cap C_2$, the tangent lines T_pC_1 and T_pC_2 are distinct. In this case, $C_1 \cap C_2$ is a discrete set of points.

Exercise 4.6. Prove that every submersion $f\colon M \to N$ is an open map. *Hint:* use the submersion theorem.

Exercise 4.7. Let $f\colon M \to \mathbb{R}$ be smooth with c a regular value. Show that the level set $f^{-1}(c)$ has a trivial normal bundle and is therefore orientable (assuming M is orientable).

Exercise 4.8. Show that $\mathrm{SL}(n, \mathbb{R})$ is connected. *Hint:* use the fact that every matrix in $\mathrm{SL}(n, \mathbb{R})$ is a product of elementary matrices.

Exercise 4.9. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (x^2 + y^2, z)$.

(a) Find all critical points and critical values of f.

(b) For which $(a, b) \in \mathbb{R}^2$ is $f^{-1}(a, b)$ a smooth manifold? Describe the topology.

Exercise 4.10. Let S_1 and S_2 be compact embedded submanifolds of $\mathbb{R}^n$ with $\dim S_1 + \dim S_2 < n$. Prove that if $S_1 \pitchfork S_2$, then $S_1 \cap S_2 = \varnothing$.

References for Chapters 3 and 4: J. M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Springer, 2013 [14]; V. Guillemin and A. Pollack, *Differential Topology*, AMS Chelsea, 2010 [15]; L. W. Tu, *An Introduction to Manifolds*, 2nd ed., Springer, 2011 [16]; M. W. Hirsch, *Differential Topology*, Springer GTM 33, 1976 [17].

Chapter 5

Transversality

Contents

Transversality is one of the most powerful concepts in differential topology. It provides a robust framework for understanding how submanifolds and maps interact in *generic* position. Where the implicit function theorem tells us when a level set is a manifold, transversality vastly generalizes this, showing that "most" geometric configurations are well-behaved. The key insight, due to Thom, is that transversal intersections are not only clean (yielding submanifolds) but also *stable* and *generic*.

5.1 Transverse maps and submanifolds

Definition 5.1.1 (Transversality of a map to a submanifold)**.** Let $f\colon M \to N$ be a smooth map between smooth manifolds, and let $W \subseteq N$ be a smooth submanifold. We say that f is **transverse** to W, written

$$f \pitchfork W,$$

if for every $x \in f^{-1}(W)$ we have

$$\mathrm{d}f_x(T_xM) + T_{f(x)}W = T_{f(x)}N.$$

In other words, the image of $\mathrm{d}f_x$ together with the tangent space of W at $f(x)$ spans the entire tangent space of N at $f(x)$.

Definition 5.1.2 (Transverse intersection of submanifolds)**.** Let M be a smooth manifold, and let $A, B \subseteq M$ be smooth submanifolds. We say that A and B **intersect transversally**, written

$$A \pitchfork B,$$

if for every $x \in A \cap B$,

$$T_xA + T_xB = T_xM.$$

Remark 5.1.3. The condition $A \pitchfork B$ is a special case of Definition 5.1.1: if $\iota\colon A \hookrightarrow M$ is the inclusion, then $A \pitchfork B$ if and only if $\iota \pitchfork B$.

Remark 5.1.4. If $f^{-1}(W) = \varnothing$, then $f \pitchfork W$ is satisfied vacuously. In particular, disjoint submanifolds are always transverse.

Remark 5.1.5. If $f\colon M \to N$ is a submersion, then $f \pitchfork W$ for every submanifold $W \subseteq N$, since $\mathrm{d}f_x(T_xM) = T_{f(x)}N$.

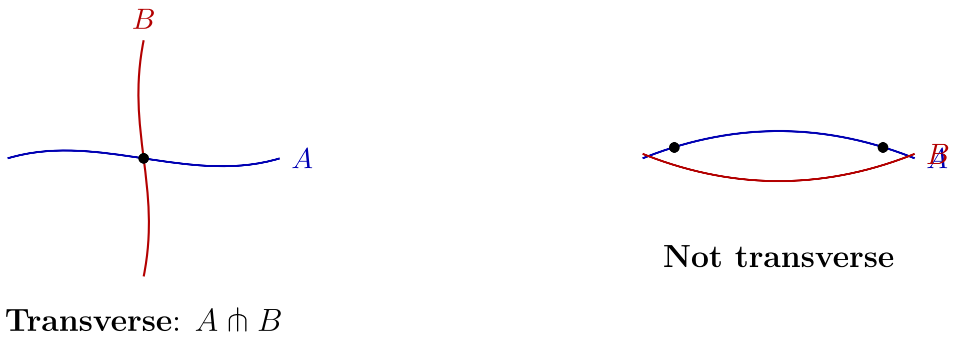


Figure 5.1: Transverse versus non-transverse intersections of curves in the plane.

Example 5.1.6. In $\mathbb{R}^2$, two lines through the origin are transverse if and only if they are distinct. If $A = \{(x, 0) : x \in \mathbb{R}\}$ and $B = \{(x, mx) : x \in \mathbb{R}\}$ with $m \neq 0$, then at the origin $T_0A + T_0B = \mathbb{R}^2$, so $A \pitchfork B$. When $m = 0$, we have $T_0A = T_0B$, so the intersection is not transverse.

Example 5.1.7. In $\mathbb{R}^3$, consider two spheres $S_1 = S^2(p_1, r_1)$ and $S_2 = S^2(p_2, r_2)$ with $S_1 \cap S_2 \neq \varnothing$. They are transverse if and only if they do not share a common tangent plane at any intersection point, which happens precisely when they are not internally tangent. When $S_1 \pitchfork S_2$, the intersection $S_1 \cap S_2$ is a circle (a 1-dimensional submanifold of $\mathbb{R}^3$).

Example 5.1.8. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth. The graph $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ is transverse to the hyperplane $W = \mathbb{R}^n \times \{c\}$ if and only if c is a regular value of f. Indeed, at a point (x_0, c) with $f(x_0) = c$, the tangent space $T_{(x_0,c)}\Gamma_f$ is spanned by the vectors $(e_i, \partial_i f(x_0))$, and $T_{(x_0,c)}W$ is $\mathbb{R}^n \times \{0\}$. These span $\mathbb{R}^{n+1}$ if and only if $\mathrm{d}f_{x_0} \neq 0$.

Example 5.1.9. Let $\Delta = \{(x, x) : x \in M\} \subset M \times M$ be the diagonal. If $f, g\colon N \to M$ are smooth, define $h\colon N \to M \times M$ by $h(x) = (f(x), g(x))$. Then $h \pitchfork \Delta$ if and only if for every $x \in N$ with $f(x) = g(x)$,

$$\mathrm{d}f_x(T_xN) + \mathrm{d}g_x(T_xN) = T_{f(x)}M.$$

This reformulation is central to intersection theory.

5.2 The Transversality Theorem

The following theorem is the fundamental structural result of transversality. It generalizes the preimage theorem (regular value theorem) to the setting of transverse maps.

Theorem 5.2.1 (Transversality theorem)**.** *Let $f\colon M \to N$ be a smooth map and $W \subseteq N$ a smooth submanifold of codimension k. If $f \pitchfork W$, then:*

(i) $f^{-1}(W)$ is a smooth submanifold of M of codimension k (hence $\dim f^{-1}(W) = \dim M - k$, provided it is nonempty).

(ii) For every $x \in f^{-1}(W)$, the tangent space satisfies

$$T_x(f^{-1}(W)) = (\mathrm{d}f_x)^{-1}(T_{f(x)}W).$$

Proof. Let $x_0 \in f^{-1}(W)$ and set $y_0 = f(x_0) \in W$. Since W is a submanifold of codimension k in N, there exists a neighborhood V of y_0 in N and a smooth submersion $\varphi\colon V \to \mathbb{R}^k$ such that

$$W \cap V = \varphi^{-1}(0).$$

Consider the composition $g = \varphi \circ f\colon f^{-1}(V) \to \mathbb{R}^k$. We claim that 0 is a regular value of g.

Let $x \in g^{-1}(0) = f^{-1}(W \cap V)$. We must show that $\mathrm{d}g_x\colon T_xM \to \mathbb{R}^k$ is surjective. By the chain rule, $\mathrm{d}g_x = \mathrm{d}\varphi_{f(x)} \circ \mathrm{d}f_x$. Let $w \in \mathbb{R}^k$. Since φ is a submersion, $\mathrm{d}\varphi_{f(x)}$ is surjective; but we need more. Since $\ker(\mathrm{d}\varphi_{f(x)}) = T_{f(x)}W$ (as $W = \varphi^{-1}(0)$ locally), the map $\mathrm{d}\varphi_{f(x)}$ induces an isomorphism

$$\overline{\mathrm{d}\varphi}_{f(x)}\colon T_{f(x)}N/T_{f(x)}W \xrightarrow{\sim} \mathbb{R}^k.$$

By the transversality condition $f \pitchfork W$, we have $\mathrm{d}f_x(T_xM) + T_{f(x)}W = T_{f(x)}N$. Passing to the quotient, the map $\mathrm{d}f_x$ composed with the projection $\pi\colon T_{f(x)}N \to T_{f(x)}N/T_{f(x)}W$ is surjective:

$$\pi \circ \mathrm{d}f_x\colon T_xM \to T_{f(x)}N/T_{f(x)}W \quad \text{is surjective.}$$

Therefore

$$\mathrm{d}g_x = \mathrm{d}\varphi_{f(x)} \circ \mathrm{d}f_x = \overline{\mathrm{d}\varphi}_{f(x)} \circ \pi \circ \mathrm{d}f_x$$

is a composition of a surjection followed by an isomorphism, hence surjective. Thus 0 is a regular value of g.

By the preimage theorem (regular value theorem), $g^{-1}(0)$ is a smooth submanifold of $f^{-1}(V)$ of codimension k. Since $g^{-1}(0) = f^{-1}(W) \cap f^{-1}(V)$, this shows that $f^{-1}(W)$ is locally (near x_0) a smooth submanifold of codimension k. As x_0 was arbitrary, $f^{-1}(W)$ is a smooth submanifold of M of codimension k.

For part (ii), the tangent space to $g^{-1}(0)$ at x is $\ker(\mathrm{d}g_x) = \ker(\mathrm{d}\varphi_{f(x)} \circ \mathrm{d}f_x)$, which consists of those $v \in T_xM$ such that $\mathrm{d}f_x(v) \in \ker(\mathrm{d}\varphi_{f(x)}) = T_{f(x)}W$. Hence $T_x(f^{-1}(W)) = (\mathrm{d}f_x)^{-1}(T_{f(x)}W)$. □

Corollary 5.2.2. *If A and B are smooth submanifolds of M with $A \pitchfork B$, then $A \cap B$ is a smooth submanifold of M with*

$$\operatorname{codim}(A \cap B) = \operatorname{codim}(A) + \operatorname{codim}(B).$$

Equivalently, $\dim(A \cap B) = \dim A + \dim B - \dim M$ *(when $A \cap B \neq \varnothing$).*

Proof. Apply Theorem 5.2.1 to the inclusion $\iota\colon A \hookrightarrow M$ and the submanifold $B \subseteq M$. Since $\iota \pitchfork B$, we have $\iota^{-1}(B) = A \cap B$ is a submanifold of A of codimension $\operatorname{codim}_M(B)$. Hence $\operatorname{codim}_M(A \cap B) = \operatorname{codim}_A(A \cap B) + \operatorname{codim}_M(A) = \operatorname{codim}_M(B) + \operatorname{codim}_M(A)$. □

Corollary 5.2.3. *If M has dimension m,* $\dim A = a$, *and* $\dim B = b$ *with $A \pitchfork B$, then* $\dim(A \cap B) = a + b - m$. *In particular, if $a + b < m$, then $A \cap B = \varnothing$ (since a manifold cannot have negative dimension).*

Example 5.2.4. Two curves (dimension 1) on a surface (dimension 2) meeting transversally intersect in a 0-dimensional submanifold, i.e., a discrete set of points. If the curves are compact, the intersection is finite.

Example 5.2.5. In $\mathbb{R}^n$, k hyperplanes in general position (pairwise transverse intersections propagating) meet in a submanifold of dimension $n - k$ (an affine subspace). Concretely, $k \leq n$ hyperplanes through the origin given by linearly independent linear forms intersect in an $(n - k)$-dimensional subspace.

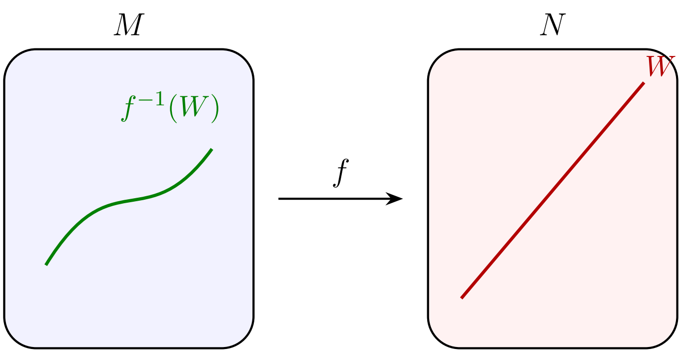


Figure 5.2: When $f \pitchfork W$, the preimage $f^{-1}(W)$ is a submanifold of M with the same codimension as W in N.

5.3 Stability of transversality

Transversality is a robust condition: it persists under small perturbations. This stability is crucial for applications and underpins the genericity results that follow.

Theorem 5.3.1 (Stability of transverse intersections). *Let $f\colon M \to N$ be a smooth map with M compact, and let $W \subseteq N$ be a closed submanifold. If $f \pitchfork W$, then there exists a neighborhood $\mathcal{U}$ of f in $C^\infty(M, N)$ (in the C^1 topology) such that every $g \in \mathcal{U}$ satisfies $g \pitchfork W$.*

Proof sketch. The transversality condition $\mathrm{d}f_x(T_xM) + T_{f(x)}W = T_{f(x)}N$ is an open condition on the 1-jet of f at each point. Since M is compact, $f^{-1}(W)$ is compact, and the finitely many local conditions can be simultaneously satisfied in a C^1-neighborhood. □

Remark 5.3.2. When M is not compact, stability still holds in the strong (Whitney) C^∞ topology, but not necessarily in the weak topology.

5.4 The Thom Transversality Theorem

The Thom transversality theorem asserts that transversality is the generic situation. This is one of the most important results in differential topology, providing a systematic way to achieve transversality by arbitrarily small perturbations.

Definition 5.4.1 (Residual and generic sets). A subset of a topological space is **residual** if it contains a countable intersection of open dense sets. A property of smooth maps $M \to N$ is called **generic** if the set of maps satisfying it is residual in $C^\infty(M, N)$ (with the strong topology). By the Baire category theorem, residual subsets of complete metric spaces are dense.

Theorem 5.4.2 (Thom Transversality Theorem). *Let M and N be smooth manifolds and $W \subseteq N$ a smooth submanifold. Then the set*

$$\pitchfork(M, N; W) = \{f \in C^\infty(M, N) : f \pitchfork W\}$$

is residual (and hence dense) in $C^\infty(M, N)$ equipped with the strong Whitney C^∞ topology. If W is closed, then $\pitchfork(M, N; W)$ is open and dense.

Proof sketch. The proof uses Sard's theorem in a parametric fashion. We sketch the main ideas.

Step 1: Local representation. Cover M by countably many coordinate charts $\{U_i\}$. It suffices to show that for each i, the set of maps f with $f|_{U_i} \pitchfork W$ is open and dense, since $\pitchfork(M, N; W)$ is the countable intersection of these sets.

Step 2: Parametric families. Working in local coordinates, one constructs a smooth family $F\colon M \times S \to N$ (where S is an open subset of some $\mathbb{R}^p$) such that $F_s = F(\cdot, s)$ gives a "perturbation" of f for each $s \in S$, and such that F itself is a submersion near $F^{-1}(W)$ (and hence $F \pitchfork W$).

Step 3: Application of Sard's theorem. By Theorem 5.2.1, $F^{-1}(W)$ is a submanifold. Consider the projection $\pi\colon F^{-1}(W) \to S$. By Sard's theorem (5.4.3 below), the set of regular values of π has full measure in S, hence is dense. One verifies that s is a regular value of π if and only if $F_s \pitchfork W$. The density of regular values gives the density of transverse maps.

Step 4: Openness. When W is closed, the openness follows from the stability of transversality (Theorem 5.3.1), applied locally and patched using compactness of the relevant subsets. □

Theorem 5.4.3 (Sard's Theorem — recalled)**.** *Let $f\colon M \to N$ be a smooth map. The set of critical values of f has measure zero in N. In particular, the set of regular values is dense.*

Remark 5.4.4. The philosophical content of the Thom transversality theorem is profound: *any smooth map can be perturbed by an arbitrarily small amount to become transverse to any given submanifold.* This means that transversality is the "expected" (generic) situation, and non-transverse intersections are "exceptional."

Corollary 5.4.5. *Let $A \subseteq M$ be a compact submanifold and $B \subseteq M$ a closed submanifold. Then A can be made transverse to B by an arbitrarily small perturbation: there exist embeddings $\iota'\colon A \to M$ arbitrarily C^∞-close to the inclusion $\iota\colon A \hookrightarrow M$ such that $\iota'(A) \pitchfork B$.*

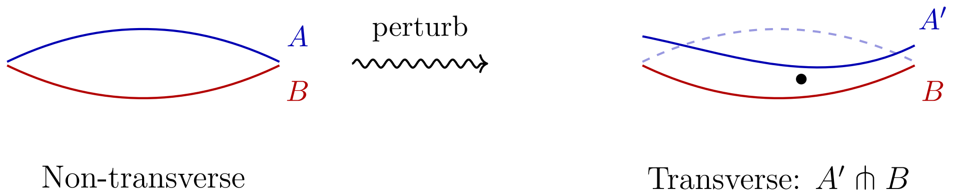


Figure 5.3: A small perturbation of A achieves transversality with B.

5.5 Genericity via Sard's theorem

The paradigm of "genericity via Sard" pervades differential topology. We present some consequences that illustrate this principle.

Proposition 5.5.1. *Let M^m be a compact smooth manifold. For $n \geq 2m$, a generic smooth map $f\colon M \to \mathbb{R}^n$ is an immersion. For $n \geq 2m+1$, a generic smooth map is an embedding.*

Proof sketch. One applies the Thom transversality theorem in jet space. Consider the 1-jet extension $j^1 f\colon M \to J^1(M, \mathbb{R}^n)$. The set $\Sigma \subset J^1(M, \mathbb{R}^n)$ of 1-jets of rank less than m is a submanifold of codimension $(n - m + 1)$ (by the determinantal variety structure). If $n \geq 2m$, this codimension exceeds $m = \dim M$, so $j^1 f \pitchfork \Sigma$ implies $(j^1 f)^{-1}(\Sigma) = \varnothing$ by dimension count, meaning f is an immersion. The embedding statement for $n \geq 2m+1$ uses additionally that self-intersections can be eliminated. □

Proposition 5.5.2 (Generic maps have only transverse self-intersections)**.** *For $n \geq 2m$, a generic immersion $f\colon M^m \to \mathbb{R}^n$ has only transverse double points and no higher-order multiple points.*

Corollary 5.5.3. *A generic immersion of a closed curve in $\mathbb{R}^2$ has only transverse double points (simple crossings). This is the generic picture of plane curves.*

5.6 Intersection numbers

Transversality allows us to define intersection numbers, which are fundamental invariants in differential topology. Throughout this section, all manifolds are assumed to be oriented unless stated otherwise.

Definition 5.6.1 (Orientation at a transverse intersection point)**.** Let M^n be an oriented manifold and let $A^a, B^b \subseteq M$ be compact oriented submanifolds with $A \pitchfork B$ and $a+b=n$. At each point $x \in A \cap B$, we have $T_xA \oplus T_xB \cong T_xM$ (as a consequence of transversality and dimension count). Define the **sign** $\varepsilon(x) = +1$ if the isomorphism $T_xA \oplus T_xB \to T_xM$ is orientation-preserving, and $\varepsilon(x) = -1$ otherwise.

Definition 5.6.2 (Intersection number)**.** With the setup above, the **intersection number** of A and B in M is

$$I(A,B) \;=\; \sum_{x \in A \cap B} \varepsilon(x) \;\in\; \mathbb{Z}.$$

Since A is compact and $A \pitchfork B$, the intersection $A \cap B$ is a compact 0-dimensional manifold, hence a finite set, so this sum is finite.

Proposition 5.6.3. *The intersection number $I(A,B)$ satisfies:*

(i) ***Homotopy invariance****: $I(A,B)$ depends only on the homology classes $[A] \in H_a(M;\mathbb{Z})$ and $[B] \in H_b(M;\mathbb{Z})$.*

(ii) ***Symmetry****: $I(A,B) = (-1)^{ab} I(B,A)$, where $a = \dim A$ and $b = \dim B$.*

(iii) ***Linearity****: $I(A_1 \sqcup A_2, B) = I(A_1,B) + I(A_2,B)$.*

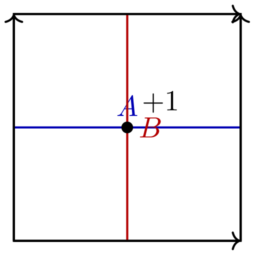
Proof. (i) If A_0 and A_1 are homologous (there exists a compact oriented $(a+1)$-manifold W with $\partial W = A_1 - A_0$), then after perturbing to achieve transversality with B, the signed count of intersection points of W with B gives a cobordism between $A_0 \cap B$ and $A_1 \cap B$, preserving the total count.

(ii) Swapping the order in the direct sum $T_xA \oplus T_xB$ introduces a sign $(-1)^{ab}$, from the permutation of basis vectors.

(iii) This follows directly from the definition. □

Example 5.6.4. On the torus $T^2 = S^1 \times S^1$, let $A = S^1 \times \{1\}$ and $B = \{1\} \times S^1$. These are transverse and meet in a single point $(1,1)$. With the standard orientations,

$I(A, B) = +1$. This reflects the fact that the "meridian" and "longitude" of the torus link once.



T^2 (fundamental domain)

Figure 5.4: Intersection of the two standard circles on the torus: $I(A, B) = +1$.

5.7 Self-intersection number

Definition 5.7.1 (Self-intersection number)**.** Let M^n be a compact oriented manifold and $A^a \subseteq M$ a compact oriented submanifold with $2a = n$. The **self-intersection number** of A is defined as $I(A, A)$, computed by perturbing one copy of A to A' so that $A \pitchfork A'$, and setting

$$I(A, A) \;=\; I(A, A').$$

By homotopy invariance, this is independent of the perturbation.

Proposition 5.7.2. *The self-intersection number $I(A, A)$ equals the Euler number of the normal bundle $\nu(A \hookrightarrow M)$:*

$$I(A, A) = e(\nu_A),$$

where $e(\nu_A) \in \mathbb{Z}$ is the evaluation of the Euler class on the fundamental class of A.

Example 5.7.3. Consider S^n as the equatorial sphere in S^{2n} (via the standard inclusion $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{2n+1}$). The normal bundle of S^n in S^{2n} is the tangent bundle TS^n (this can be seen via the diagonal embedding $\Delta\colon S^n \to S^n \times S^n$). Therefore $I(S^n, S^n) = \chi(S^n) = 1 + (-1)^n$.

Example 5.7.4. The zero section $M \hookrightarrow TM$ (viewing TM as a $2n$-dimensional manifold) has self-intersection number equal to $\chi(M)$, the Euler characteristic of M. This provides a differential-topological proof of the Poincaré–Hopf theorem.

5.8 The Lefschetz fixed-point theorem

We briefly state the connection between intersection theory and fixed-point theory.

Definition 5.8.1 (Lefschetz number)**.** Let $f\colon M \to M$ be a smooth map on a compact oriented manifold M. The **Lefschetz number** of f is

$$L(f) \;=\; \sum_{k=0}^{\dim M} (-1)^k \operatorname{tr}\Big(f_*\colon H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q})\Big).$$

Theorem 5.8.2 (Lefschetz Fixed-Point Theorem)**.** *Let $f\colon M \to M$ be a smooth map on a compact oriented manifold. If $L(f) \neq 0$, then f has a fixed point. More precisely, if the graph $\Gamma_f = \{(x, f(x)) : x \in M\}$ is transverse to the diagonal $\Delta = \{(x,x) : x \in M\}$ in $M \times M$, then*

$$L(f) = I(\Gamma_f, \Delta) = \sum_{\substack{x \in M \\ f(x)=x}} \operatorname{sign}\det(\mathrm{Id} - \mathrm{d}f_x).$$

Remark 5.8.3. When $f = \mathrm{Id}_M$, we have $L(\mathrm{Id}_M) = \chi(M)$. After perturbing the identity to a map with only non-degenerate fixed points (a Lefschetz map), we recover the Poincaré–Hopf theorem: the sum of indices of a generic vector field equals $\chi(M)$. This connects the self-intersection theory of Section 5.7 to the Lefschetz theory.

5.9 Exercises

Exercise 5.1. Show that two lines in $\mathbb{R}^3$ are transverse if and only if they are not coplanar (i.e., they are skew). Conclude that two lines in $\mathbb{R}^3$ that intersect in a point are *never* transverse.

Exercise 5.2. Let A^a and B^b be submanifolds of M^n with $A \pitchfork B$ and $a + b < n$. Show that $A \cap B = \varnothing$.

Exercise 5.3. Let $f\colon M \to N$ be a smooth map and $W \subseteq N$ a submanifold.

(a) Show that if f is a submersion, then $f \pitchfork W$.

(b) Give an example showing that the converse is false.

Exercise 5.4. Let $f\colon M \to N$ and $g\colon M' \to N'$ be smooth maps, and let $W \subseteq N$, $W' \subseteq N'$ be submanifolds. Show that if $f \pitchfork W$ and $g \pitchfork W'$, then $(f \times g) \pitchfork (W \times W')$.

Exercise 5.5. Let M be a compact manifold (not necessarily oriented) and $A, B \subseteq M$ compact submanifolds with complementary dimensions and $A \pitchfork B$. Define the **mod 2 intersection number** $I_2(A,B) = \#(A \cap B) \mod 2$. Show that $I_2(A,B)$ is a homotopy invariant.

Exercise 5.6. Let $\gamma_1, \gamma_2\colon S^1 \to \mathbb{R}^3$ be disjoint smooth embeddings. The **linking number** $\mathrm{lk}(\gamma_1,\gamma_2)$ can be defined as follows: let Σ be a compact oriented surface with

$\partial\Sigma = \gamma_1(S^1)$, chosen transverse to γ_2. Set $\mathrm{lk}(\gamma_1, \gamma_2) = I(\Sigma, \gamma_2)$.

(a) Show that this is independent of the choice of Σ.

(b) Compute the linking number of the Hopf link.

(c) Show that $\mathrm{lk}(\gamma_1, \gamma_2) = \mathrm{lk}(\gamma_2, \gamma_1)$.

Exercise 5.7. Consider $\mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$ as the submanifold defined by the last homogeneous coordinate being zero. Compute the self-intersection number $I(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^1)$ in $\mathbb{R}\mathrm{P}^2$ (working mod 2, since $\mathbb{R}\mathrm{P}^2$ is non-orientable).

Exercise 5.8. Let M^m and N^n be smooth manifolds with $n \geq 2m+1$. Using the Thom transversality theorem applied to 1-jet extensions, show that a generic smooth map $f \colon M \to N$ is an injective immersion. If M is compact, show it is an embedding.

Exercise 5.9. Let $f \colon S^{2n} \to S^{2n}$ be a smooth map of degree d.

(a) Compute the Lefschetz number $L(f)$.

(b) Show that if $d \neq (-1)^{2n+1} = -1$, then f has a fixed point.

(c) Deduce that the antipodal map on S^{2n} has a fixed point. Why does this not contradict the fact that the antipodal map has no fixed point? Resolve the apparent paradox.

Exercise 5.10. Let M be a compact oriented manifold of dimension n. Show that the self-intersection number of the diagonal $\Delta \subset M \times M$ equals the Euler characteristic:

$$I(\Delta, \Delta) = \chi(M).$$

Hint: Relate this to the Lefschetz number of the identity.

Chapter 6

Differential Forms and Integration on Manifolds

Contents

Differential forms provide the natural language for integration on manifolds. They unify and generalize many constructions from vector calculus—gradient, curl, divergence, line integrals, surface integrals, and the fundamental theorems of Green, Gauss, and Stokes—into a single elegant framework. In this chapter, we develop the theory of exterior algebra, differential forms, the exterior derivative, and integration, culminating in the machinery needed for de Rham cohomology (treated in subsequent chapters).

6.1 Exterior algebra

We begin with the algebraic foundations: the exterior algebra of a finite-dimensional real vector space.

Definition 6.1.1 (Alternating multilinear form)**.** Let V be a real vector space of dimension n. A **k-linear alternating form** (or **exterior k-form**) on V is a multilinear map

$$\omega\colon \underbrace{V \times \cdots \times V}_{k} \longrightarrow \mathbb{R}$$

that is alternating: $\omega(v_1, \dots, v_k) = 0$ whenever two of the v_i are equal. Equivalently, for any permutation $\sigma \in \mathfrak{S}_k$,

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\, \omega(v_1, \dots, v_k).$$

Definition 6.1.2 (Exterior power)**.** The space of all k-linear alternating forms on V is denoted $\Lambda^k(V^*)$ or $\Lambda^k V^*$. By convention, $\Lambda^0 V^* = \mathbb{R}$ and $\Lambda^1 V^* = V^*$. We have $\Lambda^k V^* = 0$ for $k > \dim V$. The dimension is

$$\dim \Lambda^k V^* = \binom{n}{k}.$$

Definition 6.1.3 (Wedge product)**.** The **wedge product** (or **exterior product**)

$$\wedge\colon \Lambda^k V^* \times \Lambda^\ell V^* \longrightarrow \Lambda^{k+\ell} V^*$$

is defined by

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\,\ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} \operatorname{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

Equivalently, summing only over (k,ℓ)-shuffles (permutations σ with $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+\ell)$):

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \sum_{(k,\ell)\text{-shuffles } \sigma} \operatorname{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

Proposition 6.1.4 (Properties of the wedge product)**.** *For $\alpha \in \Lambda^k V^*$, $\beta \in \Lambda^\ell V^*$, and $\gamma \in \Lambda^p V^*$:*

(i) ***Associativity****: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.*

(ii) ***Graded commutativity****: $\alpha \wedge \beta = (-1)^{k\ell}\, \beta \wedge \alpha$.*

(iii) ***Bilinearity****: $\wedge$ is bilinear in each argument.*

(iv) If $e^1, \dots, e^n$ is a basis of V^*, then $\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ is a basis of $\Lambda^k V^*$.

Proof. (i) Both sides equal the $(k+\ell+p)$-linear alternating form given by summing over all (k,ℓ,p)-shuffles.

(ii) Transposing the first k arguments past the last ℓ arguments requires $k\ell$ transpositions, each contributing a factor of -1.

(iii) Clear from the definition.

(iv) One checks that these $\binom{n}{k}$ elements are linearly independent and span $\Lambda^k V^*$ by expressing any alternating form in terms of them. □

Definition 6.1.5 (Exterior algebra)**.** The **exterior algebra** of V^* is the graded algebra

$$\Lambda^*(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*)$$

equipped with the wedge product. It has total dimension 2^n.

Example 6.1.6. For $V = \mathbb{R}^3$ with dual basis e^1, e^2, e^3:

- $\Lambda^0 = \mathbb{R}$, dimension 1.
- $\Lambda^1 = \operatorname{span}\{e^1, e^2, e^3\}$, dimension 3.
- $\Lambda^2 = \operatorname{span}\{e^1 \wedge e^2,\ e^1 \wedge e^3,\ e^2 \wedge e^3\}$, dimension 3.
- $\Lambda^3 = \operatorname{span}\{e^1 \wedge e^2 \wedge e^3\}$, dimension 1.

The isomorphisms $\Lambda^1 \cong \Lambda^2$ and $\Lambda^0 \cong \Lambda^3$ underlie the vector calculus identities relating gradient, curl, and divergence.

6.2 Differential forms on manifolds

Definition 6.2.1 (Differential k-form)**.** Let M be a smooth manifold. A **differential k-form** on M is a smooth section of the k-th exterior power of the cotangent bundle:

$$\omega \in \Gamma(\Lambda^k T^* M).$$

That is, ω assigns to each $p \in M$ an alternating k-linear form $\omega_p \in \Lambda^k(T_p^* M)$, and this assignment is smooth. The space of all smooth k-forms on M is denoted

$$\Omega^k(M) = \Gamma(\Lambda^k T^* M).$$

By convention, $\Omega^0(M) = C^\infty(M)$.

Remark 6.2.2. In local coordinates $(x^1, \ldots, x^n)$ on an open set $U \subseteq M$, any $\omega \in \Omega^k(M)$ can be written as

$$\omega\Big|_U = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \cdots i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where $\omega_{i_1 \cdots i_k} \in C^\infty(U)$ are smooth coefficient functions and $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ are the coordinate 1-forms (a local frame for T^*M).

Definition 6.2.3 (Algebra of differential forms)**.** The **de Rham algebra** of M is the graded $\mathbb{R}$-algebra

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$$

with the wedge product defined pointwise: $(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p$. It is a graded-commutative algebra: for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$,

$$\alpha \wedge \beta = (-1)^{k\ell}\, \beta \wedge \alpha.$$

Example 6.2.4. On $\mathbb{R}^3$, a 1-form is $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ where $P, Q, R \in C^\infty(\mathbb{R}^3)$. A 2-form is $\eta = A\,\mathrm{d}y \wedge \mathrm{d}z + B\,\mathrm{d}z \wedge \mathrm{d}x + C\,\mathrm{d}x \wedge \mathrm{d}y$. A 3-form is $\mu = f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

6.3 The exterior derivative

Theorem 6.3.1 (Existence and uniqueness of the exterior derivative)**.** *There exists a unique family of $\mathbb{R}$-linear maps*

$$\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M), \quad k = 0, 1, \ldots, n,$$

satisfying the following properties:

(i) ***Degree*** $+1$*:* d *maps k-forms to $(k+1)$-forms.*

(ii) $\mathrm{d}^2 = 0$*: For all $\omega \in \Omega^k(M)$, $\mathrm{d}(\mathrm{d}\omega) = 0$.*

(iii) ***Graded Leibniz rule****: For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$,*

$$\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k\, \alpha \wedge (\mathrm{d}\beta).$$

(iv) ***Agreement on functions****: For $f \in \Omega^0(M) = C^\infty(M)$, $\mathrm{d}f$ is the usual differential of f, i.e., $(\mathrm{d}f)_p(v) = v(f)$ for $v \in T_pM$.*

Proof. **Uniqueness.** Properties (iii) and (iv) force the action of d on local coordinate expressions. In coordinates $(x^1, \ldots, x^n)$, for $\omega = \sum_I \omega_I\,\mathrm{d}x^I$ (where $I = (i_1 < \cdots < i_k)$ and $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$), we must have

$$\mathrm{d}\omega = \sum_I \mathrm{d}\omega_I \wedge \mathrm{d}x^I = \sum_I \sum_{j=1}^{n} \frac{\partial \omega_I}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^I,$$

since $\mathrm{d}(\mathrm{d}x^j) = 0$ by property (ii) applied to $\mathrm{d}x^j = \mathrm{d}(x^j)$, and the Leibniz rule gives $\mathrm{d}(\omega_I\,\mathrm{d}x^I) = \mathrm{d}\omega_I \wedge \mathrm{d}x^I + \omega_I\,\mathrm{d}(\mathrm{d}x^I) = \mathrm{d}\omega_I \wedge \mathrm{d}x^I$. Here $\mathrm{d}(\mathrm{d}x^I) = 0$ follows by induction using (ii) and (iii).

Existence. Define d by the formula above in each coordinate chart. To show this is well-defined (independent of coordinates), one can verify directly that the formula transforms correctly under coordinate changes, or one can give a coordinate-free formula: for $\omega \in \Omega^k(M)$ and smooth vector fields $X_0, \ldots, X_k$,

$$\begin{aligned}(\mathrm{d}\omega)(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i\, X_i\Big(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\Big) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j}\, \omega\Big([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\Big),\end{aligned}$$

where $\widehat{X_i}$ denotes omission. This is manifestly coordinate-free. One verifies that this formula defines a $(k+1)$-form and satisfies all four properties.

The property $\mathrm{d}^2 = 0$ follows in coordinates from the symmetry of second partial derivatives:

$$\mathrm{d}^2\omega = \sum_I \sum_{j,\ell} \frac{\partial^2 \omega_I}{\partial x^\ell \partial x^j} \, \mathrm{d}x^\ell \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I = 0,$$

since $\frac{\partial^2 \omega_I}{\partial x^\ell \partial x^j}$ is symmetric in ℓ, j while $\mathrm{d}x^\ell \wedge \mathrm{d}x^j$ is antisymmetric. □

Example 6.3.2. On $\mathbb{R}^3$, the exterior derivative recovers the classical vector calculus operators:

- For $f \in \Omega^0(\mathbb{R}^3)$: $\mathrm{d}f = \frac{\partial f}{\partial x}\, \mathrm{d}x + \frac{\partial f}{\partial y}\, \mathrm{d}y + \frac{\partial f}{\partial z}\, \mathrm{d}z$ (gradient).
- For $\omega = P\, \mathrm{d}x + Q\, \mathrm{d}y + R\, \mathrm{d}z \in \Omega^1(\mathbb{R}^3)$:

  $$\mathrm{d}\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y$$

  (curl).
- For $\eta = A\, \mathrm{d}y \wedge \mathrm{d}z + B\, \mathrm{d}z \wedge \mathrm{d}x + C\, \mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$: $\mathrm{d}\eta = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ (divergence).

The identity $\mathrm{d}^2 = 0$ encodes both $\operatorname{curl}(\operatorname{grad} f) = 0$ and $\operatorname{div}(\operatorname{curl} F) = 0$.

$$\Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(M) \xrightarrow{\mathrm{d}} 0$$

Figure 6.1: The de Rham complex $(\Omega^*(M), \mathrm{d})$. The condition $\mathrm{d}^2 = 0$ makes this a cochain complex.

6.4 Pullback of differential forms

Definition 6.4.1 (Pullback)**.** Let $f\colon M \to N$ be a smooth map. The **pullback** is the $\mathbb{R}$-linear map $f^*\colon \Omega^k(N) \to \Omega^k(M)$ defined by

$$(f^*\omega)_p(v_1, \dots, v_k) = \omega_{f(p)}(\mathrm{d}f_p(v_1), \dots, \mathrm{d}f_p(v_k))$$

for $\omega \in \Omega^k(N)$, $p \in M$, and $v_1, \dots, v_k \in T_pM$.

Proposition 6.4.2 (Properties of pullback)**.** *Let $f\colon M \to N$ and $g\colon N \to P$ be smooth maps.*

(i) ***Functoriality****: $(g \circ f)^* = f^* \circ g^*$.*

(ii) ***Compatibility with wedge product****: $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$.*

(iii) ***Identity****: $(\mathrm{Id}_M)^* = \mathrm{Id}_{\Omega^*(M)}$.*

(iv) ***On functions****: For* $h \in \Omega^0(N) = C^\infty(N)$, $f^*h = h \circ f$.

Theorem 6.4.3 (Naturality of the exterior derivative)**.** *Let* $f\colon M \to N$ *be a smooth map. Then the pullback commutes with the exterior derivative:*

$$f^* \circ \mathrm{d} = \mathrm{d} \circ f^*.$$

That is, for every $\omega \in \Omega^k(N)$, $f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$.

Proof. On 0-forms (functions): for $h \in C^\infty(N)$ and $v \in T_pM$,

$$(f^*\,\mathrm{d}h)_p(v) = (\mathrm{d}h)_{f(p)}(\mathrm{d}f_p(v)) = (\mathrm{d}f_p(v))(h) = v(h \circ f) = (\mathrm{d}(h \circ f))_p(v) = (\mathrm{d}(f^*h))_p(v).$$

In local coordinates $(y^1, \dots, y^n)$ on N, any k-form can be written as $\omega = \sum_I \omega_I\, \mathrm{d}y^I$. Then

$$f^*(\mathrm{d}\omega) = f^*\Big(\sum_I \mathrm{d}\omega_I \wedge \mathrm{d}y^I\Big) = \sum_I f^*(\mathrm{d}\omega_I) \wedge f^*(\mathrm{d}y^I) = \sum_I \mathrm{d}(f^*\omega_I) \wedge \mathrm{d}(f^*y^I),$$

using the 0-form case and compatibility with wedge products. On the other hand,

$$\mathrm{d}(f^*\omega) = \mathrm{d}\Big(\sum_I (f^*\omega_I)\, f^*(\mathrm{d}y^I)\Big) = \sum_I \mathrm{d}(f^*\omega_I) \wedge f^*(\mathrm{d}y^I) + \sum_I (f^*\omega_I)\, \mathrm{d}(f^*(\mathrm{d}y^I)).$$

Now $f^*(\mathrm{d}y^{i_j}) = \mathrm{d}(f^*y^{i_j}) = \mathrm{d}(y^{i_j} \circ f)$, so $\mathrm{d}(f^*(\mathrm{d}y^{i_j})) = \mathrm{d}^2(y^{i_j} \circ f) = 0$. By the Leibniz rule for the wedge product, $\mathrm{d}(f^*(\mathrm{d}y^I)) = 0$. Hence both sides agree. □

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{\ \mathrm{d}\ } & \Omega^{k+1}(N) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega^k(M) & \xrightarrow[\ \mathrm{d}\ ]{} & \Omega^{k+1}(M) \end{array}$$

Figure 6.2: Naturality of the exterior derivative: $f^* \circ \mathrm{d} = \mathrm{d} \circ f^*$.

6.5 Interior product and Lie derivative

Definition 6.5.1 (Interior product)**.** Let X be a smooth vector field on M. The **interior product** (or **contraction**) with X is the linear map $\iota_X\colon \Omega^k(M) \to \Omega^{k-1}(M)$ defined by

$$(\iota_X\omega)(v_1, \dots, v_{k-1}) = \omega(X, v_1, \dots, v_{k-1}).$$

By convention, ι_X acts as zero on $\Omega^0(M)$.

Proposition 6.5.2. *The interior product satisfies:*

(i) ι_X *has degree* -1*: it maps* Ω^k *to* Ω^{k-1}.

(ii) $\iota_X^2 = 0$.

(iii) ***Graded Leibniz rule***: *For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$,*

$$\iota_X(\alpha \wedge \beta) = (\iota_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_X\beta).$$

That is, ι_X is an antiderivation of degree -1.

(iv) $\iota_{fX} = f\,\iota_X$ *for $f \in C^\infty(M)$.*

Proof. (ii) For $\omega \in \Omega^k(M)$, $(\iota_X^2\omega)(v_1, \dots, v_{k-2}) = \omega(X, X, v_1, \dots, v_{k-2}) = 0$ by the alternating property.

(iii) Follows by a direct computation using the definition of the wedge product, tracking the position of X among the arguments. □

Definition 6.5.3 (Lie derivative of a form)**.** Let X be a smooth vector field on M with flow ϕ_t. The **Lie derivative** of $\omega \in \Omega^k(M)$ along X is

$$\mathcal{L}_X\omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t^*\omega = \lim_{t\to 0} \frac{\phi_t^*\omega - \omega}{t}.$$

Theorem 6.5.4 (Cartan's magic formula)**.** *For any smooth vector field X on M and any $\omega \in \Omega^k(M)$,*

$$\mathcal{L}_X\omega = \iota_X(\mathrm{d}\omega) + \mathrm{d}(\iota_X\omega) = (\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X)\omega.$$

In shorthand: $\mathcal{L}_X = \iota_X\,\mathrm{d} + \mathrm{d}\,\iota_X$.

Proof. We verify the formula by checking that both sides agree on functions and exact 1-forms, then extend by the Leibniz rule.

On functions ($k = 0$): For $f \in C^\infty(M)$, $\mathcal{L}_X f = X(f)$ (the directional derivative). On the right side, $\iota_X(\mathrm{d}f) + \mathrm{d}(\iota_X f) = (\mathrm{d}f)(X) + 0 = X(f)$.

On exact 1-forms ($k = 1$, $\omega = \mathrm{d}f$): $\mathcal{L}_X(\mathrm{d}f) = \mathrm{d}(\mathcal{L}_X f) = \mathrm{d}(Xf)$ (since $\mathcal{L}_X$ commutes with d). On the right side, $\iota_X(\mathrm{d}^2 f) + \mathrm{d}(\iota_X\,\mathrm{d}f) = 0 + \mathrm{d}(Xf)$.

Leibniz rule: Both $\mathcal{L}_X$ and $\iota_X\,\mathrm{d} + \mathrm{d}\,\iota_X$ are derivations of degree 0 on $\Omega^*(M)$ (i.e., they satisfy the ungraded Leibniz rule $D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + \alpha \wedge (D\beta)$). For $\mathcal{L}_X$, this follows from $\phi_t^*(\alpha \wedge \beta) = (\phi_t^*\alpha) \wedge (\phi_t^*\beta)$. For $\iota_X\,\mathrm{d} + \mathrm{d}\,\iota_X$, this follows by expanding and using the graded Leibniz rules for d and ι_X.

Since differential forms are locally generated (over C^∞) by functions and exact 1-forms $\mathrm{d}x^i$, agreement on these generators plus the Leibniz rule implies agreement everywhere. □

Corollary 6.5.5. *The Lie derivative commutes with the exterior derivative: $\mathcal{L}_X \circ \mathrm{d} = \mathrm{d} \circ \mathcal{L}_X$.*

Proof. $\mathcal{L}_X\,\mathrm{d}\omega = \iota_X\,\mathrm{d}^2\omega + \mathrm{d}\iota_X\,\mathrm{d}\omega = \mathrm{d}\iota_X\,\mathrm{d}\omega$. Also $\mathrm{d}\mathcal{L}_X\omega = \mathrm{d}\iota_X\,\mathrm{d}\omega + \mathrm{d}^2\iota_X\omega = \mathrm{d}\iota_X\,\mathrm{d}\omega$. □

Proposition 6.5.6. *For vector fields X, Y and $\omega, \eta \in \Omega^*(M)$:*

(i) $\mathcal{L}_X(f\omega) = (Xf)\omega + f\,\mathcal{L}_X\omega$ *for $f \in C^\infty(M)$.*

(ii) $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$, *i.e.,* $\mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X = \iota_{[X,Y]}$.

(iii) $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$.

Remark 6.5.7. The three operators d (degree $+1$), ι_X (degree -1), and $\mathcal{L}_X$ (degree 0) satisfy the relations of a graded Lie algebra, with $\mathrm{d}^2 = 0$, $\iota_X^2 = 0$, and $\mathcal{L}_X = [\mathrm{d}, \iota_X]$ (where $[\cdot, \cdot]$ denotes the graded commutator). This algebraic structure is central to the Weil model of equivariant cohomology.

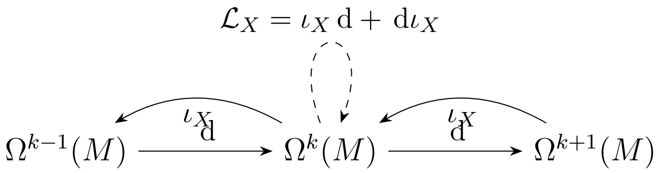


Figure 6.3: The operators d, ι_X, and $\mathcal{L}_X$ acting on $\Omega^*(M)$.

6.6 Orientation and volume forms

Definition 6.6.1 (Orientation of a manifold)**.** An **orientation** of a smooth manifold M of dimension n is a choice of a maximal atlas $\{(U_\alpha, \varphi_\alpha)\}$ such that all transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ have positive Jacobian determinant. Equivalently, an orientation is a choice of connected component of the space of nowhere-vanishing n-forms on M. A manifold is **orientable** if such a choice exists.

Proposition 6.6.2. *A connected smooth n-manifold M is orientable if and only if there exists a nowhere-vanishing n-form $\mu \in \Omega^n(M)$. Such a form is called a **volume form**. Two volume forms μ and μ' determine the same orientation if and only if $\mu' = f\mu$ for some $f \in C^\infty(M)$ with $f > 0$ everywhere.*

Example 6.6.3. The standard orientation of $\mathbb{R}^n$ is given by the volume form $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

Example 6.6.4. The unit sphere $S^n \subset \mathbb{R}^{n+1}$ is orientable. The volume form induced by the outward unit normal $\nu = x$ is

$$\mu = \iota_\nu(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n+1})\Big|_{S^n} = \sum_{i=1}^{n+1} (-1)^{i-1} x^i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{n+1}\Big|_{S^n}.$$

Example 6.6.5. The Möbius band and the Klein bottle are non-orientable: they admit no nowhere-vanishing top-degree form. Equivalently, they have transition functions with negative Jacobian determinant that cannot be eliminated by choosing a different atlas.

6.7 Integration of differential forms

6.7.1 Integration on $\mathbb{R}^n$

Definition 6.7.1. Let $\omega = f\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ be a compactly supported n-form on an open subset $U \subseteq \mathbb{R}^n$. We define

$$\int_U \omega = \int_U f\ \mathrm{d}x^1 \cdots \mathrm{d}x^n,$$

where the right side is the Lebesgue integral.

Lemma 6.7.2 (Change of variables for forms)**.** *Let $\varphi \colon U \to V$ be an orientation-preserving diffeomorphism between open subsets of $\mathbb{R}^n$. For any compactly supported n-form ω on V,*

$$\int_U \varphi^* \omega = \int_V \omega.$$

If φ is orientation-reversing, the integral picks up a sign: $\int_U \varphi^*\omega = -\int_V \omega$.

Proof. Write $\omega = f\ \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$. Then

$$\varphi^*\omega = (f \circ \varphi)\, \varphi^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n) = (f \circ \varphi)\ \det(D\varphi)\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

Therefore

$$\int_U \varphi^*\omega = \int_U (f \circ \varphi)\ \det(D\varphi)\ \mathrm{d}x^1 \cdots \mathrm{d}x^n = \int_V f\ \mathrm{d}y^1 \cdots \mathrm{d}y^n = \int_V \omega,$$

where the second equality is the change of variables formula for Lebesgue integrals (the Jacobian $|\det D\varphi|$ appears, and $\det D\varphi > 0$ since φ is orientation-preserving). □

6.7.2 Integration on manifolds

Definition 6.7.3 (Integration of n-forms on oriented manifolds)**.** Let M be an oriented smooth n-manifold, and let $\omega \in \Omega^n(M)$ be a compactly supported n-form. Choose a locally finite oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$ and a subordinate partition of unity $\{\rho_\alpha\}$. Define

$$\int_M \omega = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*(\rho_\alpha\, \omega).$$

Theorem 6.7.4. *The integral $\int_M \omega$ is well-defined: it is independent of the choice of oriented atlas and partition of unity.*

Proof. Let $\{(V_\beta, \psi_\beta)\}$ be another oriented atlas with subordinate partition of unity $\{\sigma_\beta\}$. Then

$$\begin{aligned}\sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*(\rho_\alpha\, \omega) &= \sum_\alpha \sum_\beta \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*(\rho_\alpha\, \sigma_\beta\, \omega) \\ &= \sum_\alpha \sum_\beta \int_{\psi_\beta(V_\beta)} (\psi_\beta^{-1})^*(\rho_\alpha\, \sigma_\beta\, \omega)\end{aligned}$$

where the second equality uses Lemma 6.7.2 applied to the transition map $\psi_\beta \circ \varphi_\alpha^{-1}$ (which is orientation-preserving). Summing over α gives $\sum_\beta \int (\psi_\beta^{-1})^*(\sigma_\beta \, \omega)$. □

Proposition 6.7.5 (Properties of integration)**.** *Let M be an oriented compact n-manifold.*

(i) ***Linearity****: $\int_M (a\omega + b\eta) = a\int_M \omega + b\int_M \eta$ for $a, b \in \mathbb{R}$.*

(ii) ***Orientation****: If $\overline{M}$ denotes M with reversed orientation, then $\int_{\overline{M}} \omega = -\int_M \omega$.*

(iii) ***Diffeomorphism invariance****: If $f\colon M \to N$ is an orientation-preserving diffeomorphism, then $\int_M f^*\omega = \int_N \omega$.*

Example 6.7.6. Parametrize S^1 by $\theta \mapsto (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi)$. The 1-form $\omega = -y \, \mathrm{d}x + x \, \mathrm{d}y$ restricted to S^1 pulls back to $\mathrm{d}\theta$, so $\int_{S^1} \omega = \int_0^{2\pi} \mathrm{d}\theta = 2\pi$.

Example 6.7.7. On $S^2 \subset \mathbb{R}^3$, the standard area form is $\mu = x \, \mathrm{d}y \wedge \mathrm{d}z + y \, \mathrm{d}z \wedge \mathrm{d}x + z \, \mathrm{d}x \wedge \mathrm{d}y$. Its integral $\int_{S^2} \mu = 4\pi$, which is the surface area of the unit sphere.

6.7.3 Compact support

Definition 6.7.8 (Compactly supported forms)**.** Let M be a smooth manifold. A differential form $\omega \in \Omega^k(M)$ has **compact support** if the closure of $\{p \in M : \omega_p \neq 0\}$ is compact. The space of compactly supported k-forms is denoted $\Omega^k_c(M)$.

Remark 6.7.9. Integration $\int_M \colon \Omega^n_c(M) \to \mathbb{R}$ is well-defined on any oriented manifold (not necessarily compact): the sum in Definition 7.3.1 is finite since the support of ω is compact and the atlas is locally finite.

Remark 6.7.10. The exterior derivative preserves compact support: $\mathrm{d}\colon \Omega^k_c(M) \to \Omega^{k+1}_c(M)$. Hence $(\Omega^*_c(M), \mathrm{d})$ is a subcomplex of $(\Omega^*(M), \mathrm{d})$. Its cohomology, called **compactly supported de Rham cohomology** and denoted $H^*_c(M)$, is important in Poincaré duality.

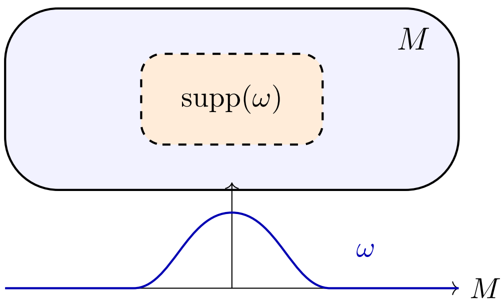


Figure 6.4: A compactly supported form on M: the form vanishes outside a compact region.

6.8 Summary: dictionary between vector calculus and forms

Vector Calculus ($\mathbb{R}^3$)		Differential Forms
scalar field f	$\longleftrightarrow$	0-form $f \in \Omega^0$
vector field $\mathbf{F}$	$\longleftrightarrow$	1-form or 2-form
∇f (gradient)	$\longleftrightarrow$	$\mathrm{d}f$
$\nabla \times \mathbf{F}$ (curl)	$\longleftrightarrow$	$\mathrm{d}\omega^1$ (as 2-form)
$\nabla \cdot \mathbf{F}$ (divergence)	$\longleftrightarrow$	$\mathrm{d}\omega^2$ (as 3-form)
$\mathrm{curl}(\mathrm{grad}) = 0$	$\longleftrightarrow$	$\mathrm{d}^2 = 0$
$\mathrm{div}(\mathrm{curl}) = 0$	$\longleftrightarrow$	$\mathrm{d}^2 = 0$

Figure 6.5: Correspondence between vector calculus on $\mathbb{R}^3$ and differential forms.

6.9 Exercises

Exercise 6.1. Let $\alpha = 3\,\mathrm{d}x - 2\,\mathrm{d}y + \mathrm{d}z$ and $\beta = \mathrm{d}x + 4\,\mathrm{d}y - 2\,\mathrm{d}z$ be 1-forms on $\mathbb{R}^3$. Compute $\alpha \wedge \beta$ and verify that $\alpha \wedge \beta = -\beta \wedge \alpha$.

Exercise 6.2. Compute the exterior derivative of the following forms on $\mathbb{R}^3$:

(a) $\omega = xy^2\,\mathrm{d}x + e^z\,\mathrm{d}y + xz\,\mathrm{d}z$.

(b) $\eta = x^2\,\mathrm{d}y \wedge \mathrm{d}z + y^2\,\mathrm{d}z \wedge \mathrm{d}x + z^2\,\mathrm{d}x \wedge \mathrm{d}y$.

(c) $f = \sin(xyz)$.

Verify that $\mathrm{d}^2 = 0$ for each.

Exercise 6.3. Let $f \colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(u,v) = (u^2, uv, v^2)$. Compute $f^*\omega$ where $\omega = x\,\mathrm{d}y - y\,\mathrm{d}z$.

Exercise 6.4. On $\mathbb{R}^3$, let $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ (the rotational vector field) and $\omega = x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z$. Verify Cartan's formula $\mathcal{L}_X\omega = \iota_X\,\mathrm{d}\omega + \mathrm{d}(\iota_X\omega)$ by computing both sides explicitly.

Exercise 6.5. Consider the 1-form on $\mathbb{R}^2 \setminus \{0\}$:

$$\omega = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}.$$

(a) Show that $\mathrm{d}\omega = 0$ (i.e., ω is closed).

(b) Show that $\int_{S^1} \omega = 2\pi$, where S^1 is the unit circle.

(c) Conclude that ω is not exact on $\mathbb{R}^2 \setminus \{0\}$.

Exercise 6.6. Show that the n-form on $\mathbb{R}^{n+1}$,

$$\mu = \sum_{i=1}^{n+1} (-1)^{i-1} x^i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{n+1},$$

restricts to a volume form on S^n, and compute $\int_{S^2} \mu$.

Exercise 6.7. Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y$.

(a) Compute $\mathcal{L}_X\omega$ using Cartan's formula.

(b) Compute $\mathcal{L}_X\omega$ directly from the flow ϕ_t of X.

(c) Verify that the results agree.

Exercise 6.8. Let $\omega = (x^2+y^2)\,\mathrm{d}x \wedge \mathrm{d}y$ on $\mathbb{R}^2$ and let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the closed unit disk. Compute $\int_D \omega$.

Exercise 6.9. Show that the Möbius band is non-orientable by showing that no nowhere-vanishing 2-form exists. *Hint*: consider a loop traversing the core circle once and track the orientation of a local frame.

Exercise 6.10. Let $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ on $\mathbb{R}^3$ and $X = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$.

(a) Compute $\iota_X\omega$.

(b) Compute $\iota_X(\iota_X\omega)$ and verify it is zero.

Chapter 7

Stokes' Theorem

Contents

The classical integral theorems of vector calculus—Green's theorem, the divergence theorem, and the classical Stokes theorem in $\mathbb{R}^3$—are all special cases of a single, elegant statement on oriented manifolds with boundary. In this chapter we develop the theory of integration on chains, prove the general Stokes theorem, and draw fundamental consequences for de Rham cohomology.

7.1 Smooth singular chains

Definition 7.1.1 (Standard k-simplex)**.** The *standard k-simplex* is the subset

$$\Delta^k = \left\{ (t_0, t_1, \ldots, t_k) \in \mathbb{R}^{k+1} \;\middle|\; t_i \geq 0,\ \sum_{i=0}^{k} t_i = 1 \right\}.$$

Equivalently, using coordinates $(t_1, \ldots, t_k)$ with $t_0 = 1 - t_1 - \cdots - t_k$, we may identify Δ^k with the convex hull of the standard basis vectors $e_0, e_1, \ldots, e_k$ in $\mathbb{R}^{k+1}$.

Definition 7.1.2 (Smooth singular simplex)**.** Let M be a smooth manifold. A *smooth singular k-simplex* in M is a smooth map $\sigma \colon \Delta^k \to M$ (smooth meaning it extends to a smooth map on an open neighbourhood of Δ^k in the affine hyperplane containing Δ^k).

Definition 7.1.3 (Chain group)**.** The *smooth singular chain group* $C_k(M)$ is the free abelian group generated by all smooth singular k-simplices in M. An element $c = \sum_i a_i \sigma_i$ with $a_i \in \mathbb{Z}$ is called a *smooth singular k-chain.*

Definition 7.1.4 (Boundary operator)**.** For a smooth singular k-simplex $\sigma\colon \Delta^k \to M$, define the *boundary* $\partial\sigma \in C_{k-1}(M)$ by

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i \, \sigma \circ F_i^k,$$

where $F_i^k\colon \Delta^{k-1} \to \Delta^k$ is the i-th face map,

$$F_i^k(t_0, \dots, t_{k-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1}).$$

Extend ∂ linearly to all of $C_k(M)$.

Proposition 7.1.5. $\partial \circ \partial = 0$.

Proof. For any smooth singular k-simplex σ one computes

$$\partial(\partial\sigma) = \sum_{i=0}^{k} \sum_{j=0}^{k-1} (-1)^{i+j} \, \sigma \circ F_i^k \circ F_j^{k-1}.$$

The standard identity $F_i^k \circ F_j^{k-1} = F_j^k \circ F_{i-1}^{k-1}$ for $j < i$ shows that each term appears twice with opposite signs, hence the sum vanishes. ☐

7.2 Integration of forms on chains

Definition 7.2.1 (Integral over a simplex)**.** Let ω be a smooth k-form on M and $\sigma\colon \Delta^k \to M$ a smooth singular k-simplex. The *integral of ω over σ* is

$$\int_\sigma \omega = \int_{\Delta^k} \sigma^*\omega,$$

where the right-hand side is the ordinary Lebesgue integral of the pulled-back form on Δ^k.

Definition 7.2.2 (Integral over a chain)**.** For a k-chain $c = \sum_i a_i \sigma_i$, define

$$\int_c \omega = \sum_i a_i \int_{\sigma_i} \omega.$$

Remark 7.2.3. The map $\omega \mapsto \int_c \omega$ is $\mathbb{R}$-linear in ω, and the map $c \mapsto \int_c \omega$ is $\mathbb{Z}$-linear (and extends to $\mathbb{R}$-linear on real chains $C_k(M; \mathbb{R})$).

Proposition 7.2.4 (Stokes for a single simplex). *Let $\omega \in \Omega^{k-1}(M)$ and $\sigma\colon \Delta^k \to M$ a smooth singular k-simplex. Then*

$$\int_\sigma \mathrm{d}\omega = \int_{\partial\sigma} \omega.$$

Proof. By the naturality of pullback and the fact that d commutes with pullback, $\sigma^*(\mathrm{d}\omega) = \mathrm{d}(\sigma^*\omega)$. Write $\sigma^*\omega = f(t)\,\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_{k-1}$ on Δ^k (using an appropriate coordinate representation). The classical Stokes theorem for $\Delta^k \subset \mathbb{R}^k$ (a domain with piecewise smooth boundary) then gives

$$\int_{\Delta^k} \mathrm{d}(\sigma^*\omega) = \int_{\partial\Delta^k} \sigma^*\omega = \sum_{i=0}^{k}(-1)^i \int_{\Delta^{k-1}} (F_i^k)^*(\sigma^*\omega) = \int_{\partial\sigma} \omega. \qquad \square$$

Corollary 7.2.5 (Stokes for chains). *For any smooth k-chain c and $\omega \in \Omega^{k-1}(M)$,*

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega.$$

7.3 Oriented manifolds and integration

We briefly recall the integration theory on oriented manifolds to state Stokes' theorem in its manifold form.

Definition 7.3.1 (Integration on an oriented manifold). Let M be an oriented smooth n-manifold (possibly with boundary), and let $\omega \in \Omega^n_c(M)$ be a compactly supported n-form. Choose a locally finite atlas $\{(U_\alpha, \varphi_\alpha)\}$ of positively oriented charts and a subordinate partition of unity $\{\rho_\alpha\}$. Define

$$\int_M \omega = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*(\rho_\alpha\,\omega).$$

This is independent of all choices.

Definition 7.3.2 (Induced boundary orientation). Let M be an oriented n-manifold with boundary. The *induced orientation* on ∂M is defined by declaring that an ordered basis $(v_1, \dots, v_{n-1})$ of $T_p(\partial M)$ is positively oriented if and only if $(-\nu, v_1, \dots, v_{n-1})$ is a positively oriented basis of T_pM, where ν is the outward-pointing normal vector at p.

7.4 Stokes' theorem: statement and proof

Theorem 7.4.1 (Stokes' theorem). *Let M be an oriented smooth n-manifold with boundary (possibly $\partial M = \emptyset$), and let $\omega \in \Omega^{n-1}_c(M)$ be a compactly supported $(n-1)$-*

form. Then

$$\boxed{\int_M \mathrm{d}\omega = \int_{\partial M} \omega,}$$

where ∂M carries the induced boundary orientation.

Proof. The proof proceeds in three steps.

Step 1: Local statement in $\mathbb{R}^n$ and in the half-space $\mathbb{H}^n$.

Write $\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$. We first prove the theorem for compactly supported forms on $\mathbb{R}^n$ and on $\mathbb{H}^n$.

Case 1: ω supported in $\mathbb{R}^n$. Write

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} f_i(x)\, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n.$$

Then

$$\mathrm{d}\omega = \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \right) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

By Fubini's theorem,

$$\int_{\mathbb{R}^n} \mathrm{d}\omega = \sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{+\infty} \frac{\partial f_i}{\partial x_i}\, \mathrm{d}x_i \right) \mathrm{d}x_1 \cdots \widehat{\mathrm{d}x_i} \cdots \mathrm{d}x_n.$$

Each inner integral equals $f_i\Big|_{x_i=-\infty}^{x_i=+\infty} = 0$ since f_i is compactly supported. Thus $\int_{\mathbb{R}^n} \mathrm{d}\omega = 0$, and the right-hand side $\int_{\partial\mathbb{R}^n} \omega = 0$ trivially (since $\partial\mathbb{R}^n = \emptyset$).

Case 2: ω supported in $\mathbb{H}^n$. The same argument applies to all terms with $i < n$: the integrals vanish because f_i is compactly supported in the x_i-direction. For the last term ($i = n$),

$$\int_{\mathbb{H}^n} \frac{\partial f_n}{\partial x_n}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_{\mathbb{R}^{n-1}} \left(\int_{0}^{+\infty} \frac{\partial f_n}{\partial x_n}\, \mathrm{d}x_n \right) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}.$$

Since f_n is compactly supported, $f_n(x', +\infty) = 0$, and hence

$$\int_{0}^{+\infty} \frac{\partial f_n}{\partial x_n}\, \mathrm{d}x_n = -f_n(x_1, \ldots, x_{n-1}, 0).$$

Therefore

$$\int_{\mathbb{H}^n} \mathrm{d}\omega = (-1)^n \cdot (-1) \int_{\mathbb{R}^{n-1}} f_n(x', 0)\, \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}.$$

On the other hand, $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\}$ with the induced orientation (the outward normal is $-e_n$), so

$$\int_{\partial\mathbb{H}^n} \omega = (-1)^{n-1} \int_{\mathbb{R}^{n-1}} f_n(x', 0)\, \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1},$$

which matches.

Step 2: Partition of unity.

Choose a locally finite atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of positively oriented charts for M such that each $\varphi_\alpha(U_\alpha)$ is either an open subset of $\mathbb{R}^n$ (for interior charts) or an open subset of $\mathbb{H}^n$ (for boundary charts). Let $\{\rho_\alpha\}$ be a subordinate smooth partition of unity. Then $\omega = \sum_\alpha \rho_\alpha \omega$ (locally finite sum), and each $\rho_\alpha \omega$ is compactly supported in U_α.

Step 3: Reassembly.

Since $\mathrm{d}(\rho_\alpha\omega)$ is also compactly supported in U_α, Step 1 gives

$$\int_M \mathrm{d}(\rho_\alpha\omega) = \int_{\partial M} \rho_\alpha\omega$$

for each α. Summing over α (the sum is locally finite, so we may interchange summation and integration):

$$\int_M \mathrm{d}\omega = \int_M \mathrm{d}\left(\sum_\alpha \rho_\alpha\omega\right) = \sum_\alpha \int_M \mathrm{d}(\rho_\alpha\omega) = \sum_\alpha \int_{\partial M} \rho_\alpha\omega = \int_{\partial M} \omega. \qquad \square$$

Remark 7.4.2. When M is a closed manifold (compact, without boundary), Stokes' theorem gives $\int_M \mathrm{d}\omega = 0$ for every $\omega \in \Omega^{n-1}(M)$. This is the starting point for de Rham cohomology.

7.5 Classical special cases

Corollary 7.5.1 (Green's theorem)**.** *Let $D \subset \mathbb{R}^2$ be a compact region with smooth boundary ∂D oriented counterclockwise, and let $P, Q\colon D \to \mathbb{R}$ be smooth. Then*

$$\oint_{\partial D} (P\ \mathrm{d}x + Q\ \mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\ \mathrm{d}y.$$

Proof. Set $\omega = P\ \mathrm{d}x + Q\ \mathrm{d}y \in \Omega^1(D)$. Then $\mathrm{d}\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y$. The result is theorem 7.4.1 with $M = D$, $n = 2$. $\square$

Corollary 7.5.2 (Divergence theorem)**.** *Let $\Omega \subset \mathbb{R}^3$ be a compact domain with smooth boundary $\partial\Omega$, and let $\mathbf{F} = (F_1, F_2, F_3)$ be a smooth vector field on Ω. Then*

$$\iiint_\Omega \operatorname{div} \mathbf{F}\ \mathrm{d}V = \oiint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}S,$$

where $\mathbf{n}$ is the outward unit normal.

Proof. Define the 2-form $\omega = F_1\ \mathrm{d}y \wedge \mathrm{d}z + F_2\ \mathrm{d}z \wedge \mathrm{d}x + F_3\ \mathrm{d}x \wedge \mathrm{d}y$. Then $\mathrm{d}\omega = \operatorname{div} \mathbf{F}\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. Apply theorem 7.4.1. $\square$

Corollary 7.5.3 (Classical Stokes theorem in $\mathbb{R}^3$)**.** *Let $S \subset \mathbb{R}^3$ be a compact oriented surface with smooth boundary ∂S, and let $\mathbf{F}$ be a smooth vector field. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ \mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Proof. Define $\omega = F_1\ \mathrm{d}x + F_2\ \mathrm{d}y + F_3\ \mathrm{d}z$. Then $\mathrm{d}\omega$ corresponds to the curl $\nabla \times \mathbf{F}$ under the standard identification of 2-forms with vector fields via the Hodge star. Apply theorem 7.4.1 with $M = S$. $\square$

Example 7.5.4. Let T^2 be the 2-torus with the standard orientation, and let $\omega \in \Omega^1(T^2)$. Since $\partial T^2 = \emptyset$, Stokes' theorem gives $\int_{T^2} \mathrm{d}\omega = 0$. In particular, if $\omega = f\,\mathrm{d}\theta_1 + g\,\mathrm{d}\theta_2$ in the standard angular coordinates, then

$$\int_{T^2} \left(\frac{\partial g}{\partial \theta_1} - \frac{\partial f}{\partial \theta_2} \right) \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 = 0.$$

Example 7.5.5 (Area via Green's theorem)**.** Let $D \subset \mathbb{R}^2$ be a compact region with smooth boundary ∂D oriented counterclockwise. Choosing $P = -y/2$ and $Q = x/2$ in theorem 7.5.1, we obtain the area formula

$$\mathrm{Area}(D) = \frac{1}{2} \oint_{\partial D} (x\,\mathrm{d}y - y\,\mathrm{d}x).$$

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, parametrise $\gamma(t) = (a\cos t, b\sin t)$ for $t \in [0, 2\pi]$. Then

$$\mathrm{Area}(D) = \frac{1}{2} \int_0^{2\pi} (a\cos t \cdot b\cos t + b\sin t \cdot a\sin t)\,\mathrm{d}t = \frac{ab}{2} \int_0^{2\pi} \mathrm{d}t = \pi ab.$$

Example 7.5.6 (Divergence theorem: flux computation)**.** Let $\mathbf{F}(x, y, z) = (x^3, y^3, z^3)$ and let Ω be the unit ball in $\mathbb{R}^3$. Then $\operatorname{div} \mathbf{F} = 3(x^2 + y^2 + z^2) = 3r^2$. By the divergence theorem,

$$\oiint_{S^2} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_{\Omega} 3r^2\,\mathrm{d}V = 3 \int_0^1 r^2 \cdot 4\pi r^2\,\mathrm{d}r = 12\pi \int_0^1 r^4\,\mathrm{d}r = \frac{12\pi}{5}.$$

7.6 De Rham cohomology

Definition 7.6.1 (Closed and exact forms)**.** A differential form $\omega \in \Omega^k(M)$ is *closed* if $\mathrm{d}\omega = 0$, and *exact* if $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(M)$. Since $\mathrm{d}^2 = 0$, every exact form is closed. We write

$$Z^k(M) = \ker(\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)), \qquad B^k(M) = \mathrm{im}(\mathrm{d}\colon \Omega^{k-1}(M) \to \Omega^k(M)).$$

Definition 7.6.2 (De Rham cohomology)**.** The *k-th de Rham cohomology group* of M is the quotient vector space

$$H^k_{\mathrm{dR}}(M) = \frac{Z^k(M)}{B^k(M)} = \frac{\ker \mathrm{d}}{\mathrm{im}\ \mathrm{d}}.$$

The equivalence class $[\omega]$ of a closed form ω is called its *cohomology class*.

Remark 7.6.3. If $f\colon M \to N$ is smooth, then pullback $f^*\colon \Omega^k(N) \to \Omega^k(M)$ satisfies $f^* \circ \mathrm{d} = \mathrm{d} \circ f^*$, hence f^* descends to a linear map $f^*\colon H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$. Moreover, $(\mathrm{Id}_M)^* = \mathrm{Id}$ and $(g \circ f)^* = f^* \circ g^*$, so H^k_{dR} is a contravariant functor from smooth manifolds to real vector spaces.

Example 7.6.4. For $M = \{\mathrm{pt}\}$, the only nonzero differential forms are 0-forms (i.e., constants), and $\mathrm{d}f = 0$ for any constant f. Hence $H^0_{\mathrm{dR}}(\mathrm{pt}) \cong \mathbb{R}$ and $H^k_{\mathrm{dR}}(\mathrm{pt}) = 0$ for $k \geq 1$.

Proposition 7.6.5. *For any smooth manifold M, $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}^{\pi_0(M)}$, where $\pi_0(M)$ denotes the set of connected components of M.*

Proof. A 0-form $f \in C^\infty(M)$ is closed if and only if $\mathrm{d}f = 0$, i.e., f is locally constant. On a connected manifold a locally constant function is constant, so $H^0_{\mathrm{dR}}(M) = Z^0(M) \cong \mathbb{R}$. For a disconnected manifold the result follows by taking one constant per component. ☐

Remark 7.6.6 (Mayer–Vietoris sequence). If $M = U \cup V$ with U, V open, there is a long exact sequence

$$\cdots \to H^{k-1}_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} H^k_{\mathrm{dR}}(M) \xrightarrow{(i_U^*, i_V^*)} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \xrightarrow{j_U^* - j_V^*} H^k_{\mathrm{dR}}(U \cap V) \to \cdots$$

where $i_U : U \hookrightarrow M$, $i_V : V \hookrightarrow M$, $j_U : U \cap V \hookrightarrow U$, $j_V : U \cap V \hookrightarrow V$ are the inclusions. This is an extremely powerful computational tool: combined with homotopy invariance, it allows one to compute the de Rham cohomology of many manifolds by induction.

Example 7.6.7 (Cohomology of S^1)**.** Cover S^1 by two open arcs U and V, each diffeomorphic to $\mathbb{R}$, with $U \cap V$ the disjoint union of two open intervals (each diffeomorphic to $\mathbb{R}$). The Mayer–Vietoris sequence in degree 0 reads

$$0 \to H^0_{\mathrm{dR}}(S^1) \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} H^1_{\mathrm{dR}}(S^1) \to 0.$$

The map $\alpha(a, b) = (a - b, a - b)$, so $\ker \alpha \cong \mathbb{R}$ and $\operatorname{im} \alpha \cong \mathbb{R}$. Hence $H^0_{\mathrm{dR}}(S^1) \cong \mathbb{R}$ and $H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}^2/\mathbb{R} \cong \mathbb{R}$. Since $\dim S^1 = 1$, $H^k_{\mathrm{dR}}(S^1) = 0$ for $k \geq 2$.

7.7 The Poincaré lemma

Definition 7.7.1 (Star-shaped domain)**.** A subset $U \subset \mathbb{R}^n$ is *star-shaped with respect to a point* $p \in U$ if for every $x \in U$, the line segment from p to x lies entirely in U.

Theorem 7.7.2 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^n$ be a star-shaped open set. Then every closed k-form on U with $k \geq 1$ is exact. Equivalently,*

$$H^k_{\mathrm{dR}}(U) = 0 \qquad \text{for all } k \geq 1.$$

Proof. Without loss of generality assume U is star-shaped with respect to the origin. We construct a *chain homotopy operator* $K : \Omega^k(U) \to \Omega^{k-1}(U)$ satisfying

$$\mathrm{d}K + K\,\mathrm{d} = \mathrm{Id} \quad \text{on } \Omega^k(U), \quad k \geq 1. \tag{7.1}$$

For a k-form $\omega = \sum_I a_I(x)\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ (the sum over increasing multi-indices I), define

$$(K\omega)(x) = \sum_I \sum_{j=1}^{k} (-1)^{j-1} \left(\int_0^1 t^{k-1}\, a_I(tx)\, \mathrm{d}t \right) x_{i_j}\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_j}} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

A direct (if lengthy) computation using the Leibniz rule and $\frac{\mathrm{d}}{\mathrm{d}t}[a_I(tx)] = \sum_\ell x_\ell \frac{\partial a_I}{\partial x_\ell}(tx)$ verifies (7.1).

Now if ω is closed ($\mathrm{d}\omega = 0$) and $k \geq 1$, then $\omega = (\mathrm{d}K + K\,\mathrm{d})\omega = \mathrm{d}(K\omega)$, so ω is exact with primitive $\eta = K\omega$. □

Corollary 7.7.3. *$H^k_{\mathrm{dR}}(\mathbb{R}^n) = 0$ for all $k \geq 1$, and $H^0_{\mathrm{dR}}(\mathbb{R}^n) \cong \mathbb{R}$.*

Example 7.7.4 (Explicit primitive via homotopy operator)**.** Consider the closed 2-form $\omega = 2xy\,\mathrm{d}x \wedge \mathrm{d}y$ on $\mathbb{R}^2$ (star-shaped with respect to the origin). The homotopy operator from the proof of theorem 7.7.2 gives

$$(K\omega)(x, y) = \left(\int_0^1 t \cdot 2(tx)(ty)\, \mathrm{d}t \right) \big(x\,\mathrm{d}y - y\,\mathrm{d}x \big) = \left(\int_0^1 2t^3 xy\, \mathrm{d}t \right) (x\,\mathrm{d}y - y\,\mathrm{d}x).$$

Evaluating the integral: $\int_0^1 2t^3\,\mathrm{d}t = \frac{1}{2}$, so

$$K\omega = \frac{xy}{2}(x\,\mathrm{d}y - y\,\mathrm{d}x) = \frac{x^2 y}{2}\,\mathrm{d}y - \frac{xy^2}{2}\,\mathrm{d}x.$$

One verifies: $\mathrm{d}(K\omega) = \mathrm{d}\Big(\frac{x^2y}{2}\,\mathrm{d}y - \frac{xy^2}{2}\,\mathrm{d}x\Big) = \Big(\frac{2xy}{2} + \frac{y^2}{2} + \frac{x^2}{2} - \frac{2xy}{2}\Big)\,\mathrm{d}x \wedge \mathrm{d}y \ldots$ which does not simplify to ω because one must use the correct formula more carefully. In fact, let us apply the formula directly for $\omega = f(x, y)\,\mathrm{d}x \wedge \mathrm{d}y$ with $f = 2xy$:

$$\begin{aligned} K\omega &= \left(\int_0^1 t \cdot f(tx, ty)\, \mathrm{d}t \right) x\,\mathrm{d}y - \left(\int_0^1 t \cdot f(tx, ty)\, \mathrm{d}t \right) y\,\mathrm{d}x \\ &= x \left(\int_0^1 2t^3 xy\, \mathrm{d}t \right) \mathrm{d}y - y \left(\int_0^1 2t^3 xy\, \mathrm{d}t \right) \mathrm{d}x \\ &= \frac{x^2 y}{2}\,\mathrm{d}y - \frac{xy^2}{2}\,\mathrm{d}x. \end{aligned}$$

Computing $\mathrm{d}(K\omega)$:

$$\mathrm{d}\left(-\frac{xy^2}{2}\,\mathrm{d}x + \frac{x^2 y}{2}\,\mathrm{d}y \right) = \left(\frac{2xy}{2} + \frac{2xy}{2} \right) \mathrm{d}x \wedge \mathrm{d}y = 2xy\,\mathrm{d}x \wedge \mathrm{d}y = \omega. \checkmark$$

Example 7.7.5. Let $\omega = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$. One checks $\mathrm{d}\omega = 0$, yet ω is not exact: $\int_{S^1} \omega = 2\pi \neq 0$ (and Stokes' theorem would give 0 if $\omega = \mathrm{d}f$). This shows $H^1_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{0\}) \neq 0$. In fact, $H^1_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$, generated by $[\omega/2\pi]$.

7.8 Homotopy invariance

Theorem 7.8.1 (Homotopy invariance of de Rham cohomology). *If $f, g\colon M \to N$ are smoothly homotopic (i.e., there exists a smooth map $H\colon M \times [0,1] \to N$ with $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$), then $f^* = g^*\colon H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ for all k.*

Proof. We construct a *cochain homotopy* $h\colon \Omega^k(N) \to \Omega^{k-1}(M)$ satisfying

$$g^* - f^* = \mathrm{d} \circ h + h \circ \mathrm{d}.$$

Let $\iota_t\colon M \hookrightarrow M \times [0,1]$, $\iota_t(p) = (p,t)$. For $\omega \in \Omega^k(N)$, set

$$h(\omega) = \int_0^1 \iota_t^*\Big(\iota_{\partial/\partial t} H^*\omega\Big)\, \mathrm{d}t,$$

where $\iota_{\partial/\partial t}$ denotes interior multiplication by $\partial/\partial t$. Cartan's magic formula gives

$$\mathcal{L}_{\partial/\partial t}(H^*\omega) = \mathrm{d}(\iota_{\partial/\partial t} H^*\omega) + \iota_{\partial/\partial t}\, \mathrm{d}(H^*\omega),$$

and integrating from $t = 0$ to $t = 1$ yields the desired identity. Hence if $\mathrm{d}\omega = 0$, then $g^*\omega - f^*\omega = \mathrm{d}(h\omega)$, so $[g^*\omega] = [f^*\omega]$. □

Example 7.8.2. The inclusion $\iota\colon S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and the retraction $r\colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, $r(x) = x/|x|$, satisfy $r \circ \iota = \mathrm{Id}_{S^{n-1}}$ and $\iota \circ r \simeq \mathrm{Id}_{\mathbb{R}^n \setminus \{0\}}$ (via the straight-line homotopy $H(x,t) = (1-t)x + t\,x/|x|$, which stays in $\mathbb{R}^n \setminus \{0\}$). Hence by homotopy invariance,

$$H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\}) \cong H^k_{\mathrm{dR}}(S^{n-1}) \cong \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 2$, this recovers $H^1_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$, generated by the angle form of theorem 7.7.5.

Corollary 7.8.3 (Homotopy equivalence). *If $f\colon M \to N$ is a smooth homotopy equivalence, then $f^*\colon H^k_{\mathrm{dR}}(N) \xrightarrow{\sim} H^k_{\mathrm{dR}}(M)$ is an isomorphism for all k.*

Corollary 7.8.4. *If M is contractible, then $H^k_{\mathrm{dR}}(M) = 0$ for $k \geq 1$ and $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$. In particular, the Poincaré lemma is a special case of homotopy invariance.*

7.9 Degree via integration

Definition 7.9.1 (Degree of a map). Let M, N be compact, connected, oriented smooth n-manifolds without boundary, and let $f\colon M \to N$ be smooth. Choose any $\omega \in \Omega^n(N)$ with $\int_N \omega \neq 0$. The *degree* of f is

$$\deg(f) = \frac{\int_M f^*\omega}{\int_N \omega}.$$

Theorem 7.9.2. *The degree* $\deg(f)$ *is a well-defined integer, independent of the choice of* ω. *Moreover,* $\deg(f)$ *is a homotopy invariant: if* $f \simeq g$, *then* $\deg f = \deg g$.

Proof. The map $f^*: H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$ is a linear map between one-dimensional vector spaces (since M and N are compact, connected, and oriented, $H^n_{\mathrm{dR}} \cong \mathbb{R}$). Hence $f^*[\omega] = \lambda[\mu]$ for some $\lambda \in \mathbb{R}$, where $[\mu]$ generates $H^n_{\mathrm{dR}}(M)$. The ratio $\int_M f^*\omega / \int_N \omega$ equals this constant λ, independent of ω. That $\lambda \in \mathbb{Z}$ follows from the local analysis: at a regular value $q \in N$, $f^{-1}(q)$ is finite and $\deg f = \sum_{p \in f^{-1}(q)} \operatorname{sign}\det \mathrm{d}f_p$, which is manifestly an integer. Homotopy invariance follows from theorem 7.8.1. □

Example 7.9.3. The map $f: S^1 \to S^1$, $f(z) = z^k$, has $\deg f = k$. Indeed, if $\omega = \mathrm{d}\theta/(2\pi)$ is the normalised volume form, then $f^*\omega = k\, \mathrm{d}\theta/(2\pi)$ and $\int_{S^1} f^*\omega = k$.

Example 7.9.4. The antipodal map $A: S^n \to S^n$, $A(x) = -x$, has $\deg A = (-1)^{n+1}$. In particular, A is homotopic to the identity if and only if n is odd.

Proposition 7.9.5 (Degree and regular values)**.** *Let* $f: M^n \to N^n$ *be a smooth map between compact connected oriented manifolds. If* $q \in N$ *is a regular value of* f, *then* $f^{-1}(q)$ *is a finite set and*

$$\deg f = \sum_{p \in f^{-1}(q)} \operatorname{sign}\det(\,\mathrm{d}f_p),$$

where $\operatorname{sign}\det(\,\mathrm{d}f_p) = +1$ *if* $\mathrm{d}f_p$ *preserves orientation and* -1 *if it reverses orientation.*

Proof. Since q is a regular value, $f^{-1}(q)$ is a 0-dimensional submanifold of the compact manifold M, hence a finite set $\{p_1, \ldots, p_r\}$. By the inverse function theorem, there exist disjoint open neighbourhoods U_i of p_i mapped diffeomorphically by f onto a common neighbourhood V of q. Choose an n-form η supported in V with $\int_V \eta = 1$. Then

$$\int_M f^*\eta = \sum_{i=1}^r \int_{U_i} f^*\eta = \sum_{i=1}^r \operatorname{sign}\det(\,\mathrm{d}f_{p_i}) \int_V \eta = \sum_{i=1}^r \operatorname{sign}\det(\,\mathrm{d}f_{p_i}),$$

since $f|_{U_i}$ is a diffeomorphism onto V. Dividing by $\int_N \eta = 1$ gives the result. □

Example 7.9.6. The Hopf map $h: S^3 \to S^2$ is not a map between manifolds of the same dimension, so the degree formula above does not directly apply. However, one can define the *Hopf invariant* using linking numbers and de Rham cohomology. We mention this as motivation for the rich interplay between topology and differential forms.

7.10 Poincaré duality

Remark 7.10.1 (Poincaré duality – informal statement). For a closed oriented n-manifold M, the wedge product followed by integration defines a nondegenerate bilinear pairing

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \longrightarrow \mathbb{R}, \qquad ([\alpha],[\beta]) \longmapsto \int_M \alpha \wedge \beta.$$

This is the *Poincaré duality* isomorphism $H^k_{\mathrm{dR}}(M) \cong \left(H^{n-k}_{\mathrm{dR}}(M)\right)^*$. When each $H^k_{\mathrm{dR}}(M)$ is finite-dimensional (as is the case for compact manifolds), this gives $\dim H^k_{\mathrm{dR}}(M) = \dim H^{n-k}_{\mathrm{dR}}(M)$. A full proof requires compactly supported cohomology and the Mayer–Vietoris sequence; we refer the reader to Bott–Tu *Differential Forms in Algebraic Topology*.

Example 7.10.2. For S^n $(n \geq 1)$, $H^k_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R} & k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$ Poincaré duality is reflected in the symmetry $H^0 \cong H^n \cong \mathbb{R}$.

7.11 Exercises

Exercise 7.1. Let $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$ with the standard orientation. Compute $\int_A \mathrm{d}\omega$ directly and via Stokes' theorem, where $\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$.

Exercise 7.2. Compute $H^k_{\mathrm{dR}}(S^1 \times \mathbb{R})$ for all k using homotopy invariance.

Exercise 7.3. Let $\omega = e^{xyz}(yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z)$ on $\mathbb{R}^3$. Verify that $\mathrm{d}\omega = 0$ and find an explicit primitive η with $\mathrm{d}\eta = \omega$ using the homotopy operator from the Poincaré lemma.

Exercise 7.4. Let $f\colon S^2 \to S^2$ be defined by $f(x,y,z) = (2xz, 2yz, 2z^2 - 1)$ (this is the map induced by $z \mapsto z^2$ under stereographic projection). Compute $\deg f$ by integrating the pullback of the area form.

Exercise 7.5. Let $T^2 = S^1 \times S^1$ with angular coordinates (θ_1, θ_2). Show that $\omega_1 = \mathrm{d}\theta_1$ and $\omega_2 = \mathrm{d}\theta_2$ are closed but not exact, and that $\{[\omega_1], [\omega_2]\}$ is a basis for $H^1_{\mathrm{dR}}(T^2)$.

Exercise 7.6. Using the fact that $S^n = U_+ \cup U_-$ where $U_\pm = S^n \setminus \{\mp e_{n+1}\}$ are each diffeomorphic to $\mathbb{R}^n$, and the Mayer–Vietoris sequence

$$\cdots \to H^{k-1}_{\mathrm{dR}}(U_+ \cap U_-) \to H^k_{\mathrm{dR}}(S^n) \to H^k_{\mathrm{dR}}(U_+) \oplus H^k_{\mathrm{dR}}(U_-) \to H^k_{\mathrm{dR}}(U_+ \cap U_-) \to \cdots$$

compute $H^k_{\mathrm{dR}}(S^n)$ by induction on n.

Exercise 7.7. Use the degree of the antipodal map and the notion of degree to prove that S^{2n} admits no nowhere-vanishing smooth vector field (the hairy ball theorem). *Hint:* A nowhere-vanishing vector field gives a homotopy from Id to the antipodal map.

Exercise 7.8. Let $\Sigma \subset \mathbb{R}^3$ be a closed oriented surface enclosing a bounded domain Ω, and let $\mathbf{F}$ be a smooth vector field on $\mathbb{R}^3$ with $\operatorname{div} \mathbf{F} = 1$ everywhere. Show that $\oiint_\Sigma \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \mathrm{Vol}(\Omega)$.

Exercise 7.9. Explain why Stokes' theorem (in its manifold form) does not directly apply to the Möbius band. What additional structure would be needed to define integration of 2-forms on a Möbius band?

Exercise 7.10. Using the fact that $S^n \to \mathbb{RP}^n$ is a 2-sheeted covering space, show that

$$H^k_{\mathrm{dR}}(\mathbb{RP}^n) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, \\ \mathbb{R} & \text{if } k = n \text{ and } n \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Hint: The pullback of forms via the covering map identifies $\Omega^k(\mathbb{RP}^n)$ with the $\mathbb{Z}/2$-invariant forms on S^n.

Exercise 7.11. Let $\omega = x \, \mathrm{d}y \wedge \mathrm{d}z + y \, \mathrm{d}z \wedge \mathrm{d}x + z \, \mathrm{d}x \wedge \mathrm{d}y$ be the standard area form on S^2 (viewed as the restriction of a 2-form on $\mathbb{R}^3$). Compute $\int_{S^2} \omega$ using the parametrisation by spherical coordinates and verify that $[\omega] \neq 0$ in $H^2_{\mathrm{dR}}(S^2)$.

Exercise 7.12. Use the notion of degree to give a topological proof that every nonconstant polynomial $p\colon \mathbb{C} \to \mathbb{C}$ has a root. *Hint:* Consider $f_t(z) = p(tz)/\,|p(tz)|$ for $z \in S^1$ and use the homotopy invariance of degree.

Chapter 8

Sard's Theorem

Contents

Sard's theorem is one of the most fundamental results in differential topology. It asserts that the set of critical values of any smooth map has measure zero, so that "almost every" value is a regular value. This seemingly analytic statement has profound topological consequences: the existence of regular values leads to the non-existence of smooth retractions $D^n \to S^{n-1}$ and hence to the Brouwer fixed-point theorem, as well as to transversality results that underpin much of modern differential topology.

8.1 Critical and regular values

Definition 8.1.1 (Critical and regular points)**.** Let $f\colon M^m \to N^n$ be a smooth map between smooth manifolds. A point $p \in M$ is a *critical point* of f if the differential $\mathrm{d}f_p\colon T_pM \to T_{f(p)}N$ is not surjective (i.e., rank $\mathrm{d}f_p < n$). If $\mathrm{d}f_p$ is surjective, p is a *regular point*.

Definition 8.1.2 (Critical and regular values)**.** A point $q \in N$ is a *critical value* of f if $q = f(p)$ for some critical point p of f. A point $q \in N$ is a *regular value* if it is not a critical value, i.e., for every $p \in f^{-1}(q)$ the differential $\mathrm{d}f_p$ is surjective.

Remark 8.1.3. A point $q \in N \setminus f(M)$ is automatically a regular value (vacuously: there are no points in $f^{-1}(q)$ to fail surjectivity).

Remark 8.1.4. If every point of M is a regular point, then f is called a *submersion*. By the implicit function theorem, a submersion $f\colon M^m \to N^n$ (with $m \geq n$) has the property that $f^{-1}(q)$ is a smooth submanifold of M of dimension $m - n$ for every $q \in f(M)$.

Example 8.1.5. Consider the height function $f\colon S^2 \to \mathbb{R}$, $f(x, y, z) = z$. The critical points are the north and south poles, $p_\pm = (0, 0, \pm 1)$, where $\mathrm{d}f_{p_\pm} = 0$. The critical values are $\{-1, +1\}$; every $c \in (-1, 1)$ is a regular value and $f^{-1}(c)$ is a circle.

Example 8.1.6. Consider the smooth map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (x^2 - y^2, 2xy)$. The Jacobian is

$$\mathrm{d}f_{(x,y,z)} = \begin{pmatrix} 2x & -2y & 0 \\ 2y & 2x & 0 \end{pmatrix}.$$

This has rank < 2 precisely when $x = y = 0$, i.e., the critical set is the z-axis $\{(0, 0, z) \mid z \in \mathbb{R}\}$. The set of critical values is $f(\{(0, 0, z)\}) = \{(0, 0)\}$, a single point—which indeed has measure zero in $\mathbb{R}^2$.

Example 8.1.7. The *Whitney umbrella* is the image of $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(u, v) = (uv, u, v^2)$. The Jacobian is $\mathrm{d}f_{(u,v)} = \left(\begin{smallmatrix} v & u \\ 1 & 0 \\ 0 & 2v \end{smallmatrix}\right)$, which has rank 2 unless $v = 0$ (giving rank $= 1$). Since $m = 2 < n = 3$, every point is critical (the differential is never surjective), so $f(\mathbb{R}^2)$ has measure zero in $\mathbb{R}^3$ by theorem 8.2.3(iii).

8.2 Measure zero in $\mathbb{R}^n$

Definition 8.2.1 (Measure zero)**.** A subset $A \subset \mathbb{R}^n$ has *(Lebesgue) measure zero* if for every $\varepsilon > 0$ there exists a countable collection of open cubes $\{Q_j\}_{j=1}^\infty$ in $\mathbb{R}^n$ such that $A \subset \bigcup_j Q_j$ and $\sum_j \mathrm{vol}(Q_j) < \varepsilon$.

Remark 8.2.2. One may equivalently use open balls, rectangles, or cubes. The notion is independent of the particular covering shapes used.

Lemma 8.2.3. *The following hold:*

(i) A countable union of sets of measure zero has measure zero.

(ii) If $A \subset \mathbb{R}^n$ has measure zero and $m > n$, then $A \times \{0\} \subset \mathbb{R}^m$ has measure zero (as a subset of $\mathbb{R}^m$).

(iii) If $A \subset \mathbb{R}^n$ has measure zero and $g\colon U \to \mathbb{R}^n$ is a smooth map defined on an open set $U \supset A$, then $g(A)$ has measure zero.

(iv) $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ has measure zero for $k < n$.

Proof. (i) is a standard $\varepsilon/2^j$ argument. (ii) follows because a cube of side ℓ in $\mathbb{R}^n$ times $\{0\}$ can be covered by a thin slab of volume ε. For (iii), on any compact subset $K \subset U$, g is Lipschitz with constant C, and g maps a cube of side ℓ into a cube of side $C\sqrt{n}\,\ell$, hence of volume $(C\sqrt{n}\,\ell)^n = C^n n^{n/2} \ell^n$. The covering of $A \cap K$ by cubes of total volume $< \varepsilon$ gives a covering of $g(A \cap K)$ by cubes of total volume $< C^n n^{n/2}\varepsilon$. Since $A = \bigcup_{j=1}^{\infty}(A \cap K_j)$ for an exhaustion by compact sets, (i) completes the proof. (iv) follows from (i) and (ii). □

Definition 8.2.4 (Measure zero in manifolds)**.** A subset A of a smooth n-manifold N has *measure zero* if for every chart (U, φ) of N, $\varphi(A \cap U)$ has measure zero in $\mathbb{R}^n$. By theorem 8.2.3(iii), this is well-defined (independent of the atlas).

Lemma 8.2.5 (Fubini-type lemma for measure zero)**.** *Let $A \subset \mathbb{R}^{n+1}$ be a subset such that for every $t \in \mathbb{R}$, the slice $A_t = \{x \in \mathbb{R}^n \mid (x, t) \in A\}$ has measure zero in $\mathbb{R}^n$. If furthermore the set $T = \{t \in \mathbb{R} \mid A_t \neq \emptyset\}$ has measure zero in $\mathbb{R}$, then A has measure zero in $\mathbb{R}^{n+1}$.*

Proof. For each $t \in T$, cover A_t by cubes in $\mathbb{R}^n$ of total n-dimensional volume less than ε_t. Taking products with thin intervals around t and using a careful covering of T by intervals of total length $< \varepsilon$, one obtains a covering of A by $(n+1)$-cubes of total volume $< C\varepsilon$. Details are left as an exercise. □

More generally, we have the following useful strengthening.

Lemma 8.2.6 (Fubini-type lemma, general form)**.** *Let $A \subset \mathbb{R}^{n+k}$. If for almost every $y \in \mathbb{R}^k$, the slice $A_y = \{x \in \mathbb{R}^n \mid (x, y) \in A\}$ has measure zero in $\mathbb{R}^n$, then A has measure zero in $\mathbb{R}^{n+k}$.*

8.3 Sard's theorem: statement and proof

Theorem 8.3.1 (Sard's theorem, 1942)**.** *Let $f\colon M^m \to N^n$ be a smooth map between smooth manifolds. Then the set of critical values of f has measure zero in N.*

The proof is by induction on m. We first treat the Euclidean case.

Theorem 8.3.2 (Sard's theorem, Euclidean version)**.** *Let $U \subset \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}^n$ smooth. Let $C = \{x \in U \mid \operatorname{rank} \mathrm{d}f_x < n\}$ be the set of critical points. Then $f(C)$ has measure zero in $\mathbb{R}^n$.*

Proof. We proceed by induction on m.

Base case $m = 0$: f is a constant map and $f(U)$ is a single point (or empty). If $n \geq 1$, a single point has measure zero. If $n = 0$, there are no critical points.

Inductive step: Assume the theorem holds for all smooth maps from open subsets of $\mathbb{R}^{m-1}$. We stratify the critical set C as follows. For $j \geq 1$, define

$$C_j = \left\{ x \in U \;\middle|\; \frac{\partial^\alpha f}{\partial x^\alpha}(x) = 0 \text{ for all multi-indices } |\alpha| \leq j \right\}.$$

(Here we adopt the convention that $C_0 = C$.) We have a descending filtration

$$C = C_0 \supset C_1 \supset C_2 \supset \cdots \supset C_{m-n+1} \supset \cdots$$

where each C_j is closed in U.

The proof is completed in three claims.

Claim 1: $f(C \setminus C_1)$ has measure zero.

Let $x_0 \in C \setminus C_1$. Then some first partial derivative of a component of f is nonzero at x_0 but rank $\mathrm{d}f_{x_0} < n$. Without loss of generality (reordering coordinates), assume $\frac{\partial f_1}{\partial x_1}(x_0) \neq 0$. Define $g\colon U \to \mathbb{R}^m$ by

$$g(x) = (f_1(x), x_2, \ldots, x_m).$$

By the inverse function theorem, g is a diffeomorphism on a neighbourhood V of x_0. Then $h = f \circ g^{-1}\colon g(V) \to \mathbb{R}^n$ has the form $h(y_1, y_2, \ldots, y_m) = (y_1, h_2(y), \ldots, h_n(y))$. A point $y \in g(V)$ is a critical point of h if and only if the map $(y_2, \ldots, y_m) \mapsto (h_2(y_1, y_2, \ldots, y_m), \ldots, h_n(y_1, y_2, \ldots, y_m))$ has non-surjective differential (since the first component is just y_1).

For fixed $y_1 = c$, define $h^c\colon \mathbb{R}^{m-1} \to \mathbb{R}^{n-1}$ by $h^c(y_2, \ldots, y_m) = (h_2(c, y_2, \ldots, y_m), \ldots, h_n(c, y_2, \ldots, y_m))$. Then y is a critical point of h if and only if $(y_2, \ldots, y_m)$ is a critical point of h^c. By the induction hypothesis (applied in dimension $m-1$), $h^c(\text{crit. pts. of } h^c)$ has measure zero in $\mathbb{R}^{n-1}$ for each c. Thus, by theorem 8.2.5, $h(\text{crit. pts. of } h \text{ in } g(V))$ has measure zero in $\mathbb{R}^n$. Since $f(C \cap V) = h(g(C \cap V))$, and $C \setminus C_1$ is covered by countably many such neighbourhoods V, we conclude $f(C \setminus C_1)$ has measure zero.

Claim 2: $f(C_j \setminus C_{j+1})$ has measure zero for each $j \geq 1$.

Let $x_0 \in C_j \setminus C_{j+1}$. Then all partial derivatives of all components of f up to order j vanish at x_0, but some $(j+1)$-st partial derivative is nonzero. Without loss of generality, assume $\frac{\partial^{j+1} f_1}{\partial x_1 \partial x^\beta}(x_0) \neq 0$ for some multi-index β with $|\beta| = j$. Set $w(x) = \frac{\partial^j f_1}{\partial x^\beta}(x)$; then $w(x_0) = 0$ and $\frac{\partial w}{\partial x_1}(x_0) \neq 0$.

Define $g(x) = (w(x), x_2, \ldots, x_m)$. By the inverse function theorem, g is a local diffeomorphism near x_0. On the set $g(C_j \cap V)$, we have $y_1 = 0$ (since w vanishes on C_j locally). Thus $g(C_j \cap V) \subset \{0\} \times \mathbb{R}^{m-1}$, and we can apply the induction hypothesis to the restricted map $f \circ g^{-1}\big|_{\{0\}\times\mathbb{R}^{m-1}}$ (an $(m-1)$-dimensional problem). By induction, $f(C_j \cap V)$ has measure zero, and the result follows by a countable cover.

Claim 3: $f(C_k)$ has measure zero for $k \geq m-n+1$ (in fact for k sufficiently large).

We use a Taylor expansion argument. Let $x_0 \in C_k$ and work in a small cube Q of side δ centered at x_0. By the vanishing of all derivatives up to order k, Taylor's theorem gives

$$|f(x) - f(x_0)| \leq C\,|x - x_0|^{k+1} \leq C'\delta^{k+1}$$

for $x \in Q$. Hence $f(C_k \cap Q)$ is contained in a cube of side $O(\delta^{k+1})$ in $\mathbb{R}^n$, with n-dimensional volume $O(\delta^{n(k+1)})$.

Now subdivide Q into N^m subcubes of side δ/N. The image of C_k intersected with each subcube lies in a cube of volume $O((\delta/N)^{n(k+1)})$. The total volume is bounded by

$$N^m \cdot O\left(\left(\frac{\delta}{N}\right)^{n(k+1)}\right) = O(N^{m-n(k+1)}).$$

If $k+1 > m/n$, i.e., $k \geq \lceil m/n \rceil$, then $m - n(k+1) < 0$ and this tends to 0 as $N \to \infty$. In particular, the condition $k \geq m - n + 1$ suffices (since $m - n + 1 \geq m/n$ when $n \geq 1$). Hence $f(C_k)$ has measure zero.

Combining Claims 1, 2, and 3:

$$f(C) = f(C \setminus C_1) \cup f(C_1 \setminus C_2) \cup \cdots \cup f(C_{k-1} \setminus C_k) \cup f(C_k)$$

is a finite union of sets of measure zero, hence has measure zero. □

Proof of theorem A.2.1 (manifold version). Cover M by countably many charts $(U_\alpha, \varphi_\alpha)$ and N by charts (V_β, ψ_β). The critical values of f are

$$f(C_f) = \bigcup_{\alpha,\beta} \psi_\beta^{-1}\Big((\psi_\beta \circ f \circ \varphi_\alpha^{-1})(C_{\psi_\beta \circ f \circ \varphi_\alpha^{-1}})\Big).$$

By theorem 8.3.2, each set $(\psi_\beta \circ f \circ \varphi_\alpha^{-1})(C_{\dots})$ has measure zero. Since ψ_β^{-1} is a diffeomorphism, the preimage has measure zero in N (theorem 8.2.3(iii)). A countable union of measure zero sets has measure zero. □

8.4 Regular values are dense

Corollary 8.4.1 (Regular values are dense)**.** *Let $f\colon M \to N$ be smooth. Then the set of regular values of f is dense in N. More precisely, it is a residual set (countable intersection of open dense sets) and in particular dense by the Baire category theorem.*

Proof. A set of measure zero in $\mathbb{R}^n$ has empty interior (since open sets have positive measure). Hence its complement is dense. The set of regular values, being the complement of a measure-zero set, is dense in N.

For the residual statement, write $M = \bigcup_j K_j$ with K_j compact. The set of critical values contributed by K_j is compact (as the image of a compact set) and of measure zero, hence closed and nowhere dense. Its complement is open and dense. The set of regular values contains $\bigcap_j (N \setminus f(C_f \cap K_j))$, a countable intersection of open dense sets. □

Remark 8.4.2. Sard's theorem requires f to be sufficiently smooth. For $f \in C^r$ with $r \geq \max(m-n+1, 1)$, the conclusion still holds (this is the sharp regularity threshold due to Whitney). In the C^∞ category we work in, this is never an issue. The necessity of the smoothness condition is illustrated by the following: there exist C^1 maps $f\colon [0,1] \to [0,1]$ whose set of critical values has positive measure (see Whitney's 1935 example).

Remark 8.4.3 (Sard's theorem and the Brown–de Rham theorem). Sard's theorem has far-reaching consequences beyond those discussed here. Combined with the implicit function theorem, it implies:

(i) Every smooth manifold admits a Morse function (functions whose critical points are all nondegenerate).

(ii) The set of embeddings $M \hookrightarrow \mathbb{R}^N$ is dense for $N \geq 2\dim M + 1$ (Whitney embedding theorem).

(iii) Every smooth map can be approximated by maps transverse to any given submanifold.

These results form the foundation of differential topology.

8.5 Existence of regular values

Corollary 8.5.1. *Let $f\colon M^m \to N^n$ be smooth with $m \geq n$. Then there exists a regular value $q \in N$. If $q \in f(M)$, then $f^{-1}(q)$ is a smooth submanifold of M of dimension $m - n$.*

Proof. By theorem 8.4.1 regular values are dense, hence exist. The preimage theorem (implicit function theorem on manifolds) gives the submanifold conclusion. □

Example 8.5.2. Consider $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x) = |x|^2$. Then $\mathrm{d}f_x = 2x^T$, which is surjective for $x \neq 0$. Hence every $c > 0$ is a regular value and $f^{-1}(c) = S^n(\sqrt{c})$ is a smooth n-manifold.

Example 8.5.3. Consider $f\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{Sym}(n,\mathbb{R})$ defined by $f(A) = A^T A$. One shows that the identity matrix I is a regular value of f. Hence $f^{-1}(I) = O(n)$ is a smooth submanifold of $\mathrm{GL}(n,\mathbb{R})$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

8.6 No retraction theorem and Brouwer fixed point

Lemma 8.6.1 (Classification of compact 1-manifolds)**.** *Every compact connected smooth 1-manifold with boundary is diffeomorphic to either S^1 (if $\partial M = \emptyset$) or $[0,1]$ (if $\partial M \neq \emptyset$). In particular, every compact 1-manifold with boundary has an even number of boundary points.*

Proof sketch. Let M be a compact connected 1-manifold with boundary. Choose a non-vanishing vector field X on M (which exists since M is 1-dimensional). The flow of X starting at any interior point traces out an injective curve; by compactness this curve must eventually return to its starting point (giving S^1) or reach a boundary point (giving $[0,1]$). Since each connected component diffeomorphic to $[0,1]$ contributes exactly 2 boundary points and each component diffeomorphic to S^1 contributes 0, the total number of boundary points is even. □

Definition 8.6.2 (Retraction)**.** A *smooth retraction* of D^n onto S^{n-1} is a smooth map $r\colon D^n \to S^{n-1}$ such that $r(x) = x$ for all $x \in S^{n-1}$.

Theorem 8.6.3 (No retraction theorem)**.** *There is no smooth retraction of D^n onto S^{n-1} for any $n \geq 1$.*

Proof. Suppose for contradiction that $r\colon D^n \to S^{n-1}$ is a smooth retraction. By Sard's theorem (theorem A.2.1), r has a regular value $q \in S^{n-1}$.

Consider $r^{-1}(q)$. Since q is a regular value, $r^{-1}(q)$ is a smooth 1-dimensional submanifold of D^n (since $\dim D^n - \dim S^{n-1} = n - (n-1) = 1$). Moreover, $r^{-1}(q)$ is compact (as a closed subset of the compact set D^n), so it consists of finitely many circles and arcs.

Since $r|_{S^{n-1}} = \mathrm{Id}_{S^{n-1}}$, the point $q \in S^{n-1}$ has exactly one preimage on the boundary, namely q itself. The arcs have their endpoints on $\partial D^n = S^{n-1}$, and the total number of boundary endpoints must be even (each arc contributes 0 or 2 boundary points, except for arcs with one interior and one boundary endpoint).

More precisely, $r^{-1}(q)$ is a compact 1-manifold with boundary $r^{-1}(q) \cap S^{n-1}$. By the classification of compact 1-manifolds with boundary, each component is either a circle or a closed interval. The boundary points of $r^{-1}(q)$ (as a manifold with boundary) are exactly the points of $r^{-1}(q) \cap S^{n-1}$, which is just $\{q\}$ (since $r|_{S^{n-1}} = \mathrm{Id}$). But a compact 1-manifold with boundary has an *even* number of boundary points (each interval component contributes exactly 2). This contradicts the fact that $r^{-1}(q) \cap S^{n-1} = \{q\}$ has exactly one point. □

Theorem 8.6.4 (Brouwer fixed-point theorem, smooth version)**.** *Every smooth map $f\colon D^n \to D^n$ has a fixed point.*

Proof. Suppose f has no fixed point, i.e., $f(x) \neq x$ for all $x \in D^n$. Define $r\colon D^n \to S^{n-1}$ by letting $r(x)$ be the point where the ray from $f(x)$ through x intersects S^{n-1}. Explicitly, $r(x) = x + t(x)(x - f(x))$ where $t(x) \geq 0$ is determined by $|r(x)| = 1$.

One checks that t depends smoothly on x (by the implicit function theorem applied to $|x + t(x - f(x))|^2 = 1$), so r is smooth. For $x \in S^{n-1}$: since $|x| = 1$ and $|f(x)| \leq 1$ with $f(x) \neq x$, the ray from $f(x)$ through x exits the disk at x itself, so $r(x) = x$.

Thus r is a smooth retraction $D^n \to S^{n-1}$, contradicting theorem 8.6.3. □

Corollary 8.6.5 (Brouwer fixed-point theorem, continuous version)**.** *Every continuous map $f\colon D^n \to D^n$ has a fixed point.*

Proof. By the Weierstrass approximation theorem, we can approximate any continuous $f\colon D^n \to D^n$ uniformly by smooth maps $f_k\colon D^n \to \mathbb{R}^n$ with $f_k \to f$ uniformly. For k sufficiently large, $g_k(x) = f_k(x)/\max(1, |f_k(x)|)$ maps D^n to D^n and is smooth (or at worst Lipschitz). One can refine this argument via a smooth approximation that stays in D^n, and then apply theorem 8.6.4 to obtain fixed points x_k of g_k. By compactness, $x_k \to x$ (along a subsequence), and continuity gives $f(x) = x$. □

Remark 8.6.6. An alternative approach uses Stokes' theorem directly. If $r\colon D^n \to S^{n-1}$ were a smooth retraction, then the volume form ω of S^{n-1} would satisfy $\int_{S^{n-1}} \omega = \int_{S^{n-1}} r^*\omega = \int_{D^n} \mathrm{d}(r^*\omega) = \int_{D^n} r^*(\mathrm{d}\omega) = 0$ (since $\mathrm{d}\omega = 0$ as a form on S^{n-1}, which is a top-degree form). But $\int_{S^{n-1}} \omega \neq 0$, giving the contradiction. This avoids the classification of 1-manifolds but uses the integration theory from chapter 7.

8.7 Transversality

Definition 8.7.1 (Transversality)**.** Let $f\colon M \to N$ be smooth and $W \subset N$ a submanifold. We say f is *transverse to* W, written $f \pitchfork W$, if for every $p \in f^{-1}(W)$,

$$\mathrm{d}f_p(T_pM) + T_{f(p)}W = T_{f(p)}N.$$

Two submanifolds $X, Y \subset N$ are *transverse*, $X \pitchfork Y$, if the inclusion $\iota\colon X \hookrightarrow N$ is transverse to Y, i.e., $T_pX + T_pY = T_pN$ for all $p \in X \cap Y$.

Remark 8.7.2. If $f(M) \cap W = \emptyset$, then $f \pitchfork W$ vacuously.

Theorem 8.7.3 (Preimage under transverse maps)**.** *If $f\colon M^m \to N^n$ is smooth and $f \pitchfork W^w \subset N$, then $f^{-1}(W)$ is a smooth submanifold of M of dimension $m - n + w = m - \operatorname{codim}(W)$.*

Proof. Let $p \in f^{-1}(W)$ and $q = f(p) \in W$. Choose a submanifold chart (V, ψ) for W near q such that $\psi(W \cap V) = \psi(V) \cap (\mathbb{R}^w \times \{0\})$. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-w}$ be the projection onto the last $n - w$ coordinates. Then $g = \pi \circ \psi \circ f$ satisfies $g^{-1}(0) = f^{-1}(W)$ near p, and transversality ensures $\mathrm{d}g_p$ is surjective. By the implicit function theorem, $g^{-1}(0)$ is a submanifold of dimension $m - (n - w)$. □

Example 8.7.4. Two curves in $\mathbb{R}^3$ are generically disjoint (since transversality requires $T_pC_1 + T_pC_2 = T_p\mathbb{R}^3$, which fails when $\dim T_pC_1 + \dim T_pC_2 = 2 < 3$, so transversality forces $C_1 \cap C_2 = \emptyset$). Two surfaces in $\mathbb{R}^3$ generically intersect in a curve.

Example 8.7.5. Let $f\colon M \to N$ be smooth and consider the graph $\Gamma_f = \{(p, f(p)) \mid p \in M\} \subset M \times N$. The map f is transverse to $W \subset N$ if and only if $\Gamma_f \pitchfork (M \times W)$ in $M \times N$.

Theorem 8.7.6 (Transversality theorem, parametric version)**.** *Let $F\colon M \times S \to N$ be smooth, and let $W \subset N$ be a submanifold. If $F \pitchfork W$, then for almost every $s \in S$ (i.e., all s outside a set of measure zero), the map $f_s = F(\cdot, s)\colon M \to N$ satisfies $f_s \pitchfork W$.*

Proof. By theorem 8.7.3, $Z = F^{-1}(W)$ is a submanifold of $M \times S$. Let $\pi\colon Z \to S$ be the projection. By Sard's theorem, almost every $s \in S$ is a regular value of π. One checks that s is a regular value of π if and only if $f_s \pitchfork W$. □

Corollary 8.7.7 (Density of transverse maps)**.** *Let $W \subset N$ be a closed submanifold. Then the set of smooth maps $f\colon M \to N$ that are transverse to W is dense (in the strong C^∞ topology) in $C^\infty(M, N)$.*

Proof sketch. By the tubular neighbourhood theorem, embed a neighbourhood of W in N as the total space of the normal bundle νW. Given $f\colon M \to N$, define $F\colon M \times B^k_\varepsilon \to N$ (where $k = \operatorname{codim} W$ and B^k_ε is a small ball) by translating f in the normal directions. One shows $F \pitchfork W$; then theorem 8.7.6 gives $f_s \pitchfork W$ for almost every s, and choosing s small makes f_s C^∞-close to f. □

Remark 8.7.8 (Transversality and stability). If $W \subset N$ is a closed submanifold and $f\colon M \to N$ satisfies $f \pitchfork W$, then any map sufficiently C^1-close to f also satisfies transversality to W. In other words, the set of maps transverse to a closed submanifold is not only dense but also open in $C^\infty(M, N)$. Combined with density, transversality is *generic* (it holds on an open dense set).

Example 8.7.9 (Intersection numbers via transversality)**.** Let M^n be a compact oriented manifold and $X^p, Y^q \subset M$ be compact oriented submanifolds with $p + q = n$ and $X \pitchfork Y$. Then $X \cap Y$ is a finite set of points, and one defines the *intersection number*

$$X \cdot Y = \sum_{p \in X \cap Y} \varepsilon(p),$$

where $\varepsilon(p) = +1$ if the orientation of $T_pX \oplus T_pY$ agrees with that of T_pM, and $\varepsilon(p) = -1$ otherwise. This integer depends only on the homology classes of X and Y.

8.8 Exercises

Exercise 8.1. Let $f\colon M_n(\mathbb{R}) \to \mathbb{R}$ be the determinant map $f(A) = \det A$. Find the critical points of f and describe $f^{-1}(1) = \mathrm{SL}(n, \mathbb{R})$ as a smooth submanifold.

Exercise 8.2. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be smooth with $m < n$. Prove that $f(\mathbb{R}^m)$ has measure zero in $\mathbb{R}^n$. *Hint:* Every point of $\mathbb{R}^m$ is a critical point since $\mathrm{d}f_x$ cannot be surjective.

Exercise 8.3. Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial. Show that the set of critical values of p is finite. *Hint:* The critical set is an algebraic variety; combine with Sard's theorem and the structure of semi-algebraic sets.

Exercise 8.4. Show that there *does* exist a smooth retraction from the annulus $A = \{x \in \mathbb{R}^2 \mid 1 \le |x| \le 2\}$ onto the inner circle $S^1 = \{x \mid |x| = 1\}$. Why does this not contradict theorem 8.6.3?

Exercise 8.5. Use the Brouwer fixed-point theorem to prove that the game of Hex has a winning strategy for the first player. *Hint:* Show that every Hex game has a winner (there are no draws), which is equivalent to a topological connectivity statement related to Brouwer.

Exercise 8.6. Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ (a cylinder) and $Y = \{(x, y, z) \in \mathbb{R}^3 \mid x + z = 1\}$ (a plane). Show that $X \pitchfork Y$ and describe the intersection

$X \cap Y$ as a submanifold.

Exercise 8.7. Let $X, Y \subset N$ be transverse submanifolds. Show that the normal bundle of $X \cap Y$ in N satisfies $\nu^N_{X\cap Y} \cong \nu^X_{X\cap Y} \oplus \nu^Y_{X\cap Y}$.

Exercise 8.8. A smooth function $f\colon M \to \mathbb{R}$ on a compact manifold is called a *Morse function* if all its critical points are nondegenerate (the Hessian matrix in local coordinates is nonsingular). Use Sard's theorem to prove that Morse functions are dense in $C^\infty(M, \mathbb{R})$. *Hint:* For an embedding $\iota\colon M \hookrightarrow \mathbb{R}^N$, consider the family of functions $f_a(x) = |x - a|^2$ parametrised by $a \in \mathbb{R}^N$.

Exercise 8.9. Let M^m be a compact smooth manifold. Using Sard's theorem and transversality, show that M can be immersed in $\mathbb{R}^{2m}$ and embedded in $\mathbb{R}^{2m+1}$. *Hint:* Start with an embedding $M \hookrightarrow \mathbb{R}^N$ for some large N (e.g., via Whitney) and project to lower-dimensional subspaces, using Sard to find projections that preserve injectivity and immersion properties.

Exercise 8.10. Prove the classification of compact connected 1-manifolds with boundary: every such manifold is diffeomorphic to either S^1 or $[0, 1]$. Use this to give a self-contained proof that a compact 1-manifold with boundary has an even number of boundary points.

Exercise 8.11. Show that there is no smooth surjection $f\colon \mathbb{R} \to \mathbb{R}^2$. *Hint:* Since $1 < 2$, every point of $\mathbb{R}$ is a critical point. Contrast this with the existence of continuous (Peano) space-filling curves.

Exercise 8.12. Let $\mathrm{Gr}_k(\mathbb{R}^n)$ denote the Grassmannian of k-planes in $\mathbb{R}^n$. Consider the smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^{n-k}$ given by projection onto a $(n-k)$-dimensional subspace $V \in \mathrm{Gr}_{n-k}(\mathbb{R}^n)$. Show that if $M^m \subset \mathbb{R}^n$ is a smooth submanifold, then for almost every V, the restriction $f|_M\colon M \to V \cong \mathbb{R}^{n-k}$ is transverse to $\{0\} \subset \mathbb{R}^{n-k}$.

Exercise 8.13. Let $\gamma_1\colon S^1 \to \mathbb{R}^3$ and $\gamma_2\colon S^1 \to \mathbb{R}^3$ be two disjoint smooth embeddings. The *linking number* is defined as

$$\mathrm{lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\,\mathrm{d}\mathbf{r}_1 \times \,\mathrm{d}\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}.$$

Show that this equals the degree of the Gauss map $G\colon S^1 \times S^1 \to S^2$, $G(s,t) = \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|}$, and that it is an integer.

Chapter 9

Morse Theory — Introduction

Contents

Morse theory establishes a profound connection between the analysis of smooth real-valued functions on a manifold and the topology of the manifold itself. The key insight, due to Marston Morse in the 1920s and subsequently refined by Smale, Thom, Milnor, and others, is that the critical points of a "generic" smooth function encode enough information to reconstruct the homotopy type of the manifold.

The philosophy is simple yet powerful: as we "sweep" through the level sets $f^{-1}(t)$ of a smooth function $f\colon M \to \mathbb{R}$, the topology of the sublevel set $f^{-1}((-\infty, t])$ changes only when t passes through a critical value, and the nature of the change is controlled by the *index* of the critical point. This idea leads to a cell decomposition of M, to fundamental inequalities relating critical points and Betti numbers, and ultimately to the Morse complex, a precursor to Floer homology and modern symplectic topology.

This chapter develops the foundations: Morse functions, the Morse lemma, handle attachments, and the weak and strong Morse inequalities.

9.1 Morse Functions and Critical Points

Definition 9.1.1 (Critical point, critical value)**.** Let M be a smooth manifold and $f\colon M \to \mathbb{R}$ a smooth function. A point $p \in M$ is a **critical point** of f if $\mathrm{d}f_p = 0$, i.e., in any local coordinates $(x_1, \ldots, x_n)$ centered at p,

$$\frac{\partial f}{\partial x_i}(p) = 0, \qquad i = 1, \ldots, n.$$

The value $f(p) \in \mathbb{R}$ is then called a **critical value**. A point of $\mathbb{R}$ that is not a critical value is a **regular value**.

Definition 9.1.2 (Non-degenerate critical point, Hessian)**.** A critical point p of $f\colon M \to \mathbb{R}$ is **non-degenerate** if the **Hessian matrix**

$$H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{1 \le i,j \le n}$$

is non-singular (i.e., $\det H_f(p) \neq 0$) in some (hence any) local coordinate system centered at p.

Remark 9.1.3. Although the Hessian matrix $H_f(p)$ depends on the choice of coordinates, its non-degeneracy does not: a change of coordinates $y = \varphi(x)$ transforms $H_f(p)$ by congruence $H \mapsto J^T H J$ where J is the Jacobian of φ at p, and congruence preserves non-singularity.

Definition 9.1.4 (Index of a critical point)**.** The **index** of a non-degenerate critical point p of $f\colon M \to \mathbb{R}$ is the number of negative eigenvalues of $H_f(p)$ (counted with multiplicity). We denote it $\lambda(p)$ or $\mathrm{ind}(p)$.

Definition 9.1.5 (Morse function)**.** A smooth function $f\colon M \to \mathbb{R}$ is a **Morse function** if every critical point of f is non-degenerate. If M is compact, a Morse function has finitely many critical points.

Example 9.1.6 (Height function on S^n)**.** Consider $S^n = \left\{x \in \mathbb{R}^{n+1} : |x|^2 = 1\right\}$ and $f\colon S^n \to \mathbb{R}$, $f(x_0, \ldots, x_n) = x_n$. The critical points are the north pole $N = (0, \ldots, 0, 1)$ and south pole $S = (0, \ldots, 0, -1)$.
To verify non-degeneracy, use stereographic coordinates at S: $\varphi(x_0, \ldots, x_n) = \frac{1}{1+x_n}(x_0, \ldots, x_{n-1})$ with inverse giving $x_n = \frac{|u|^2-1}{|u|^2+1}$. Then $f \circ \varphi^{-1}(u) = \frac{|u|^2-1}{|u|^2+1}$, whose Hessian at $u = 0$ is $\frac{4}{4} I_n = I_n > 0$, confirming index 0 at S. Similarly, N has index n. Thus f is a Morse function with exactly two critical points.

Example 9.1.7 (Height function on the torus T^2)**.** Embed the torus T^2 in $\mathbb{R}^3$ in the standard way:

$$(u, v) \mapsto \big((2 + \cos v) \cos u,\ (2 + \cos v) \sin u,\ \sin v\big).$$

The height function $f(x, y, z) = z$ restricted to T^2 gives $f(u, v) = \sin v$. This has four critical points:

- $v = -\pi/2$ (minimum, index 0),
- two saddle points at $v = 0$ with suitable u values (index 1),
- $v = \pi/2$ (maximum, index 2).

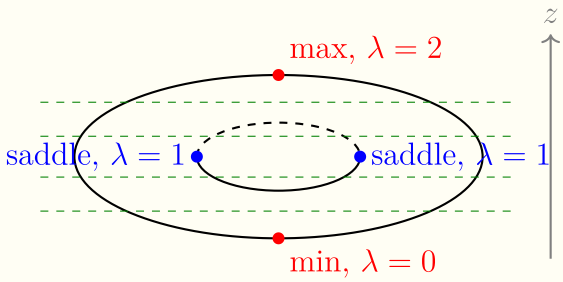


Figure 9.1: The torus T^2 with height function. Critical points are shown with their indices. Dashed lines indicate level sets.

Example 9.1.8 (Height on $\mathbb{R}P^2$)**.** Realize $\mathbb{R}P^2$ as the quotient of S^2 by the antipodal map. The function $f[x_0 : x_1 : x_2] = \frac{x_2^2}{x_0^2+x_1^2+x_2^2}$ is a well-defined Morse function on $\mathbb{R}P^2$ with three critical points of indices 0, 1, and 2. This is consistent with $\chi(\mathbb{R}P^2) = 1 - 1 + 1 = 1$ and the CW structure $\mathbb{R}P^2 = e^0 \cup e^1 \cup e^2$.

Example 9.1.9 (A non-Morse function)**.** Consider $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2y^2$. The origin is a critical point with Hessian $H_f(0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, which is degenerate. The origin is not an isolated critical point (the entire coordinate axes are critical), confirming that f is not a Morse function. By contrast, $g(x, y) = x^2 + y^2 + x^2y^2$ has a non-degenerate minimum at the origin.

9.2 The Morse Lemma

The Morse lemma states that near a non-degenerate critical point, a Morse function has a canonical quadratic form up to smooth change of coordinates.

Theorem 9.2.1 (Morse Lemma)**.** *Let $f : M \to \mathbb{R}$ be smooth and let p be a non-degenerate critical point of f with index λ. Then there exist local coordinates $(y_1, \ldots, y_n)$ centered at p such that*

$$f(y) = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2.$$

Proof. We may assume $p = 0 \in \mathbb{R}^n$ and $f(0) = 0$. By Taylor's theorem with integral remainder, since $df_0 = 0$, we can write

$$f(x) = \sum_{i,j=1}^{n} x_i x_j \, h_{ij}(x),$$

where $h_{ij}\colon U \to \mathbb{R}$ are smooth functions with $h_{ij}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)$. By symmetrizing we may assume $h_{ij} = h_{ji}$.

We proceed by induction. Suppose that after a coordinate change we have achieved

$$f(x) = \pm x_1^2 \pm \cdots \pm x_{k-1}^2 + \sum_{i,j \geq k} x_i x_j \, g_{ij}(x)$$

with $g_{ij} = g_{ji}$ smooth.

Case 1: $g_{kk}(0) \neq 0$. After possibly changing the sign, assume $g_{kk}(0) > 0$. Since g_{kk} is continuous, $g_{kk}(x) > 0$ near 0. Define

$$y_k = \sqrt{g_{kk}(x)}\Big(x_k + \sum_{j>k} \frac{g_{kj}(x)}{g_{kk}(x)} x_j\Big), \qquad y_i = x_i \text{ for } i \neq k.$$

Then $\frac{\partial y_k}{\partial x_k}(0) = \sqrt{g_{kk}(0)} \neq 0$, so by the inverse function theorem $(y_1, \ldots, y_n)$ is a valid coordinate system near 0. A direct computation shows

$$f = \pm x_1^2 \pm \cdots \pm x_{k-1}^2 + y_k^2 + \sum_{i,j>k} x_i x_j \, g'_{ij}(x)$$

for new smooth functions g'_{ij}, completing the induction step.

Case 2: $g_{kk}(0) = 0$ but $g_{k\ell}(0) \neq 0$ for some $\ell > k$. The linear substitution $x_k \mapsto x_k + x_\ell$, $x_\ell \mapsto x_k - x_\ell$ produces a nonzero diagonal coefficient, reducing to Case 1.

Since $H_f(0)$ is non-degenerate, at each step we can always enter Case 1 or Case 2. After n steps we obtain the desired form. The number of negative signs equals the index λ by Sylvester's law of inertia. ☐

Corollary 9.2.2. *Non-degenerate critical points are isolated.*

Proof. In Morse coordinates $(y_1, \ldots, y_n)$, the gradient of $f(y) = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2$ vanishes only at $y = 0$. ☐

Corollary 9.2.3. *If M is compact, every Morse function $f\colon M \to \mathbb{R}$ has finitely many critical points.*

Proof. The set of critical points is a closed subset of the compact manifold M, hence compact. But each critical point is isolated by Theorem 9.2.2. A compact discrete set is finite. ☐

Example 9.2.4 (Morse lemma in dimension 1)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ have a non-degenerate critical point at 0 with $f(0) = 0$ and $f''(0) \neq 0$. The Morse lemma gives a coordinate y with $f(y) = \pm y^2$. If $f''(0) > 0$ (minimum), then $f = y^2$ (index 0); if $f''(0) < 0$ (maximum), then $f = -y^2$ (index 1). The coordinate change $y = y(x)$ is determined by $f(x) = x^2 h(x) = (x\sqrt{h(x)})^2$, so $y = x\sqrt{h(x)}$ where $h(0) = f''(0)/2 > 0$.

Example 9.2.5 (Morse lemma in dimension 2)**.** For $f\colon \mathbb{R}^2 \to \mathbb{R}$ with a saddle point at the origin (index 1), the Morse lemma gives coordinates (y_1, y_2) such that $f = -y_1^2 + y_2^2$. The level sets $f^{-1}(c)$ near the origin are:

- for $c = 0$: a pair of lines $y_1 = \pm y_2$ (the "cross"),

- for $c > 0$: hyperbolas opening along the y_2-axis,
- for $c < 0$: hyperbolas opening along the y_1-axis.

This local picture governs the topology change at a saddle critical value.

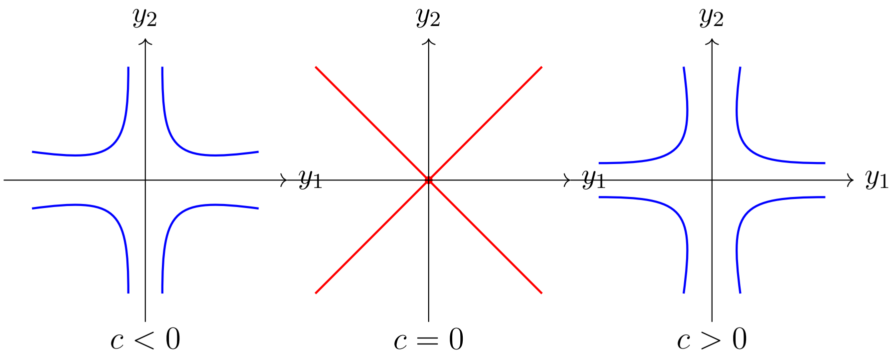


Figure 9.2: Level sets of $f = -y_1^2 + y_2^2$ near a saddle point (index 1). At $c = 0$ the level set degenerates into two crossing lines.

9.3 Density of Morse Functions

Theorem 9.3.1 (Morse functions are generic). *Let M be a smooth manifold (without boundary). Then the set of Morse functions is dense (and in fact residual) in $C^\infty(M, \mathbb{R})$ with the strong (Whitney) C^2 topology.*
More precisely, if $f\colon M \to \mathbb{R}$ is any smooth function and $\varepsilon > 0$, there exists a Morse function g with $|f(x) - g(x)| < \varepsilon$ for all $x \in M$.

Proof sketch. If $M \subset \mathbb{R}^N$ is embedded, consider for $a \in \mathbb{R}^N$ the function $f_a(x) = f(x) + \langle a, x\rangle$. The map $F\colon M \times \mathbb{R}^N \to T^*M$ given by $F(x, a) = \mathrm{d}(f_a)_x$ is a submersion. By the parametric transversality theorem (Theorem A.4.3), for almost every $a \in \mathbb{R}^N$ (in the sense of Lebesgue measure), f_a is a Morse function. Choosing $|a|$ small gives the desired approximation. □

Remark 9.3.2. On a compact manifold M, every Morse function is automatically proper. On non-compact manifolds, one typically requires Morse functions to be proper and to have distinct critical values, which can also be arranged by a small perturbation.

9.4 Passing through Critical Levels

Definition 9.4.1 (Sublevel set). For $f\colon M \to \mathbb{R}$ and $a \in \mathbb{R}$, define the **sublevel set** $M^a = f^{-1}((-\infty, a]) = \{x \in M : f(x) \le a\}$.

Theorem 9.4.2 (Regular interval theorem). *Let $f\colon M \to \mathbb{R}$ be a smooth proper function on a manifold without boundary. If $f^{-1}([a, b])$ is compact and contains no* *critical points of f, then M^a is diffeomorphic to M^b. Moreover, M^a is a deformation retract of M^b.*

Proof. Since $[a,b]$ contains no critical values, $\nabla f \neq 0$ on $f^{-1}([a,b])$ (using any Riemannian metric). Define the vector field

$$X = \frac{\nabla f}{|\nabla f|^2}$$

on a neighborhood of $f^{-1}([a,b])$, multiplied by a suitable bump function to have compact support. The flow of X moves the level set $f^{-1}(a)$ to $f^{-1}(b)$ in time $b-a$, since $X(f)=1$. The time-$(b-a)$ map of the flow restricts to a diffeomorphism $M^a \xrightarrow{\sim} M^b$. The retraction is given by flowing backwards along X. □

9.5 Handle Attachments

Definition 9.5.1 (λ-handle)**.** A **λ-handle** of dimension n is a copy of $D^\lambda \times D^{n-\lambda}$, where D^k denotes the closed unit disk in $\mathbb{R}^k$. We say that M^b is obtained from M^a by **attaching a λ-handle** if there is an embedding $\varphi\colon S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow \partial M^a$ and a diffeomorphism

$$M^b \cong M^a \cup_\varphi (D^\lambda \times D^{n-\lambda}).$$

Theorem 9.5.2 (Handle attachment at a critical point)**.** *Let $f\colon M \to \mathbb{R}$ be a Morse function and p a critical point of index λ with $f(p)=c$. Suppose $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains no critical point other than p. Then for sufficiently small $\varepsilon > 0$, $M^{c+\varepsilon}$ is diffeomorphic to $M^{c-\varepsilon}$ with a λ-handle attached. In particular, $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell e^λ attached.*

Proof sketch. By the Morse lemma, in a neighborhood of p we have coordinates $(y_1,\dots,y_n)$ with

$$f = c - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2.$$

Write $u = (y_1,\dots,y_\lambda)$ and $v = (y_{\lambda+1},\dots,y_n)$, so $f = c - |u|^2 + |v|^2$. The region $f^{-1}([c-\varepsilon, c+\varepsilon])$ in Morse coordinates becomes $\left\{(u,v) : -\varepsilon \le -|u|^2 + |v|^2 \le \varepsilon\right\}$. One constructs a modified function $\tilde f$ that agrees with f outside a small neighborhood, such that $\tilde f^{-1}((-\infty, c-\varepsilon])$ is diffeomorphic to $M^{c-\varepsilon} \cup (D^\lambda \times D^{n-\lambda})$. See Milnor [1] for the complete construction. □

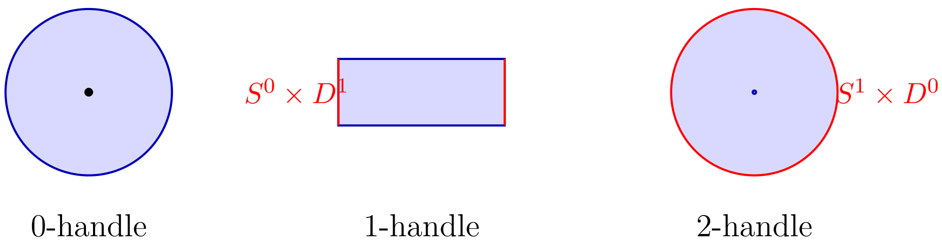


Figure 9.3: Handles in dimension $n=2$: a 0-handle (disk), a 1-handle (strip attached along two arcs), and a 2-handle (disk glued along its boundary circle).

9.6 CW Structure from a Morse Function

Theorem 9.6.1 (CW decomposition)**.** *Let M be a compact smooth manifold and $f : M \to \mathbb{R}$ a Morse function with critical points $p_1, \dots, p_k$ of indices $\lambda_1, \dots, \lambda_k$. Then M has the homotopy type of a CW complex with one cell of dimension λ_i for each critical point p_i.*

Proof. Arrange the critical values $c_1 < c_2 < \cdots < c_k$ (by a small perturbation we may assume all critical values are distinct). Choose $a_0 < c_1 < a_1 < c_2 < \cdots < a_{k-1} < c_k < a_k$. Then:

(i) $M^{a_0} = \emptyset$,

(ii) by Theorem 9.5.2, M^{a_i} has the homotopy type of $M^{a_{i-1}}$ with a cell e^{λ_i} attached,

(iii) $M^{a_k} = M$.

By induction, M has the homotopy type of a CW complex with cells $e^{\lambda_1}, \dots, e^{\lambda_k}$. □

Example 9.6.2. The sphere S^n with the height function (Theorem 9.1.6) has two critical points of indices 0 and n. Hence S^n has the homotopy type of a CW complex $e^0 \cup e^n$, recovering the standard CW structure $S^n = D^n/\partial D^n$.

Example 9.6.3. The torus T^2 with the height function (Theorem 9.1.7) has critical points of indices 0, 1, 1, 2. Hence $T^2 \simeq e^0 \cup e^1 \cup e^1 \cup e^2$, consistent with $\chi(T^2) = 1 - 2 + 1 = 0$.

Example 9.6.4 (CW structure on $\mathbb{C}\mathrm{P}^n$)**.** Consider the function $f : \mathbb{C}\mathrm{P}^n \to \mathbb{R}$ defined by

$$f[z_0 : \cdots : z_n] = \frac{\sum_{k=0}^n k\, |z_k|^2}{\sum_{k=0}^n |z_k|^2}.$$

This is a Morse function with $n+1$ critical points $p_k = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ (the 1 in position k) of index $2k$. Hence $\mathbb{C}\mathrm{P}^n$ has the homotopy type of a CW complex with one cell in each even dimension $0, 2, 4, \dots, 2n$:

$$\mathbb{C}\mathrm{P}^n \simeq e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}.$$

This explains why $H_{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \mathbb{Z}$ and $H_{\mathrm{odd}}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) = 0$.

Remark 9.6.5 (Handle decomposition vs. CW structure)*.* The handle decomposition of a compact manifold from a Morse function is stronger than just a CW structure: it gives a smooth structure on each "thickened cell." In particular, the handles can be rearranged and cancelled in pairs (if a λ-handle and a $(\lambda+1)$-handle meet along a single λ-sphere), leading to simplified decompositions. This is the starting point for the h-cobordism theorem and Smale's proof of the generalized Poincaré conjecture in dimensions ≥ 5.

9.7 Morse Inequalities

Let $f\colon M \to \mathbb{R}$ be a Morse function on a compact manifold M. Denote by c_λ the number of critical points of index λ, and by $b_\lambda = \dim H_\lambda(M;\mathbb{R})$ the λ-th Betti number.

Theorem 9.7.1 (Weak Morse inequalities)**.** *For each $\lambda \geq 0$,*

$$c_\lambda \geq b_\lambda.$$

In particular, $\sum_\lambda c_\lambda \geq \sum_\lambda b_\lambda = \chi(M) + 2\sum_{odd\ \lambda} b_\lambda$.

Proof. By Theorem 9.6.1, M has the homotopy type of a CW complex with c_λ cells of dimension λ. The cellular chain group C_λ is a free abelian group of rank c_λ, so

$$b_\lambda = \dim H_\lambda(M;\mathbb{R}) \leq \dim(C_\lambda \otimes \mathbb{R}) = c_\lambda. \qquad \square$$

Theorem 9.7.2 (Strong Morse inequalities)**.** *For each $k \geq 0$,*

$$\sum_{j=0}^{k}(-1)^{k-j}\, c_j \;\geq\; \sum_{j=0}^{k}(-1)^{k-j}\, b_j.$$

Equality holds when $k = n = \dim M$, giving the Euler characteristic formula:

$$\chi(M) = \sum_{j=0}^{n}(-1)^j\, c_j = \sum_{j=0}^{n}(-1)^j\, b_j.$$

Proof. Let $C_\bullet$ be the cellular chain complex from the CW structure, with $\partial_j\colon C_j \to C_{j-1}$. Set $d_j = \operatorname{rank}(\partial_j)$. Then $c_j = b_j + d_j + d_{j+1}$ (the rank-nullity relation in each degree). For the alternating sum:

$$\sum_{j=0}^{k}(-1)^{k-j}c_j = \sum_{j=0}^{k}(-1)^{k-j}(b_j + d_j + d_{j+1}) = \sum_{j=0}^{k}(-1)^{k-j}b_j + d_{k+1},$$

where the telescoping of the d_j terms leaves only $d_{k+1} \geq 0$. This gives the inequality. When $k = n$, $d_{n+1} = 0$, yielding equality. $\square$

Example 9.7.3 (Morse inequalities for T^2)**.** For the torus T^2, the Betti numbers are $b_0 = 1$, $b_1 = 2$, $b_2 = 1$ and the height function yields $c_0 = 1$, $c_1 = 2$, $c_2 = 1$. The weak inequalities $c_\lambda \geq b_\lambda$ are all equalities. Such a Morse function is called **perfect**.

Corollary 9.7.4. *Every Morse function on a compact manifold M has at least $\sum_\lambda b_\lambda(M)$ critical points.*

9.8 The Morse Complex

Remark 9.8.1 (Connection to the Morse complex). Given a Morse function f and a Riemannian metric g on M, one can refine the CW structure into the **Morse complex** $(C_\bullet^{\text{Morse}}, \partial)$ where:

- C_k^{Morse} is the free abelian group generated by critical points of index k,
- the boundary operator counts (with signs) gradient flow lines of $-\nabla_g f$ connecting critical points of consecutive index.

For a **Morse–Smale pair** (f, g) (where stable and unstable manifolds intersect transversally), the homology of this complex is isomorphic to the singular homology of M. This construction is the finite-dimensional precursor to Floer homology.

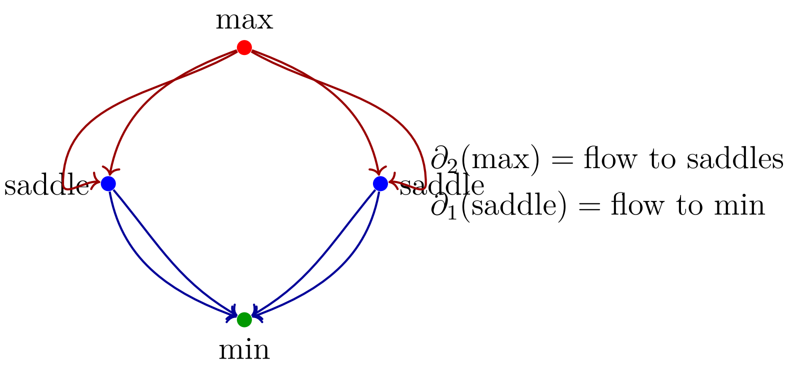


Figure 9.4: Schematic gradient flow lines for the height function on T^2, illustrating the Morse complex boundary operator.

9.9 Exercises

Exercise 9.1. Show that $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = |x|^4$ has a single critical point at the origin which is degenerate. Conclude that f is not a Morse function.

Exercise 9.2. Let A be a real symmetric $n \times n$ matrix and define $f\colon S^{n-1} \to \mathbb{R}$ by $f(x) = x^T Ax$. Show that f is a Morse function if and only if A has distinct eigenvalues. Determine the critical points and their indices.

Exercise 9.3. Let $M = S^1 \times S^1$ and $f(e^{i\theta}, e^{i\varphi}) = \cos\theta + \cos\varphi$. Verify that f is a Morse function, find all critical points, compute their indices, and verify the Euler characteristic formula.

Exercise 9.4. Prove that the Morse lemma implies non-degenerate critical points are isolated, directly from the normal form.

Exercise 9.5. Show that $\mathbb{R}\mathrm{P}^n$ admits a Morse function with exactly $n+1$ critical points, one of each index $0, 1, \ldots, n$. *Hint:* use $f[x_0 : \cdots : x_n] = \frac{\sum_i a_i x_i^2}{\sum_i x_i^2}$ with distinct a_i.

Exercise 9.6. Let Σ_g be the closed orientable surface of genus g. Determine the minimum number of critical points for any Morse function on Σ_g. Is a perfect Morse function always achievable?

Exercise 9.7. (Reeb's theorem) Let M be a compact manifold admitting a Morse function with exactly two critical points. Prove that M is homeomorphic to S^n. *Hint:* the two critical points must have indices 0 and n; use the regular interval theorem to show $M \cong D^n \cup_{S^{n-1}} D^n$.

Exercise 9.8. Verify the strong Morse inequalities for $\mathbb{C}\mathrm{P}^n$ equipped with the Morse function arising from the norm-square of a generic linear functional. *Hint:* $\mathbb{C}\mathrm{P}^n$ has Betti numbers $b_{2k} = 1$ for $0 \le k \le n$ and $b_{\mathrm{odd}} = 0$.

Exercise 9.9. Let $f\colon M \to \mathbb{R}$ be Morse on a compact manifold. Show that if all critical points have even index, then M has the homotopy type of a CW complex with no odd-dimensional cells. Deduce that $H_{\mathrm{odd}}(M; \mathbb{Z})$ is torsion-free.

Exercise 9.10. Let $f\colon M \to \mathbb{R}$ be a Morse function on a compact manifold and let $-f$ denote the function $p \mapsto -f(p)$. Show that $-f$ is also a Morse function, and that p has index λ for f if and only if p has index $n - \lambda$ for $-f$ (where $n = \dim M$). Deduce the Poincaré duality relation $c_\lambda(f) = c_{n-\lambda}(-f)$ for the number of critical points.

Exercise 9.11. (Gradient-like vector fields) A vector field X on M is **gradient-like** for a Morse function f if $X(f) > 0$ outside the critical set and X agrees with $-\nabla_g f$ in a Morse-coordinate neighborhood of each critical point. Show that gradient-like vector fields exist for any Morse function (use a partition of unity).

Chapter 10

Degree of a Smooth Map and Applications

Contents

The notion of degree assigns an integer to a smooth map between compact manifolds of the same dimension, measuring "how many times" the domain wraps around the target. This is one of the most fundamental invariants in topology: it is computable, homotopy-invariant, and yet powerful enough to prove major theorems.

Degree theory provides a unifying framework for many classical results, including the Brouwer fixed point theorem, the hairy ball theorem, the Borsuk–Ulam theorem, and the fundamental theorem of algebra. We develop both the mod 2 and the oriented (integer-valued) degree, prove their key properties, and showcase a wide range of applications.

10.1 Mod 2 Degree

Definition 10.1.1 (Mod 2 degree)**.** Let $f\colon M \to N$ be a smooth map between compact connected manifolds of the same dimension n (without boundary). For a regular value $q \in N$, define the **mod 2 degree** of f as

$$\deg_2(f) = \#f^{-1}(q) \mod 2 \;\in\; \mathbb{Z}/2\mathbb{Z}.$$

Theorem 10.1.2 (Well-definedness of mod 2 degree)**.** *The mod 2 degree* $\deg_2(f)$ *is independent of the choice of regular value* q*.*

Proof. Let q_0 and q_1 be two regular values of f. By Sard's theorem (Theorem A.2.1), the set of regular values is dense and open in N. Since N is connected, there is a smooth path $\gamma\colon [0,1] \to N$ from q_0 to q_1 consisting entirely of regular values (after a small perturbation of γ using Sard's theorem applied to the map $f \times \mathrm{Id}_{[0,1]}$).

Consider the pullback $W = \{(p,t) \in M \times [0,1] : f(p) = \gamma(t)\}$. Since $\gamma(t)$ is a regular value for each t, W is a compact 1-manifold with boundary

$$\partial W = f^{-1}(q_0) \times \{0\} \sqcup f^{-1}(q_1) \times \{1\}.$$

A compact 1-manifold with boundary has an even number of boundary points, so $\#f^{-1}(q_0) + \#f^{-1}(q_1) \equiv 0 \pmod 2$, whence $\#f^{-1}(q_0) \equiv \#f^{-1}(q_1) \pmod 2$. ☐

Remark 10.1.3. The compactness of M ensures that $f^{-1}(q)$ is a finite set for any regular value q (it is a discrete subset of a compact space). Without compactness, the cardinality could be infinite.

Proposition 10.1.4 (Homotopy invariance of mod 2 degree)**.** *If* $f, g\colon M \to N$ *are smoothly homotopic, then* $\deg_2(f) = \deg_2(g)$*.*

Proof. Let $F\colon M \times [0,1] \to N$ be a smooth homotopy with $F_0 = f$, $F_1 = g$. By Sard's theorem, choose a regular value q of F (which is then a regular value of both f and g). Then $F^{-1}(q)$ is a compact 1-manifold with boundary $f^{-1}(q) \sqcup g^{-1}(q)$, so $\#f^{-1}(q) \equiv \#g^{-1}(q) \pmod 2$. ☐

10.2 Oriented Degree

Definition 10.2.1 (Oriented degree)**.** Let $f\colon M \to N$ be a smooth map between compact connected *oriented* manifolds of the same dimension n. For a regular value $q \in N$, define the **degree** of f as

$$\deg(f) = \sum_{p \in f^{-1}(q)} \operatorname{sgn}(\det \mathrm{d}f_p),$$

where $\operatorname{sgn}(\det \mathrm{d}f_p) = +1$ if $\mathrm{d}f_p\colon T_pM \to T_qN$ preserves orientation, and -1 if it reverses orientation.

Theorem 10.2.2 (Well-definedness and homotopy invariance)**.** *The integer* $\deg(f)$ *is independent of the regular value* q *and is a homotopy invariant: if* $f \simeq g$ *(smoothly homotopic), then* $\deg(f) = \deg(g)$*.*

Proof. **Independence of regular value.** Let q_0, q_1 be regular values of f. As in the mod 2 case, connect them by a path of regular values and consider the 1-manifold $W = \{(p,t) : f(p) = \gamma(t)\}$. Now W inherits an orientation from the orientations of M and N. The signed count of boundary points in an oriented compact 1-manifold is zero, giving $\sum_{p \in f^{-1}(q_0)} \operatorname{sgn} - \sum_{p \in f^{-1}(q_1)} \operatorname{sgn} = 0$.

Homotopy invariance. Given a smooth homotopy $F\colon M \times [0,1] \to N$, choose q a regular value of F. Then $F^{-1}(q)$ is a compact oriented 1-manifold with boundary. The boundary contributions with signs give $\deg(g) - \deg(f) = 0$. ☐

Remark 10.2.3. Key properties of the degree:

(a) $\deg(\mathrm{Id}_M) = 1$.

(b) If $\deg(f) \neq 0$, then f is surjective.

(c) Reversing the orientation of M or N changes the sign of $\deg(f)$.

(d) For non-orientable manifolds, only the mod 2 degree is defined.

10.3 Degree via Differential Forms

Theorem 10.3.1 (Degree formula via integration)**.** *Let $f\colon M \to N$ be a smooth map between compact connected oriented n-manifolds. For any n-form ω on N with $\int_N \omega \neq 0$,*

$$\deg(f) = \frac{\int_M f^*\omega}{\int_N \omega}.$$

Proof sketch. By de Rham theory, $H^n_{\mathrm{dR}}(N) \cong \mathbb{R}$ for a compact connected oriented n-manifold. The pullback f^* acts on H^n by multiplication by an integer (the degree). Since $\int_N$ provides an isomorphism $H^n(N) \xrightarrow{\sim} \mathbb{R}$, we get $\int_M f^*\omega = \deg(f) \cdot \int_N \omega$. One checks that this integer agrees with the signed count of preimages. ☐

10.4 Computations

Example 10.4.1 (Degree of z^n)**.** The map $f\colon S^1 \to S^1$, $f(z) = z^n$ (for $n \in \mathbb{Z}$) has $\deg(f) = n$. Indeed, for any regular value $q \in S^1$, the preimage $f^{-1}(q)$ consists of $|n|$ points, each with sign $\mathrm{sgn}(n)$.
Alternatively, letting $\omega = d\theta/(2\pi)$ be the standard 1-form on S^1, we have $f^*\omega = n\, d\theta/(2\pi)$, and $\int_{S^1} f^*\omega = n$.

Example 10.4.2 (Antipodal map)**.** The antipodal map $\alpha\colon S^n \to S^n$, $\alpha(x) = -x$, has $\deg(\alpha) = (-1)^{n+1}$. This is because α is the composition of $(n+1)$ reflections, each of degree -1. In particular, $\deg(\alpha) = -1$ when n is even, and $\deg(\alpha) = +1$ when n is odd.

Example 10.4.3 (Degree of a covering map)**.** Let $\pi\colon \tilde{M} \to M$ be a smooth k-sheeted covering map between compact connected oriented manifolds. Then $\deg(\pi) = k$ (if π preserves orientation). Indeed, every point $q \in M$ is a regular value of π (since π is a local diffeomorphism), and $\#\pi^{-1}(q) = k$ with all signs positive.
For instance, the double cover $\pi\colon S^n \to \mathbb{RP}^n$ has $\deg(\pi) = 2$ when n is odd (so $\mathbb{RP}^n$ is orientable and π preserves orientation). When n is even, $\mathbb{RP}^n$ is non-orientable and

only $\deg_2(\pi) = 0$ is defined.

Example 10.4.4 (Degree and winding number)**.** For a smooth map $f: S^1 \to S^1$, the degree $\deg(f)$ coincides with the classical **winding number**:

$$\deg(f) = \frac{1}{2\pi} \int_0^{2\pi} f'(\theta)\, \mathrm{d}\theta$$

where we identify f with a function $\mathbb{R} \to \mathbb{R}$ satisfying $f(\theta + 2\pi) = f(\theta) + 2\pi \deg(f)$. This is the archetypal example of the degree-via-forms formula.

Proposition 10.4.5. *For smooth maps $f: M \to N$ and $g: N \to P$ between compact connected oriented manifolds of the same dimension,* $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

Proof. Choose a regular value q of $g \circ f$ that is also a regular value of g (using Sard). Then

$$\deg(g\circ f) = \sum_{p \in (g\circ f)^{-1}(q)} \operatorname{sgn}\det \mathrm{d}(g\circ f)_p = \sum_{r \in g^{-1}(q)} \operatorname{sgn}\det \mathrm{d}g_r \sum_{p \in f^{-1}(r)} \operatorname{sgn}\det \mathrm{d}f_p = \deg(g)\cdot\deg(f).$$

□

10.5 Applications of Degree Theory

Theorem 10.5.1 (Brouwer fixed point theorem)**.** *Every continuous map $f: D^n \to D^n$ has a fixed point.*

Proof. Suppose f has no fixed point. Define a retraction $r: D^n \to S^{n-1}$ by sending x to the point where the ray from $f(x)$ through x hits S^{n-1}. Then r is continuous and $r|_{S^{n-1}} = \mathrm{Id}_{S^{n-1}}$. Consider $i: S^{n-1} \hookrightarrow D^n$ the inclusion. Then $r \circ i = \mathrm{Id}$, so $\deg(r \circ i) = 1$. But r extends over D^n, so by homotopy invariance (since D^n is contractible) r is homotopic to a constant, hence $\deg(r \circ i) = 0$, a contradiction. □

Theorem 10.5.2 (Hairy ball theorem)**.** *There is no continuous nowhere-vanishing tangent vector field on S^{2k} (for any $k \geq 1$).*

Proof. Suppose $v: S^{2k} \to \mathbb{R}^{2k+1}$ is a continuous nowhere-vanishing tangent vector field. Normalize it to $\hat{v}(x) = v(x)/|v(x)|$, so $\hat{v}(x) \perp x$ and $|\hat{v}(x)| = 1$. Define the homotopy

$$H_t(x) = (\cos \pi t)\, x + (\sin \pi t)\, \hat{v}(x), \qquad t \in [0, 1].$$

One checks $|H_t(x)| = 1$ for all t (since $x \perp \hat{v}(x)$ and both are unit vectors). Thus $H_t: S^{2k} \to S^{2k}$, with $H_0 = \mathrm{Id}$ and $H_1 = -\mathrm{Id}$ (the antipodal map). By homotopy invariance, $\deg(\mathrm{Id}) = \deg(-\mathrm{Id})$, i.e., $1 = (-1)^{2k+1} = -1$, a contradiction. □

Theorem 10.5.3 (Borsuk–Ulam theorem)**.** *For every continuous map $f: S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

Proof. Suppose $f(x) \neq f(-x)$ for all x. Define $g\colon S^n \to S^{n-1}$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

Then $g(-x) = -g(x)$, i.e., g is an odd (equivariant) map.

We show by induction on n that an odd map $S^n \to S^{n-1}$ cannot exist. For $n = 1$: an odd map $g\colon S^1 \to S^0 = \{\pm 1\}$ would satisfy $g(-x) = -g(x)$, but S^1 is connected and S^0 is not, so g must be constant, contradicting oddness.

For $n \geq 2$: restrict g to the equatorial $S^{n-1} \subset S^n$. This gives an odd map $g|_{S^{n-1}}\colon S^{n-1} \to S^{n-1}$. An odd map $S^{n-1} \to S^{n-1}$ has odd degree (a standard result using the homology of the quotient $\mathbb{RP}^{n-1}$), so $\deg(g|_{S^{n-1}})$ is odd, hence nonzero. But $g|_{S^{n-1}}$ extends to the upper hemisphere D^n_+, so it is null-homotopic, giving $\deg(g|_{S^{n-1}}) = 0$, a contradiction. □

Corollary 10.5.4 (Ham sandwich theorem)**.** *Given n bounded measurable sets in $\mathbb{R}^n$, there exists a hyperplane that bisects each of them simultaneously.*

10.6 Lefschetz Fixed Point Theory

Definition 10.6.1 (Lefschetz number)**.** Let $f\colon M \to M$ be a continuous map on a compact manifold. The **Lefschetz number** of f is

$$L(f) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}\Big(f_*\colon H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q})\Big).$$

Theorem 10.6.2 (Lefschetz fixed point theorem)**.** *If $L(f) \neq 0$, then f has a fixed point.*

Proof sketch. One shows that $L(f)$ equals the intersection number of the graph $\Gamma_f = \{(x, f(x))\}$ with the diagonal $\Delta = \{(x,x)\}$ in $M \times M$. If f has no fixed point, then $\Gamma_f \cap \Delta = \emptyset$, so the intersection number is zero, hence $L(f) = 0$. For a transversal intersection, each fixed point p contributes ± 1 to $L(f)$, and the sign is the **local Lefschetz index**, which equals $\operatorname{sgn}\det(\mathrm{Id} - \mathrm{d}f_p)$. □

Corollary 10.6.3. *Since $L(\mathrm{Id}_M) = \chi(M)$, any map homotopic to the identity on a manifold with $\chi(M) \neq 0$ has a fixed point. In particular, this recovers the Brouwer theorem for contractible manifolds.*

Example 10.6.4 (Lefschetz number computations)**.**

(a) For $f = \mathrm{Id}_{S^{2k}}\colon S^{2k} \to S^{2k}$, we have $L(f) = 1 + (-1)^{2k} \cdot 1 = 2 = \chi(S^{2k}) \neq 0$, so Id has a fixed point (obviously). More interestingly, *every* map $f\colon S^{2k} \to S^{2k}$ with $\deg(f) \neq (-1)^{2k+1} = -1$ has $L(f) = 1 + \deg(f) \neq 0$ and hence has a fixed point.

(b) For the torus T^2, $\chi(T^2) = 0$, so $L(\mathrm{Id}) = 0$; the Lefschetz theorem gives no information. Indeed, translations on T^2 have no fixed points.

(c) For the antipodal map α on S^n, $L(\alpha) = 1 + (-1)^n(-1)^{n+1} = 1 - 1 = 0$ when n

is even (consistent with α having no fixed points on S^{2k}).

10.7 Intersection Number as Degree

Definition 10.7.1 (Intersection number)**.** Let M be a compact oriented n-manifold, and let $X^k, Y^{n-k} \subset M$ be compact oriented submanifolds intersecting transversally. The **intersection number** $I(X, Y) \in \mathbb{Z}$ is defined as

$$I(X, Y) = \sum_{p \in X \cap Y} \varepsilon(p),$$

where $\varepsilon(p) = +1$ if the orientation of $T_pX \oplus T_pY$ agrees with that of T_pM, and $\varepsilon(p) = -1$ otherwise.

Proposition 10.7.2. *The intersection number $I(X, Y)$ depends only on the homology classes $[X] \in H_k(M; \mathbb{Z})$ and $[Y] \in H_{n-k}(M; \mathbb{Z})$, and agrees with the homological intersection pairing*

$$H_k(M; \mathbb{Z}) \otimes H_{n-k}(M; \mathbb{Z}) \to H_0(M; \mathbb{Z}) \cong \mathbb{Z}.$$

Proposition 10.7.3. *Let $f\colon M^n \to N^n$ be a smooth map between compact oriented manifolds. For a regular value $q \in N$,*

$$\deg(f) = I(M, \{q\})$$

where the intersection is computed in the graph: the intersection number of the graph $\Gamma_f \subset M \times N$ with $M \times \{q\}$ equals $\deg(f)$.

Example 10.7.4 (Self-intersection number)**.** Consider $\mathbb{CP}^1 \subset \mathbb{CP}^2$ (a complex line). Its self-intersection number $I(\mathbb{CP}^1, \mathbb{CP}^1) = +1$. This can be computed by perturbing one copy slightly: two generic complex lines in $\mathbb{CP}^2$ intersect in exactly one point, transversally, with positive sign. More generally, two algebraic curves of degrees d and e in $\mathbb{CP}^2$ have intersection number de (Bézout's theorem).

10.8 Exercises

Exercise 10.1. Compute $\deg_2(f)$ and $\deg(f)$ for $f\colon S^1 \to S^1$, $f(e^{i\theta}) = e^{3i\theta}$.

Exercise 10.2. Let $f\colon S^n \to S^n$ be a smooth map with no fixed points. Show that $\deg(f) = (-1)^{n+1}$. *Hint:* f is homotopic to the antipodal map via $H_t(x) = \frac{(1-t)f(x)-tx}{|(1-t)f(x)-tx|}$.

Exercise 10.3. Show that $\deg\colon [S^n, S^n] \to \mathbb{Z}$ is a group isomorphism (where $[S^n, S^n]$ denotes homotopy classes of based maps with the group structure given by the co-H-

space structure of S^n). Deduce that $\pi_n(S^n) \cong \mathbb{Z}$.

Exercise 10.4. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a complex polynomial. Show that for R sufficiently large, $p/|p| : S^1_R \to S^1$ has degree n (where S^1_R is the circle of radius R). Deduce the fundamental theorem of algebra.

Exercise 10.5. Use degree theory to prove that every map $f\colon S^{2n} \to S^{2n}$ has either a fixed point or sends some point to its antipode.

Exercise 10.6. Let $f\colon T^n \to T^n$ be a smooth map of the n-torus. Express $\deg(f)$ in terms of the matrix $A \in M_n(\mathbb{Z})$ induced by f_* on $H_1(T^n;\mathbb{Z}) \cong \mathbb{Z}^n$. Verify for the map $(z_1, z_2) \mapsto (z_1^2 z_2, z_1 z_2^3)$ on T^2.

Exercise 10.7. Show that the Hopf map $h\colon S^3 \to S^2$, $h(z_1, z_2) = [z_1 : z_2] \in \mathbb{C}\mathrm{P}^1 \cong S^2$, has the property that $\#h^{-1}(q)$ is constant for all regular values q, but the degree formula does not apply directly. Why?

Exercise 10.8. Let $f\colon \Sigma_g \to S^2$ be a smooth map from a surface of genus $g \geq 1$. Show that $\deg(f) = 0$ if and only if f is null-homotopic. *Hint:* use the cohomological degree formula and $H^2(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}$.

Exercise 10.9. (Gauss map and degree) Let $S \subset \mathbb{R}^3$ be a compact oriented surface and $\nu\colon S \to S^2$ its Gauss map. Show that $\deg(\nu) = \frac{1}{2}\chi(S)$. *Hint:* use the Gauss–Bonnet theorem.

Chapter 11

Whitney Embedding Theorem

Contents

A fundamental question in differential topology asks: can every abstract smooth manifold be realized as a submanifold of some Euclidean space? The Whitney embedding theorem provides a definitive affirmative answer, with sharp dimension bounds.

Historically, manifolds were first studied as subsets of Euclidean space (Riemann, Poincaré). The intrinsic definition of a manifold via charts and atlases, due to Weyl and Whitney, raised the question of whether the abstract viewpoint is truly more general. Whitney's landmark 1936 paper showed that it is not: every abstract smooth manifold admits a smooth embedding into some $\mathbb{R}^N$.

This chapter proves the "easy" Whitney theorem (compact n-manifold embeds in $\mathbb{R}^{2n+1}$) in full detail, states the "hard" Whitney theorem (embedding in $\mathbb{R}^{2n}$), develops the Whitney approximation theorems, and discusses applications including the classification of surfaces and exotic phenomena.

11.1 Review of Embeddings and Immersions

Definition 11.1.1 (Immersion)**.** A smooth map $f\colon M^m \to N^n$ is an **immersion** if $\mathrm{d}f_p\colon T_pM \to T_{f(p)}N$ is injective for every $p \in M$. Necessarily $m \le n$.

Definition 11.1.2 (Embedding)**.** A smooth map $f\colon M \to N$ is a (smooth) **embedding** if:

(i) f is an immersion,

(ii) f is a topological embedding, i.e., $f\colon M \to f(M)$ is a homeomorphism (where $f(M)$ has the subspace topology).

If M is compact, condition (ii) is equivalent to f being injective.

Definition 11.1.3 (Proper map)**.** A continuous map $f\colon M \to N$ is **proper** if the preimage of every compact set is compact. A proper injective immersion is an embedding.

Remark 11.1.4. If $f\colon M \hookrightarrow N$ is an embedding, then $f(M)$ is a (regular/embedded) submanifold of N, and f is a diffeomorphism onto $f(M)$.

Example 11.1.5 (Immersions that are not embeddings)**.** (a) The figure-eight $\gamma\colon (-\pi,\pi) \to \mathbb{R}^2$, $\gamma(t) = (\sin 2t, \sin t)$, is an injective immersion but not an embedding (the image in $\mathbb{R}^2$ has a self-intersection at the origin in the closure).

(b) The map $\iota\colon \mathbb{R} \to T^2$ given by $t \mapsto (e^{it}, e^{i\alpha t})$ with α irrational is an injective immersion whose image is dense in T^2. This is an immersion that is far from being an embedding.

11.2 The Weak Whitney Embedding Theorem

Theorem 11.2.1 (Weak Whitney embedding theorem)**.** *Every compact smooth n-manifold can be embedded in $\mathbb{R}^{2n+1}$.*

We break the proof into several steps.

Lemma 11.2.2 (Embedding in Euclidean space of some dimension)**.** *Every compact smooth n-manifold M can be embedded in $\mathbb{R}^N$ for some N.*

Proof. Since M is compact, choose a finite atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha=1}^k$ and a subordinate partition of unity $\{\rho_\alpha\}$. Define $F\colon M \to \mathbb{R}^{k(n+1)}$ by

$$F(p) = \big(\rho_1(p)\varphi_1(p),\ \rho_1(p),\ \ldots,\ \rho_k(p)\varphi_k(p),\ \rho_k(p)\big),$$

where $\rho_\alpha(p)\varphi_\alpha(p)$ is extended by zero outside U_α. We claim F is an embedding:

- **Injectivity:** If $F(p) = F(q)$, then $\rho_\alpha(p) = \rho_\alpha(q)$ for all α. Pick α with $\rho_\alpha(p) > 0$. Then $\rho_\alpha(p)\varphi_\alpha(p) = \rho_\alpha(q)\varphi_\alpha(q)$ implies $\varphi_\alpha(p) = \varphi_\alpha(q)$, so $p = q$.
- **Immersion:** If $v \in T_pM \setminus \{0\}$, pick α with $\rho_\alpha(p) > 0$. The component $\rho_\alpha\varphi_\alpha$ has differential $\rho_\alpha(p)\, \mathrm{d}(\varphi_\alpha)_p + \varphi_\alpha(p)\, \mathrm{d}(\rho_\alpha)_p$, and by choosing appropriate α this is nonzero on v.

- Since M is compact, an injective immersion is an embedding. □

Lemma 11.2.3 (Dimension reduction by projection)**.** *If $M^n \subset \mathbb{R}^N$ is a compact embedded submanifold with $N > 2n+1$, then there exists a unit vector $v \in S^{N-1}$ such that the orthogonal projection $\pi_v \colon \mathbb{R}^N \to v^\perp \cong \mathbb{R}^{N-1}$ restricts to an embedding $\pi_v|_M \colon M \hookrightarrow \mathbb{R}^{N-1}$.*

Proof. We must show that for a generic direction v, the projection $\pi_v|_M$ is both injective and an immersion.

Step 1: Injectivity. Define $\alpha \colon M \times M \setminus \Delta \to S^{N-1}$ by

$$\alpha(p,q) = \frac{p-q}{|p-q|}.$$

The projection $\pi_v|_M$ fails to be injective if and only if v lies in the image of α. The domain has dimension $2n$, and the codomain has dimension $N-1$. Since $N > 2n+1$, we have $2n < N-1$, so by Sard's theorem the image of α has measure zero in S^{N-1}.

Step 2: Immersion. Define $\beta \colon TM \setminus \{0\text{-section}\} \to S^{N-1}$ by $\beta(p,w) = w/|w|$ (viewing $T_pM \subset \mathbb{R}^N$ via the embedding). The projection $\pi_v|_M$ fails to be an immersion at p if and only if $v \in T_pM$, i.e., v is in the image of β restricted to the unit sphere bundle of TM, which has dimension $2n-1$. Since $2n-1 < N-1$, the image of β also has measure zero by Sard's theorem.

Step 3: Conclusion. The set of "bad" directions v (where $\pi_v|_M$ fails to be an embedding) is contained in $\mathrm{im}(\alpha) \cup \mathrm{im}(\beta)$, a set of measure zero in S^{N-1}. Hence for almost every v, $\pi_v|_M$ is an embedding into $\mathbb{R}^{N-1}$. □

Proof of Theorem 11.2.1. By Theorem 11.2.2, embed $M^n \hookrightarrow \mathbb{R}^N$ for some N. If $N > 2n+1$, apply Theorem 11.2.3 repeatedly to reduce the ambient dimension one step at a time until we reach $\mathbb{R}^{2n+1}$. □

Corollary 11.2.4 (Immersion in $\mathbb{R}^{2n}$)**.** *Every compact smooth n-manifold can be immersed in $\mathbb{R}^{2n}$.*

Proof. For immersion we only need Step 2 of Theorem 11.2.3: the "bad" directions for the immersion condition form a set of dimension $\leq 2n-1$ in S^{N-1}. We can project to $\mathbb{R}^{N-1}$ preserving the immersion property as long as $2n-1 < N-1$, i.e., $N > 2n$. Starting from some N and projecting repeatedly, we reach $\mathbb{R}^{2n}$. □

Remark 11.2.5. The immersion dimension $2n$ from Theorem 11.2.4 is not sharp. The sharp result is the **Hirsch–Smale immersion theorem**, combined with obstruction theory, which shows that every compact n-manifold immerses in $\mathbb{R}^{2n-1}$ (for $n > 1$). For specific manifolds, even lower dimensions are possible: $\mathbb{RP}^2$ immerses in $\mathbb{R}^3$ (Boy's surface), and all orientable surfaces immerse in $\mathbb{R}^3$.

11.3 The Strong Whitney Embedding Theorem

Theorem 11.3.1 (Strong Whitney embedding theorem)**.** *Every smooth n-manifold (with $n > 0$) admits a proper embedding into $\mathbb{R}^{2n}$.*

Remark 11.3.2. The dimension $2n$ is sharp: the non-orientable manifold $\mathbb{RP}^n$ (for n a power of 2) cannot be embedded in $\mathbb{R}^{2n-1}$ by Stiefel–Whitney class obstructions. The proof of the strong Whitney theorem is substantially harder than the weak version. The key new ingredient is the **Whitney trick** : pairs of intersection points of opposite sign can be cancelled by an isotopy, provided $2n \geq 5$ (i.e., $n \geq 3$). For $n = 1$ the result is trivial (embed $S^1 \hookrightarrow \mathbb{R}^2$), and $n = 2$ requires separate (difficult) arguments. We refer to [9] and [10] for the full proof.

11.4 Whitney Approximation Theorems

Theorem 11.4.1 (Whitney approximation — functions)**.** *Let M be a smooth manifold (possibly non-compact) and $f\colon M \to \mathbb{R}^N$ a continuous map. Then f can be uniformly approximated by smooth maps. More precisely, for any continuous $\varepsilon\colon M \to (0, \infty)$, there exists a smooth map $g\colon M \to \mathbb{R}^N$ with $|f(p) - g(p)| < \varepsilon(p)$ for all p. If f is already smooth on a closed set A, then g can be chosen to agree with f on A.*

Theorem 11.4.2 (Whitney approximation — maps between manifolds)**.** *Every continuous map between smooth manifolds is homotopic to a smooth map. If two smooth maps are continuously homotopic, they are smoothly homotopic.*

Remark 11.4.3. Theorem 11.4.2 has the profound consequence that the **smooth and continuous homotopy categories of manifolds are equivalent**. In particular, any homotopy-theoretic invariant (fundamental group, homology, homotopy groups) can be computed using either smooth or continuous maps. This justifies the free passage between smooth and continuous settings throughout differential topology.

Definition 11.4.4 (Isotopy)**.** An **isotopy** of embeddings $f_0, f_1\colon M \hookrightarrow N$ is a smooth map $F\colon M \times [0,1] \to N$ such that $F_0 = f_0$, $F_1 = f_1$, and F_t is an embedding for each $t \in [0,1]$. An **ambient isotopy** is a smooth family of diffeomorphisms $\Phi_t\colon N \to N$ with $\Phi_0 = \mathrm{Id}$ and $\Phi_1 \circ f_0 = f_1$.

Theorem 11.4.5 (Isotopy extension theorem)**.** *Let M be compact and N a manifold without boundary. Every isotopy of embeddings $f_t\colon M \hookrightarrow N$ extends to an ambient isotopy $\Phi_t\colon N \to N$.*

11.5 Applications and Examples

Example 11.5.1 (Standard embeddings)**.** Some classical embeddings:

(a) $S^n \hookrightarrow \mathbb{R}^{n+1}$ as the unit sphere (codimension 1).

(b) $T^2 = S^1 \times S^1 \hookrightarrow \mathbb{R}^3$: $(e^{i\theta}, e^{i\varphi}) \mapsto \big((2+\cos\varphi)\cos\theta,\ (2+\cos\varphi)\sin\theta,\ \sin\varphi\big)$.

(c) $T^n = (S^1)^n \hookrightarrow \mathbb{R}^{2n}$ via $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mapsto (\cos\theta_1, \sin\theta_1, \ldots, \cos\theta_n, \sin\theta_n)$, which is an embedding ("flat torus").

Example 11.5.2 ($\mathbb{R}\mathrm{P}^2$ in $\mathbb{R}^4$)**.** The real projective plane $\mathbb{R}\mathrm{P}^2$ does not embed in $\mathbb{R}^3$ (since it is non-orientable and compact without boundary). However, it embeds in $\mathbb{R}^4$. An explicit embedding is given by

$$[x : y : z] \mapsto (xy,\ xz,\ y^2 - z^2,\ 2yz),$$

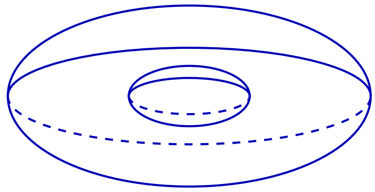
restricted to $S^2 \to \mathbb{R}^4$, which descends to a well-defined embedding $\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{R}^4$.

Example 11.5.3 (Boy's surface — an immersion of $\mathbb{R}\mathrm{P}^2$ in $\mathbb{R}^3$)**.** Although $\mathbb{R}\mathrm{P}^2$ cannot be embedded in $\mathbb{R}^3$, it can be *immersed* in $\mathbb{R}^3$. The famous **Boy surface** is an immersion $\mathbb{R}\mathrm{P}^2 \looparrowright \mathbb{R}^3$ with a single triple point and no pinch points. It was discovered by Werner Boy (a student of Hilbert) in 1901, disproving Hilbert's conjecture that no such immersion exists. An explicit parameterization can be given using the Apéry formula; see [4] for a discussion.

Example 11.5.4 (Non-orientable surfaces cannot embed in $\mathbb{R}^3$)**.** No compact non-orientable surface (without boundary) embeds in $\mathbb{R}^3$. This follows from the **Jordan–Brouwer separation theorem**: a compact connected hypersurface $\Sigma^{n-1} \subset \mathbb{R}^n$ separates $\mathbb{R}^n$ into two connected components (a bounded "inside" and an unbounded "outside"). The outward normal then gives a continuous nowhere-vanishing normal field, which orients Σ.

Proposition 11.5.5 (Embedding dimension and Stiefel–Whitney classes)**.** *If a compact n-manifold M embeds in $\mathbb{R}^{n+k}$, then the $(k+1)$-st through n-th Stiefel–Whitney classes vanish: $\overline{w}_i(M) = 0$ for $i > k$ (where $\overline{w}$ denotes the dual Stiefel–Whitney classes). This provides obstructions to low-codimension embeddings.*

Example 11.5.6. The non-immersion $\mathbb{R}\mathrm{P}^{2^k} \not\looparrowright \mathbb{R}^{2^{k+1}-2}$ can be proved using Stiefel–Whitney classes. For instance, $\mathbb{R}\mathrm{P}^4$ does not immerse in $\mathbb{R}^6$, showing that Whitney's bound of $\mathbb{R}^{2n}$ is close to optimal for this family.



$T^2 \hookrightarrow \mathbb{R}^3$

Figure 11.1: Schematic of the standard embedding of the torus T^2 in $\mathbb{R}^3$.

11.6 Classification of Compact Surfaces

Theorem 11.6.1 (Classification of compact surfaces)**.** *Every compact connected surface (2-manifold) without boundary is diffeomorphic to exactly one of the following:*

(i) The sphere S^2 (genus 0, orientable),

(ii) The connected sum $\Sigma_g = T^2 \# \cdots \# T^2$ (g copies) for $g \geq 1$ (genus g, orientable),

(iii) The connected sum $N_k = \mathbb{R}\mathrm{P}^2 \# \cdots \# \mathbb{R}\mathrm{P}^2$ (k copies) for $k \geq 1$ (non-orientable).

The invariants are the orientability and the Euler characteristic $\chi(\Sigma_g) = 2 - 2g$ and $\chi(N_k) = 2 - k$.

Remark 11.6.2. The embedding dimensions for compact surfaces are:

- Every orientable surface Σ_g embeds in $\mathbb{R}^3$.
- Every non-orientable surface N_k embeds in $\mathbb{R}^4$ (but not in $\mathbb{R}^3$). In particular, the Klein bottle N_2 embeds in $\mathbb{R}^4$.
- By the weak Whitney theorem, all surfaces embed in $\mathbb{R}^5$; by the strong theorem, in $\mathbb{R}^4$.

11.7 Exotic Spheres — A Preview

Remark 11.7.1 (Exotic spheres). A natural question is whether the smooth structure on S^n is unique. Remarkably, Milnor [2] showed in 1956 that S^7 admits exotic smooth structures: there exist smooth manifolds homeomorphic but not diffeomorphic to S^7. The group Θ_n of exotic n-spheres (up to orientation-preserving diffeomorphism, under connected sum) has been computed for many n. Notable facts:

- $\Theta_7 \cong \mathbb{Z}/28\mathbb{Z}$ (there are 28 smooth structures on S^7),
- $\Theta_n = 0$ for $n \leq 6$,
- Θ_4 is unknown (the smooth Poincaré conjecture in dimension 4 is open).

The construction of exotic spheres uses Morse theory and handle decompositions in an essential way: Milnor's original examples arise as boundaries of certain disk bundles over S^4, analyzed via Morse-theoretic handle attachments.

11.8 Exercises

Exercise 11.1. Show that the Klein bottle $K = \mathbb{RP}^2 \# \mathbb{RP}^2$ does not embed in $\mathbb{R}^3$. *Hint:* a compact surface without boundary embedded in $\mathbb{R}^3$ separates $\mathbb{R}^3$ (Jordan–Brouwer) and hence is orientable.

Exercise 11.2. Construct an explicit embedding of $S^1 \times S^1$ in $\mathbb{R}^3$ and verify it is an embedding (injective immersion, homeomorphism onto its image).

Exercise 11.3. The **Veronese embedding** $\nu\colon \mathbb{RP}^2 \to \mathbb{R}^6$ is defined by $\nu[x : y : z] = (x^2, y^2, z^2, \sqrt{2}\,xy, \sqrt{2}\,xz, \sqrt{2}\,yz)$ (on S^2, modulo the antipodal identification). Verify that ν is a well-defined smooth embedding.

Exercise 11.4. Prove that any compact 1-manifold (without boundary) is diffeomorphic to a finite disjoint union of circles S^1. Deduce that it embeds in $\mathbb{R}^2$.

Exercise 11.5. Let M^n be a compact parallelizable manifold (i.e., TM is trivial). Show that M immerses in $\mathbb{R}^{n+1}$. *Hint:* use the trivialization to define a map to S^n (Gauss map) and the immersion theorem.

Exercise 11.6. Show that $\mathbb{RP}^n$ can be embedded in $\mathbb{R}^{2n}$ for all $n \geq 1$. For $n = 1$, give an explicit embedding $\mathbb{RP}^1 \cong S^1 \hookrightarrow \mathbb{R}^2$.

Exercise 11.7. Prove that if M^n embeds in $\mathbb{R}^{n+1}$ (codimension 1), then M is orientable if and only if the normal bundle is trivial.

Exercise 11.8. Show that the connected sum $M_1 \# M_2$ of two compact n-manifolds embeds in $\mathbb{R}^{2n+1}$ if both M_1 and M_2 do. Does the same hold for $\mathbb{R}^{2n}$?

Exercise 11.9. (Whitney–Graustein theorem) Two immersions $\gamma_0, \gamma_1\colon S^1 \looparrowright \mathbb{R}^2$ are regularly homotopic if and only if they have the same **turning number** $\tau(\gamma) = \deg(\gamma'/\left|\gamma'\right| : S^1 \to S^1)$. Verify this for the figure-eight curve ($\tau = 0$) and the standard circle ($\tau = 1$).

Appendix A

Review of Analysis and Topology

This appendix collects standard results from analysis and point-set topology used throughout the text. Proofs are generally omitted; see [6] or [13] for details.

A.1 Inverse and Implicit Function Theorems

Theorem A.1.1 (Inverse function theorem). *Let $U \subset \mathbb{R}^n$ be open and $F\colon U \to \mathbb{R}^n$ be C^k $(k \geq 1)$. If $\det(\mathrm{d}F_p) \neq 0$ at some $p \in U$, then there exist open neighborhoods V of p and W of $F(p)$ such that $F|_V\colon V \xrightarrow{\sim} W$ is a C^k diffeomorphism.*

Theorem A.1.2 (Implicit function theorem). *Let $U \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $F\colon U \to \mathbb{R}^m$ be C^k. Suppose $F(a,b) = 0$ and the partial derivative $\frac{\partial F}{\partial y}(a,b)$ is invertible. Then there exist neighborhoods $V \ni a$ in $\mathbb{R}^n$ and $W \ni b$ in $\mathbb{R}^m$, and a unique C^k map $g\colon V \to W$ with $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in V$.*

A.2 Sard's Theorem

Theorem A.2.1 (Sard's theorem). *Let $f\colon M^m \to N^n$ be a smooth map between smooth manifolds. The set of critical values of f has Lebesgue measure zero in N. Equivalently, the set of regular values is dense in N (in fact, residual in the sense of Baire).*

A.3 Partitions of Unity

Theorem A.3.1 (Existence of partitions of unity). *Let M be a smooth manifold and $\{U_\alpha\}_{\alpha \in A}$ an open cover of M. There exists a smooth partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$, i.e., $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$, $\rho_\alpha \geq 0$, and $\sum_\alpha \rho_\alpha = 1$ (locally finite sum).*

A.4 Transversality

Definition A.4.1 (Transversality)**.** A smooth map $f\colon M \to N$ is **transverse** to a submanifold $S \subset N$ (written $f \pitchfork S$) if for every $p \in f^{-1}(S)$,

$$\mathrm{d}f_p(T_pM) + T_{f(p)}S = T_{f(p)}N.$$

Two submanifolds $X, Y \subset N$ are **transverse** if the inclusion $X \hookrightarrow N$ is transverse to Y.

Theorem A.4.2 (Transversality theorem)**.** *If $f \pitchfork S$, then $f^{-1}(S)$ is a submanifold of M with* $\operatorname{codim}(f^{-1}(S)) = \operatorname{codim}(S)$*.*

Theorem A.4.3 (Parametric transversality theorem)**.** *Let $F\colon M \times S \to N$ be smooth and $F \pitchfork W$ for a submanifold $W \subset N$. Then for almost every $s \in S$ (a residual set of full measure), the map $F_s = F(-, s)\colon M \to N$ satisfies $F_s \pitchfork W$.*

A.5 Compact Manifolds and Exhaustion Functions

Theorem A.5.1. *Every smooth manifold M admits a proper smooth function $\rho\colon M \to [0, \infty)$ (an **exhaustion function**). If M is compact, ρ can be taken to be any smooth function.*

Theorem A.5.2 (Ehresmann's fibration theorem)**.** *Let $f\colon M \to N$ be a proper smooth submersion between manifolds without boundary. Then f is a locally trivial fiber bundle.*

A.6 Smooth Bump Functions and Cutoffs

Theorem A.6.1 (Existence of bump functions)**.** *For any point p in a smooth manifold M and any open neighborhood $U \ni p$, there exists a smooth function $\rho\colon M \to [0, 1]$ with $\rho(p) = 1$ and* $\operatorname{supp}(\rho) \subset U$*.*

Remark A.6.2. The existence of bump functions has several important consequences:

(i) Smooth manifolds are normal topological spaces (Urysohn-type separation via smooth functions).

(ii) Any closed subset of a smooth manifold is the zero set of some smooth non-negative function.

(iii) Smooth functions separate points: for any $p \neq q$ in M, there exists $\rho \in C^\infty(M)$ with $\rho(p) \neq \rho(q)$.

A.7 Tubular Neighborhoods

Theorem A.7.1 (Tubular neighborhood theorem)**.** *Let $S \subset M$ be a compact embedded submanifold. Then S has a* ***tubular neighborhood****: there exist an open neighborhood U of S in M and a diffeomorphism $\Phi\colon \nu(S) \xrightarrow{\sim} U$ from (a neighborhood of the zero section in) the normal bundle $\nu(S)$ to U, with $\Phi|_S = \mathrm{Id}_S$.*

Remark A.7.2. The tubular neighborhood is unique up to isotopy: any two tubular neighborhood embeddings $\Phi_0, \Phi_1\colon \nu(S) \to M$ that agree on S are isotopic through tubular neighborhood embeddings. This is a consequence of the isotopy extension theorem.

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