

Algebraic Topology

Lecture Notes

Master M1 — 2025–2026

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“Algebraic topology is the art of turning geometry problems into algebra problems.”

— Henri Poincaré

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Notation

The following notation is used throughout this text.

Symbol	Meaning
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Natural numbers, integers, rationals, reals, complex numbers
$I = [0, 1]$	The unit interval
S^n	The n -sphere $\{x \in \mathbb{R}^{n+1} : x = 1\}$
D^n	The closed n -disk $\{x \in \mathbb{R}^n : x \leq 1\}$
$\mathbb{R}P^n, \mathbb{C}P^n$	Real and complex projective n -space
T^n	The n -torus $\underbrace{S^1 \times \dots \times S^1}_n$
$X \times Y$	Product of topological spaces
X/\sim	Quotient space by equivalence relation \sim
$X \vee Y$	Wedge sum (one-point union)
$X \simeq Y$	X is homotopy equivalent to Y
$X \cong Y$	X is homeomorphic to Y
$f \simeq g$	f is homotopic to g
$[f]$	Homotopy class of the map f
$\pi_1(X, x_0)$	Fundamental group of X at basepoint x_0
$\pi_n(X, x_0)$	n -th homotopy group of X at basepoint x_0
$H_n(X), H^n(X)$	n -th singular homology and cohomology groups
f_*, f^*	Induced map on homology, cohomology
$\text{Hom}(A, B)$	Set (or group) of homomorphisms from A to B
$\text{im } f, \text{ker } f$	Image and kernel of a homomorphism f
$\text{coker } f$	Cokernel of f : quotient of codomain by $\text{im } f$
$\text{Aut}(G)$	Automorphism group of G
Id_X	Identity map on X
Top	Category of topological spaces and continuous maps
hTop	Homotopy category of topological spaces
Grp	Category of groups and group homomorphisms
Ab	Category of abelian groups

Symbol	Meaning
Set	Category of sets and functions
\star	Concatenation of paths
$\bar{\gamma}$	Reverse path: $\bar{\gamma}(t) = \gamma(1 - t)$
c_{x_0}	Constant path at x_0

Preface

Algebraic topology is the study of topological spaces by means of algebraic invariants. Its central theme is the translation of topological problems—often geometric and intuitive, yet notoriously hard to resolve by purely topological methods—into algebraic ones, where powerful computational machinery is available.

This course is intended for beginning graduate students who have a solid background in point-set topology and abstract algebra. We assume familiarity with topological spaces, continuous maps, compactness, connectedness, quotient spaces, groups, rings, modules, and exact sequences. Some exposure to category theory is helpful but not strictly required; we introduce the necessary categorical language as it arises.

The first chapter provides a rapid review of point-set topology, with an emphasis on the concepts and constructions that are most relevant to the algebraic theory: homotopy, retracts, CW complexes, and the homotopy category. The second chapter introduces the fundamental group π_1 , our first and most classical algebraic invariant. We prove its basic properties, compute $\pi_1(S^1) \cong \mathbb{Z}$ from first principles, and derive the two-dimensional Brouwer fixed-point theorem as a striking application.

Subsequent chapters will develop covering space theory, the Seifert–van Kampen theorem, singular homology, cohomology, and further topics. Throughout, we emphasize rigorous proofs, concrete computations, and the interplay between algebra and geometry.

Prerequisites. Point-set topology (at the level of Munkres, Chapters 1–4) and abstract algebra (groups, rings, homomorphisms, quotient groups).

Recommended references.

- A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
- J. P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999.
- W. S. Massey, *A Basic Course in Algebraic Topology*, Springer, 1991.
- G. E. Bredon, *Topology and Geometry*, Springer, 1993.
- E. H. Spanier, *Algebraic Topology*, Springer, 1966.

Chapter 1

Topology Review and Motivations

The purpose of this chapter is twofold. First, we recall the basic notions of point-set topology that form the foundation for everything that follows. Second, we introduce the language and the guiding philosophy of algebraic topology, emphasizing the passage from topological spaces to algebraic invariants.

1.1 Topological Spaces and Continuous Maps

We begin by recalling the fundamental definitions.

Definition 1.1 (Topological space). A *topological space* is a pair (X, τ) where X is a set and $\tau \subseteq \mathcal{P}(X)$ is a collection of subsets, called *open sets*, satisfying:

- (i) $\emptyset \in \tau$ and $X \in \tau$.
- (ii) If $\{U_\alpha\}_{\alpha \in A} \subseteq \tau$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
- (iii) If $U_1, \dots, U_n \in \tau$, then $U_1 \cap \dots \cap U_n \in \tau$.

Definition 1.2 (Continuous map). A map $f: X \rightarrow Y$ between topological spaces is *continuous* if for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X .

Definition 1.3 (Homeomorphism). A *homeomorphism* is a continuous bijection $f: X \rightarrow Y$ whose inverse $f^{-1}: Y \rightarrow X$ is also continuous. If such a map exists, we write $X \cong Y$ and say that X and Y are *homeomorphic*.

Remark 1.4. Homeomorphism is an equivalence relation on topological spaces. A topological invariant is any property or quantity that is preserved under homeomorphism. One of the central goals of topology is to find invariants that are powerful enough to distinguish non-homeomorphic spaces.

Example 1.5 (Metric spaces). Every metric space (X, d) carries a natural topology whose open sets are unions of open balls $B(x, r) = \{y \in X : d(x, y) < r\}$. In particular, \mathbb{R}^n with the Euclidean metric, the spheres S^n , and the disks D^n are all topological spaces in this way.

Example 1.6 (Quotient topology). Given a topological space X and an equivalence relation \sim on X , the *quotient space* X/\sim carries the finest topology making the projection $q: X \rightarrow X/\sim$ continuous. That is, a set $U \subseteq X/\sim$ is open if and only if $q^{-1}(U)$ is open in X . Many important spaces arise as quotients: for instance, the torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, and the real projective plane $\mathbb{R}P^2 \cong S^2/(x \sim -x)$.

1.2 Connectedness and Path-Connectedness

Definition 1.7 (Connected space). A topological space X is *connected* if it cannot be written as a disjoint union $X = U \sqcup V$ of two nonempty open sets. Equivalently, the only subsets of X that are both open and closed are \emptyset and X itself.

Definition 1.8 (Path-connected space). A space X is *path-connected* if for every pair of points $x, y \in X$, there exists a continuous map $\gamma: I \rightarrow X$ (a *path*) with $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 1.9 (Path-connected implies connected). *Every path-connected space is connected.*

Proof. Suppose X is path-connected and $X = U \sqcup V$ with U, V open and nonempty. Pick $x \in U$ and $y \in V$, and let $\gamma: I \rightarrow X$ be a path from x to y . Then $I = \gamma^{-1}(U) \sqcup \gamma^{-1}(V)$ is a decomposition of I into two disjoint nonempty open sets, contradicting the connectedness of I . Thus no such decomposition exists. \square

Example 1.10 (Connected but not path-connected). The *topologist's sine curve*

$$X = \{(x, \sin(1/x)) : x \in (0, 1]\} \cup \{(0, y) : y \in [-1, 1]\}$$

is connected but not path-connected. The closure of the graph of $\sin(1/x)$ is connected (being the closure of a connected set), but there is no path in X from $(0, 0)$ to $(1, \sin 1)$.

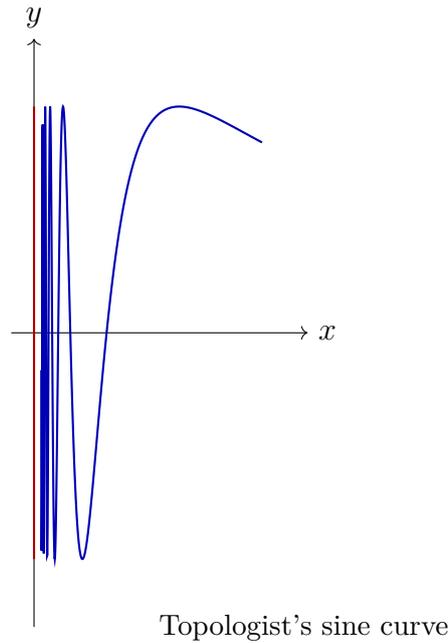


Figure 1.1: The topologist's sine curve: connected but not path-connected.

1.3 Compactness and the Hausdorff Property

Definition 1.11 (Compact space). A topological space X is *compact* if every open cover of X admits a finite subcover: whenever $X = \bigcup_{\alpha \in A} U_\alpha$ with each U_α open, there exist $\alpha_1, \dots, \alpha_n \in A$ such that $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$.

Definition 1.12 (Hausdorff space). A space X is *Hausdorff* (or T_2) if for any two distinct points $x, y \in X$, there exist disjoint open sets U, V with $x \in U$ and $y \in V$.

Theorem 1.13 (Compact subsets of Hausdorff spaces are closed). *Let X be a Hausdorff space and let $K \subseteq X$ be compact. Then K is closed in X .*

Proof. We show that $X \setminus K$ is open. Let $x \in X \setminus K$. For each $y \in K$, choose disjoint open sets $U_y \ni x$ and $V_y \ni y$. The collection $\{V_y\}_{y \in K}$ covers K , so by compactness there exist y_1, \dots, y_n with $K \subseteq V_{y_1} \cup \dots \cup V_{y_n}$. Then $U = U_{y_1} \cap \dots \cap U_{y_n}$ is an open neighborhood of x disjoint from K . Hence $X \setminus K$ is open. \square

Proposition 1.14 (Continuous image of a compact set is compact). *If $f: X \rightarrow Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$. Then $\{f^{-1}(V_\alpha)\}$ is an open cover of X . By compactness of X , finitely many $f^{-1}(V_{\alpha_1}), \dots, f^{-1}(V_{\alpha_n})$ cover X , whence $V_{\alpha_1}, \dots, V_{\alpha_n}$ cover $f(X)$. \square

Corollary 1.15 (Bijective continuous map from compact to Hausdorff is a homeomorphism). *If $f: X \rightarrow Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof. It suffices to show that f is a closed map. Let $C \subseteq X$ be closed. Since X is compact, C is compact, so $f(C)$ is compact by Proposition 1.14, hence closed in Y by Theorem 1.13. \square

1.4 Homotopy and Homotopy Equivalence

The key insight of algebraic topology is that homeomorphism is too rigid a notion of equivalence for many purposes. *Homotopy equivalence* is a coarser but more flexible relation that captures the essential “shape” of a space.

Definition 1.16 (Homotopy). Let $f, g: X \rightarrow Y$ be continuous maps. A *homotopy* from f to g is a continuous map $H: X \times I \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. We write $f \simeq g$ and say f is *homotopic* to g .

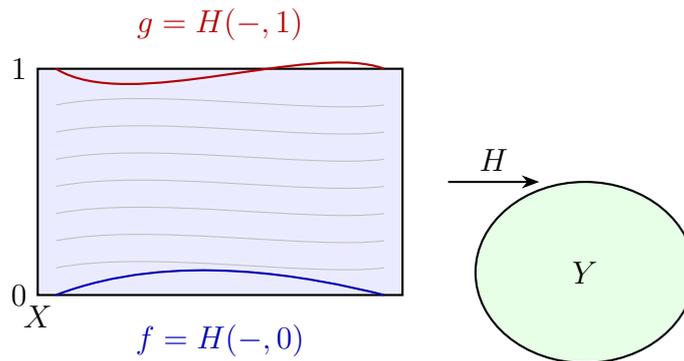


Figure 1.2: A homotopy $H: X \times I \rightarrow Y$ deforming f into g .

Proposition 1.17 (Homotopy is an equivalence relation). *For any spaces X and Y , homotopy \simeq is an equivalence relation on the set of continuous maps from X to Y .*

Proof. Reflexivity: $H(x, t) = f(x)$ is a homotopy from f to itself.

Symmetry: If H is a homotopy from f to g , then $\bar{H}(x, t) = H(x, 1 - t)$ is a homotopy from g to f .

Transitivity: If H is a homotopy from f to g and K is a homotopy from g to h , define

$$L(x, t) = \begin{cases} H(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ K(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $H(x, 1) = g(x) = K(x, 0)$, the map L is well-defined and continuous by the pasting lemma. It is a homotopy from f to h . \square

Definition 1.18 (Homotopy equivalence). A continuous map $f: X \rightarrow Y$ is a *homotopy equivalence* if there exists a continuous map $g: Y \rightarrow X$ such that $g \circ f \simeq \text{Id}_X$ and $f \circ g \simeq \text{Id}_Y$. In this case, g is called a *homotopy inverse* of f , and we write $X \simeq Y$.

Definition 1.19 (Contractible space). A space X is *contractible* if it is homotopy equivalent to a point, i.e., $X \simeq \{\star\}$. Equivalently, the identity map Id_X is homotopic to a constant map.

Example 1.20 (Contractible spaces). The Euclidean space \mathbb{R}^n is contractible: the homotopy $H(x, t) = (1 - t)x$ contracts \mathbb{R}^n to the origin. More generally, any convex subset of \mathbb{R}^n is contractible. The open disk, the closed disk D^n , and any star-shaped region are all contractible.

1.5 Retracts and Deformation Retracts

Definition 1.21 (Retraction and retract). Let $A \subseteq X$ be a subspace. A *retraction* of X onto A is a continuous map $r: X \rightarrow A$ such that $r(a) = a$ for all $a \in A$ (equivalently, $r \circ \iota = \text{Id}_A$ where $\iota: A \hookrightarrow X$ is the inclusion). If such a map exists, A is called a *retract* of X .

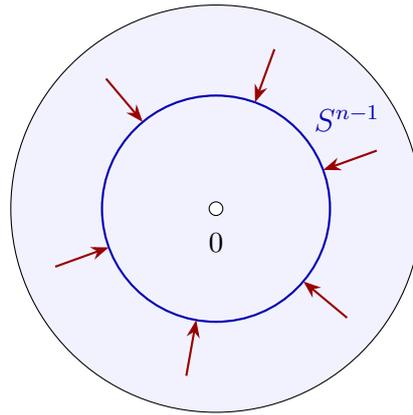
Definition 1.22 (Deformation retract). A subspace $A \subseteq X$ is a *deformation retract* of X if there exists a retraction $r: X \rightarrow A$ such that $\iota \circ r \simeq \text{Id}_X$, i.e., the composition $X \xrightarrow{r} A \xrightarrow{\iota} X$ is homotopic to the identity on X .

The retract is a *strong deformation retract* if the homotopy $H: X \times I \rightarrow X$ from Id_X to $\iota \circ r$ can be chosen so that $H(a, t) = a$ for all $a \in A$ and all $t \in I$.

Example 1.23 (S^{n-1} as a deformation retract of $\mathbb{R}^n \setminus \{0\}$). The map $r: \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ defined by $r(x) = x/|x|$ is a retraction. The homotopy

$$H(x, t) = (1 - t)x + t \frac{x}{|x|} = \left(1 - t + t/|x|\right)x$$

shows that S^{n-1} is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$. In particular, $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$.



$$r(x) = x/|x|$$

Figure 1.3: Radial retraction of $\mathbb{R}^n \setminus \{0\}$ onto S^{n-1} .

Proposition 1.24 (Deformation retract implies homotopy equivalence). *If A is a deformation retract of X , then $A \simeq X$.*

Proof. The inclusion $\iota: A \hookrightarrow X$ and the retraction $r: X \rightarrow A$ satisfy $r \circ \iota = \text{Id}_A$ and $\iota \circ r \simeq \text{Id}_X$, which is precisely the definition of a homotopy equivalence. \square

1.6 CW Complexes

CW complexes form the most important class of spaces in algebraic topology. They are built inductively by attaching cells of increasing dimension, providing a combinatorial handle on their topology.

Definition 1.25 (CW complex). A *CW complex* is a topological space X together with a filtration

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots \subseteq X = \bigcup_{n \geq 0} X^n,$$

where:

- (i) X^0 is a discrete set of points (the *0-cells*).
- (ii) For each $n \geq 1$, the *n-skeleton* X^n is obtained from X^{n-1} by attaching *n-cells*: for each cell e_α^n , there is a continuous *attaching map* $\varphi_\alpha: S^{n-1} \rightarrow X^{n-1}$ and

$$X^n = X^{n-1} \cup_{\{\varphi_\alpha\}} \bigsqcup_{\alpha} D_\alpha^n,$$

where the *n-disk* D_α^n is glued to X^{n-1} along its boundary via φ_α .

- (iii) X has the *weak topology*: a subset $A \subseteq X$ is closed if and only if $A \cap X^n$ is closed in X^n for each n .

The letters “CW” stand for *Closure-finite* (each cell meets only finitely many other cells) and *Weak topology*.

Example 1.26 (CW structure on S^n). The sphere S^n has a CW structure with exactly two cells: one 0-cell e^0 (a point) and one n -cell e^n , attached via the map $\varphi: S^{n-1} \rightarrow e^0$ that collapses the entire boundary to the point. This gives $S^n \cong D^n/\partial D^n$.

Example 1.27 (CW structure on the torus T^2). The torus T^2 admits a CW structure with one 0-cell v , two 1-cells a and b (both loops based at v), and one 2-cell e^2 attached via the word $aba^{-1}b^{-1}$.

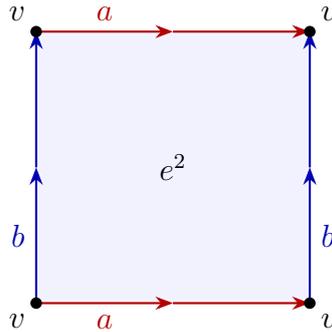


Figure 1.4: The standard CW structure on the torus: one vertex, two edges, one face.

Example 1.28 (Real projective space $\mathbb{R}P^n$). The real projective space $\mathbb{R}P^n$ has a CW structure with exactly one cell in each dimension $0, 1, \dots, n$. The k -cell is attached to $\mathbb{R}P^{k-1}$ via the quotient map $S^{k-1} \rightarrow \mathbb{R}P^{k-1}$, which identifies antipodal points. Thus

$$\mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cup \dots \cup e^n.$$

Remark 1.29. CW complexes enjoy many pleasant properties. They are always Hausdorff and paracompact. A compact subset of a CW complex meets only finitely many cells. Moreover, any CW complex is homotopy equivalent to a space built from simplices (a simplicial complex), but the CW decomposition is typically far more economical: for instance, S^n requires only 2 cells as a CW complex but at least $2(n+1)$ simplices as a simplicial complex.

1.7 The Homotopy Category

We briefly introduce the categorical perspective, which clarifies the functorial nature of algebraic topology.

Definition 1.30 (Category **Top**). The category **Top** has topological spaces as objects and continuous maps as morphisms. Composition is the usual composition of functions.

Definition 1.31 (Homotopy category **hTop**). The *homotopy category* **hTop** has the same objects as **Top**, but the morphisms from X to Y are homotopy classes $[X, Y]$

of continuous maps. Composition is induced by the composition of representatives:

$$[g] \circ [f] = [g \circ f].$$

This is well-defined because homotopy is compatible with composition.

Remark 1.32. The isomorphisms in **hTop** are precisely the homotopy equivalences. Thus two spaces are “the same” in **hTop** if and only if they are homotopy equivalent.

A functor $F: \mathbf{hTop} \rightarrow \mathbf{Grp}$ assigns a group $F(X)$ to each space X and a group homomorphism $F([f]): F(X) \rightarrow F(Y)$ to each homotopy class $[f]$, preserving identity and composition. The fundamental group π_1 is the first example of such a functor; homology groups H_n provide further examples.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 g \downarrow & & \downarrow h \\
 Z & \xrightarrow{k} & W
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(X) & \xrightarrow{F(f)} & F(Y) \\
 F(g) \downarrow & & \downarrow F(h) \\
 F(Z) & \xrightarrow{F(k)} & F(W)
 \end{array}$$

Figure 1.5: A functor F maps a commutative diagram in **hTop** to a commutative diagram in **Grp**.

1.8 Motivations for Algebraic Topology

The fundamental question driving algebraic topology is:

When are two topological spaces “essentially the same,” and how can we tell them apart?

Purely topological arguments can be remarkably difficult. Consider the following questions:

- (1) Is \mathbb{R}^2 homeomorphic to \mathbb{R}^3 ? (No—but this is surprisingly hard to prove without algebraic tools.)
- (2) Is S^1 a retract of D^2 ? (No—proved below using the fundamental group.)
- (3) Does every continuous map $f: D^2 \rightarrow D^2$ have a fixed point? (Yes—the Brouwer fixed-point theorem, proved in Chapter 2.)
- (4) Can the 2-sphere S^2 be combed? That is, does there exist a nonvanishing continuous tangent vector field on S^2 ? (No—the hairy ball theorem.)

The strategy of algebraic topology is to associate to each space X an algebraic object—a group, a ring, a module—in such a way that continuous maps induce homomorphisms, and homotopy equivalent spaces receive isomorphic invariants. If two spaces have non-isomorphic invariants, they cannot be homotopy equivalent (and a fortiori cannot be homeomorphic).

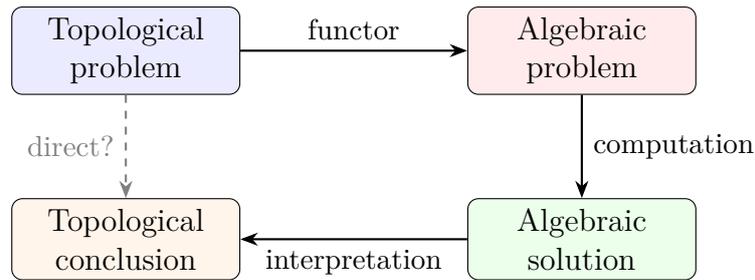


Figure 1.6: The algebraic topology paradigm: translate, compute, interpret.

The main invariants we will study are:

- The **fundamental group** $\pi_1(X, x_0)$: captures information about loops in X up to continuous deformation.
- **Higher homotopy groups** $\pi_n(X, x_0)$: higher-dimensional analogues of π_1 .
- **Singular homology** $H_n(X; G)$: abelian groups measuring n -dimensional “holes” in X .
- **Singular cohomology** $H^n(X; G)$: the dual theory, with a rich multiplicative structure (the cup product).

1.9 Exercises

Exercise 1.1. Let X be a topological space with the discrete topology (every subset is open). Show that X is connected if and only if $|X| \leq 1$.

Exercise 1.2. Let $f: X \rightarrow Y$ be continuous, and suppose X is connected. Show that $f(X)$ is connected.

Exercise 1.3. Prove that S^n is path-connected for all $n \geq 1$. *Hint:* For $n = 1$, parameterize the circle. For $n \geq 2$, use the fact that $S^n \setminus \{p\} \cong \mathbb{R}^n$ (stereographic projection) and that \mathbb{R}^n is path-connected.

Exercise 1.4. Show that the product $X \times Y$ is connected (resp. path-connected) if and only if both X and Y are connected (resp. path-connected).

Exercise 1.5. Let X and Y be homotopy equivalent. Show that if X is path-connected, then so is Y .

Exercise 1.6. Show that the Möbius band is homotopy equivalent to S^1 . *Hint:* Exhibit the central circle as a deformation retract.

Exercise 1.7. Describe a CW structure on the Klein bottle K with one 0-cell, two 1-cells, and one 2-cell. What is the attaching word for the 2-cell?

Exercise 1.8. Show that the punctured plane $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 , and that $\mathbb{R}^3 \setminus \{0\}$ is homotopy equivalent to S^2 . Generalize to $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$.

Exercise 1.9. Let $X = S^1 \vee S^1$ be the wedge (one-point union) of two circles. Describe a CW structure on X and compute the Euler characteristic $\chi(X) = c_0 - c_1 + c_2 - \dots$ where c_k is the number of k -cells.

Exercise 1.10 (Mapping cylinder). Given a continuous map $f: X \rightarrow Y$, the *mapping cylinder* is the quotient space

$$M_f = ((X \times I) \sqcup Y) / ((x, 1) \sim f(x)).$$

Show that Y is a strong deformation retract of M_f . Conclude that any continuous map is homotopic to an inclusion, up to homotopy equivalence.

Chapter 2

The Fundamental Group — Definition and First Properties

The fundamental group is the oldest and most intuitive invariant of algebraic topology, introduced by Henri Poincaré in 1895. It captures the essential information about loops in a space up to continuous deformation.

2.1 Paths and Loops

Throughout this chapter, $I = [0, 1]$ denotes the unit interval.

Definition 2.1 (Path). A *path* in a topological space X is a continuous map $\gamma: I \rightarrow X$. The point $\gamma(0)$ is the *initial point* and $\gamma(1)$ is the *terminal point*. We say γ is a path from $\gamma(0)$ to $\gamma(1)$.

Definition 2.2 (Loop). A *loop* in X based at a point $x_0 \in X$ is a path $\gamma: I \rightarrow X$ with $\gamma(0) = \gamma(1) = x_0$. The point x_0 is called the *basepoint*.

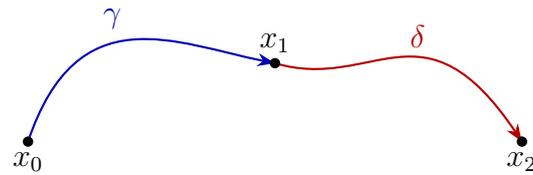
Definition 2.3 (Constant path). For any $x_0 \in X$, the *constant path* at x_0 is the map $c_{x_0}: I \rightarrow X$ defined by $c_{x_0}(t) = x_0$ for all t .

Definition 2.4 (Reverse path). Given a path $\gamma: I \rightarrow X$, its *reverse* is the path $\bar{\gamma}: I \rightarrow X$ defined by $\bar{\gamma}(t) = \gamma(1 - t)$.

Definition 2.5 (Concatenation of paths). Given paths $\gamma, \delta: I \rightarrow X$ with $\gamma(1) = \delta(0)$, the *concatenation* $\gamma \star \delta: I \rightarrow X$ is defined by

$$(\gamma \star \delta)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \delta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

This is continuous by the pasting lemma, since $\gamma(1) = \delta(0)$.



$\gamma \star \delta$: traverse γ then δ , at double speed

Figure 2.1: Concatenation of paths γ (from x_0 to x_1) and δ (from x_1 to x_2).

2.2 Path Homotopy

We now introduce the key equivalence relation on paths.

Definition 2.6 (Path homotopy). Two paths $\gamma_0, \gamma_1: I \rightarrow X$ with the same initial point $\gamma_0(0) = \gamma_1(0) = x_0$ and the same terminal point $\gamma_0(1) = \gamma_1(1) = x_1$ are *path-homotopic* if there exists a continuous map $H: I \times I \rightarrow X$ such that:

- (i) $H(s, 0) = \gamma_0(s)$ and $H(s, 1) = \gamma_1(s)$ for all $s \in I$,
- (ii) $H(0, t) = x_0$ and $H(1, t) = x_1$ for all $t \in I$.

We write $\gamma_0 \simeq \gamma_1 \text{ rel } \partial I$ and denote the homotopy class of γ by $[\gamma]$.

Condition (ii) is what distinguishes path homotopy from free homotopy: the endpoints are held fixed throughout the deformation.

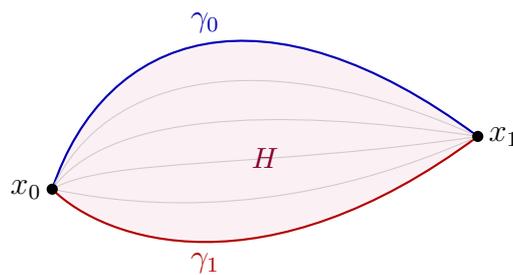


Figure 2.2: A path homotopy H deforming γ_0 into γ_1 with fixed endpoints.

Proposition 2.7 (Path homotopy is an equivalence relation). *Path homotopy is an equivalence relation on the set of paths from x_0 to x_1 .*

Proof. The proof is analogous to that of Proposition 1.17. Reflexivity uses $H(s, t) = \gamma(s)$. Symmetry uses $\bar{H}(s, t) = H(s, 1 - t)$. Transitivity uses the concatenation of homotopies along the t -parameter with the pasting lemma. \square

2.3 The Fundamental Group

Theorem 2.8 (The fundamental group $\pi_1(X, x_0)$). *Let X be a topological space and $x_0 \in X$. The set of path-homotopy classes of loops based at x_0 , equipped with the operation*

$$[\gamma] \cdot [\delta] = [\gamma \star \delta],$$

is a group, denoted $\pi_1(X, x_0)$ and called the fundamental group of X at the basepoint x_0 .

Proof. We must verify well-definedness, associativity, the existence of an identity element, and the existence of inverses.

Well-definedness. We must show that if $\gamma_0 \simeq \gamma_1 \text{ rel } \partial I$ and $\delta_0 \simeq \delta_1 \text{ rel } \partial I$, then $\gamma_0 \star \delta_0 \simeq \gamma_1 \star \delta_1 \text{ rel } \partial I$. Let H be a homotopy from γ_0 to γ_1 and K a homotopy from δ_0 to δ_1 . Define

$$L(s, t) = \begin{cases} H(2s, t) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ K(2s - 1, t) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then L is a well-defined path homotopy from $\gamma_0 \star \delta_0$ to $\gamma_1 \star \delta_1$.

Associativity. We need $(\gamma \star \delta) \star \varepsilon \simeq \gamma \star (\delta \star \varepsilon) \text{ rel } \partial I$. Define

$$H(s, t) = \begin{cases} \gamma\left(\frac{4s}{1+t}\right) & \text{if } 0 \leq s \leq \frac{1+t}{4}, \\ \delta(4s - 1 - t) & \text{if } \frac{1+t}{4} \leq s \leq \frac{2+t}{4}, \\ \varepsilon\left(\frac{4s - 2 - t}{2 - t}\right) & \text{if } \frac{2+t}{4} \leq s \leq 1. \end{cases}$$

One checks that H is continuous (by the pasting lemma applied to the three pieces), that $H(s, 0) = ((\gamma \star \delta) \star \varepsilon)(s)$, that $H(s, 1) = (\gamma \star (\delta \star \varepsilon))(s)$, and that $H(0, t) = x_0$ and $H(1, t) = \varepsilon(1)$ for all t . The key idea is that the reparameterization amounts to changing where the “time” is spent along each of the three paths.

Identity. The constant loop c_{x_0} serves as the identity: we claim $\gamma \star c_{x_0} \simeq \gamma \simeq c_{x_0} \star \gamma \text{ rel } \partial I$. For the first, define

$$H(s, t) = \begin{cases} \gamma\left(\frac{2s}{1+t}\right) & \text{if } 0 \leq s \leq \frac{1+t}{2}, \\ x_0 & \text{if } \frac{1+t}{2} \leq s \leq 1. \end{cases}$$

Then $H(s, 0) = (\gamma \star c_{x_0})(s)$ and $H(s, 1) = \gamma(s)$. The other case is similar.

Inverses. For any loop γ based at x_0 , we claim $[\gamma \star \bar{\gamma}] = [c_{x_0}]$. Define

$$H(s, t) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq \frac{t}{2}, \\ \gamma(t) & \text{if } \frac{t}{2} \leq s \leq 1 - \frac{t}{2}, \\ \gamma(2 - 2s) & \text{if } 1 - \frac{t}{2} \leq s \leq 1. \end{cases}$$

At $t = 0$, this is the constant path c_{x_0} . At $t = 1$, this is $\gamma \star \bar{\gamma}$. Thus $\gamma \star \bar{\gamma} \simeq c_{x_0}$. A similar construction shows $\bar{\gamma} \star \gamma \simeq c_{x_0}$. \square

Remark 2.9. The fundamental group $\pi_1(X, x_0)$ depends on the choice of basepoint x_0 . However, as we will see shortly, for path-connected spaces the groups at different basepoints are isomorphic (though not canonically so).

Notation 2.10. When X is path-connected and the basepoint is unimportant (e.g., when stating that the group is trivial or abelian), we may write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition 2.11 (Simply connected space). A path-connected space X is *simply connected* if $\pi_1(X, x_0) = \{e\}$ for some (hence every) basepoint $x_0 \in X$. Equivalently, X is simply connected if it is path-connected and every loop in X is path-homotopic to a constant loop.

2.4 Functoriality of π_1

The fundamental group is not merely an assignment of a group to each pointed space; it is a *functor* from the category of pointed topological spaces to the category of groups.

Definition 2.12 (Pointed space and pointed map). A *pointed topological space* is a pair (X, x_0) where X is a topological space and $x_0 \in X$ is a chosen basepoint. A *pointed map* $f: (X, x_0) \rightarrow (Y, y_0)$ is a continuous map $f: X \rightarrow Y$ with $f(x_0) = y_0$.

Theorem 2.13 (Functoriality of π_1). *Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a pointed map. The assignment*

$$f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0), \quad f_*([\gamma]) = [f \circ \gamma],$$

is a well-defined group homomorphism, called the induced homomorphism. Moreover:

(i) $(\text{Id}_X)_* = \text{Id}_{\pi_1(X, x_0)}$.

(ii) $(g \circ f)_* = g_* \circ f_*$.

In other words, π_1 is a (covariant) functor from the category of pointed topological spaces to the category of groups.

Proof. Well-definedness. If $\gamma_0 \simeq \gamma_1 \text{ rel } \partial I$ via a homotopy H , then $f \circ H$ is a path homotopy from $f \circ \gamma_0$ to $f \circ \gamma_1$.

Homomorphism. We have $f \circ (\gamma \star \delta) = (f \circ \gamma) \star (f \circ \delta)$, since

$$(f \circ (\gamma \star \delta))(t) = f((\gamma \star \delta)(t)) = \begin{cases} f(\gamma(2t)) & \text{if } t \leq \frac{1}{2}, \\ f(\delta(2t - 1)) & \text{if } t \geq \frac{1}{2}, \end{cases} = ((f \circ \gamma) \star (f \circ \delta))(t).$$

Hence $f_*([\gamma] \cdot [\delta]) = f_*([\gamma]) \cdot f_*([\delta])$.

Properties (i) and (ii) are immediate from the definitions. □

The functoriality is neatly expressed by the following commutative diagram. If $f: (X, x_0) \rightarrow (Y, y_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$ are pointed maps, then:

$$\begin{array}{ccc}
 (X, x_0) & \xrightarrow{f} & (Y, y_0) \\
 \searrow^{g \circ f} & & \downarrow g \\
 & & (Z, z_0)
 \end{array}
 \xrightarrow{\pi_1}
 \begin{array}{ccc}
 \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\
 \searrow^{(g \circ f)_*} & & \downarrow g_* \\
 & & \pi_1(Z, z_0)
 \end{array}$$

 Figure 2.3: Functoriality of π_1 : the functor preserves composition.

2.5 Change of Basepoint

Theorem 2.14 (Change of basepoint isomorphism). *Let X be a topological space and let $\alpha: I \rightarrow X$ be a path from x_0 to x_1 . The map*

$$\beta_\alpha: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1), \quad \beta_\alpha([\gamma]) = [\bar{\alpha} * \gamma * \alpha],$$

is a group isomorphism.

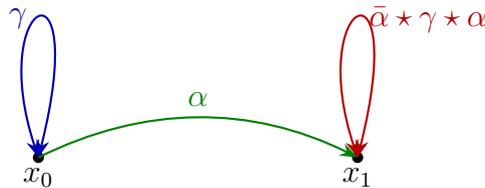
Proof. First, β_α is well-defined: if $\gamma \simeq \gamma' \text{ rel } \partial I$, then $\bar{\alpha} * \gamma * \alpha \simeq \bar{\alpha} * \gamma' * \alpha \text{ rel } \partial I$.

Next, β_α is a homomorphism:

$$\begin{aligned}
 \beta_\alpha([\gamma] \cdot [\delta]) &= [\bar{\alpha} * (\gamma * \delta) * \alpha] \\
 &= [\bar{\alpha} * \gamma * (c_{x_0}) * \delta * \alpha] \\
 &\simeq [\bar{\alpha} * \gamma * (\alpha * \bar{\alpha}) * \delta * \alpha] \\
 &= [(\bar{\alpha} * \gamma * \alpha) * (\bar{\alpha} * \delta * \alpha)] \\
 &= \beta_\alpha([\gamma]) \cdot \beta_\alpha([\delta]).
 \end{aligned}$$

Finally, $\beta_{\bar{\alpha}}: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ is the inverse of β_α :

$$\beta_{\bar{\alpha}} \circ \beta_\alpha([\gamma]) = [\alpha * (\bar{\alpha} * \gamma * \alpha) * \bar{\alpha}] = [(\alpha * \bar{\alpha}) * \gamma * (\alpha * \bar{\alpha})] = [c_{x_0} * \gamma * c_{x_0}] = [\gamma]. \quad \square$$


 Figure 2.4: Change of basepoint: a loop γ at x_0 is conjugated by the path α to yield a loop at x_1 .

Corollary 2.15. *If X is path-connected, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any two points $x_0, x_1 \in X$.*

Remark 2.16 (Non-canonical isomorphism). The isomorphism β_α depends on the choice of path α : different paths from x_0 to x_1 may yield different isomorphisms. The isomorphism is canonical (independent of α) if and only if $\pi_1(X, x_0)$ is abelian. Indeed,

if α and α' are two paths from x_0 to x_1 , then $\beta_{\alpha'} \circ \beta_{\alpha}^{-1}$ is conjugation by the loop $\alpha \star \bar{\alpha}' \in \pi_1(X, x_1)$.

2.6 Homotopy Invariance

The fundamental group is a homotopy invariant: homotopy equivalent spaces have isomorphic fundamental groups. This is one of the most important properties of π_1 , and we give a complete proof.

Theorem 2.17 (π_1 is a homotopy invariant). *Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a homotopy equivalence. Then*

$$f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

is a group isomorphism.

To prove this, we first establish a key lemma relating the induced maps of homotopic maps.

Lemma 2.18 (Homotopic maps induce conjugate homomorphisms). *Let $f, g: (X, x_0) \rightarrow Y$ be continuous maps (not necessarily pointed), and let $H: X \times I \rightarrow Y$ be a homotopy from f to g . Define the path $\alpha(t) = H(x_0, t)$ from $f(x_0)$ to $g(x_0)$. Then for every loop γ based at x_0 :*

$$g_*([\gamma]) = [\bar{\alpha}] \cdot f_*([\gamma]) \cdot [\alpha] \quad \text{in } \pi_1(Y, g(x_0)).$$

That is, $g_ = \beta_{\bar{\alpha}} \circ f_*$ where $\beta_{\bar{\alpha}}$ is the change-of-basepoint isomorphism.*

Proof. Consider the map $\Gamma: I \times I \rightarrow Y$ defined by $\Gamma(s, t) = H(\gamma(s), t)$. The four boundary segments of $I \times I$ map as follows:

- Bottom ($t = 0$): $\Gamma(s, 0) = f(\gamma(s))$, the loop $f \circ \gamma$.
- Top ($t = 1$): $\Gamma(s, 1) = g(\gamma(s))$, the loop $g \circ \gamma$.
- Left ($s = 0$): $\Gamma(0, t) = H(x_0, t) = \alpha(t)$, the path α .
- Right ($s = 1$): $\Gamma(1, t) = H(x_0, t) = \alpha(t)$, the path α .

By reparameterizing the boundary of the square, we obtain a path homotopy $\alpha \star (f \circ \gamma) \star \bar{\alpha} \simeq g \circ \gamma \text{ rel } \partial I$. Hence $[g \circ \gamma] = [\alpha] \cdot [f \circ \gamma] \cdot [\bar{\alpha}]$, i.e., $g_*([\gamma]) = [\alpha] \cdot f_*([\gamma]) \cdot [\bar{\alpha}]$.

Equivalently, $[\bar{\alpha}] \cdot g_*([\gamma]) \cdot [\alpha] = f_*([\gamma])$, which gives $g_* = \beta_{\bar{\alpha}} \circ f_*$ with the convention stated. \square

Proof of Theorem 2.17. Let $g: Y \rightarrow X$ be a homotopy inverse to f , so $g \circ f \simeq \text{Id}_X$ and $f \circ g \simeq \text{Id}_Y$.

Step 1: $g_ \circ f_*$ is an isomorphism.* By Lemma 2.18 applied to the homotopy $g \circ f \simeq \text{Id}_X$, there is a path α in X from $g(f(x_0))$ to x_0 such that

$$(\text{Id}_X)_* = \beta_{\bar{\alpha}} \circ (g \circ f)_* = \beta_{\bar{\alpha}} \circ g_* \circ f_*.$$

Since $(\text{Id}_X)_* = \text{Id}_{\pi_1(X, x_0)}$ and $\beta_{\bar{\alpha}}$ is an isomorphism, $g_* \circ f_*$ is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(X, g(f(x_0)))$.

Step 2: $f_ \circ g_*$ is an isomorphism.* Similarly, the homotopy $f \circ g \simeq \text{Id}_Y$ yields a path β in Y such that $f_* \circ g_* = \beta_\beta$, which is an isomorphism.

Step 3: f_ is an isomorphism.* Since $g_* \circ f_*$ is injective (as a composition equal to an isomorphism up to conjugation), f_* is injective. Since $f_* \circ g_*$ is surjective, f_* is surjective. Hence f_* is a bijective homomorphism, i.e., an isomorphism. \square

Corollary 2.19. *If X is contractible, then $\pi_1(X, x_0)$ is trivial for every $x_0 \in X$.*

Proof. A contractible space is homotopy equivalent to a point, and $\pi_1(\{*\}) = \{e\}$. \square

Corollary 2.20 ($\pi_1(\mathbb{R}^n) = 0$). *For all $n \geq 1$, $\pi_1(\mathbb{R}^n, x_0) = \{e\}$ for any basepoint x_0 .*

Proof. \mathbb{R}^n is contractible (via the straight-line homotopy to the origin), so the result follows from Corollary 2.19. \square

2.7 The Fundamental Group of the Circle

The computation of $\pi_1(S^1) \cong \mathbb{Z}$ is one of the cornerstones of algebraic topology. We present the classical proof using lifting theory for the covering map $p: \mathbb{R} \rightarrow S^1$.

Definition 2.21 (Covering map $p: \mathbb{R} \rightarrow S^1$). Define $p: \mathbb{R} \rightarrow S^1$ by $p(t) = (\cos 2\pi t, \sin 2\pi t)$, viewing $S^1 \subseteq \mathbb{R}^2$. This is a covering map: every point of S^1 has a neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets, each mapped homeomorphically onto U by p .

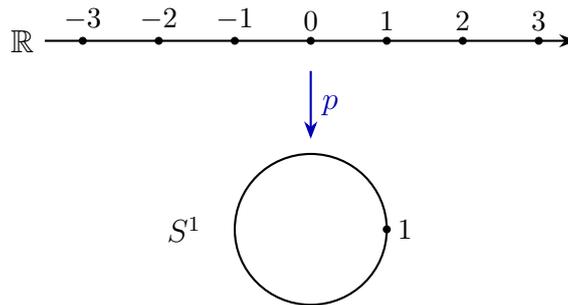


Figure 2.5: The covering map $p: \mathbb{R} \rightarrow S^1$, $t \mapsto e^{2\pi it}$.

Lemma 2.22 (Path lifting). *Let $\gamma: I \rightarrow S^1$ be a path with $\gamma(0) = x_0 \in S^1$, and let $\tilde{x}_0 \in p^{-1}(x_0)$. Then there exists a unique path $\tilde{\gamma}: I \rightarrow \mathbb{R}$ such that $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{x}_0$.*

Proof. Existence. Cover S^1 by open sets $\{U_j\}$ that are evenly covered by p . Since I is compact, we can find a partition $0 = t_0 < t_1 < \dots < t_m = 1$ such that each $\gamma([t_{k-1}, t_k])$ lies in some U_{j_k} . We lift γ inductively on each subinterval $[t_{k-1}, t_k]$: since U_{j_k} is evenly covered,

$p^{-1}(U_{j_k}) = \bigsqcup_{\ell} \tilde{U}_{j_k, \ell}$ with $p|_{\tilde{U}_{j_k, \ell}}$ a homeomorphism. The lift of $\gamma|_{[t_{k-1}, t_k]}$ is determined by choosing the sheet containing $\tilde{\gamma}(t_{k-1})$.

Uniqueness. If $\tilde{\gamma}$ and $\tilde{\gamma}'$ are two lifts with the same initial point, then $\{t \in I : \tilde{\gamma}(t) = \tilde{\gamma}'(t)\}$ is both open (by local homeomorphism) and closed (by continuity), hence is all of I since it contains 0 and I is connected. \square

Lemma 2.23 (Homotopy lifting). *Let $H: I \times I \rightarrow S^1$ be a path homotopy (i.e., $H(0, t) = x_0$ and $H(1, t) = x_1$ for all t), and let $\tilde{x}_0 \in p^{-1}(x_0)$. Then there exists a unique lift $\tilde{H}: I \times I \rightarrow \mathbb{R}$ with $p \circ \tilde{H} = H$ and $\tilde{H}(0, 0) = \tilde{x}_0$. Moreover, \tilde{H} is a path homotopy in \mathbb{R} : $\tilde{H}(0, t) = \tilde{x}_0$ and $\tilde{H}(1, t) = \tilde{H}(1, 0)$ for all t .*

Proof. The existence and uniqueness follow by the same Lebesgue number argument applied to the square $I \times I$. Subdivide $I \times I$ into small squares, each mapping into an evenly covered neighborhood, and lift inductively.

For the boundary conditions: $t \mapsto \tilde{H}(0, t)$ is a lift of the constant path $t \mapsto x_0$ starting at \tilde{x}_0 , hence is constant (by uniqueness of lifts). Similarly, $t \mapsto \tilde{H}(1, t)$ is a lift of the constant path at x_1 , hence is constant. \square

Definition 2.24 (Degree of a loop). For a loop γ in S^1 based at $1 = (1, 0)$, let $\tilde{\gamma}$ be the unique lift to \mathbb{R} with $\tilde{\gamma}(0) = 0$. The integer $\tilde{\gamma}(1) \in \mathbb{Z}$ is called the *degree* (or *winding number*) of γ , denoted $\deg(\gamma)$.

Remark 2.25. Since $p(\tilde{\gamma}(1)) = \gamma(1) = 1 = p(0)$ and $p^{-1}(1) = \mathbb{Z}$, we have $\tilde{\gamma}(1) \in \mathbb{Z}$. By Lemma 4.9, if $\gamma \simeq \gamma' \text{ rel } \partial I$, then $\deg(\gamma) = \deg(\gamma')$, so the degree depends only on the homotopy class.

Theorem 2.26 ($\pi_1(S^1) \cong \mathbb{Z}$). *The map $\Phi: \pi_1(S^1, 1) \rightarrow \mathbb{Z}$ defined by $\Phi([\gamma]) = \deg(\gamma)$ is a group isomorphism.*

Proof. We verify that Φ is a well-defined group homomorphism and a bijection.

Well-definedness. If $[\gamma] = [\gamma']$, then $\gamma \simeq \gamma' \text{ rel } \partial I$. By Lemma 4.9, the lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ (both starting at 0) satisfy $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$. So $\deg(\gamma) = \deg(\gamma')$.

Homomorphism. Let γ and δ be loops at 1, with lifts $\tilde{\gamma}$ and $\tilde{\delta}$ starting at 0. Write $n = \tilde{\gamma}(1)$ and $m = \tilde{\delta}(1)$. Define $\tilde{\delta}_n(t) = \tilde{\delta}(t) + n$; this is a lift of δ starting at n . The concatenation $\tilde{\gamma} \star \tilde{\delta}_n$ is a path in \mathbb{R} from 0 to $m+n$, lifting $\gamma \star \delta$. By uniqueness, $\widetilde{\gamma \star \delta} = \tilde{\gamma} \star \tilde{\delta}_n$, and so

$$\Phi([\gamma \star \delta]) = \widetilde{\gamma \star \delta}(1) = m + n = \Phi([\gamma]) + \Phi([\delta]).$$

Surjectivity. For each $n \in \mathbb{Z}$, let $\omega_n(t) = (\cos 2\pi nt, \sin 2\pi nt)$. This is a loop at 1, and its lift starting at 0 is $\tilde{\omega}_n(t) = nt$, which ends at n . Hence $\Phi([\omega_n]) = n$.

Injectivity. Suppose $\Phi([\gamma]) = 0$, i.e., $\tilde{\gamma}(1) = 0$. Then $\tilde{\gamma}$ is a loop in \mathbb{R} based at 0. Since \mathbb{R} is simply connected (being contractible), $\tilde{\gamma} \simeq c_0 \text{ rel } \partial I$ in \mathbb{R} . Applying p to this homotopy, we get $\gamma = p \circ \tilde{\gamma} \simeq p \circ c_0 = c_1 \text{ rel } \partial I$ in S^1 . Hence $[\gamma] = [c_1]$, the identity element. \square

Remark 2.27 (Geometric interpretation). The integer $n = \Phi([\gamma])$ is the *winding number*: it counts how many times γ wraps around S^1 , with sign indicating direction. The loop ω_n wraps n times counterclockwise (or $|n|$ times clockwise if $n < 0$).

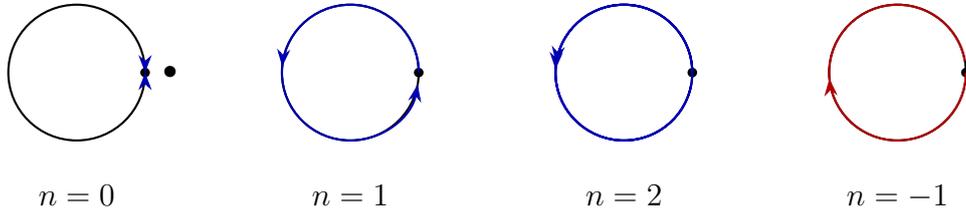


Figure 2.6: Loops on S^1 with winding numbers 0, 1, 2, and -1 .

2.8 Applications: Brouwer Fixed-Point Theorem in Dimension 2

As a striking application of the computation $\pi_1(S^1) \cong \mathbb{Z}$, we prove the Brouwer fixed-point theorem for the disk D^2 .

Lemma 2.28 (No retraction $D^2 \rightarrow S^1$). *There is no retraction $r: D^2 \rightarrow S^1$, i.e., no continuous map r with $r|_{S^1} = \text{Id}_{S^1}$.*

Proof. Suppose for contradiction that $r: D^2 \rightarrow S^1$ is a retraction. Let $\iota: S^1 \hookrightarrow D^2$ be the inclusion. Then $r \circ \iota = \text{Id}_{S^1}$, so by functoriality of π_1 :

$$r_* \circ \iota_* = (\text{Id}_{S^1})_* = \text{Id}_{\pi_1(S^1)}.$$

Now consider the diagram of induced homomorphisms:

$$\begin{array}{ccccc} \pi_1(S^1, x_0) & \xrightarrow{\iota_*} & \pi_1(D^2, x_0) & \xrightarrow{r_*} & \pi_1(S^1, x_0) \\ & & \searrow & \nearrow & \\ & & \text{Id} & & \end{array}$$

Since D^2 is contractible, $\pi_1(D^2, x_0) = \{e\}$. Hence ι_* maps $\pi_1(S^1) \cong \mathbb{Z}$ to the trivial group, and $r_* \circ \iota_*$ is the zero map. But we need $r_* \circ \iota_* = \text{Id}$, which is impossible since $\mathbb{Z} \neq \{0\}$. Contradiction. \square

Theorem 2.29 (Brouwer fixed-point theorem, dimension 2). *Every continuous map $f: D^2 \rightarrow D^2$ has a fixed point.*

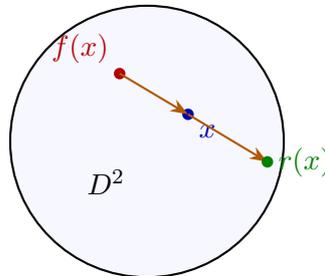
Proof. Suppose for contradiction that f has no fixed point, i.e., $f(x) \neq x$ for all $x \in D^2$. For each $x \in D^2$, consider the ray starting at $f(x)$, passing through x , and continuing until it hits S^1 . Define $r(x)$ to be this intersection point.

Explicitly: $r(x) = x + t(x) \cdot (x - f(x))$ where $t(x) \geq 0$ is the unique non-negative real number such that $|r(x)| = 1$. To find $t(x)$, we solve

$$|x + t(x - f(x))|^2 = 1,$$

which is a quadratic in t with a unique non-negative solution (given by the quadratic formula, since $|x - f(x)| > 0$ by hypothesis). The function $t(x)$ is continuous (by continuity of the quadratic formula), hence r is continuous.

Moreover, if $x \in S^1$, then $|x| = 1$, so $t(x) = 0$ and $r(x) = x$. Thus $r: D^2 \rightarrow S^1$ is a retraction, contradicting Lemma 2.28. \square



Ray from $f(x)$ through x to S^1

Figure 2.7: Construction of the retraction r in the proof of the Brouwer fixed-point theorem.

Remark 2.30. The Brouwer fixed-point theorem holds in all dimensions: every continuous map $f: D^n \rightarrow D^n$ has a fixed point. For $n \geq 3$, the proof requires homology theory (since $\pi_1(S^{n-1}) = 0$ for $n \geq 3$ and the argument above breaks down). We will return to this in the chapter on singular homology.

Corollary 2.31 (No-retraction theorem). S^1 is not a retract of D^2 .

Proof. This is precisely Lemma 2.28. \square

2.9 Further First Computations

Using the tools developed so far, we can compute the fundamental groups of several basic spaces.

Proposition 2.32 ($\pi_1(S^n) = 0$ for $n \geq 2$). For $n \geq 2$, the sphere S^n is simply connected: $\pi_1(S^n, x_0) = \{e\}$.

Proof. Let $\gamma: I \rightarrow S^n$ be a loop based at x_0 . We claim γ is homotopic to a loop that misses some point $q \in S^n$. If γ is not surjective, this is immediate. If γ is surjective, we use the compactness of I and the fact that γ cannot fill S^n in a sufficiently strong sense: by a Lebesgue number argument and a small perturbation (for $n \geq 2$, the complement of the image of a curve in S^n is dense), γ can be homotoped to a non-surjective loop.

Once γ misses some point q , it lies in $S^n \setminus \{q\} \cong \mathbb{R}^n$ (via stereographic projection). Since \mathbb{R}^n is contractible, γ is null-homotopic in $S^n \setminus \{q\}$, hence in S^n . \square

Proposition 2.33 (Fundamental group of a product). For any pointed spaces (X, x_0)

and (Y, y_0) ,

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. The projections $p_1: X \times Y \rightarrow X$ and $p_2: X \times Y \rightarrow Y$ induce a homomorphism

$$\Psi: \pi_1(X \times Y) \rightarrow \pi_1(X) \times \pi_1(Y), \quad [\gamma] \mapsto ((p_1)_*[\gamma], (p_2)_*[\gamma]).$$

The inverse is given by $\Psi^{-1}([\alpha], [\beta]) = [\alpha \times \beta]$, where $(\alpha \times \beta)(t) = (\alpha(t), \beta(t))$. One checks directly that Ψ and Ψ^{-1} are inverse homomorphisms. \square

Corollary 2.34 (Fundamental group of the torus). $\pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2$. More generally, $\pi_1(T^n) \cong \mathbb{Z}^n$.

Example 2.35 (Fundamental group of the punctured plane). Since $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 (via radial retraction, Example 1.23), we have $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$ by homotopy invariance. The generator is the loop that winds once around the origin.

Example 2.36 (Cylinder and Möbius band). The cylinder $S^1 \times I$ deformation retracts onto $S^1 \times \{0\}$, so $\pi_1(S^1 \times I) \cong \mathbb{Z}$. The Möbius band also deformation retracts onto its central circle (which is a copy of S^1), so it too has fundamental group \mathbb{Z} .

2.10 Topological Consequences of Functoriality

The functoriality of π_1 has many immediate and powerful consequences. We collect several here.

Proposition 2.37 (Retract inherits fundamental group). *Let A be a retract of X with retraction $r: X \rightarrow A$, and let $\iota: A \hookrightarrow X$ be the inclusion. Then for any $a_0 \in A$:*

- (i) $\iota_*: \pi_1(A, a_0) \rightarrow \pi_1(X, a_0)$ is injective.
- (ii) $r_*: \pi_1(X, a_0) \rightarrow \pi_1(A, a_0)$ is surjective.
- (iii) $\pi_1(X, a_0) \cong \pi_1(A, a_0) \oplus \ker(r_*)$ when $\pi_1(X, a_0)$ is abelian (in general, $\pi_1(A, a_0)$ is a retract of $\pi_1(X, a_0)$ in the category of groups).

Proof. Since $r \circ \iota = \text{Id}_A$, functoriality gives $r_* \circ \iota_* = \text{Id}_{\pi_1(A, a_0)}$. This immediately implies that ι_* is injective and r_* is surjective. For (iii), when $\pi_1(X, a_0)$ is abelian, the map $\iota_* \circ r_*$ is an idempotent endomorphism of $\pi_1(X, a_0)$, giving the splitting. \square

Proposition 2.38 (Homeomorphic spaces have isomorphic π_1). *If $f: X \rightarrow Y$ is a homeomorphism with $f(x_0) = y_0$, then $f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is an isomorphism.*

Proof. Since f is a homeomorphism, it is in particular a homotopy equivalence (with homotopy inverse f^{-1}). By Theorem 2.17, f_* is an isomorphism. Alternatively, one can check directly: $(f^{-1})_*$ is a two-sided inverse of f_* by functoriality. \square

Proposition 2.39 ($\mathbb{R}^2 \not\cong \mathbb{R}^n$ for $n \neq 2$). *The Euclidean spaces \mathbb{R}^2 and \mathbb{R}^n are not homeomorphic for $n \neq 2$.*

Proof. Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}^n$ were a homeomorphism. Restricting to complements of a point, f would give a homeomorphism $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}^n \setminus \{f(0)\}$. But $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$, whereas for $n = 1$, $\mathbb{R} \setminus \{p\}$ is disconnected (so not even path-connected), and for $n \geq 3$, $\pi_1(\mathbb{R}^n \setminus \{p\}) \cong \pi_1(S^{n-1}) = \{e\}$ by Proposition 2.32. In each case the fundamental groups are non-isomorphic, so no such homeomorphism can exist. \square

Remark 2.40. The general result that $\mathbb{R}^m \cong \mathbb{R}^n$ implies $m = n$ is called *invariance of domain* (or invariance of dimension). A full proof for all m, n requires homology theory. The argument above settles the case $m = 2$.

Theorem 2.41 (Fundamental theorem of algebra, topological proof sketch). *Every non-constant polynomial $p(z) \in \mathbb{C}[z]$ has a root in \mathbb{C} .*

Proof sketch. Write $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ with $n \geq 1$. Suppose for contradiction that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Define $f_R: S^1 \rightarrow S^1$ by

$$f_R(e^{i\theta}) = \frac{p(Re^{i\theta})}{|p(Re^{i\theta})|}.$$

For R large, $p(Re^{i\theta}) \approx R^n e^{in\theta}$, so f_R is homotopic to the map $e^{i\theta} \mapsto e^{in\theta}$, which has degree n . For $R = 0$, f_0 is the constant map $p(0)/|p(0)|$, which has degree 0. Since the degree is a homotopy invariant and $R \mapsto f_R$ gives a homotopy from f_0 to f_R (using the assumption that p has no zeros), we obtain $n = 0$, a contradiction. \square

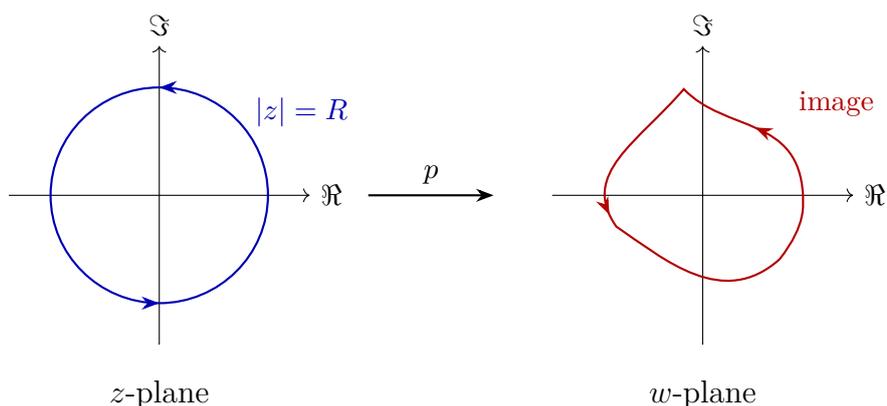


Figure 2.8: The image of a large circle under $p(z)$ winds n times around the origin.

Proposition 2.42 (Fundamental group of a topological group is abelian). *If G is a topological group with identity element e , then $\pi_1(G, e)$ is abelian.*

Proof. Let γ and δ be loops in G based at e . Define

$$H(s, t) = \gamma(s) \cdot \delta(t),$$

where \cdot denotes the group operation in G . This is continuous since G is a topological group. We compute:

$$\begin{aligned} H(s, 0) &= \gamma(s) \cdot e = \gamma(s), & H(s, 1) &= \gamma(s) \cdot e = \gamma(s), \\ H(0, t) &= e \cdot \delta(t) = \delta(t), & H(1, t) &= e \cdot \delta(t) = \delta(t). \end{aligned}$$

Now consider the two paths along the boundary of $I \times I$ from $(0, 0)$ to $(1, 1)$: the path along the bottom then right gives $\gamma \star \delta$, and the path along the left then top gives $\delta \star \gamma$. The map H provides a homotopy between these two paths (relative to endpoints), showing $[\gamma] \cdot [\delta] = [\delta] \cdot [\gamma]$.

More precisely, define $\alpha(t) = H(t, t) = \gamma(t) \cdot \delta(t)$ and note that the paths $s \mapsto H(s, 0)$ then $t \mapsto H(1, t)$ and $t \mapsto H(0, t)$ then $s \mapsto H(s, 1)$ give, respectively, $\gamma \star \delta$ and $\delta \star \gamma$. Since H is a homotopy of the square with the right boundary values, one obtains $[\gamma \star \delta] = [\delta \star \gamma]$. \square

Remark 2.43. The proof above is an instance of the *Eckmann–Hilton argument*: when a set carries two binary operations, both with the same unit, and each distributing over the other, then the two operations coincide and are commutative. Here the two operations are concatenation of loops and pointwise multiplication in G .

2.11 Summary and Outlook

Let us summarize the main results of this chapter.

Space X	$\pi_1(X)$	Justification
\mathbb{R}^n	$\{e\}$	Contractible
D^n	$\{e\}$	Contractible
$S^n, n \geq 2$	$\{e\}$	Proposition 2.32
S^1	\mathbb{Z}	Theorem 2.26
$T^n = (S^1)^n$	\mathbb{Z}^n	Corollary 3.22
$\mathbb{R}^2 \setminus \{0\}$	\mathbb{Z}	Homotopy equivalent to S^1
$S^2 \times S^1$	\mathbb{Z}	Product formula
Möbius band	\mathbb{Z}	Deformation retract onto S^1
Cylinder $S^1 \times I$	\mathbb{Z}	Deformation retract onto S^1

Table 2.1: Fundamental groups computed in Chapter 2.

In the next chapter, we will develop the theory of covering spaces, which provides a powerful geometric tool for computing and understanding the fundamental group. We will then prove the Seifert–van Kampen theorem, which allows us to compute the fundamental group of a space built from simpler pieces whose fundamental groups are known. These tools will enable us to compute π_1 of surfaces, wedge sums, and many other spaces.

2.12 Exercises

Exercise 2.1. Let $\gamma, \delta: I \rightarrow X$ be two paths from x_0 to x_1 . Show that $\gamma \simeq \delta \text{ rel } \partial I$ if and only if $\gamma \star \bar{\delta}$ is null-homotopic (i.e., $[\gamma \star \bar{\delta}] = e$ in $\pi_1(X, x_0)$).

Exercise 2.2. Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a pointed map and let $A \subseteq X$ be a retract of X with $x_0 \in A$. Show that $f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ maps $\iota_*(\pi_1(A, x_0))$ injectively into $\pi_1(Y, y_0)$, where $\iota: A \hookrightarrow X$ is the inclusion. In particular, $\pi_1(A, x_0)$ is (isomorphic to) a subgroup of $\pi_1(X, x_0)$.

Exercise 2.3. Show that S^1 is not simply connected by proving directly (without using $\pi_1(S^1) \cong \mathbb{Z}$) that the loop $\omega_1(t) = (\cos 2\pi t, \sin 2\pi t)$ is not null-homotopic. *Hint:* Use the degree/lifting approach.

Exercise 2.4. Let $f: S^1 \rightarrow S^1$ be a continuous map. The *degree* of f , denoted $\deg(f)$, is the integer n such that $f_*: \pi_1(S^1) \rightarrow \pi_1(S^1)$ is multiplication by n (i.e., $f_*(k) = nk$ under the identification $\pi_1(S^1) \cong \mathbb{Z}$). Show that:

- (a) If f has no fixed point, then $\deg(f) = -1$. *Hint:* show $f \simeq -\text{Id}$ (the antipodal map).
- (b) The antipodal map $a: S^1 \rightarrow S^1$, $a(z) = -z$, has degree -1 .
- (c) If f is not surjective, then $\deg(f) = 0$.

Exercise 2.5. (*Borsuk–Ulam in dimension 1.*) Show that for every continuous map $f: S^1 \rightarrow \mathbb{R}$, there exists a point $x \in S^1$ with $f(x) = f(-x)$. *Hint:* Consider $g(x) = f(x) - f(-x)$ and use the intermediate value theorem.

Exercise 2.6. Let X be a path-connected space. Show that $\pi_1(X, x_0)$ is abelian if and only if for all paths α, β from x_0 to x_1 , the change of basepoint isomorphisms β_α and β_β agree (i.e., $\beta_\alpha = \beta_\beta: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$).

Exercise 2.7. Compute $\pi_1(X)$ for the following spaces (justify your answers using results from this chapter):

- (a) $X = \mathbb{R}^3 \setminus \{0\}$.
- (b) $X = S^1 \times \mathbb{R}^2$.
- (c) $X = S^2 \times S^1$.
- (d) $X = \mathbb{R}^2 \setminus \{p_1, \dots, p_k\}$ for k distinct points (just state the answer; a full computation requires the Seifert–van Kampen theorem).

Exercise 2.8. Use the Brouwer fixed-point theorem (dimension 2) to show that every 2×2 real matrix A with all entries positive has a positive real eigenvalue. *Hint:* Consider the action of A on the simplex $\Delta = \{(x, y) \in \mathbb{R}^2 : x + y = 1, x \geq 0, y \geq 0\}$.

Exercise 2.9. Let $f, g: X \rightarrow S^1$ be continuous maps. Define $f \cdot g: X \rightarrow S^1$ by $(f \cdot g)(x) = f(x) \cdot g(x)$, using the group multiplication in $S^1 \subseteq \mathbb{C}$. Show that $(f \cdot g)_* = f_* + g_*$ as homomorphisms $\pi_1(X, x_0) \rightarrow \pi_1(S^1) \cong \mathbb{Z}$.

Exercise 2.10. Let $p: \mathbb{R} \rightarrow S^1$ be the standard covering map. Show that for any simply connected, locally path-connected space Y and any continuous map $f: Y \rightarrow S^1$, there exists a lift $\tilde{f}: Y \rightarrow \mathbb{R}$ with $p \circ \tilde{f} = f$. *Hint:* Fix a basepoint $y_0 \in Y$, choose $\tilde{f}(y_0) \in p^{-1}(f(y_0))$, and for any $y \in Y$ define $\tilde{f}(y)$ using the lift of $f \circ \gamma$ where γ is a path from y_0 to y . Show this is well-defined using simple connectivity.

Chapter 3

The Seifert–Van Kampen Theorem

The computation of the fundamental group of a circle, while beautiful, relied on the particular geometry of S^1 . For more complicated spaces—wedge sums, surfaces, CW complexes—one needs a systematic tool that expresses $\pi_1(X)$ in terms of simpler pieces. The *Seifert–Van Kampen theorem* is precisely such a tool: it describes the fundamental group of a union $X = U_1 \cup U_2$ in terms of $\pi_1(U_1)$, $\pi_1(U_2)$, and $\pi_1(U_1 \cap U_2)$.

The algebraic prerequisite is the notion of *free product* and, more generally, *amalgamated free product* of groups. We begin with these constructions.

3.1 Free groups and free products

Definition 3.1 (Free group). Let S be a set. The **free group** on S , denoted $F(S)$, is a group $F(S)$ together with a map $\iota: S \rightarrow F(S)$ satisfying the following universal property: for every group G and every map $f: S \rightarrow G$, there exists a unique group homomorphism $\varphi: F(S) \rightarrow G$ such that $\varphi \circ \iota = f$.

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F(S) \\ & \searrow f & \downarrow \exists! \varphi \\ & & G \end{array}$$

Remark 3.2 (Existence and explicit construction). Concretely, $F(S)$ consists of all reduced words in the alphabet $S \cup S^{-1}$: finite sequences $s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n}$ with $s_i \in S$, $\varepsilon_i \in \{+1, -1\}$, and no adjacent pair $s s^{-1}$ or $s^{-1} s$. The group operation is concatenation followed by free reduction. The empty word serves as the identity. The universal property is then easily verified.

Definition 3.3 (Free product of groups). Let $\{G_\alpha\}_{\alpha \in A}$ be a family of groups. The **free product** $\ast_{\alpha \in A} G_\alpha$ is a group G together with homomorphisms $\iota_\alpha: G_\alpha \rightarrow G$ satisfying the following universal property: for every group H and every family of homomorphisms $\varphi_\alpha: G_\alpha \rightarrow H$, there exists a unique homomorphism $\Phi: G \rightarrow H$ such

that $\Phi \circ \iota_\alpha = \varphi_\alpha$ for all α .

$$\begin{array}{ccc}
 G_\alpha & \xrightarrow{\iota_\alpha} & *_{\alpha} G_\alpha \\
 & \searrow \varphi_\alpha & \downarrow \exists! \Phi \\
 & & H
 \end{array}$$

Remark 3.4 (Explicit construction). Elements of $G_1 * G_2$ are represented by reduced words $g_1 g_2 g_3 \cdots g_n$ where each g_i belongs to one of the factors G_1 or G_2 , successive letters come from *different* factors, and no letter is the identity. Multiplication is concatenation followed by amalgamation of adjacent letters from the same factor and deletion of resulting identity elements.

Example 3.5 (Free product $\mathbb{Z} * \mathbb{Z}$). The free product $\mathbb{Z} * \mathbb{Z}$ is the free group on two generators. Writing a for the generator of the first copy and b for the second, a typical reduced word is $a^2 b^{-1} a^3 b^2$. Note that $\mathbb{Z} * \mathbb{Z}$ is *not* abelian: $ab \neq ba$.

Proposition 3.6 (Basic properties of free products). *Let G and H be groups.*

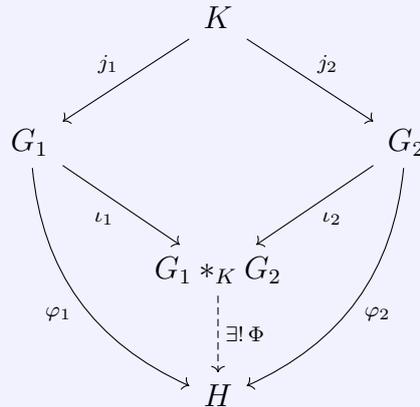
- (i) *The inclusions $\iota_G: G \hookrightarrow G * H$ and $\iota_H: H \hookrightarrow G * H$ are injective.*
- (ii) *$G * H \cong H * G$.*
- (iii) *$G * \{e\} \cong G$.*
- (iv) *The abelianization of $G * H$ is $G^{\text{ab}} \oplus H^{\text{ab}}$.*

Proof. Parts (i)–(iii) are straightforward from the reduced-word description. For (iv), the abelianization is the largest abelian quotient. The universal property of $G * H$, combined with the fact that every pair of homomorphisms into an abelian group A factors uniquely, yields $(G * H)^{\text{ab}} \cong G^{\text{ab}} \oplus H^{\text{ab}}$. \square

3.2 Amalgamated free products

Definition 3.7 (Amalgamated free product). Let G_1, G_2 be groups and let K be a group with homomorphisms $j_1: K \rightarrow G_1$ and $j_2: K \rightarrow G_2$. The **amalgamated free product** $G_1 *_K G_2$ is a group G together with homomorphisms $\iota_1: G_1 \rightarrow G$ and $\iota_2: G_2 \rightarrow G$ satisfying $\iota_1 \circ j_1 = \iota_2 \circ j_2$ and the following universal property: for every group H and homomorphisms $\varphi_1: G_1 \rightarrow H$, $\varphi_2: G_2 \rightarrow H$ with $\varphi_1 \circ j_1 = \varphi_2 \circ j_2$, there

exists a unique homomorphism $\Phi: G \rightarrow H$ making the diagram commute.



Equivalently, $G_1 *_K G_2$ is the **pushout** in the category of groups:

$$\begin{array}{ccc}
 K & \xrightarrow{j_2} & G_2 \\
 j_1 \downarrow & \lrcorner & \downarrow l_2 \\
 G_1 & \xrightarrow{l_1} & G_1 *_K G_2
 \end{array}$$

Remark 3.8 (Explicit construction). The amalgamated free product can be constructed as

$$G_1 *_K G_2 = (G_1 * G_2) / N,$$

where N is the normal closure of $\{j_1(k)j_2(k)^{-1} : k \in K\}$. In other words, we take the free product and impose the relations $j_1(k) = j_2(k)$ for all $k \in K$.

Example 3.9 (Trivial amalgamation). When $K = \{e\}$, the amalgamated free product $G_1 *_{\{e\}} G_2$ is just the ordinary free product $G_1 * G_2$.

Example 3.10 (Amalgamation with $K = \mathbb{Z}$). Let $G_1 = G_2 = \mathbb{Z}$ with $j_1(1) = 2$ and $j_2(1) = 3$. Then $G_1 *_K G_2 = \langle a, b \mid a^2 = b^3 \rangle$, which is the trefoil knot group.

3.3 Statement and proof of the Seifert–Van Kampen theorem

We now state the central theorem of this chapter.

Theorem 3.11 (Seifert–Van Kampen). *Let X be a topological space and let $U_1, U_2 \subseteq X$ be open subsets such that $X = U_1 \cup U_2$ and $U_0 := U_1 \cap U_2$ is path-connected and nonempty. Choose a basepoint $x_0 \in U_0$. Let $j_k: \pi_1(U_0, x_0) \rightarrow \pi_1(U_k, x_0)$ and $\iota_k: \pi_1(U_k, x_0) \rightarrow \pi_1(X, x_0)$ be the homomorphisms induced by the inclusions $U_0 \hookrightarrow U_k$*

and $U_k \hookrightarrow X$. Then the diagram

$$\begin{array}{ccc} \pi_1(U_0, x_0) & \xrightarrow{j_2} & \pi_1(U_2, x_0) \\ j_1 \downarrow & \lrcorner & \downarrow \iota_2 \\ \pi_1(U_1, x_0) & \xrightarrow{\iota_1} & \pi_1(X, x_0) \end{array}$$

is a pushout in the category of groups. Equivalently,

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) *_{\pi_1(U_0, x_0)} \pi_1(U_2, x_0).$$

Remark 3.12 (Hypotheses). The hypothesis that $U_0 = U_1 \cap U_2$ is path-connected is essential. If U_0 has multiple path components, a more general version of the theorem holds involving groupoids (see Exercise 3.10). Some authors also require each of U_1, U_2, U_0 to be path-connected; this is sufficient for most applications.

Proof of Theorem 3.11. We must show that $\pi_1(X, x_0)$ satisfies the universal property of the amalgamated free product $\pi_1(U_1) *_{\pi_1(U_0)} \pi_1(U_2)$.

Step 1: Surjectivity. We show that the natural map $\Phi: \pi_1(U_1) *_{\pi_1(U_0)} \pi_1(U_2) \rightarrow \pi_1(X, x_0)$ is surjective. Let $[\gamma] \in \pi_1(X, x_0)$. By the Lebesgue number lemma, we can subdivide $[0, 1]$ into subintervals $[t_{i-1}, t_i]$, $i = 1, \dots, n$, such that each $\gamma|_{[t_{i-1}, t_i]}$ has image contained in either U_1 or U_2 .

For each division point t_i with $0 < i < n$, if $\gamma(t_i) \notin U_0$ then the preceding and succeeding subpaths must lie in the same U_k , and we can merge them. So we may assume $\gamma(t_i) \in U_1 \cup U_2$ for all i , and whenever the path crosses from U_1 to U_2 (or vice versa), the crossing point lies in U_0 .

Choose a path α_i in U_0 from x_0 to $\gamma(t_i)$ for each i with $\gamma(t_i) \in U_0$ (this is possible since U_0 is path-connected). Then γ is homotopic to a product of loops $\alpha_{i-1} \cdot \gamma_i \cdot \bar{\alpha}_i$, each of which lies entirely in U_1 or U_2 . Hence $[\gamma]$ is in the image of Φ .

Step 2: Injectivity. Suppose $w = g_1 g_2 \cdots g_m$ is a reduced word in $\pi_1(U_1) * \pi_1(U_2)$ that maps to the identity in $\pi_1(X, x_0)$. We must show that w can be reduced to the empty word using only the amalgamation relations $j_1(k) = j_2(k)$, $k \in \pi_1(U_0)$.

Represent each g_i by a loop γ_i in U_1 or U_2 , so that $\gamma_1 \cdot \gamma_2 \cdots \gamma_m$ is null-homotopic in X via some homotopy $H: [0, 1] \times [0, 1] \rightarrow X$.

By the Lebesgue number lemma applied to the open cover $\{H^{-1}(U_1), H^{-1}(U_2)\}$ of $[0, 1]^2$, we can subdivide the square into small rectangles, each mapping into U_1 or U_2 . The restriction of H to each horizontal strip provides a homotopy between consecutive “layers” of loops.

Tracking these subdivisions carefully, one shows that the word w can be transformed to the empty word by a sequence of moves, each of which either:

- inserts or deletes a pair $g g^{-1}$ within the same factor, or
- replaces $j_1(k)$ by $j_2(k)$ (or vice versa) for some $k \in \pi_1(U_0)$.

These are precisely the relations defining the amalgamated free product. Hence w represents the identity in $\pi_1(U_1) *_{\pi_1(U_0)} \pi_1(U_2)$, proving injectivity.

Step 3: Universal property. Steps 1 and 2 together show that Φ is an isomorphism. The universal property of $\pi_1(X, x_0)$ as a pushout now follows from the universal property of the amalgamated free product. \square

3.4 Applications

Remark 3.13 (Presentations from the Seifert–Van Kampen theorem). In practice, the Seifert–Van Kampen theorem is often used in the following form. Suppose $\pi_1(U_1) = \langle S_1 \mid R_1 \rangle$, $\pi_1(U_2) = \langle S_2 \mid R_2 \rangle$, and $\pi_1(U_0) = \langle S_0 \mid R_0 \rangle$ with j_1 and j_2 expressed on generators. Then

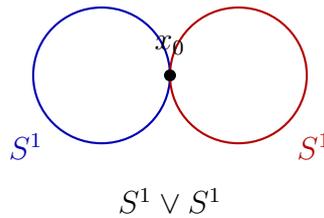
$$\pi_1(X) \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup \{j_1(s) = j_2(s) : s \in S_0\} \rangle.$$

That is, we combine all generators and all relations, and add new relations equating the images of the generators of $\pi_1(U_0)$ in the two factors.

Corollary 3.14 (Collapsing a simply connected subspace). *Let $A \subseteq X$ be a simply connected subspace such that A is closed, X is a nice space, and A has a path-connected open neighborhood that deformation-retracts onto A . Then the quotient map $q: X \rightarrow X/A$ induces an isomorphism $\pi_1(X) \cong \pi_1(X/A)$.*

3.4.1 Fundamental group of the wedge sum

Definition 3.15 (Wedge sum). Let (X, x_0) and (Y, y_0) be pointed spaces. The **wedge sum** (or one-point union) is $X \vee Y := (X \sqcup Y)/(x_0 \sim y_0)$.



Theorem 3.16 (Fundamental group of a wedge sum). *Let X and Y be path-connected spaces with non-degenerate basepoints (meaning each basepoint has a contractible open neighborhood). Then*

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

Proof. Let p denote the wedge point. By the non-degeneracy hypothesis, there exist contractible open neighborhoods $V_X \ni x_0$ in X and $V_Y \ni y_0$ in Y . Set

$$U_1 = X \cup V_Y, \quad U_2 = V_X \cup Y, \quad U_0 = U_1 \cap U_2 = V_X \cup V_Y,$$

where the unions are taken inside $X \vee Y$. Then U_0 is contractible (it is the wedge of two contractible sets along a point), so $\pi_1(U_0) = \{e\}$. Moreover, U_1 deformation-retracts onto X and U_2 deformation-retracts onto Y . By the Seifert–Van Kampen theorem,

$$\pi_1(X \vee Y) \cong \pi_1(U_1) *_{\{e\}} \pi_1(U_2) \cong \pi_1(X) * \pi_1(Y). \quad \square$$

Corollary 3.17 ($\pi_1(S^1 \vee S^1)$). $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$, the free group on two generators. More generally, $\pi_1(\bigvee_n S^1) \cong F_n$, the free group of rank n .

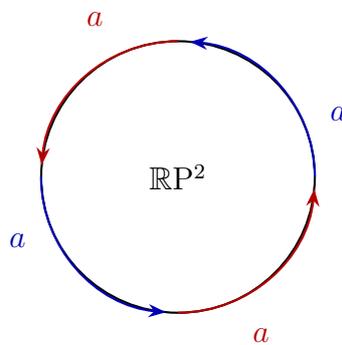
3.4.2 The real projective plane

Theorem 3.18 ($\pi_1(\mathbb{RP}^2)$). $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Recall that \mathbb{RP}^2 is obtained from D^2 by identifying antipodal points on the boundary S^1 . Equivalently, \mathbb{RP}^2 has a CW structure with one 0-cell, one 1-cell, and one 2-cell attached by a map of degree 2. Concretely, let a denote the loop generating π_1 of the 1-skeleton (which is S^1). The 2-cell is attached along a^2 .

By the Seifert–Van Kampen theorem (cell attachment version, Proposition 3.25),

$$\pi_1(\mathbb{RP}^2) \cong \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}. \quad \square$$

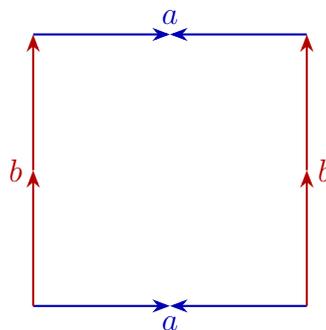


3.4.3 The Klein bottle

Theorem 3.19 (π_1 of the Klein bottle). The Klein bottle K satisfies

$$\pi_1(K) \cong \langle a, b \mid abab^{-1} = 1 \rangle.$$

Proof. The Klein bottle is obtained from a square by the identifications shown below:



The Klein bottle has a CW structure with one 0-cell, two 1-cells a and b , and one 2-cell attached along the word $abab^{-1}$. The 1-skeleton is $S^1 \vee S^1$, so π_1 of the 1-skeleton is $F(a, b)$. Attaching the 2-cell kills the element $abab^{-1}$, giving

$$\pi_1(K) \cong \langle a, b \mid abab^{-1} \rangle. \quad \square$$

Remark 3.20 (Klein bottle group). The group $\pi_1(K) = \langle a, b \mid aba = b \rangle$ is a non-abelian infinite group. Its abelianization is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In particular, K is not orientable (the torsion detects non-orientability).

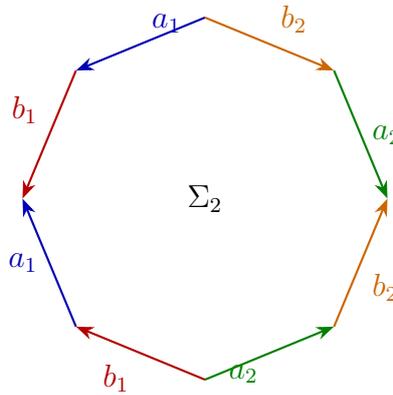
3.4.4 Orientable surfaces

Theorem 3.21 (Fundamental group of orientable surfaces). *Let Σ_g denote the closed orientable surface of genus $g \geq 1$. Then*

$$\pi_1(\Sigma_g) \cong \left\langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \right\rangle,$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

Proof. The surface Σ_g is obtained from a $4g$ -gon by identifying edges in pairs according to the word $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. All vertices are identified to a single point.



The CW structure has one 0-cell, $2g$ oriented 1-cells $a_1, b_1, \dots, a_g, b_g$, and one 2-cell. The 1-skeleton is a wedge of $2g$ circles, so π_1 of the 1-skeleton is $F(a_1, b_1, \dots, a_g, b_g)$. Attaching the 2-cell along the boundary word $\prod_{i=1}^g [a_i, b_i]$ gives the result by the cell attachment proposition. \square

Corollary 3.22 (π_1 of the torus). $\pi_1(T^2) = \pi_1(\Sigma_1) \cong \langle a, b \mid [a, b] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Remark 3.23 (Euler characteristic check). The surface Σ_g has a CW structure with 1 vertex, $2g$ edges, and 1 face, so $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$. The abelianization of $\pi_1(\Sigma_g)$ is \mathbb{Z}^{2g} (see Exercise 3.7), which is consistent with $H_1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ and the relation $\chi = \sum (-1)^k \text{rank}(H_k) = 1 - 2g + 1 = 2 - 2g$.

Proposition 3.24 (Fundamental group of connected sums). *Let M and N be closed connected n -manifolds with $n \geq 3$. Then*

$$\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N).$$

For surfaces ($n = 2$), the same formula holds at the level of the 1-skeleton, but the

extra 2-cell relation must be accounted for.

Proof. The connected sum $M\#N$ is obtained by removing open balls $B_M \subset M$ and $B_N \subset N$ and gluing along the resulting boundary spheres S^{n-1} . Take $U_1 = M \setminus \{p\}$ and $U_2 = N \setminus \{q\}$ (where p, q are interior points of the removed balls), thickened slightly so that $U_0 = U_1 \cap U_2$ is homotopy equivalent to S^{n-1} . For $n \geq 3$, $\pi_1(S^{n-1}) = \{e\}$, so the Seifert–Van Kampen theorem gives a free product. \square

3.4.5 Cell attachments

Proposition 3.25 (Effect of attaching a 2-cell). *Let X be a path-connected space and let $Y = X \cup_{\varphi} e^2$ be obtained by attaching a 2-cell along a map $\varphi: S^1 \rightarrow X$. Let $[\varphi] \in \pi_1(X, x_0)$ be the class of the attaching map (after choosing a path from $\varphi(1)$ to x_0 if needed). Then*

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0) / \langle\langle [\varphi] \rangle\rangle,$$

where $\langle\langle [\varphi] \rangle\rangle$ denotes the normal closure.

Proof. We apply the Seifert–Van Kampen theorem. Let $U_1 = X \cup$ (open collar of e^2) and let U_2 be the open disk e^2 . Then U_1 deformation-retracts onto X , U_2 is contractible, and $U_0 = U_1 \cap U_2$ deformation-retracts onto S^1 . The map $\pi_1(U_0) \rightarrow \pi_1(U_1)$ sends the generator of $\pi_1(S^1) \cong \mathbb{Z}$ to $[\varphi]$, while $\pi_1(U_0) \rightarrow \pi_1(U_2) = \{e\}$ is trivial. Therefore,

$$\pi_1(Y) \cong \pi_1(X) *_Z \{e\} \cong \pi_1(X) / \langle\langle [\varphi] \rangle\rangle. \quad \square$$

Corollary 3.26 (Fundamental group of CW complexes). *Let X be a CW complex. Then $\pi_1(X)$ depends only on the 2-skeleton $X^{(2)}$ and is given by*

$$\pi_1(X) \cong \langle 1\text{-cells} \mid \text{attaching words of 2-cells} \rangle.$$

Moreover, attaching cells of dimension ≥ 3 does not change π_1 .

Proof. The 1-skeleton $X^{(1)}$ is a graph, so $\pi_1(X^{(1)})$ is a free group (on generators corresponding to 1-cells not in a maximal tree, by Proposition 3.6). Attaching 2-cells imposes the stated relations by iterated application of Proposition 3.25.

For cells of dimension $n \geq 3$: the attaching map $S^{n-1} \rightarrow X^{(n-1)}$ satisfies $\pi_1(S^{n-1}) = 0$ for $n - 1 \geq 2$, so the amalgamation is trivial and π_1 does not change. \square

Proposition 3.27 (Every finitely presented group is realizable). *For every finitely presented group $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$, there exists a compact CW complex X with $\pi_1(X) \cong G$.*

Proof. Take $X^{(1)} = \bigvee_{i=1}^n S^1$ and attach m copies of 2-cells along the relators r_1, \dots, r_m . By Corollary 3.26, $\pi_1(X) \cong G$. \square

3.5 Exercises

Exercise 3.1. Use the Seifert–Van Kampen theorem to show that $\pi_1(S^n) = \{e\}$ for $n \geq 2$.

Hint: Write S^n as the union of two open sets, each homeomorphic to \mathbb{R}^n , whose intersection is homotopy equivalent to S^{n-1} (which is connected for $n \geq 2$).

Exercise 3.2. Give a direct proof using the Seifert–Van Kampen theorem that $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$, by decomposing the torus into two suitable open sets. Compare with the CW complex approach.

Exercise 3.3. Compute $\pi_1(\mathbb{R}P^2 \vee S^1)$.

Exercise 3.4. Let N_k denote the connected sum of k copies of $\mathbb{R}P^2$ (the non-orientable surface of genus k). Show that

$$\pi_1(N_k) \cong \langle c_1, \dots, c_k \mid c_1^2 c_2^2 \cdots c_k^2 = 1 \rangle.$$

Verify that $N_1 = \mathbb{R}P^2$ and N_2 is the Klein bottle.

Exercise 3.5. Let Θ be the *theta curve*: the space formed by two vertices p, q joined by three distinct arcs. Compute $\pi_1(\Theta)$.

Exercise 3.6. The **dunce hat** is obtained from a triangle by identifying all three edges via $a \rightarrow a \rightarrow a$ (all edges are identified with the same orientation). Show that the dunce hat is simply connected.

Exercise 3.7. Let $G = \pi_1(\Sigma_g)$. Show that $G^{\text{ab}} \cong \mathbb{Z}^{2g}$. Deduce that surfaces of different genus are not homotopy equivalent.

Exercise 3.8. Let Σ_2 be the closed orientable surface of genus 2. Show that Σ_2 can be decomposed as the union of two tori-with-a-hole. Use the Seifert–Van Kampen theorem directly (not via the polygon model) to recover the presentation $\langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle$.

Exercise 3.9. Let $f: S^1 \rightarrow S^1$ be the map $z \mapsto z^n$. The **mapping torus** M_f is obtained from $S^1 \times [0, 1]$ by identifying $(z, 0) \sim (f(z), 1)$. Show that $\pi_1(M_f) \cong \langle a, t \mid tat^{-1} = a^n \rangle$.

Exercise 3.10. (Groupoid version.) Formulate and prove a version of the Seifert–Van Kampen theorem when $U_1 \cap U_2$ has two path components. Apply it to compute $\pi_1(S^1)$ by decomposing S^1 into two open arcs.

Chapter 4

Covering Spaces and the Fundamental Group

Covering space theory provides one of the most elegant interactions between algebra and topology. The key result—the *Galois correspondence*—establishes a bijection between connected coverings of a “nice” space X and subgroups of $\pi_1(X)$, paralleling the correspondence between field extensions and subgroups of a Galois group. Along the way, we develop the crucial lifting properties that make the theory work.

4.1 Definitions and first examples

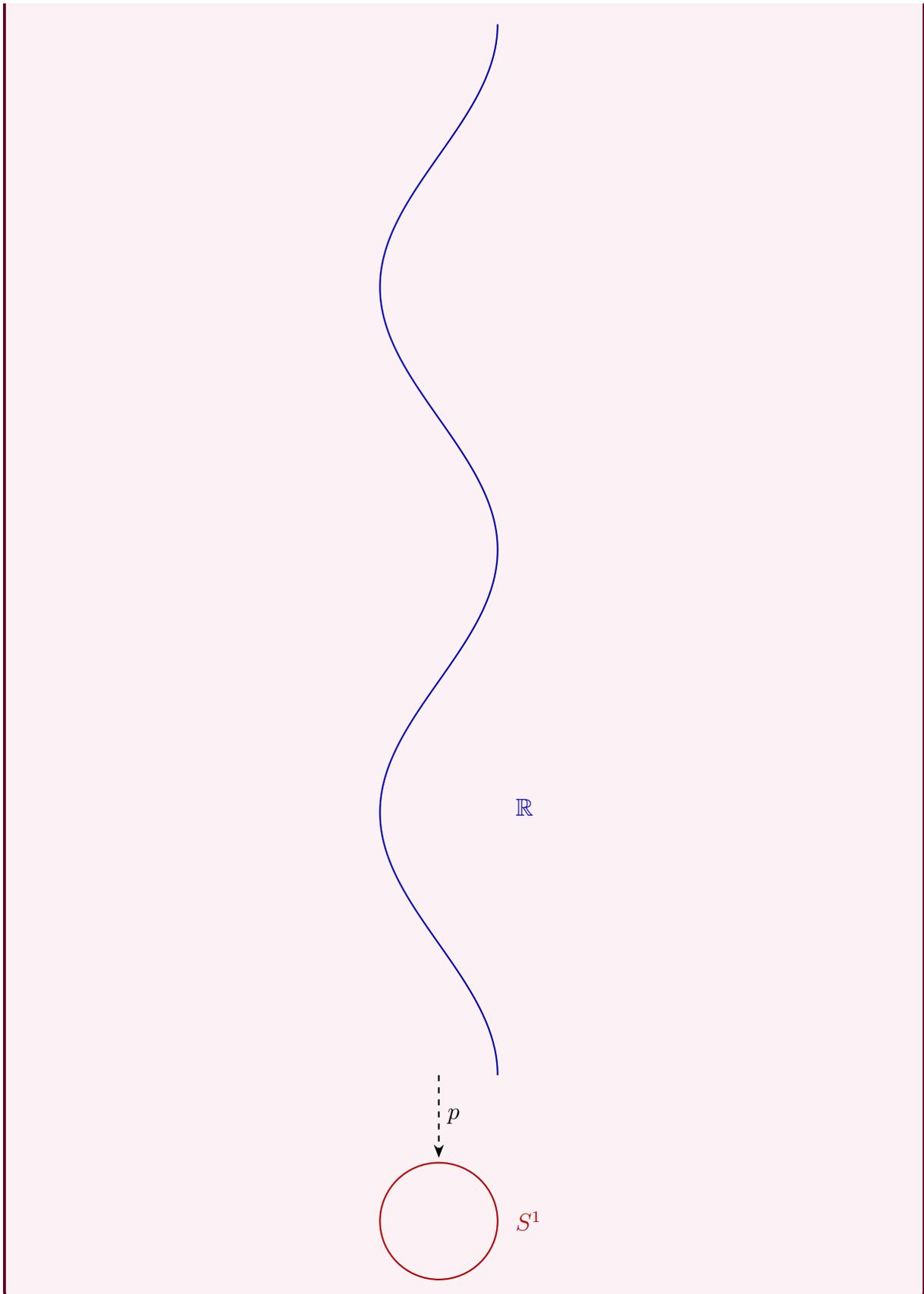
Definition 4.1 (Covering space). A **covering space** (or **cover**) of a topological space X is a topological space \widetilde{X} together with a surjective continuous map $p: \widetilde{X} \rightarrow X$ such that for every $x \in X$, there exists an open neighborhood U of x (called an **evenly covered neighborhood**) for which

$$p^{-1}(U) = \bigsqcup_{\alpha \in A} V_\alpha,$$

where each V_α is open in \widetilde{X} and $p|_{V_\alpha}: V_\alpha \xrightarrow{\sim} U$ is a homeomorphism. The sets V_α are called **sheets** over U .

Definition 4.2 (Degree of a covering). If X is connected, the cardinality $|p^{-1}(x)|$ is the same for all $x \in X$. This common value is called the **degree** (or **number of sheets**) of the covering.

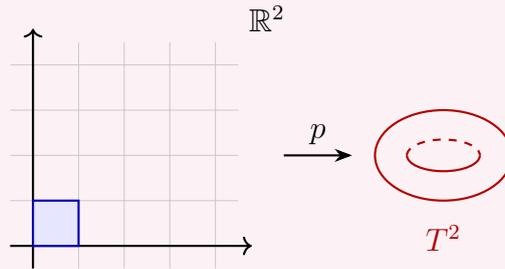
Example 4.3 (The exponential map). The map $p: \mathbb{R} \rightarrow S^1$ defined by $p(t) = e^{2\pi it}$ is a covering of infinite degree. For any open arc $U \subset S^1$ of length less than 2π , the preimage $p^{-1}(U)$ is a disjoint union of open intervals in \mathbb{R} , each mapped homeomorphically onto U .



Example 4.4 (Finite cyclic coverings of S^1). The map $p_n: S^1 \rightarrow S^1$ given by $z \mapsto z^n$ is an n -sheeted covering.

Example 4.5 (Covering of \mathbb{RP}^2). The quotient map $p: S^2 \rightarrow \mathbb{RP}^2$ sending each point to its equivalence class under the antipodal identification is a 2-sheeted covering. Here S^2 is simply connected, and the two sheets over any evenly covered set correspond to a point and its antipode.

Example 4.6 (Covering of the torus). The map $p: \mathbb{R}^2 \rightarrow T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the universal covering. For each $n \geq 1$, the torus T^2 also has an n^2 -sheeted covering $T^2 \rightarrow T^2$ induced by the linear map $(x, y) \mapsto (nx, ny)$.



Definition 4.7 (Morphism of coverings). Let $p_1: \tilde{X}_1 \rightarrow X$ and $p_2: \tilde{X}_2 \rightarrow X$ be coverings of X . A **morphism of coverings** is a continuous map $f: \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $p_2 \circ f = p_1$:

$$\begin{array}{ccc}
 \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\
 & \searrow p_1 & \swarrow p_2 \\
 & X &
 \end{array}$$

An isomorphism of coverings is a morphism that is also a homeomorphism.

4.2 Lifting properties

The fundamental property of covering spaces is the ability to lift paths and homotopies.

Lemma 4.8 (Path lifting). Let $p: \tilde{X} \rightarrow X$ be a covering, let $\gamma: [0, 1] \rightarrow X$ be a path, and let $\tilde{x}_0 \in p^{-1}(\gamma(0))$. Then there exists a unique path $\tilde{\gamma}: [0, 1] \rightarrow \tilde{X}$ such that $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{x}_0$.

$$\begin{array}{ccc}
 & & \tilde{X} \\
 & \nearrow \exists! \tilde{\gamma} & \downarrow p \\
 [0, 1] & \xrightarrow{\gamma} & X
 \end{array}$$

Proof. Existence. Cover $\gamma([0, 1])$ by evenly covered neighborhoods. By compactness of $[0, 1]$, find a partition $0 = t_0 < t_1 < \dots < t_n = 1$ such that each $\gamma([t_{i-1}, t_i])$ is contained in an evenly covered set U_i .

We construct $\tilde{\gamma}$ inductively. On $[0, t_1]$: the preimage $p^{-1}(U_1)$ decomposes into sheets; exactly one sheet V_1 contains \tilde{x}_0 . Define $\tilde{\gamma} = (p|_{V_1})^{-1} \circ \gamma$ on $[0, t_1]$. Proceeding step by step, each lift is determined by the requirement that $\tilde{\gamma}$ be continuous.

Uniqueness. If $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two lifts with the same initial point, the set $\{t : \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)\}$ is both open and closed in $[0, 1]$ (by the homeomorphism property of sheets), nonempty (it contains 0), hence equal to $[0, 1]$. \square

Lemma 4.9 (Homotopy lifting). *Let $p: \tilde{X} \rightarrow X$ be a covering, let $H: [0, 1] \times [0, 1] \rightarrow X$ be a homotopy with $H(0, s) = x_0$ for all s , and let $\tilde{x}_0 \in p^{-1}(x_0)$. Then there exists a unique lift $\tilde{H}: [0, 1] \times [0, 1] \rightarrow \tilde{X}$ with $p \circ \tilde{H} = H$ and $\tilde{H}(0, s) = \tilde{x}_0$ for all s .*

$$\begin{array}{ccc} [0, 1] \times \{0\} & \xrightarrow{\tilde{\gamma}_0} & \tilde{X} \\ \downarrow & \nearrow \tilde{H} & \downarrow p \\ [0, 1] \times [0, 1] & \xrightarrow{H} & X \end{array}$$

Proof. The argument is analogous to path lifting. Subdivide $[0, 1]^2$ into small squares, each mapping into an evenly covered set, and lift inductively, one square at a time, working row by row or column by column. Uniqueness follows from the same open-and-closed argument. \square

Corollary 4.10 (Lifting loops and p_*). *Let $p: \tilde{X} \rightarrow X$ be a covering and $\tilde{x}_0 \in p^{-1}(x_0)$. A loop γ at x_0 lifts to a loop at \tilde{x}_0 if and only if $[\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Proof. The lift $\tilde{\gamma}$ starts at \tilde{x}_0 and ends at some point $\tilde{x}_1 \in p^{-1}(x_0)$. By definition, $\tilde{\gamma}$ is a loop if and only if $\tilde{x}_1 = \tilde{x}_0$, which happens if and only if $[\gamma]$ is in the image of p_* . \square

Example 4.11 (Lifting loops to \mathbb{R}). Consider the covering $p: \mathbb{R} \rightarrow S^1$, $t \mapsto e^{2\pi it}$, with basepoint lift $\tilde{x}_0 = 0$. The loop $\gamma_n(t) = e^{2\pi int}$ (winding n times) lifts to the path $\tilde{\gamma}_n(t) = nt$ in \mathbb{R} . This lift is a loop (i.e., $\tilde{\gamma}_n(1) = \tilde{\gamma}_n(0) = 0$) only when $n = 0$, consistent with $p_*(\pi_1(\mathbb{R})) = \{0\} \leq \mathbb{Z}$.

For the k -sheeted covering $p_k: S^1 \rightarrow S^1$, $z \mapsto z^k$, the loop γ_n lifts to a loop if and only if $k \mid n$, consistent with $p_{k*}(\pi_1(S^1)) = k\mathbb{Z} \leq \mathbb{Z}$.

Theorem 4.12 (Monodromy theorem). *Let $p: \tilde{X} \rightarrow X$ be a covering and let γ_0, γ_1 be paths in X from x_0 to x_1 that are homotopic relative to endpoints. Let $\tilde{\gamma}_0, \tilde{\gamma}_1$ be their lifts starting at the same point $\tilde{x}_0 \in p^{-1}(x_0)$. Then $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$, and the lifts are homotopic relative to endpoints in \tilde{X} .*

Proof. Let H be a homotopy from γ_0 to γ_1 relative to endpoints. By the homotopy lifting lemma (Lemma 4.9), H lifts to \tilde{H} with $\tilde{H}(\cdot, 0) = \tilde{\gamma}_0$. Since $H(1, s) = x_1$ for all s , the path $s \mapsto \tilde{H}(1, s)$ lies in the discrete fiber $p^{-1}(x_1)$; by continuity, it is constant. Thus $\tilde{\gamma}_0(1) = \tilde{H}(1, 0) = \tilde{H}(1, 1) = \tilde{\gamma}_1(1)$. Similarly, $\tilde{H}(\cdot, 1)$ is the unique lift of γ_1 starting at \tilde{x}_0 , so $\tilde{H}(\cdot, 1) = \tilde{\gamma}_1$. Hence \tilde{H} is a homotopy from $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$ relative to endpoints. \square

4.3 The monodromy action

Definition 4.13 (Monodromy action). Let $p: \widetilde{X} \rightarrow X$ be a covering and let $x_0 \in X$. The **monodromy action** of $\pi_1(X, x_0)$ on the fiber $F = p^{-1}(x_0)$ is defined by

$$[\gamma] \cdot \tilde{x}_0 := \tilde{\gamma}(1),$$

where $\tilde{\gamma}$ is the unique lift of γ starting at \tilde{x}_0 . This is well-defined by the monodromy theorem (Theorem 4.12).

Proposition 4.14 (Properties of the monodromy action). *The monodromy action is a right action of $\pi_1(X, x_0)$ on F . Moreover:*

- (i) *If \widetilde{X} is path-connected, the action is **transitive**.*
- (ii) *The stabilizer of $\tilde{x}_0 \in F$ is the subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$.*

Proof. The constant loop lifts to the constant path, so the identity acts trivially. If γ and δ are loops at x_0 , then the lift of $\gamma \cdot \delta$ starting at \tilde{x}_0 is the concatenation of the lift of γ starting at \tilde{x}_0 with the lift of δ starting at $\tilde{\gamma}(1)$. This verifies the action axiom.

(i) If \widetilde{X} is path-connected and $\tilde{x}_0, \tilde{x}_1 \in F$, choose a path $\tilde{\alpha}$ in \widetilde{X} from \tilde{x}_0 to \tilde{x}_1 . Then $\gamma := p \circ \tilde{\alpha}$ is a loop at x_0 and $[\gamma] \cdot \tilde{x}_0 = \tilde{x}_1$.

(ii) $[\gamma] \cdot \tilde{x}_0 = \tilde{x}_0$ if and only if the lift $\tilde{\gamma}$ is a loop at \tilde{x}_0 , i.e., $[\gamma] = p_*([\tilde{\gamma}]) \in p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. \square

Corollary 4.15 (Index formula). *If $p: \widetilde{X} \rightarrow X$ is a connected covering, then*

$$|p^{-1}(x_0)| = [\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \tilde{x}_0))].$$

In particular, the number of sheets equals the index of the image of $\pi_1(\widetilde{X})$ in $\pi_1(X)$.

Proof. By the orbit-stabilizer theorem applied to the transitive monodromy action, $|F| = [\pi_1(X, x_0) : \text{Stab}(\tilde{x}_0)] = [\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \tilde{x}_0))]$. \square

4.4 The induced homomorphism and the lifting criterion

Proposition 4.16 (Injectivity of p_*). *Let $p: \widetilde{X} \rightarrow X$ be a covering and $\tilde{x}_0 \in p^{-1}(x_0)$. The induced homomorphism $p_*: \pi_1(\widetilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.*

Proof. Suppose $p_*([\tilde{\gamma}]) = [p \circ \tilde{\gamma}] = [c_{x_0}]$ in $\pi_1(X, x_0)$. Then $p \circ \tilde{\gamma}$ is null-homotopic in X via some homotopy H . Lift H to \tilde{H} in \widetilde{X} . Since $H(1, s) = x_0$ for all s , the path $s \mapsto \tilde{H}(1, s)$ lies in the discrete fiber $p^{-1}(x_0)$ and is therefore constant. Hence \tilde{H} is a null-homotopy of $\tilde{\gamma}$ in \widetilde{X} . \square

Theorem 4.17 (General lifting criterion). *Let $p: \widetilde{X} \rightarrow X$ be a covering with \widetilde{X} path-connected. Let Y be a connected and locally path-connected space, $y_0 \in Y$, and $f: (Y, y_0) \rightarrow (X, x_0)$ a continuous map. Choose $\tilde{x}_0 \in p^{-1}(x_0)$. Then there exists a lift $\tilde{f}: (Y, y_0) \rightarrow (\widetilde{X}, \tilde{x}_0)$ with $p \circ \tilde{f} = f$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \tilde{x}_0)).$$

Moreover, when it exists, such a lift is unique.

$$\begin{array}{ccc} & & (\widetilde{X}, \tilde{x}_0) \\ & \nearrow \tilde{f} & \downarrow p \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}$$

Proof. Necessity. If \tilde{f} exists, then $f_* = p_* \circ \tilde{f}_*$, so $f_*(\pi_1(Y)) \subseteq p_*(\pi_1(\widetilde{X}))$.

Sufficiency. For $y \in Y$, choose a path α in Y from y_0 to y . Then $f \circ \alpha$ is a path in X starting at x_0 . Lift it to $\tilde{f} \circ \alpha$ starting at \tilde{x}_0 , and define $\tilde{f}(y) := \tilde{f} \circ \alpha(1)$.

We must show this is well-defined. If β is another path from y_0 to y , then $\alpha \cdot \bar{\beta}$ is a loop at y_0 . Its image $f_*([\alpha \cdot \bar{\beta}])$ lies in $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ by hypothesis. By the monodromy theorem, the lifts of $f \circ \alpha$ and $f \circ \beta$ have the same endpoint.

Continuity. Local path-connectedness of Y ensures that for any $y \in Y$ and any evenly covered neighborhood U of $f(y)$, we can find a path-connected neighborhood W of y mapping into U , on which \tilde{f} is continuous (it equals $(p|_V)^{-1} \circ f$ on W for an appropriate sheet V).

Uniqueness. Two lifts \tilde{f}_1, \tilde{f}_2 agreeing at y_0 agree everywhere by the same open-and-closed argument as in Lemma 4.8. \square

4.5 The universal covering space

Definition 4.18 (Simply connected covering). A covering $p: \widetilde{X} \rightarrow X$ is called a **universal covering** (or **universal cover**) if \widetilde{X} is simply connected.

Theorem 4.19 (Existence of the universal cover). *Let X be a connected, locally path-connected, and semi-locally simply connected space. Then X admits a universal covering $p: \widetilde{X} \rightarrow X$, unique up to isomorphism of coverings.*

Proof sketch. Fix $x_0 \in X$. Define \widetilde{X} as the set of homotopy classes (relative to endpoints) of paths in X starting at x_0 :

$$\widetilde{X} = \{[\gamma] : \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0\},$$

with $p([\gamma]) = \gamma(1)$. The topology on \widetilde{X} is generated by sets of the form

$$U_{[\gamma], V} = \{[\gamma \cdot \eta] : \eta \text{ is a path in } V \text{ with } \eta(0) = \gamma(1)\},$$

where V is a path-connected open subset of X such that $\pi_1(V) \rightarrow \pi_1(X)$ is trivial (guaranteed by the semi-local simple connectivity hypothesis).

One checks that p is a covering map, \widetilde{X} is simply connected, and any two universal covers are isomorphic via the lifting criterion. \square

Remark 4.20 (Semi-local simple connectivity). A space X is **semi-locally simply connected** if every point has a neighborhood U such that the inclusion-induced map $\pi_1(U) \rightarrow \pi_1(X)$ is trivial. This does *not* require U to be simply connected. The Hawaiian earring is the standard example of a space that fails this condition and has no universal cover.

Example 4.21 (Universal covers). (i) $\mathbb{R} \rightarrow S^1, t \mapsto e^{2\pi it}$, is the universal cover of S^1 .

(ii) $S^2 \rightarrow \mathbb{R}P^2$ is the universal cover of $\mathbb{R}P^2$.

(iii) $S^n \rightarrow \mathbb{R}P^n$ for all $n \geq 1$.

(iv) $\mathbb{R}^2 \rightarrow T^2 = S^1 \times S^1$ is the universal cover of the torus.

(v) The universal cover of $S^1 \vee S^1$ is an infinite 4-regular tree (the Cayley graph of F_2).

Proposition 4.22 (Universal property of the universal cover). *Let $p: \widetilde{X} \rightarrow X$ be a universal covering and let $q: Y \rightarrow X$ be any connected covering. Then there exists a covering map $r: \widetilde{X} \rightarrow Y$ with $q \circ r = p$:*

$$\begin{array}{ccc} \widetilde{X} & \overset{r}{\dashrightarrow} & Y \\ & \searrow p & \swarrow q \\ & X & \end{array}$$

Proof. Since \widetilde{X} is simply connected, $p_*(\pi_1(\widetilde{X})) = \{e\} \subseteq q_*(\pi_1(Y))$. By the lifting criterion (Theorem 4.17), p lifts to $r: \widetilde{X} \rightarrow Y$. One checks that r is itself a covering map. \square

4.6 The Galois correspondence

We now state the main classification theorem.

Definition 4.23 (Locally path-connected and semi-locally simply connected). We say that a space X is **nice** if it is connected, locally path-connected, and semi-locally simply connected. All CW complexes and manifolds are nice.

Theorem 4.24 (Classification of coverings — Galois correspondence). *Let X be a*

nice space with basepoint x_0 . There is a bijection

$$\left\{ \begin{array}{l} \text{Connected coverings} \\ p: (\widetilde{X}, \tilde{x}_0) \rightarrow (X, x_0) \\ \text{up to isomorphism} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{Subgroups of } \pi_1(X, x_0) \\ \text{up to conjugacy} \end{array} \right\}$$

given by $p \mapsto p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. If we fix a basepoint lift \tilde{x}_0 , then the correspondence is with actual subgroups (not conjugacy classes).

Under this correspondence:

- (i) The universal cover corresponds to the trivial subgroup $\{e\}$.
- (ii) The trivial covering $\text{Id}: X \rightarrow X$ corresponds to the whole group $\pi_1(X, x_0)$.
- (iii) The degree of the covering equals the index of the subgroup: $|p^{-1}(x_0)| = [\pi_1(X) : H]$ where $H = p_*(\pi_1(\widetilde{X}))$.
- (iv) Larger subgroups correspond to “smaller” (fewer sheets) coverings.

Proof. Injectivity. If p_1 and p_2 are connected coverings with $p_{1*}(\pi_1(\widetilde{X}_1, \tilde{x}_{0,1})) = p_{2*}(\pi_1(\widetilde{X}_2, \tilde{x}_{0,2}))$, then by the lifting criterion applied in both directions, there exist lifts $f: \widetilde{X}_1 \rightarrow \widetilde{X}_2$ and $g: \widetilde{X}_2 \rightarrow \widetilde{X}_1$ with $g \circ f = \text{Id}$ and $f \circ g = \text{Id}$ (by uniqueness of lifts). Hence $p_1 \cong p_2$.

Surjectivity. Given a subgroup $H \leq \pi_1(X, x_0)$, we construct a covering with $p_*(\pi_1(\widetilde{X})) = H$. Start with the universal cover $\tilde{p}: \widetilde{X} \rightarrow X$ and define $\widetilde{X}_H = \widetilde{X}/\sim_H$, where $[\gamma_1] \sim_H [\gamma_2]$ if and only if $\gamma_1(1) = \gamma_2(1)$ and $[\gamma_1 \cdot \bar{\gamma}_2] \in H$. The projection $p_H: \widetilde{X}_H \rightarrow X$ is a covering with $p_{H*}(\pi_1(\widetilde{X}_H)) = H$. \square

Example 4.25 (Coverings of S^1). Since $\pi_1(S^1) \cong \mathbb{Z}$, the subgroups are $n\mathbb{Z}$ for $n \geq 0$.

- $n = 0$: subgroup $\{0\}$, giving the universal cover $\mathbb{R} \rightarrow S^1$.
- $n \geq 1$: subgroup $n\mathbb{Z}$, giving the n -sheeted cover $S^1 \xrightarrow{z \mapsto z^n} S^1$.

Since \mathbb{Z} is abelian, every subgroup is normal, and there are no conjugacy ambiguities.

Example 4.26 (Coverings of $S^1 \vee S^1$). Since $\pi_1(S^1 \vee S^1) \cong F_2$, which has a vast collection of subgroups (including free groups of every countable rank, by the Nielsen–Schreier theorem), the figure eight has a very rich collection of coverings. For instance:

- The commutator subgroup $[F_2, F_2]$ gives the covering corresponding to the maximal abelian quotient \mathbb{Z}^2 ; this is an infinite-sheeted cover whose group of deck transformations is \mathbb{Z}^2 .
- Each normal subgroup $N \trianglelefteq F_2$ with F_2/N finite gives a finite regular covering.

Example 4.27 (Lattice of coverings of $\mathbb{R}P^2 \vee S^1$). Since $\pi_1(\mathbb{R}P^2 \vee S^1) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$, which is an infinite non-abelian group, the space $\mathbb{R}P^2 \vee S^1$ has a rich family of coverings.

The subgroup lattice of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$ includes:

- The trivial subgroup $\{e\}$: universal cover (infinite tree-like structure with S^2 -shaped “bubbles”).
- The commutator subgroup $[\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}, \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}]$: an infinite-index normal subgroup.
- Finite-index subgroups giving finite-sheeted coverings.

Remark 4.28 (Functoriality of the correspondence). The Galois correspondence is order-reversing with respect to inclusion: if $H_1 \leq H_2 \leq \pi_1(X)$ with corresponding coverings p_1 and p_2 , then there exists a covering map $\tilde{X}_1 \rightarrow \tilde{X}_2$ making the diagram commute:

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\quad} & \tilde{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

The number of sheets of $\tilde{X}_1 \rightarrow \tilde{X}_2$ is $[H_2 : H_1]$.

4.7 Deck transformations

Definition 4.29 (Deck transformation). Let $p: \tilde{X} \rightarrow X$ be a covering. A **deck transformation** (or **covering transformation**) is a homeomorphism $\varphi: \tilde{X} \rightarrow \tilde{X}$ such that $p \circ \varphi = p$. The set of all deck transformations forms a group under composition, denoted $\text{Aut}(p)$ or $\text{Deck}(\tilde{X}/X)$.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\quad \varphi \quad} & \tilde{X} \\ & \searrow p & \swarrow p \\ & X & \end{array}$$

Proposition 4.30 (Fixed-point-free action). *If \tilde{X} is connected, then every non-identity deck transformation acts without fixed points: if $\varphi(\tilde{x}) = \tilde{x}$ for some $\tilde{x} \in \tilde{X}$, then $\varphi = \text{Id}$.*

Proof. The set $\{\tilde{x} \in \tilde{X} : \varphi(\tilde{x}) = \tilde{x}\}$ is closed (as \tilde{X} is Hausdorff) and open (locally, φ permutes the sheets, so if it fixes a point it fixes the entire sheet). By connectedness, this set is either empty or all of \tilde{X} . \square

Proposition 4.31 (Structure of the deck group). *Let $p: \tilde{X} \rightarrow X$ be a connected covering with $\tilde{x}_0 \in p^{-1}(x_0)$, and let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Then:*

- (i) $\text{Aut}(p)$ is isomorphic to $N(H)/H$, where $N(H) = \{g \in \pi_1(X, x_0) : gHg^{-1} = H\}$

is the normalizer of H .

(ii) $|\text{Aut}(p)| \leq |p^{-1}(x_0)|$, with equality if and only if H is normal in $\pi_1(X, x_0)$.

Proof. (i) A deck transformation φ is determined by $\varphi(\tilde{x}_0)$, which must lie in $p^{-1}(x_0)$. Given $[\gamma] \in \pi_1(X, x_0)$, define $\varphi_{[\gamma]}(\tilde{x}_0) = \tilde{\gamma}(1)$ (the endpoint of the lift of γ). For $\varphi_{[\gamma]}$ to extend to a deck transformation, one needs $\varphi_{[\gamma]}$ to be well-defined on all of \tilde{X} , which happens precisely when $[\gamma]H[\gamma]^{-1} = H$, i.e., $[\gamma] \in N(H)$. The map $[\gamma] \mapsto \varphi_{[\gamma]}$ has kernel H , giving $\text{Aut}(p) \cong N(H)/H$.

(ii) By the orbit-stabilizer theorem, $|\text{Aut}(p)| = |N(H)/H| \leq [\pi_1(X) : H] = |p^{-1}(x_0)|$. Equality holds if and only if $N(H) = \pi_1(X, x_0)$, i.e., $H \trianglelefteq \pi_1(X, x_0)$. \square

4.8 Normal (regular) coverings

Definition 4.32 (Normal covering). A connected covering $p: \tilde{X} \rightarrow X$ is called **normal** (or **regular** or **Galois**) if $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$. Equivalently, $\text{Aut}(p)$ acts transitively on each fiber.

Theorem 4.33 (Characterization of normal coverings). For a connected covering $p: \tilde{X} \rightarrow X$ with X nice, the following are equivalent:

- (i) p is normal.
- (ii) $H = p_*(\pi_1(\tilde{X}))$ is normal in $\pi_1(X)$.
- (iii) $\text{Aut}(p)$ acts transitively on every fiber $p^{-1}(x)$.
- (iv) $\tilde{X}/\text{Aut}(p) \cong X$.
- (v) For every loop γ in X , either every lift of γ is a loop, or no lift is a loop.

When these hold, $\text{Aut}(p) \cong \pi_1(X)/H$.

Proof. The equivalence (i) \Leftrightarrow (ii) is by definition.

(ii) \Leftrightarrow (iii): By Proposition 4.31, $|\text{Aut}(p)| = [N(H) : H]$, and $\text{Aut}(p)$ acts transitively on fibers if and only if $|\text{Aut}(p)| = [G : H]$ (where $G = \pi_1(X)$), i.e., $N(H) = G$, i.e., $H \trianglelefteq G$.

(iii) \Rightarrow (iv): If $\text{Aut}(p)$ acts transitively on fibers, the natural map $\tilde{X}/\text{Aut}(p) \rightarrow X$ is a homeomorphism.

(iv) \Rightarrow (iii) is clear.

(ii) \Leftrightarrow (v): A lift of γ starting at \tilde{x} is a loop if and only if $[\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}))$. As \tilde{x} varies over the fiber, the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}))$ runs over the conjugates of H . The condition that all lifts are simultaneously loops (or non-loops) is equivalent to H being invariant under conjugation, i.e., H is normal. \square

Example 4.34 (Normal coverings). (i) Every universal covering is normal (the trivial subgroup is always normal). For the universal cover, $\text{Aut}(p) \cong \pi_1(X)$.

- (ii) $S^1 \xrightarrow{z^n} S^1$ is normal with $\text{Aut}(p) \cong \mathbb{Z}/n\mathbb{Z}$ (rotation by $2\pi/n$).
- (iii) $S^2 \rightarrow \mathbb{R}P^2$ is normal with $\text{Aut}(p) \cong \mathbb{Z}/2\mathbb{Z}$ (the antipodal map).
- (iv) If $\pi_1(X)$ is abelian, every connected covering of X is normal.

Proposition 4.35 (Recovering π_1 from the universal cover). *If $p: \widetilde{X} \rightarrow X$ is the universal cover of a nice space X , then*

$$\pi_1(X, x_0) \cong \text{Aut}(p).$$

Proof. Since \widetilde{X} is simply connected, $H = p_*(\pi_1(\widetilde{X})) = \{e\}$, which is normal. By Theorem 4.33, $\text{Aut}(p) \cong \pi_1(X)/\{e\} = \pi_1(X)$. \square

Example 4.36 (Deck transformations of $\mathbb{R} \rightarrow S^1$). The deck transformations of $p: \mathbb{R} \rightarrow S^1$, $t \mapsto e^{2\pi it}$, are the translations $t \mapsto t + n$ for $n \in \mathbb{Z}$. Hence $\text{Aut}(p) \cong \mathbb{Z} \cong \pi_1(S^1)$.

4.9 Applications and constructions

Proposition 4.37 (Coverings of graphs). *Every covering space of a graph is a graph. Every subgroup of a free group is free.*

Proof. A graph is a 1-dimensional CW complex. The preimage of a CW structure under a covering map inherits a CW structure (sheets of cells are cells). Hence coverings of graphs are graphs.

Now let $F = \pi_1(\Gamma)$ for a graph Γ , and let $H \leq F$. By the Galois correspondence, $H = p_*(\pi_1(\widetilde{\Gamma}))$ for some covering $\widetilde{\Gamma}$ of Γ . Since $\widetilde{\Gamma}$ is also a graph, $\pi_1(\widetilde{\Gamma})$ is free. Since p_* is injective, $H \cong \pi_1(\widetilde{\Gamma})$ is free. \square

Corollary 4.38 (Nielsen–Schreier theorem). *Every subgroup of a free group is free. Moreover, if F has finite rank n and H has finite index d , then H has rank $d(n-1) + 1$.*

Proof. The first statement was proved above. For the rank formula: if $F = \pi_1(\Gamma)$ where Γ has one vertex and n edges (so $\chi(\Gamma) = 1 - n$), and H corresponds to a d -sheeted covering $\widetilde{\Gamma}$, then $\chi(\widetilde{\Gamma}) = d \cdot \chi(\Gamma) = d(1 - n)$. Since $\widetilde{\Gamma}$ is a graph with Euler characteristic $1 - \text{rank}(H)$, we get $\text{rank}(H) = d(n - 1) + 1$. \square

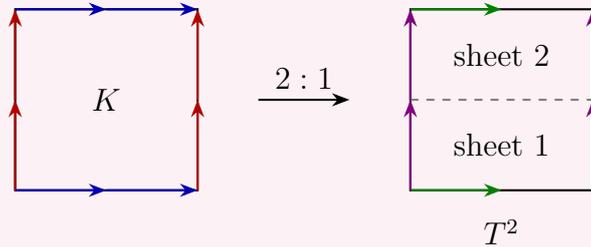
Proposition 4.39 (Coverings and group actions). *Let G be a group acting on a topological space \widetilde{X} by homeomorphisms. Suppose the action is **properly discontinuous**: every $\tilde{x} \in \widetilde{X}$ has a neighborhood U such that $g \cdot U \cap U = \emptyset$ for all $g \neq e$. Then the quotient map $p: \widetilde{X} \rightarrow \widetilde{X}/G$ is a normal covering with $\text{Aut}(p) \cong G$.*

Proof. The properly discontinuous condition ensures that p is a covering: each U as

above is evenly covered, with sheets $\{g \cdot U\}_{g \in G}$. Clearly each $g \in G$ defines a deck transformation, and the action is transitive on fibers by construction. Hence p is normal with $\text{Aut}(p) = G$. \square

Example 4.40 (Lens spaces). Let p, q be coprime positive integers with $1 \leq q < p$. Consider $S^3 \subset \mathbb{C}^2$ and the $\mathbb{Z}/p\mathbb{Z}$ -action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$. This action is properly discontinuous and free, so $S^3 \rightarrow L(p, q) := S^3/(\mathbb{Z}/p\mathbb{Z})$ is a covering. Since S^3 is simply connected, $\pi_1(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$.

Example 4.41 (Coverings of the Klein bottle). Recall $\pi_1(K) = \langle a, b \mid abab^{-1} \rangle$. The abelianization is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The subgroup $\langle a, b^2, bab^{-1} \rangle$ has index 2 and corresponds to the torus T^2 as a 2-sheeted covering of K .



Proposition 4.42 (Euler characteristic and coverings). If $p: \widetilde{X} \rightarrow X$ is an n -sheeted covering of a finite CW complex X , then $\chi(\widetilde{X}) = n \cdot \chi(X)$.

Proof. Each k -cell of X lifts to exactly n copies of k -cells in \widetilde{X} . If c_k denotes the number of k -cells of X , then \widetilde{X} has $n \cdot c_k$ cells of dimension k . Hence

$$\chi(\widetilde{X}) = \sum_k (-1)^k (n \cdot c_k) = n \sum_k (-1)^k c_k = n \cdot \chi(X). \quad \square$$

4.10 Summary: the covering space dictionary

The following table summarizes the Galois correspondence.

Topology	Algebra
Connected covering $p: \widetilde{X} \rightarrow X$	Subgroup $H \leq \pi_1(X)$
Universal cover	Trivial subgroup $\{e\}$
Identity covering Id_X	Whole group $\pi_1(X)$
Number of sheets $ p^{-1}(x_0) $	Index $[\pi_1(X) : H]$
Normal (regular) covering	Normal subgroup $H \trianglelefteq \pi_1(X)$
Deck transformations $\text{Aut}(p)$	$N_{\pi_1(X)}(H)/H$; $\pi_1(X)/H$ if normal
Intermediate covering $\widetilde{X}_1 \rightarrow \widetilde{X}_2 \rightarrow X$	$H_1 \leq H_2 \leq \pi_1(X)$

4.11 Exercises

Exercise 4.1. Let $p: \mathbb{R}^2 \rightarrow T^2$ be the universal covering of the torus.

- Describe explicitly the deck transformations of p .
- For each subgroup $H \leq \mathbb{Z}^2 = \pi_1(T^2)$, describe the corresponding covering space.
- Which coverings of T^2 are connected? What are their fundamental groups?

Exercise 4.2. Let $p: S^2 \rightarrow \mathbb{R}P^2$ be the standard 2-sheeted covering.

- Describe all coverings of $\mathbb{R}P^2$.
- Let $f: T^2 \rightarrow \mathbb{R}P^2$ be a continuous map. Show that f lifts to a map $\tilde{f}: T^2 \rightarrow S^2$.

Exercise 4.3. Classify all connected 3-sheeted coverings of $S^1 \vee S^1$ (up to isomorphism). Draw the covering graphs and determine which are normal.

Hint: A 3-sheeted covering corresponds to a transitive action of F_2 on a set of 3 elements, i.e., a transitive homomorphism $F_2 \rightarrow S_3$.

Exercise 4.4. Let G be a finite group acting freely on a Hausdorff space X . Show that the action is properly discontinuous. Conclude that $X \rightarrow X/G$ is a covering.

Exercise 4.5. Show that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by constructing the universal cover of $X \times Y$ from the universal covers of X and Y .

Exercise 4.6. Let K be the Klein bottle.

- Show that T^2 is a 2-sheeted covering of K .
- Find a normal covering of K with deck group \mathbb{Z} .
- Classify all connected coverings of K with at most 4 sheets.

Exercise 4.7. Let F_n be the free group of rank n and $H \leq F_n$ a subgroup of index $d < \infty$. Use the covering space proof of the Nielsen–Schreier theorem to show that $\text{rank}(H) = d(n-1) + 1$. Apply this to find the rank of the commutator subgroup $[F_2, F_2]$ (is it finitely generated?).

Exercise 4.8. Let $G = \langle S \mid R \rangle$ be a finitely presented group. The **Cayley complex** \tilde{X}_G is the universal cover of the standard 2-complex X_G (with one vertex, one edge per generator, one 2-cell per relator). Show that \tilde{X}_G is simply connected and $\text{Aut}(\tilde{X}_G/X_G) \cong G$. Describe \tilde{X}_G explicitly for $G = \mathbb{Z}/n\mathbb{Z}$ and $G = \mathbb{Z} \times \mathbb{Z}$.

Exercise 4.9. Use covering space theory to prove the following: there is no continuous map $f: S^2 \rightarrow S^1$ satisfying $f(-x) = -f(x)$ for all $x \in S^2$.

Hint: Such an f would induce a map $\tilde{f}: \mathbb{R}P^2 \rightarrow \mathbb{R}P^1 = S^1$. Analyze \tilde{f}_* on fundamental groups.

Exercise 4.10. Let Σ_g be the closed orientable surface of genus $g \geq 2$.

- Show that the universal cover of Σ_g is contractible. (In fact, it is homeomorphic to \mathbb{R}^2 —this is much harder to prove.)

- (b) Construct a covering $\Sigma_h \rightarrow \Sigma_g$ for suitable $h > g$. What is the relationship between h , g , and the number of sheets?

Hint: Use Proposition 4.42: if the covering has d sheets, then $2 - 2h = d(2 - 2g)$, giving $h = d(g - 1) + 1$.

Exercise 4.11. Let $p: \widetilde{X} \rightarrow X$ be a connected covering with $\widetilde{X} \neq X$ (i.e., more than one sheet). Show that there is no continuous map $s: X \rightarrow \widetilde{X}$ with $p \circ s = \text{Id}_X$ (i.e., p has no section) unless X is simply connected.

Hint: If s exists, what does $s_* \circ p_*$ tell you about $\pi_1(\widetilde{X})$ and $\pi_1(X)$?

Exercise 4.12. Let $X = S^1 \vee S^1$ and let $p: \widetilde{X} \rightarrow X$ be the covering corresponding to the commutator subgroup $[F_2, F_2]$ of $\pi_1(X) \cong F_2 = \langle a, b \rangle$.

- (a) What is $\text{Aut}(p)$?
- (b) Show that \widetilde{X} is an infinite graph. Describe it as a subset of \mathbb{R}^2 : it is the 1-skeleton of the standard tiling of \mathbb{R}^2 by unit squares, with the \mathbb{Z}^2 -action by translation.
- (c) What is $\pi_1(\widetilde{X})$? Is it finitely generated?

Hint: The commutator subgroup $[F_2, F_2]$ has infinite index in F_2 (since $F_2/[F_2, F_2] \cong \mathbb{Z}^2$), so the covering is infinite-sheeted and $\pi_1(\widetilde{X}) \cong [F_2, F_2]$ is free of infinite rank.

Exercise 4.13. Let $p: Y \rightarrow X$ and $q: Z \rightarrow Y$ be coverings.

- (a) Show that $p \circ q: Z \rightarrow X$ is not necessarily a covering in general. Give a counterexample.

Hint: Consider $\mathbb{R} \xrightarrow{p_1} S^1 \xrightarrow{p_2} S^1$ where $p_2(z) = z^2$. Is $p_2 \circ p_1$ a covering? What goes wrong?

- (b) Show that if p has finitely many sheets, then $p \circ q$ is indeed a covering.

Exercise 4.14. Use covering space theory to give another proof that every continuous map $f: D^2 \rightarrow D^2$ has a fixed point (Brouwer's theorem for $n = 2$).

Hint: If f has no fixed point, construct a retraction $D^2 \rightarrow S^1$ and derive a contradiction using π_1 .

Exercise 4.15. Let $X = \mathbb{R}^2 \setminus \{0\}$ and let \mathbb{Z} act on X by $n \cdot (x, y) = (2^n x, 2^n y)$.

- (a) Verify that this action is properly discontinuous.
- (b) Identify the quotient space X/\mathbb{Z} .

Hint: It is a familiar compact surface.

- (c) Compute $\pi_1(X/\mathbb{Z})$ using covering space theory.

Chapter 5

Singular Homology — Definition and Properties

The passage from topology to algebra is achieved by associating to each topological space X a sequence of abelian groups $H_n(X)$, called its *homology groups*. The construction is entirely functorial and, remarkably, depends only on the homotopy type of X . In this chapter we develop the foundations of singular homology theory.

5.1 The standard simplex and singular simplices

Definition 5.1 (Standard n -simplex). The **standard n -simplex** is the subspace of \mathbb{R}^{n+1} defined by

$$\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0 \text{ for all } i \right\}.$$

We write e_0, e_1, \dots, e_n for the vertices of Δ^n , where e_i is the point whose i -th coordinate is 1 and all other coordinates are 0.

Definition 5.2 (Face maps). For $0 \leq i \leq n$, the **i -th face map** is the affine embedding

$$\delta_i^n: \Delta^{n-1} \longrightarrow \Delta^n, \quad (t_0, \dots, t_{n-1}) \longmapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$

Thus δ_i^n maps Δ^{n-1} onto the face of Δ^n opposite the vertex e_i .

Lemma 5.3 (Cosimplicial identity). For $i < j$ we have

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n.$$

Proof. Both sides send (t_0, \dots, t_{n-1}) to the point $(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-2}, 0, t_{j-1}, \dots, t_{n-1})$. A direct coordinate check confirms the identity. \square

Definition 5.4 (Singular n -simplex). Let X be a topological space. A **singular n -simplex** in X is a continuous map

$$\sigma: \Delta^n \longrightarrow X.$$

We denote the set of all singular n -simplices in X by $\text{Sing}_n(X)$.

Example 5.5 (Singular simplices in low dimensions). • A singular 0-simplex is simply a point of X .

- A singular 1-simplex is a continuous path $\sigma: [0, 1] \rightarrow X$.
- A singular 2-simplex is a continuous map from a triangle into X .

5.2 Chain groups and the boundary operator

Definition 5.6 (Singular chain group). The **singular chain group** $C_n(X)$ is the free abelian group generated by $\text{Sing}_n(X)$:

$$C_n(X) = \bigoplus_{\sigma \in \text{Sing}_n(X)} \mathbb{Z} \cdot \sigma.$$

An element of $C_n(X)$ is called a **singular n -chain** and is a finite formal sum $\sum_i n_i \sigma_i$ with $n_i \in \mathbb{Z}$. We set $C_n(X) = 0$ for $n < 0$.

Definition 5.7 (Boundary operator). The **boundary operator** $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$ is the homomorphism defined on generators by

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i \sigma \circ \delta_i^n,$$

and extended linearly to all of $C_n(X)$.

Example 5.8 (Boundary of a 1-simplex). If $\sigma: \Delta^1 \rightarrow X$ is a singular 1-simplex (a path from $\sigma(e_0)$ to $\sigma(e_1)$), then

$$\partial_1(\sigma) = \sigma \circ \delta_0^1 - \sigma \circ \delta_1^1 = \sigma(e_1) - \sigma(e_0).$$

Example 5.9 (Boundary of a 2-simplex). For a singular 2-simplex $\sigma: \Delta^2 \rightarrow X$,

$$\partial_2(\sigma) = \sigma \circ \delta_0^2 - \sigma \circ \delta_1^2 + \sigma \circ \delta_2^2,$$

which is the alternating sum of the three edges of the triangle.

The following theorem is the cornerstone of homological algebra.

Theorem 5.10 ($\partial^2 = 0$). For every n , the composition $\partial_{n-1} \circ \partial_n = 0$. Equivalently, $\text{im } \partial_n \subseteq \ker \partial_{n-1}$.

Proof. It suffices to check on a generator $\sigma \in \text{Sing}_n(X)$. We compute

$$\begin{aligned} \partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1}\left(\sum_{i=0}^n (-1)^i \sigma \circ \delta_i^n\right) = \sum_{i=0}^n (-1)^i \sum_{j=0}^{n-1} (-1)^j \sigma \circ \delta_i^n \circ \delta_j^{n-1} \\ &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} \sigma \circ \delta_i^n \circ \delta_j^{n-1} + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} \sigma \circ \delta_i^n \circ \delta_j^{n-1}. \end{aligned}$$

In the first sum, the cosimplicial identity (Lemma 5.3) gives $\delta_i^n \circ \delta_j^{n-1} = \delta_j^n \circ \delta_{i-1}^{n-1}$ for $j < i$. Substituting $i' = j$, $j' = i - 1$ (so $i' + j' = i + j - 1$ and $j' \geq i'$), this first sum becomes

$$\sum_{0 \leq i' \leq j' \leq n-1} (-1)^{i'+j'+1} \sigma \circ \delta_{i'}^n \circ \delta_{j'}^{n-1},$$

which is exactly the negative of the second sum. Hence the total is 0. \square

5.3 The chain complex and homology groups

Definition 5.11 (Singular chain complex). The **singular chain complex** of X is the sequence

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0.$$

We write $C_\bullet(X) = (C_n(X), \partial_n)_{n \geq 0}$ for this chain complex.

Definition 5.12 (Cycles, boundaries, homology). Let $n \geq 0$. Define:

- The group of **n -cycles**: $Z_n(X) = \ker \partial_n \subseteq C_n(X)$.
- The group of **n -boundaries**: $B_n(X) = \text{im } \partial_{n+1} \subseteq C_n(X)$.

Since $\partial^2 = 0$ we have $B_n(X) \subseteq Z_n(X)$, and the **n -th singular homology group** of X is the quotient

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\ker \partial_n}{\text{im } \partial_{n+1}}.$$

Remark 5.13 (Intuitive meaning). An n -cycle is a chain “without boundary,” while an n -boundary is a chain that bounds an $(n+1)$ -chain. Two cycles that differ by a boundary represent the same homology class. Thus $H_n(X)$ detects n -dimensional “holes” in X that are not filled in.

5.4 Functoriality

Definition 5.14 (Induced chain map). Let $f: X \rightarrow Y$ be a continuous map. For each n , define $f_{\#}: C_n(X) \rightarrow C_n(Y)$ on generators by $f_{\#}(\sigma) = f \circ \sigma$, extended linearly.

Proposition 5.15 (Chain map property). *The map $f_{\#}$ is a chain map: $\partial_n \circ f_{\#} = f_{\#} \circ \partial_n$ for every n . Consequently, $f_{\#}$ descends to a well-defined homomorphism*

$$f_*: H_n(X) \longrightarrow H_n(Y).$$

Proof. On a generator $\sigma \in \text{Sing}_n(X)$:

$$\partial_n(f_{\#}(\sigma)) = \partial_n(f \circ \sigma) = \sum_{i=0}^n (-1)^i (f \circ \sigma) \circ \delta_i^n = \sum_{i=0}^n (-1)^i f \circ (\sigma \circ \delta_i^n) = f_{\#}(\partial_n(\sigma)).$$

Since $f_{\#}$ maps cycles to cycles and boundaries to boundaries, it induces f_* on quotients. \square

Proposition 5.16 (Functoriality of homology). *Singular homology defines a functor from the category **Top** of topological spaces to the category **Ab** of abelian groups:*

- (i) $(\text{Id}_X)_* = \text{Id}_{H_n(X)}$.
- (ii) $(g \circ f)_* = g_* \circ f_*$.

Proof. Both identities follow immediately from the corresponding identities at the chain level: $(\text{Id}_X)_{\#} = \text{Id}_{C_n(X)}$ and $(g \circ f)_{\#} = g_{\#} \circ f_{\#}$. \square

Corollary 5.17 (Homeomorphism invariance). *If $f: X \rightarrow Y$ is a homeomorphism, then $f_*: H_n(X) \xrightarrow{\sim} H_n(Y)$ is an isomorphism for every n .*

5.5 Homotopy invariance

The deepest property of singular homology is that it depends only on the homotopy type of the space.

Definition 5.18 (Chain homotopy). Let $f_{\#}, g_{\#}: C_{\bullet}(X) \rightarrow C_{\bullet}(Y)$ be chain maps. A **chain homotopy** from $f_{\#}$ to $g_{\#}$ is a sequence of homomorphisms $P_n: C_n(X) \rightarrow C_{n+1}(Y)$ satisfying

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = g_{\#} - f_{\#}: C_n(X) \rightarrow C_n(Y).$$

The key construction is the *prism operator*.

Definition 5.19 (Prism operator). Let $F: X \times [0, 1] \rightarrow Y$ be a homotopy from f to g . For a singular n -simplex $\sigma: \Delta^n \rightarrow X$, consider the prism $\Delta^n \times [0, 1]$. We triangulate

this prism into $(n+1)$ -simplices as follows. For $0 \leq i \leq n$, define

$$\lambda_i: \Delta^{n+1} \rightarrow \Delta^n \times [0, 1]$$

by

$$\lambda_i(e_j) = \begin{cases} (e_j, 0) & \text{if } j \leq i, \\ (e_{j-1}, 1) & \text{if } j > i. \end{cases}$$

The **prism operator** $P_n: C_n(X) \rightarrow C_{n+1}(Y)$ is defined on generators by

$$P_n(\sigma) = \sum_{i=0}^n (-1)^i F \circ (\sigma \times \text{Id}) \circ \lambda_i.$$

Theorem 5.20 (Homotopy invariance). *If $f, g: X \rightarrow Y$ are homotopic, then $f_* = g_*: H_n(X) \rightarrow H_n(Y)$ for all n .*

Proof. Let $F: X \times [0, 1] \rightarrow Y$ be a homotopy with $F(-, 0) = f$ and $F(-, 1) = g$. We claim the prism operator satisfies the chain homotopy identity

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = g_{\#} - f_{\#}.$$

By linearity it suffices to verify on a generator $\sigma \in \text{Sing}_n(X)$. For each prism piece $F \circ (\sigma \times \text{Id}) \circ \lambda_i$, one computes ∂_{n+1} of this $(n+1)$ -simplex. The faces split into three types:

- (a) The *top* face (coming from $t = 1$), which yields a term in $g_{\#}(\sigma)$.
- (b) The *bottom* face (coming from $t = 0$), which yields a term in $f_{\#}(\sigma)$.
- (c) *Lateral* faces, which cancel in pairs between consecutive prism pieces λ_i and λ_{i+1} , thanks to the alternating signs.

More precisely, one verifies the telescoping identity:

$$\partial_{n+1} \left(\sum_{i=0}^n (-1)^i F \circ (\sigma \times \text{Id}) \circ \lambda_i \right) = g_{\#}(\sigma) - f_{\#}(\sigma) - \sum_{i=0}^n (-1)^i P_{n-1}(\sigma \circ \delta_i^n).$$

Since $P_{n-1}(\partial_n(\sigma)) = \sum_{i=0}^n (-1)^i P_{n-1}(\sigma \circ \delta_i^n)$, we obtain $\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = g_{\#} - f_{\#}$.

Therefore, for any cycle $\alpha \in Z_n(X)$,

$$g_{\#}(\alpha) - f_{\#}(\alpha) = \partial_{n+1}(P_n(\alpha)) + P_{n-1}(\underbrace{\partial_n(\alpha)}_{=0}) = \partial_{n+1}(P_n(\alpha)) \in B_n(Y),$$

so $[g_{\#}(\alpha)] = [f_{\#}(\alpha)]$ in $H_n(Y)$, i.e., $f_* = g_*$. □

Corollary 5.21 (Homotopy equivalence invariance). *If $f: X \rightarrow Y$ is a homotopy equivalence, then $f_*: H_n(X) \xrightarrow{\sim} H_n(Y)$ is an isomorphism for all n . In particular, contractible spaces have the homology of a point.*

Proof. Let $g: Y \rightarrow X$ be a homotopy inverse. Then $g \circ f \simeq \text{Id}_X$ implies $(g \circ f)_* = g_* \circ f_* = \text{Id}$, and similarly $f_* \circ g_* = \text{Id}$. So f_* is an isomorphism with inverse g_* . □

5.6 Reduced homology

Definition 5.22 (Augmented chain complex). The **augmentation map** $\varepsilon: C_0(X) \rightarrow \mathbb{Z}$ is defined by $\varepsilon(\sigma) = 1$ for every singular 0-simplex σ , extended linearly. One checks $\varepsilon \circ \partial_1 = 0$, so we obtain the **augmented chain complex**

$$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Definition 5.23 (Reduced homology). The **reduced homology groups** of X are

$$\widetilde{H}_n(X) = \begin{cases} H_n(X) & \text{if } n \geq 1, \\ \ker \varepsilon / \operatorname{im} \partial_1 & \text{if } n = 0. \end{cases}$$

Equivalently, $\widetilde{H}_n(X)$ is the homology of the augmented complex. For non-empty X , we have $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

Remark 5.24 (When to use reduced homology). Reduced homology eliminates the “trivial” copy of \mathbb{Z} in degree 0 that every non-empty path-connected space carries. This simplifies many formulas, especially for suspensions and exact sequences.

5.7 First computations

Proposition 5.25 (Homology of a point). *Let $\text{pt} = \{*\}$ be a one-point space. Then*

$$H_n(\text{pt}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n \geq 1. \end{cases}$$

Equivalently, $\widetilde{H}_n(\text{pt}) = 0$ for all n .

Proof. There is exactly one singular n -simplex $\sigma_n: \Delta^n \rightarrow \{*\}$ for every n . Hence $C_n(\text{pt}) \cong \mathbb{Z}$ for all $n \geq 0$, generated by σ_n . The boundary is $\partial_n(\sigma_n) = \sum_{i=0}^n (-1)^i \sigma_{n-1}$. The sum $\sum_{i=0}^n (-1)^i$ equals 1 when n is even and 0 when n is odd. Therefore the chain complex takes the form

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

where the maps alternate between Id and 0. Computing:

- $H_0(\text{pt}) = \ker(\partial_0) / \operatorname{im}(\partial_1) = \mathbb{Z}/0 = \mathbb{Z}$.
- For n odd ($n \geq 1$): $\partial_n = 0$ so $\ker \partial_n = \mathbb{Z}$, and $\partial_{n+1} = \text{Id}$ so $\operatorname{im} \partial_{n+1} = \mathbb{Z}$, giving $H_n = 0$.
- For n even ($n \geq 2$): $\partial_n = \text{Id}$ so $\ker \partial_n = 0$, giving $H_n = 0$.

Hence $H_n(\text{pt}) = 0$ for all $n \geq 1$. □

Theorem 5.26 (Homology of spheres). For $n \geq 0$,

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise,} \end{cases} \quad \widetilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The case $n = 0$: $S^0 = \{-1, +1\}$ consists of two points, so $H_0(S^0) \cong \mathbb{Z}^2$ and $\widetilde{H}_0(S^0) \cong \mathbb{Z}$.

For $n \geq 1$, we use the Mayer–Vietoris sequence or the long exact sequence of the pair (to be established in Chapter 6). We defer the complete inductive proof to Corollary 6.12. \square

Proposition 5.27 (Homology and path components). For any space X , $H_0(X)$ is the free abelian group on the set $\pi_0(X)$ of path components of X . In particular, $H_0(X) \cong \mathbb{Z}^{|\pi_0(X)|}$.

Proof. Every 0-chain is a finite formal sum $\sum n_i x_i$ with $x_i \in X$. The augmentation $\varepsilon: C_0(X) \rightarrow \mathbb{Z}$ sends this to $\sum n_i$. A 1-chain σ has $\partial_1 \sigma = \sigma(e_1) - \sigma(e_0)$, so the boundary of any 1-chain is a sum of differences of points connected by paths.

Two points $x, y \in X$ define 0-chains. They represent the same class in H_0 if and only if their difference $y - x$ lies in $B_0(X) = \text{im } \partial_1$, i.e., there is a finite sequence of paths connecting x to y . This means x and y lie in the same path component. Hence $H_0(X) \cong \bigoplus_{\alpha \in \pi_0(X)} \mathbb{Z}$. \square

Example 5.28 (Homology of $\mathbb{R}^n \setminus \{0\}$). Since the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is a homotopy equivalence (with homotopy inverse $x \mapsto x/|x|$), we have

$$H_k(\mathbb{R}^n \setminus \{0\}) \cong H_k(S^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

This will be crucial for proving invariance of dimension.

Example 5.29 (Homology of disjoint unions). For a disjoint union $X = \bigsqcup_{\alpha} X_{\alpha}$, every singular simplex $\sigma: \Delta^n \rightarrow X$ has its image contained in a single component X_{α} (since Δ^n is connected). Therefore

$$C_n(X) = \bigoplus_{\alpha} C_n(X_{\alpha}), \quad H_n(X) \cong \bigoplus_{\alpha} H_n(X_{\alpha}).$$

5.8 Relative homology

Definition 5.30 (Relative chain groups). Let $A \subseteq X$ be a subspace. The inclusion $A \hookrightarrow X$ induces an injection $C_n(A) \hookrightarrow C_n(X)$. The **relative chain group** is

$$C_n(X, A) = C_n(X) / C_n(A).$$

Since ∂_n maps $C_n(A)$ to $C_{n-1}(A)$, it descends to a boundary operator $\bar{\partial}_n: C_n(X, A) \rightarrow$

$C_{n-1}(X, A)$ with $\bar{\partial}_{n-1} \circ \bar{\partial}_n = 0$.

Definition 5.31 (Relative homology groups). The **relative homology groups** are

$$H_n(X, A) = H_n(C_\bullet(X, A)) = \frac{\ker \bar{\partial}_n}{\text{im } \bar{\partial}_{n+1}}.$$

Remark 5.32 (Interpretation). A class in $H_n(X, A)$ is represented by a chain in X whose boundary lies in A . We think of relative homology as measuring the topology of X “modulo” A . The relative group $H_n(X, A)$ detects “holes in X relative to A ”: cycles that bound in A are considered trivial.

Example 5.33 (Relative homology $H_n(X, \emptyset)$). When $A = \emptyset$, we have $C_n(X, \emptyset) = C_n(X)$, so $H_n(X, \emptyset) = H_n(X)$. When $A = X$, the quotient $C_n(X, X) = 0$, so $H_n(X, X) = 0$ for all n .

Proposition 5.34 (Functoriality of relative homology). *A continuous map of pairs $f: (X, A) \rightarrow (Y, B)$ (meaning $f(A) \subseteq B$) induces homomorphisms $f_*: H_n(X, A) \rightarrow H_n(Y, B)$ for all n . These satisfy functoriality: $(\text{Id})_* = \text{Id}$ and $(g \circ f)_* = g_* \circ f_*$. Moreover, the connecting homomorphisms ∂_* in the long exact sequences of the pairs are natural with respect to maps of pairs.*

Proof. The map $f_\# : C_n(X) \rightarrow C_n(Y)$ sends $C_n(A)$ to $C_n(B)$, hence descends to $\bar{f}_\# : C_n(X, A) \rightarrow C_n(Y, B)$. This is a chain map, and functoriality follows from the chain level. Naturality of ∂_* is a consequence of Theorem 6.5. \square

Proposition 5.35 (Short exact sequence of chain complexes). *For a pair (X, A) , the sequence*

$$0 \longrightarrow C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j_\#} C_n(X, A) \longrightarrow 0$$

is exact for every n , where $i: A \hookrightarrow X$ is the inclusion and $j_\#$ is the quotient map.

Proof. Exactness at $C_n(A)$: the map $i_\#$ is injective since $C_n(A)$ is a free abelian group on $\text{Sing}_n(A) \subseteq \text{Sing}_n(X)$. Exactness at $C_n(X, A)$: by definition of quotient. Exactness at $C_n(X)$: $\ker j_\# = \text{im } i_\# = C_n(A)$. \square

5.9 Exercises

Exercise 5.1. Let $\sigma: \Delta^2 \rightarrow X$ be a singular 2-simplex. Verify by direct computation that $\partial_1(\partial_2(\sigma)) = 0$.

Exercise 5.2. Let X be a set with the discrete topology. Compute $H_n(X)$ for all n .

Exercise 5.3. Show that if X is contractible then $\widetilde{H}_n(X) = 0$ for all n .

Exercise 5.4. Show that a chain homotopy equivalence $f_{\#}: C_{\bullet}(X) \rightarrow C_{\bullet}(Y)$ induces isomorphisms on homology. Deduce that $H_n(X) \cong H_n(Y)$ if $X \simeq Y$.

Exercise 5.5. Let $CX = (X \times [0, 1]) / (X \times \{1\})$ be the cone on X . Show that CX is contractible, and hence $\widetilde{H}_n(CX) = 0$ for all n .

Exercise 5.6. Let $x_0 \in X$. Show that $H_n(X, \{x_0\}) \cong \widetilde{H}_n(X)$ for all $n \geq 0$.

Exercise 5.7. Let $F: [0, 1] \times [0, 1] \rightarrow Y$ be a homotopy of paths. Write out the prism operator P_1 explicitly and verify the chain homotopy identity $\partial_2 \circ P_1 + P_0 \circ \partial_1 = g_{\#} - f_{\#}$ for a singular 1-simplex.

Exercise 5.8. Let $A \subseteq X$ be a retract (i.e., there exists $r: X \rightarrow A$ with $r \circ i = \text{Id}_A$). Show that $i_*: H_n(A) \rightarrow H_n(X)$ is injective and $r_*: H_n(X) \rightarrow H_n(A)$ is surjective for all n .

Exercise 5.9. Prove that $H_n(X \sqcup Y) \cong H_n(X) \oplus H_n(Y)$ for all n .

Exercise 5.10. Let C_{\bullet} be a chain complex of finite-dimensional vector spaces over a field k , with $C_n = 0$ for all but finitely many n . Prove that

$$\sum_{n \geq 0} (-1)^n \dim_k C_n = \sum_{n \geq 0} (-1)^n \dim_k H_n(C_{\bullet}).$$

Hint: use the rank–nullity theorem for the short exact sequences $0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0$ and $0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0$.

Chapter 6

Long Exact Sequences and Excision

In this chapter we develop the main computational tools of singular homology: the long exact sequence of a pair and the excision theorem. Together, they allow the inductive computation of homology groups for a wide class of spaces.

6.1 Short exact sequences of chain complexes

Definition 6.1 (Short exact sequence of chain complexes). A **short exact sequence of chain complexes** is a diagram

$$0 \longrightarrow A_{\bullet} \xrightarrow{i} B_{\bullet} \xrightarrow{p} C_{\bullet} \longrightarrow 0$$

where i and p are chain maps such that for every n ,

$$0 \longrightarrow A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \longrightarrow 0$$

is a short exact sequence of abelian groups.

6.2 The connecting homomorphism

Definition 6.2 (Connecting homomorphism). Given a short exact sequence of chain complexes $0 \rightarrow A_{\bullet} \xrightarrow{i} B_{\bullet} \xrightarrow{p} C_{\bullet} \rightarrow 0$, the **connecting homomorphism** $\partial_*: H_n(C_{\bullet}) \rightarrow H_{n-1}(A_{\bullet})$ is defined as follows.

Let $[c] \in H_n(C_{\bullet})$ with $\partial_n^C c = 0$.

Step 1. Since p_n is surjective, choose $b \in B_n$ with $p_n(b) = c$.

Step 2. Then $p_{n-1}(\partial_n^B b) = \partial_n^C(p_n(b)) = \partial_n^C c = 0$.

Step 3. By exactness at B_{n-1} , there exists a unique $a \in A_{n-1}$ with $i_{n-1}(a) = \partial_n^B b$.

Step 4. One checks $\partial_{n-1}^A a = 0$ (since $i_{n-2}(\partial_{n-1}^A a) = \partial_{n-1}^B(\partial_n^B b) = 0$ and i_{n-2} is injective).

Step 5. Set $\partial_*[c] = [a] \in H_{n-1}(A_{\bullet})$.

The following diagram summarizes the construction:

$$\begin{array}{ccccc}
 & & b & \xrightarrow{p_n} & c \\
 & & \downarrow \partial_n^B & & \downarrow \partial_n^C \\
 a & \xrightarrow{i_{n-1}} & \partial_n^B b & \xrightarrow{p_{n-1}} & 0
 \end{array}$$

Lemma 6.3 (Well-definedness of ∂_*). *The class $[a]$ does not depend on the choice of b .*

Proof. Suppose b' is another lift with $p_n(b') = c$. Then $p_n(b - b') = 0$, so $b - b' = i_n(a'')$ for some $a'' \in A_n$. Then

$$i_{n-1}(a - a') = \partial_n^B b - \partial_n^B b' = \partial_n^B(b - b') = \partial_n^B(i_n(a'')) = i_{n-1}(\partial_n^A a'').$$

Since i_{n-1} is injective, $a - a' = \partial_n^A a''$, so $[a] = [a'] \in H_{n-1}(A_\bullet)$. \square

6.3 The Snake Lemma

The connecting homomorphism is a special case of the following fundamental result in homological algebra.

Lemma 6.4 (Snake Lemma). *Consider a commutative diagram of abelian groups with exact rows:*

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
 \end{array}$$

Then there is an exact sequence

$$\ker \alpha \longrightarrow \ker \beta \longrightarrow \ker \gamma \xrightarrow{\delta} \operatorname{coker} \alpha \longrightarrow \operatorname{coker} \beta \longrightarrow \operatorname{coker} \gamma,$$

*where $\delta: \ker \gamma \rightarrow \operatorname{coker} \alpha$ is the **connecting map** (or “snake map”) defined by the diagram chase $c \mapsto [(\alpha \circ (f)^{-1} \circ (\beta)^{-1} \circ g)(c)]$, made precise as follows: given $c \in \ker \gamma$, lift to $b \in B$ with $g(b) = c$; then $g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = 0$, so $\beta(b) = f'(a')$ for unique $a' \in A'$; set $\delta(c) = [a'] \in \operatorname{coker} \alpha$.*

Proof. The proof is a diagram chase. We verify exactness at each term.

Exactness at $\ker \beta$: The map $\ker \alpha \rightarrow \ker \beta$ is induced by f . Its image consists of elements $f(a)$ with $\alpha(a) = 0$. The map $\ker \beta \rightarrow \ker \gamma$ is induced by g . If $b \in \ker \beta$ maps to 0 in $\ker \gamma$, then $g(b) = 0$, so $b = f(a)$ for some a by exactness of the top row. Then $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$, and f' is injective, so $\alpha(a) = 0$, i.e., $a \in \ker \alpha$.

Exactness at $\ker \gamma$: If $c = g(b)$ with $b \in \ker \beta$, then $\beta(b) = 0 = f'(0)$, so $\delta(c) = [0] = 0$. Conversely, if $\delta(c) = 0$, then the a' from the construction satisfies $a' = \alpha(a)$ for some a . Replace b by $b - f(a)$; then $g(b - f(a)) = c$ and $\beta(b - f(a)) = \beta(b) - f'(\alpha(a)) = f'(a') - f'(a') = 0$, so $b - f(a) \in \ker \beta$ maps to c .

Exactness at coker α : First, $\text{im } \delta \subseteq \ker(\text{coker } \alpha \rightarrow \text{coker } \beta)$: if $c \in \ker \gamma$ with $\delta(c) = [a']$, then by construction $f'(a') = \beta(b)$ for some b , so $[f'(a')] = [\beta(b)] = 0$ in $\text{coker } \beta$.

Conversely, suppose $[a'] \in \text{coker } \alpha$ maps to 0 in $\text{coker } \beta$, i.e., $f'(a') = \beta(b)$ for some $b \in B$. Set $c = g(b)$. Then $\gamma(c) = \gamma(g(b)) = g'(\beta(b)) = g'(f'(a')) = 0$ (since $g' \circ f' = 0$ by exactness of the bottom row at B'). So $c \in \ker \gamma$. The lift of c is b , and $\beta(b) = f'(a')$, so $\delta(c) = [a']$.

Exactness at coker β : The map $\text{coker } \alpha \rightarrow \text{coker } \beta$ sends $[a']$ to $[f'(a')]$. The map $\text{coker } \beta \rightarrow \text{coker } \gamma$ sends $[b']$ to $[g'(b')]$. If $[b'] = [f'(a')]$, then $g'(b') = g'(f'(a')) = 0$, so $[g'(b')] = 0$. Conversely, if $[g'(b')] = 0$, then $g'(b') = \gamma(c_0)$ for some c_0 . Pick b_0 with $g(b_0) = c_0$; then $g'(b' - \beta(b_0)) = g'(b') - \gamma(g(b_0)) = 0$. By exactness at B' , $b' - \beta(b_0) = f'(a')$ for some a' . Hence $[b'] = [f'(a') + \beta(b_0)] = [f'(a')]$ in $\text{coker } \beta$.

Exactness at $\text{coker } \gamma$ is similar and left to the reader. \square

6.4 The long exact sequence in homology

Theorem 6.5 (Long exact sequence of a short exact sequence of chain complexes).
 Given a short exact sequence of chain complexes

$$0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{p} C_\bullet \longrightarrow 0,$$

there is a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \longrightarrow \cdots$$

extending to the right as $\cdots \rightarrow H_1(C) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(B) \xrightarrow{p_*} H_0(C) \rightarrow 0$. Moreover, this construction is natural: a morphism of short exact sequences induces a morphism of long exact sequences.

Proof. We must prove exactness at three places: at $H_n(B)$, at $H_n(C)$, and at $H_{n-1}(A)$.

Exactness at $H_n(B)$: $\text{im } i_* \subseteq \ker p_*$.

We have $p_* \circ i_* = (p \circ i)_* = 0$ since $p \circ i = 0$.

Conversely, let $[b] \in \ker p_*$, so $p_n(b) = \partial_{n+1}^C(c')$ for some $c' \in C_{n+1}$. Since p_{n+1} is surjective, pick $b' \in B_{n+1}$ with $p_{n+1}(b') = c'$. Then $p_n(b - \partial_{n+1}^B b') = p_n(b) - \partial_{n+1}^C c' = 0$. By exactness, $b - \partial_{n+1}^B b' = i_n(a)$ for some $a \in A_n$. Now $i_{n-1}(\partial_n^A a) = \partial_n^B(i_n(a)) = \partial_n^B(b) - 0 = 0$ (since b is a cycle and $\partial^B \circ \partial^B = 0$), so $\partial_n^A a = 0$ by injectivity of i_{n-1} . Thus $a \in Z_n(A)$ and $[b] = [i_n(a)] = i_*[a]$.

Exactness at $H_n(C)$: $\text{im } p_* \subseteq \ker \partial_*$.

If $[c] = p_*[b]$, then $c = p_n(b)$ with $\partial_n^B b = 0$. In the construction of ∂_* , we may choose this b as our lift, giving $\partial_n^B b = 0 = i_{n-1}(0)$, so $\partial_*[c] = [0] = 0$.

Conversely, suppose $\partial_*[c] = 0$. Using the construction, pick b with $p_n(b) = c$ and let $a \in A_{n-1}$ satisfy $i_{n-1}(a) = \partial_n^B b$. If $[a] = 0$ in $H_{n-1}(A)$, then $a = \partial_n^A(a')$ for some $a' \in A_n$. Set $\tilde{b} = b - i_n(a')$. Then $p_n(\tilde{b}) = c$ and $\partial_n^B \tilde{b} = \partial_n^B b - i_{n-1}(\partial_n^A a') = i_{n-1}(a) - i_{n-1}(a) = 0$, so $\tilde{b} \in Z_n(B)$ and $p_*[\tilde{b}] = [c]$.

Exactness at $H_{n-1}(A)$: $\text{im } \partial_* \subseteq \ker i_*$.

By construction, $i_{n-1}(a) = \partial_n^B b \in B_{n-1}(B)$, so $i_*[a] = [\partial_n^B b] = 0$ in $H_{n-1}(B)$.

Conversely, suppose $[a] \in H_{n-1}(A)$ with $i_*[a] = 0$, i.e., $i_{n-1}(a) = \partial_n^B(b)$ for some $b \in B_n$. Set $c = p_n(b)$. Then $\partial_n^C c = p_{n-1}(\partial_n^B b) = p_{n-1}(i_{n-1}(a)) = 0$, so $c \in Z_n(C)$. By the construction of ∂_* (using this b as the lift), $\partial_*[c] = [a]$.

Naturality: Given a morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_\bullet & \longrightarrow & B_\bullet & \longrightarrow & C_\bullet \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A'_\bullet & \longrightarrow & B'_\bullet & \longrightarrow & C'_\bullet \longrightarrow 0 \end{array}$$

one checks that $\partial_* \circ \gamma_* = \alpha_* \circ \partial_*$ by tracing the diagram chase through both compositions. The remaining squares commute by functoriality of homology. \square

6.5 The long exact sequence of a pair

Theorem 6.6 (Long exact sequence of a pair). *For any pair (X, A) with $A \subseteq X$, there is a long exact sequence*

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \longrightarrow \cdots$$

which is natural in the pair (X, A) .

Proof. Apply Theorem 6.5 to the short exact sequence of chain complexes from Proposition 5.35:

$$0 \rightarrow C_\bullet(A) \xrightarrow{i_\#} C_\bullet(X) \xrightarrow{j_\#} C_\bullet(X, A) \rightarrow 0. \quad \square$$

Example 6.7 (The pair (D^n, S^{n-1})). Consider the pair (D^n, S^{n-1}) where D^n is the closed unit disk. Since D^n is contractible, $\widetilde{H}_k(D^n) = 0$ for all k . The long exact sequence gives

$$\cdots \longrightarrow H_k(D^n) \longrightarrow H_k(D^n, S^{n-1}) \xrightarrow{\partial_*} H_{k-1}(S^{n-1}) \longrightarrow H_{k-1}(D^n) \longrightarrow \cdots$$

For $k \geq 2$, the outer terms vanish, yielding $H_k(D^n, S^{n-1}) \cong H_{k-1}(S^{n-1})$.

6.6 The excision theorem

Theorem 6.8 (Excision). *Let X be a topological space and let $Z \subseteq A \subseteq X$ be subspaces such that $\overline{Z} \subseteq \text{int}(A)$. Then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A) \quad \text{for all } n.$$

Equivalently, if $A, B \subseteq X$ are subspaces whose interiors cover X , i.e., $X = \text{int}(A) \cup \text{int}(B)$, then $H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$.

The proof uses *barycentric subdivision* to make chains “small” enough that they lie entirely in A or in $X \setminus Z$.

Definition 6.9 (Barycentric subdivision). The **barycentric subdivision operator** $\text{sd}_n: C_n(X) \rightarrow C_n(X)$ is defined inductively. For a singular 0-simplex σ , set $\text{sd}_0(\sigma) = \sigma$. For $n \geq 1$ and $\sigma: \Delta^n \rightarrow X$, let $b_\sigma = \sigma(\text{barycenter of } \Delta^n)$ and define

$$\text{sd}_n(\sigma) = b_\sigma \cdot \text{sd}_{n-1}(\partial_n \sigma),$$

where $b \cdot c$ denotes the “cone from b ” operation on chains, extended linearly.

Lemma 6.10 (Properties of barycentric subdivision). (i) *sd is a chain map: $\partial_n \circ \text{sd}_n = \text{sd}_{n-1} \circ \partial_n$.*

(ii) *sd is chain homotopic to the identity: there exist homomorphisms $T_n: C_n(X) \rightarrow C_{n+1}(X)$ with $\partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n = \text{sd}_n - \text{Id}$.*

(iii) *After sufficiently many iterations, $\text{sd}^m(\sigma)$ has arbitrarily small image for each simplex σ : the diameter of each simplex in $\text{sd}^m(\Delta^n)$ is at most $\left(\frac{n}{n+1}\right)^m \text{diam}(\Delta^n)$.*

Proof sketch. Part (i) follows by induction using the cone construction and the identity $\partial(b \cdot c) = c - b \cdot \partial c$ (valid when ∂c is defined and the cone makes sense).

Part (ii) uses the same cone technique: $T_n(\sigma) = b_\sigma \cdot T_{n-1}(\partial_n \sigma) + b_\sigma \cdot (\text{sd}_n(\sigma) - \sigma)$, adjusted to make the identity work by induction.

Part (iii) is a standard estimate. Each barycentric subdivision replaces an n -simplex with simplices whose diameter is at most $\frac{n}{n+1}$ times the original. After m iterations, the factor is $\left(\frac{n}{n+1}\right)^m \rightarrow 0$. \square

Proof of the Excision Theorem 6.8. We use the formulation: $\{A, B\}$ is an open cover with $X = \text{int}(A) \cup \text{int}(B)$. Let $C_n^{\{A, B\}}(X) \subseteq C_n(X)$ denote the subgroup of chains that are sums of simplices each mapping into A or into B . We need to show that the inclusion $C_\bullet^{\{A, B\}}(X)/C_\bullet(A) \hookrightarrow C_\bullet(X)/C_\bullet(A)$ induces isomorphisms on homology.

Step 1. By the Lebesgue number lemma (in the compact setting) or direct argument, for every singular simplex $\sigma: \Delta^n \rightarrow X$, there exists $m \geq 0$ such that every simplex in $\text{sd}^m(\sigma)$ maps into A or into B . That is, $\text{sd}^m(\sigma) \in C_n^{\{A, B\}}(X)$.

Step 2. By Lemma 6.10(ii), sd^m is chain homotopic to the identity. More precisely, there exists $D_n: C_n(X) \rightarrow C_{n+1}(X)$ with $\text{sd}^m - \text{Id} = \partial_{n+1} \circ D_n + D_{n-1} \circ \partial_n$.

Step 3 (Surjectivity). Given a relative cycle $\alpha \in C_n(X)$ with $\partial_n \alpha \in C_{n-1}(A)$, we show $[\alpha]$ lies in the image. Choose m large enough so that $\text{sd}^m(\alpha) \in C_n^{\{A, B\}}(X)$. Then $\text{sd}^m(\alpha) - \alpha = \partial_{n+1}(D_n \alpha) + D_{n-1}(\partial_n \alpha)$. Since $\partial_n \alpha \in C_{n-1}(A)$, the term $D_{n-1}(\partial_n \alpha)$ lies in $C_n(A)$. Hence $\text{sd}^m(\alpha) \equiv \alpha + \partial_{n+1}(D_n \alpha) \pmod{C_n(A)}$, showing $[\alpha] = [\text{sd}^m(\alpha)]$ in $H_n(X, A)$. Since $\text{sd}^m(\alpha) \in C_n^{\{A, B\}}(X)$, surjectivity follows.

Step 4 (Injectivity). Suppose $\alpha \in C_n^{\{A, B\}}(X)$ represents 0 in $H_n(X, A)$, i.e., $\alpha = \partial_{n+1} \beta + \gamma$ with $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$. Choose m so that $\text{sd}^m(\beta) \in C_{n+1}^{\{A, B\}}(X)$. Then $\text{sd}^m(\alpha) = \partial_{n+1}(\text{sd}^m \beta) + \text{sd}^m(\gamma)$ and $\alpha - \text{sd}^m(\alpha) = \partial D \alpha + D \partial \alpha$. Since $\partial \alpha \in C_{n-1}(A)$ (as $\alpha - \gamma = \partial \beta$ implies $\partial \alpha = \partial \gamma \in C_{n-1}(A)$), the chain homotopy terms stay controlled, and one concludes $[\alpha] = 0$ in $H_n(C_\bullet^{\{A, B\}}(X)/C_\bullet(A))$.

Finally, the natural identification $C_{\bullet}^{\{A,B\}}(X)/C_{\bullet}(A) \cong C_{\bullet}(B)/C_{\bullet}(A \cap B)$ (since a simplex in $C^{\{A,B\}}$ not in $C(A)$ must map into B) yields the desired isomorphism $H_n(B, A \cap B) \cong H_n(X, A)$. \square

6.7 Relative homology and quotient spaces

Theorem 6.11 (Collapsing a subspace). *Let (X, A) be a pair such that A is a closed subspace and is a deformation retract of some open neighborhood in X (i.e., A is a “good pair”). Then the quotient map $q: (X, A) \rightarrow (X/A, A/A)$ induces isomorphisms*

$$q_*: H_n(X, A) \xrightarrow{\cong} H_n(X/A, A/A) \cong \widetilde{H}_n(X/A) \quad \text{for all } n.$$

Proof. Let U be an open neighborhood of A that deformation retracts onto A . By homotopy invariance, $H_n(U, A) \cong H_n(A, A) = 0$. The long exact sequence of the triple (X, U, A) then gives $H_n(X, A) \cong H_n(X, U)$. By excision (removing A from both):

$$H_n(X \setminus A, U \setminus A) \cong H_n(X, U).$$

The quotient map q restricts to a homeomorphism $X \setminus A \cong X/A \setminus \{*\}$, and the analogous excision argument for the quotient gives

$$H_n(X/A, U/A) \cong H_n(X/A \setminus \{*\}, U/A \setminus \{*\}).$$

Since U/A is contractible (it deformation retracts to the point A/A), $H_n(X/A, U/A) \cong \widetilde{H}_n(X/A)$. Combining all isomorphisms yields the result. \square

Corollary 6.12 (Homology of spheres, complete computation). *For $n \geq 1$,*

$$\widetilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use the long exact sequence of the pair (D^n, S^{n-1}) . Since D^n is contractible, $\widetilde{H}_k(D^n) = 0$ for all k , and the long exact sequence gives

$$\widetilde{H}_k(D^n) \rightarrow H_k(D^n, S^{n-1}) \xrightarrow{\partial_*} \widetilde{H}_{k-1}(S^{n-1}) \rightarrow \widetilde{H}_{k-1}(D^n),$$

so $H_k(D^n, S^{n-1}) \cong \widetilde{H}_{k-1}(S^{n-1})$. By Theorem 6.11, $H_k(D^n, S^{n-1}) \cong \widetilde{H}_k(D^n/S^{n-1}) \cong \widetilde{H}_k(S^n)$. Therefore $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$. Induction on n with base case $\widetilde{H}_0(S^0) \cong \mathbb{Z}$ completes the proof. \square

6.8 Applications

6.8.1 Degree of maps between spheres

Definition 6.13 (Degree). Let $f: S^n \rightarrow S^n$ be a continuous map with $n \geq 1$. Since $H_n(S^n) \cong \mathbb{Z}$, the induced map $f_*: H_n(S^n) \rightarrow H_n(S^n)$ is multiplication by an integer

$d \in \mathbb{Z}$. This integer is the **degree** of f , denoted $\deg f$.

Proposition 6.14 (Properties of degree). (i) $\deg(\text{Id}_{S^n}) = 1$.

(ii) $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

(iii) If $f \simeq g$, then $\deg f = \deg g$.

(iv) The antipodal map $a: S^n \rightarrow S^n$, $x \mapsto -x$, has $\deg a = (-1)^{n+1}$.

(v) A reflection of S^n (negating one coordinate) has degree -1 .

Proof. (i) Functoriality: $(\text{Id}_{S^n})_* = \text{Id}_{H_n(S^n)}$, which is multiplication by 1.

(ii) $(g \circ f)_* = g_* \circ f_*$ is multiplication by $\deg g \cdot \deg f$.

(iii) Homotopy invariance (Theorem 5.20).

(iv) Let $r_i: S^n \rightarrow S^n$ be the reflection $(x_0, \dots, x_n) \mapsto (x_0, \dots, -x_i, \dots, x_n)$. The antipodal map is $a = r_0 \circ r_1 \circ \dots \circ r_n$, a composition of $(n+1)$ reflections. By (ii), $\deg a = (\deg r_i)^{n+1}$. So it suffices to show each r_i has degree -1 .

(v) We verify $\deg r_i = -1$ for reflections. Consider the generator $[\mu_n] \in H_n(S^n) \cong \mathbb{Z}$ arising from the long exact sequence of (D_+^n, S^{n-1}) . The reflection r_i reverses the orientation of the fundamental class: the induced map on the relative homology $H_n(D^n, S^{n-1})$ is multiplication by -1 (since r_i reverses orientation of \mathbb{R}^{n+1} restricted to the relevant hemisphere). Following the isomorphisms through the connecting homomorphism yields $\deg r_i = -1$.

Combining (iv) and (v): $\deg a = (-1)^{n+1}$. \square

Example 6.15 (Degree and surjectivity). If $f: S^n \rightarrow S^n$ has $\deg f \neq 0$, then f is surjective. Indeed, if $y \notin f(S^n)$, then f factors through $S^n \setminus \{y\} \cong \mathbb{R}^n$, which is contractible, so $f_* = 0$ and $\deg f = 0$.

6.8.2 Brouwer fixed point theorem

Theorem 6.16 (Brouwer Fixed Point Theorem). *Every continuous map $f: D^n \rightarrow D^n$ has a fixed point, for all $n \geq 1$.*

Proof. Suppose f has no fixed point. Define $r: D^n \rightarrow S^{n-1}$ by letting $r(x)$ be the point where the ray from $f(x)$ through x meets S^{n-1} . Then r is continuous and $r|_{S^{n-1}} = \text{Id}_{S^{n-1}}$, so r is a retraction.

Let $i: S^{n-1} \hookrightarrow D^n$ be the inclusion. Then $r \circ i = \text{Id}_{S^{n-1}}$, so $(r \circ i)_* = r_* \circ i_* = \text{Id}$ on $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$. In particular, i_* is injective.

But D^n is contractible, so $H_{n-1}(D^n) = 0$ for $n \geq 2$, meaning $i_*: \mathbb{Z} \rightarrow 0$ is the zero map—a contradiction.

For $n = 1$: $H_0(S^0) \cong \mathbb{Z}^2$ but the retraction forces $r_* \circ i_* = \text{Id}$ on $\widetilde{H}_0(S^0) \cong \mathbb{Z}$, while $\widetilde{H}_0(D^1) = 0$, giving the same contradiction. \square

Corollary 6.17 (No-retraction theorem). *There is no retraction $r: D^n \rightarrow S^{n-1}$ for $n \geq 1$.*

Proof. This was established in the proof above: such a retraction leads to a contradiction on homology groups. \square

6.8.3 Invariance of domain

Theorem 6.18 (Invariance of Domain). *Let $U \subseteq \mathbb{R}^n$ be open and let $f: U \rightarrow \mathbb{R}^n$ be a continuous injection. Then $f(U)$ is open in \mathbb{R}^n .*

Proof. It suffices to show that $f(U)$ is open at each point. Let $x \in U$; we show $f(x)$ is an interior point of $f(U)$. Choose a closed ball $\bar{B} \subseteq U$ centered at x . By the following key lemma, $f(\bar{B})$ contains a neighborhood of $f(x)$, so $f(x) \in \text{int}(f(U))$. \square

Lemma 6.19 (Key lemma for invariance of domain). *Let $f: D^n \rightarrow \mathbb{R}^n$ be a continuous injection. Then $f(x) \notin f(S^{n-1})$ implies $f(x) \in \text{int}(f(D^n))$ for any $x \in \text{int}(D^n)$. More precisely, $\mathbb{R}^n \setminus f(S^{n-1})$ has exactly two connected components (for $n \geq 2$), and the one containing $f(x)$ lies in $f(D^n)$.*

Proof sketch. Suppose $y \notin f(D^n)$. Define $g: S^{n-1} \rightarrow S^{n-1}$ by $g(z) = \frac{f(z)-y}{|f(z)-y|}$. The map g extends to $D^n \rightarrow S^{n-1}$ via $\hat{g}(z) = \frac{f(z)-y}{|f(z)-y|}$ (well-defined since $y \notin f(D^n)$). Thus $g = \hat{g} \circ i$ factors through the contractible space D^n , so $\deg g = 0$. On the other hand, for $y = f(x_0)$ with $x_0 \in \text{int}(D^n)$, one can show $\deg g \neq 0$ by a deformation argument (deforming f to the identity), contradicting the above. Hence $f(x_0) \notin \mathbb{R}^n \setminus f(D^n)$, and since f is injective, $f(x_0) \in f(\text{int}(D^n)) \subseteq \text{int}(f(D^n))$. \square

Corollary 6.20 (Topological invariance of dimension). *If $m \neq n$, then \mathbb{R}^m is not homeomorphic to \mathbb{R}^n .*

Proof. If $m < n$, embed $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ as $\mathbb{R}^m \times \{0\}$. This image is not open in \mathbb{R}^n , contradicting invariance of domain if there were a homeomorphism $\mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^m \hookrightarrow \mathbb{R}^n$.

Alternatively, one can argue directly via homology: removing a point, $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$, so $H_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$ while $H_{m-1}(\mathbb{R}^m \setminus \{0\}) \cong \mathbb{Z}$ for different values of $m-1 \neq n-1$. A homeomorphism $\mathbb{R}^m \cong \mathbb{R}^n$ would restrict to a homeomorphism $\mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\}$, yielding $\mathbb{Z} \cong H_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong H_{n-1}(\mathbb{R}^m \setminus \{0\}) = 0$ for $n-1 > m-1$, a contradiction. \square

Proposition 6.21 (Hairy ball theorem, homological version). *The sphere S^{2n} admits no continuous nowhere-vanishing tangent vector field.*

Proof. A nowhere-vanishing tangent vector field on S^{2n} provides a homotopy from the identity to the antipodal map (via $x \mapsto (\cos t)x + (\sin t)v(x)$ for the normalized field v). But $\deg(\text{Id}) = 1$ and $\deg(a) = (-1)^{2n+1} = -1$, contradicting homotopy invariance of degree. \square

6.9 The Mayer–Vietoris sequence

As an important consequence of excision, we obtain a tool for computing homology from decompositions.

Theorem 6.22 (Mayer–Vietoris sequence). *Let $X = A \cup B$ where $\text{int}(A) \cup \text{int}(B) = X$. Let $i_A: A \cap B \hookrightarrow A$, $i_B: A \cap B \hookrightarrow B$, $j_A: A \hookrightarrow X$, $j_B: B \hookrightarrow X$ be the inclusions. There is a long exact sequence*

$$\cdots \rightarrow H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \rightarrow \cdots$$

where $\Phi(\alpha) = ((i_A)_*(\alpha), -(i_B)_*(\alpha))$ and $\Psi(\alpha, \beta) = (j_A)_*(\alpha) + (j_B)_*(\beta)$.

Proof. Consider the short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A \cap B) \xrightarrow{\Phi_\#} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{\Psi_\#} C_\bullet^{\{A,B\}}(X) \longrightarrow 0,$$

where $C_\bullet^{\{A,B\}}(X) = C_\bullet(A) + C_\bullet(B) \subseteq C_\bullet(X)$. Exactness at each position is straightforward: $\Phi_\#$ is injective, $\ker \Psi_\# = \text{im } \Phi_\#$ (a chain in both A and B lives in $A \cap B$), and $\Psi_\#$ is surjective by definition.

By the proof of the excision theorem, the inclusion $C_\bullet^{\{A,B\}}(X) \hookrightarrow C_\bullet(X)$ is a chain homotopy equivalence, hence induces isomorphisms on homology. Applying Theorem 6.5 to the short exact sequence above and replacing $H_n(C_\bullet^{\{A,B\}}(X))$ with $H_n(X)$ yields the Mayer–Vietoris sequence. \square

Example 6.23 (Homology of S^n via Mayer–Vietoris). Write $S^n = U_+ \cup U_-$ where U_+ and U_- are the upper and lower open hemispheres slightly enlarged to overlap. Then $U_+ \simeq \text{pt} \simeq U_-$ and $U_+ \cap U_- \simeq S^{n-1}$. The Mayer–Vietoris sequence in reduced homology gives

$$\cdots \rightarrow \widetilde{H}_k(U_+) \oplus \widetilde{H}_k(U_-) \rightarrow \widetilde{H}_k(S^n) \xrightarrow{\partial_*} \widetilde{H}_{k-1}(S^{n-1}) \rightarrow \widetilde{H}_{k-1}(U_+) \oplus \widetilde{H}_{k-1}(U_-) \rightarrow \cdots$$

Since $\widetilde{H}_*(\text{pt}) = 0$, we get isomorphisms $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ for all k . Induction gives $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_k(S^n) = 0$ for $k \neq n$, confirming Theorem 7.5.

Example 6.24 (Homology of the torus). The torus $T^2 = S^1 \times S^1$ can be decomposed as a union of two overlapping cylinders. The Mayer–Vietoris sequence yields

$$H_k(T^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = 2, \\ \mathbb{Z}^2 & \text{if } k = 1, \\ 0 & \text{if } k \geq 3. \end{cases}$$

We leave the details as Exercise 6.5.

Example 6.25 (Homology of the Klein bottle). The Klein bottle K can be decomposed as a union of two Möbius bands A and B , with $A \cap B \simeq S^1 \sqcup S^1$. A careful analysis of the Mayer–Vietoris sequence (tracking the maps on H_1) yields

$$H_k(K) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

The torsion element in $H_1(K)$ reflects the non-orientability of K .

Remark 6.26 (Mayer–Vietoris in reduced homology). There is also a Mayer–Vietoris sequence in reduced homology:

$$\cdots \rightarrow \widetilde{H}_n(A \cap B) \rightarrow \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \rightarrow \widetilde{H}_n(X) \xrightarrow{\partial_*} \widetilde{H}_{n-1}(A \cap B) \rightarrow \cdots$$

This is obtained by applying the same construction to the augmented chain complexes. The reduced version is often more convenient since $\widetilde{H}_0(\text{pt}) = 0$.

6.10 Exercises

Exercise 6.1. Verify directly that the connecting homomorphism $\partial_*: H_n(C_\bullet) \rightarrow H_{n-1}(A_\bullet)$ is a group homomorphism (i.e., $\partial_*([c] + [c']) = \partial_*[c] + \partial_*[c']$).

Exercise 6.2. Prove the **Five Lemma**: given a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

with exact rows, if $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are isomorphisms, then so is α_3 . Show that “ α_2, α_4 surjective and α_1, α_5 injective” suffices if we only want α_3 surjective; state and prove the dual for injectivity.

Exercise 6.3. Let $X = A \vee B$ be the wedge sum (identifying base points). Assuming that the base points are “good” (deformation retracts of neighborhoods), use excision to show that $\widetilde{H}_n(A \vee B) \cong \widetilde{H}_n(A) \oplus \widetilde{H}_n(B)$ for all n .

Exercise 6.4. Let ΣX denote the (unreduced) suspension of X . Using the Mayer–Vietoris sequence, prove that $\widetilde{H}_n(\Sigma X) \cong \widetilde{H}_{n-1}(X)$ for all n . Deduce the homology of S^n by induction.

Exercise 6.5. Compute the homology of the torus T^2 using the Mayer–Vietoris sequence. Decompose T^2 as $A \cup B$ where A and B are each homeomorphic to $S^1 \times (0, 1)$ and $A \cap B$ is the disjoint union of two copies of $S^1 \times (0, \frac{1}{2})$. Carefully determine the maps Φ and Ψ and extract $H_k(T^2)$ for all k .

Exercise 6.6. Use the Brouwer fixed point theorem to prove: if $K \subseteq \mathbb{R}^n$ is a compact convex set and $f: K \rightarrow K$ is continuous, then f has a fixed point. *Hint:* K is homeomorphic to D^n (or a lower-dimensional disk) if it has non-empty interior.

Exercise 6.7. (a) Show that the map $z \mapsto z^k: S^1 \rightarrow S^1$ (identifying $S^1 \subseteq \mathbb{C}$) has degree k .

(b) More generally, show that if $f: S^n \rightarrow S^n$ has no fixed point, then $\deg f = (-1)^{n+1}$.

(c) Deduce that a map $f: S^{2n} \rightarrow S^{2n}$ with $\deg f \neq (-1)^{2n+1} = -1$ must have a fixed point.

Exercise 6.8. Prove the **Borsuk–Ulam theorem in dimension 2**: for every continuous map $f: S^2 \rightarrow \mathbb{R}^2$ there exists $x \in S^2$ with $f(x) = f(-x)$.

Hint: suppose not, and consider $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}: S^2 \rightarrow S^1$. Show $g(-x) = -g(x)$, and use degree theory to derive a contradiction.

Exercise 6.9. Let $A \subseteq B \subseteq X$. Establish the **long exact sequence of the triple**:

$$\cdots \rightarrow H_n(B, A) \rightarrow H_n(X, A) \rightarrow H_n(X, B) \xrightarrow{\partial_*} H_{n-1}(B, A) \rightarrow \cdots$$

Hint: apply the long exact sequence theorem to the short exact sequence $0 \rightarrow C_\bullet(B, A) \rightarrow C_\bullet(X, A) \rightarrow C_\bullet(X, B) \rightarrow 0$.

Exercise 6.10. For $x \in X$, the **local homology groups** are defined as $H_n(X, X \setminus \{x\})$. Show that local homology is a topological invariant: if $f: X \rightarrow Y$ is a homeomorphism, then $H_n(X, X \setminus \{x\}) \cong H_n(Y, Y \setminus \{f(x)\})$. Compute $H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\})$ and use it to give another proof that $\mathbb{R}^m \not\cong \mathbb{R}^n$ for $m \neq n$.

Chapter 7

The Mayer–Vietoris Theorem

The Mayer–Vietoris sequence is one of the most powerful computational tools in algebraic topology. It allows us to compute the homology of a space X that is written as the union of two subspaces A and B , provided we know the homology of A , B , and their intersection $A \cap B$. In spirit it is the homological analogue of the Seifert–van Kampen theorem for the fundamental group.

7.1 Statement of the Mayer–Vietoris sequence

Throughout this chapter, all homology groups are taken with coefficients in an abelian group G (typically $G = \mathbb{Z}$).

Theorem 7.1 (Mayer–Vietoris exact sequence). *Let X be a topological space and let $A, B \subseteq X$ be subspaces whose interiors cover X , i.e. $X = \mathring{A} \cup \mathring{B}$. Let $i_A: A \cap B \hookrightarrow A$, $i_B: A \cap B \hookrightarrow B$, $j_A: A \hookrightarrow X$, and $j_B: B \hookrightarrow X$ denote the inclusion maps. Then there is a long exact sequence*

$$\cdots \xrightarrow{\partial_{n+1}} H_n(A \cap B) \xrightarrow{\Phi_n} H_n(A) \oplus H_n(B) \xrightarrow{\Psi_n} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \xrightarrow{\Phi_{n-1}} \cdots$$

where

- $\Phi_n(\alpha) = ((i_A)_*(\alpha), -(i_B)_*(\alpha))$,
- $\Psi_n(\alpha, \beta) = (j_A)_*(\alpha) + (j_B)_*(\beta)$,
- ∂_n is the connecting homomorphism.

The sequence terminates at the right:

$$\cdots \rightarrow H_0(A \cap B) \xrightarrow{\Phi_0} H_0(A) \oplus H_0(B) \xrightarrow{\Psi_0} H_0(X) \rightarrow 0.$$

Remark 7.2 (The interior condition). The condition $X = \mathring{A} \cup \mathring{B}$ is essential. It is automatically satisfied when A and B are open subsets of X , or when X is a CW complex and A, B are subcomplexes with $A \cup B = X$. More generally, it suffices that the inclusion $C_\bullet(A) + C_\bullet(B) \hookrightarrow C_\bullet(X)$ induces an isomorphism on homology, which is

guaranteed by the excision theorem.

The Mayer–Vietoris sequence is often displayed as a diagram. The following commutative diagram illustrates the core short exact sequence of chain complexes from which it arises:

$$0 \longrightarrow C_\bullet(A \cap B) \xrightarrow{\Phi} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{\Psi} C_\bullet(A) + C_\bullet(B) \longrightarrow 0$$

7.2 Proof from excision

We now give the full proof of Theorem 7.1, relying on the excision theorem and the long exact sequence of a pair.

Proof of Theorem 7.1. Step 1: The short exact sequence of chain complexes.

Define chain maps

$$\begin{aligned} \Phi_\bullet: C_\bullet(A \cap B) &\longrightarrow C_\bullet(A) \oplus C_\bullet(B), & \sigma &\longmapsto ((i_A)_\#(\sigma), -(i_B)_\#(\sigma)), \\ \Psi_\bullet: C_\bullet(A) \oplus C_\bullet(B) &\longrightarrow C_\bullet(A) + C_\bullet(B), & (\sigma, \tau) &\longmapsto (j_A)_\#(\sigma) + (j_B)_\#(\tau). \end{aligned}$$

Here $C_\bullet(A) + C_\bullet(B)$ denotes the subcomplex of $C_\bullet(X)$ generated by singular simplices whose image lies in A or in B .

Φ_\bullet is injective. If $\Phi_\bullet(\sigma) = (0, 0)$, then $(i_A)_\#(\sigma) = 0$ and $(i_B)_\#(\sigma) = 0$. Since i_A and i_B are inclusions, $\sigma = 0$.

Ψ_\bullet is surjective. Every generator of $C_n(A) + C_n(B)$ is a singular simplex with image in A or in B . If $\sigma: \Delta^n \rightarrow A$, then $\Psi_n(\sigma, 0) = \sigma$; similarly for simplices in B .

$\text{im } \Phi_\bullet = \ker \Psi_\bullet$. We have $\Psi_\bullet \circ \Phi_\bullet(\sigma) = (j_A)_\# \circ (i_A)_\#(\sigma) - (j_B)_\# \circ (i_B)_\#(\sigma) = \sigma - \sigma = 0$, so $\text{im } \Phi_\bullet \subseteq \ker \Psi_\bullet$. Conversely, if $\Psi_n(\sigma, \tau) = 0$ in $C_n(A) + C_n(B)$, then $\sigma = -\tau$ in $C_n(X)$, so every simplex appearing in σ has image in both A and B , hence in $A \cap B$. Writing γ for this chain viewed in $C_n(A \cap B)$, we get $\Phi_n(\gamma) = (\sigma, -\tau) = (\sigma, \sigma) \dots$ but the sign gives $(\sigma, -(-\sigma))$; more carefully, $\sigma = (i_A)_\#(\gamma)$ and $-\tau = (i_B)_\#(\gamma)$ shows $(\sigma, \tau) = \Phi_n(\gamma)$.

Thus we have a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A \cap B) \xrightarrow{\Phi_\bullet} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{\Psi_\bullet} C_\bullet(A) + C_\bullet(B) \longrightarrow 0.$$

Step 2: Applying the snake lemma. By the fundamental theorem on short exact sequences of chain complexes (the zig-zag lemma), the short exact sequence above induces a long exact sequence in homology:

$$\cdots \rightarrow H_n(A \cap B) \xrightarrow{\Phi_*} H_n(A) \oplus H_n(B) \xrightarrow{\Psi_*} H_n(C_\bullet(A) + C_\bullet(B)) \xrightarrow{\partial} H_{n-1}(A \cap B) \rightarrow \cdots$$

Step 3: Identifying the middle term via excision. The inclusion $C_\bullet(A) + C_\bullet(B) \hookrightarrow C_\bullet(X)$ induces an isomorphism on homology. This is precisely the content of the theorem on small simplices: if $\mathcal{U} = \{A, B\}$ is an open cover of X (or more generally if $X = \mathring{A} \cup \mathring{B}$), then the inclusion $C_\bullet^\mathcal{U}(X) \hookrightarrow C_\bullet(X)$ is a chain homotopy equivalence, where $C_\bullet^\mathcal{U}(X)$ consists of chains whose simplices are \mathcal{U} -small. Since $C_\bullet^\mathcal{U}(X) = C_\bullet(A) + C_\bullet(B)$, we obtain

$$H_n(C_\bullet(A) + C_\bullet(B)) \cong H_n(X) \quad \text{for all } n \geq 0.$$

Substituting into the long exact sequence from Step 2 yields the Mayer–Vietoris exact sequence as stated.

Step 4: The connecting homomorphism. We describe $\partial_n: H_n(X) \rightarrow H_{n-1}(A \cap B)$ explicitly. Let $[z] \in H_n(X)$. Using the isomorphism from Step 3, represent z by a cycle $z = \alpha + \beta$ where $\alpha \in C_n(A)$ and $\beta \in C_n(B)$. Then $\partial\alpha + \partial\beta = \partial z = 0$, so $\partial\alpha = -\partial\beta$. Since $\partial\alpha \in C_{n-1}(A)$ and $-\partial\beta \in C_{n-1}(B)$, this common chain lies in $C_{n-1}(A \cap B)$. The connecting homomorphism is $\partial_n[z] = [\partial\alpha] = [-\partial\beta] \in H_{n-1}(A \cap B)$. \square

7.3 The reduced Mayer–Vietoris sequence

For computations it is often convenient to use reduced homology.

Theorem 7.3 (Reduced Mayer–Vietoris sequence). *Under the same hypotheses as Theorem 7.1, and assuming $A \cap B \neq \emptyset$, there is a long exact sequence of reduced homology groups:*

$$\cdots \rightarrow \tilde{H}_n(A \cap B) \xrightarrow{\Phi_n} \tilde{H}_n(A) \oplus \tilde{H}_n(B) \xrightarrow{\Psi_n} \tilde{H}_n(X) \xrightarrow{\partial_n} \tilde{H}_{n-1}(A \cap B) \rightarrow \cdots \rightarrow \tilde{H}_0(X) \rightarrow 0.$$

Proof. The augmentation maps $\varepsilon: C_0(\cdot) \rightarrow \mathbb{Z}$ are compatible with the inclusions, yielding a short exact sequence of augmented chain complexes. The argument proceeds identically to the unreduced case. The nonemptiness of $A \cap B$ ensures that all augmentation maps are surjective, so the short exact sequence at the augmented level remains exact. \square

Remark 7.4. When $A \cap B = \emptyset$, the space X is disconnected as $X = A \sqcup B$ (with A and B open), and the Mayer–Vietoris sequence degenerates: $\tilde{H}_n(X) \cong \tilde{H}_n(A) \oplus \tilde{H}_n(B)$ for all n .

7.4 Homology of spheres by induction

Theorem 7.5 (Homology of spheres). *For $n \geq 1$,*

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

Proof. We proceed by induction on n .

Base case $n = 1$. Decompose S^1 as $A = S^1 \setminus \{N\}$ and $B = S^1 \setminus \{S\}$, where $N = (0, 1)$ and $S = (0, -1)$ are the north and south poles. Then $A \simeq * \simeq B$ (both are contractible arcs), and $A \cap B$ is homotopy equivalent to a two-point discrete space $\{p, q\}$, so $\tilde{H}_0(A \cap B) \cong \mathbb{Z}$. The reduced Mayer–Vietoris sequence gives

$$\cdots \rightarrow \underbrace{\tilde{H}_1(A) \oplus \tilde{H}_1(B)}_{=0} \rightarrow \tilde{H}_1(S^1) \xrightarrow{\partial} \underbrace{\tilde{H}_0(A \cap B)}_{\cong \mathbb{Z}} \xrightarrow{\Phi_0} \underbrace{\tilde{H}_0(A) \oplus \tilde{H}_0(B)}_{=0} \rightarrow \cdots$$

Exactness forces $\partial: \tilde{H}_1(S^1) \xrightarrow{\sim} \mathbb{Z}$. For $k \geq 2$ the sequence gives $\tilde{H}_k(S^1) = 0$ since $\tilde{H}_k(A) = \tilde{H}_k(B) = \tilde{H}_{k-1}(A \cap B) = 0$.

Inductive step. Assume the result for S^{n-1} , $n \geq 2$. Write $S^n \subseteq \mathbb{R}^{n+1}$ and let

$$A = S^n \setminus \{N\}, \quad B = S^n \setminus \{S\},$$

where $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$. Both A and B are contractible (via stereographic projection onto \mathbb{R}^n), and $A \cap B = S^n \setminus \{N, S\} \simeq S^{n-1}$ (deformation retraction onto the equatorial sphere).

The reduced Mayer–Vietoris sequence reads

$$\tilde{H}_k(A) \oplus \tilde{H}_k(B) \rightarrow \tilde{H}_k(S^n) \xrightarrow{\partial} \tilde{H}_{k-1}(A \cap B) \rightarrow \tilde{H}_{k-1}(A) \oplus \tilde{H}_{k-1}(B).$$

Since $\tilde{H}_*(A) = \tilde{H}_*(B) = 0$, exactness gives isomorphisms

$$\tilde{H}_k(S^n) \xrightarrow[\cong]{\partial} \tilde{H}_{k-1}(S^{n-1}) \quad \text{for all } k.$$

By the induction hypothesis, $\tilde{H}_{k-1}(S^{n-1}) \cong \mathbb{Z}$ if $k-1 = n-1$ (i.e. $k = n$) and vanishes otherwise. The result follows. \square

7.5 Homology of surfaces

7.5.1 The torus

Example 7.6 (Homology of the torus T^2). Recall that $T^2 = S^1 \times S^1$. We compute its homology using Mayer–Vietoris. Think of T^2 as a square with opposite sides identified. Cut the torus along a longitudinal circle to obtain:

- A = an open neighborhood of the “left” cylinder,
- B = an open neighborhood of the “right” cylinder,

so that $A \simeq S^1 \simeq B$ (each piece deformation retracts onto a circle), and $A \cap B \simeq S^1 \sqcup S^1$. The reduced Mayer–Vietoris sequence in low degrees gives:

$$0 \rightarrow \tilde{H}_2(T^2) \xrightarrow{\partial} \tilde{H}_1(A \cap B) \xrightarrow{\Phi} \tilde{H}_1(A) \oplus \tilde{H}_1(B) \xrightarrow{\Psi} \tilde{H}_1(T^2) \xrightarrow{\partial} \tilde{H}_0(A \cap B) \xrightarrow{\Phi} \tilde{H}_0(A) \oplus \tilde{H}_0(B)$$

Substituting known values:

$$0 \rightarrow \tilde{H}_2(T^2) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Phi} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Psi} \tilde{H}_1(T^2) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\Phi} 0$$

The map $\Phi_0: \tilde{H}_0(A \cap B) \cong \mathbb{Z} \rightarrow \tilde{H}_0(A) \oplus \tilde{H}_0(B) = 0$ is the zero map, so exactness gives $\partial: \tilde{H}_1(T^2) \rightarrow \mathbb{Z}$.

For $\Phi_1: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$, one computes that it sends the generators of the two circle components of $A \cap B$ to the same generator of $\tilde{H}_1(A)$ and $\tilde{H}_1(B)$ respectively. An analysis of the inclusion maps shows Φ_1 has matrix $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, so $\ker \Phi_1 \cong \mathbb{Z}$ and $\text{im } \Phi_1 \cong \mathbb{Z}$.

From the exact sequence:

- $\tilde{H}_2(T^2) \cong \ker \Phi_1 \cong \mathbb{Z}$,
- $\tilde{H}_1(T^2) \cong (\mathbb{Z}^2 / \text{im } \Phi_1) \oplus \ker(\Phi_0) = \mathbb{Z} \oplus \mathbb{Z}$,
- $\tilde{H}_k(T^2) = 0$ for $k \geq 3$.

In summary:

$$H_k(T^2) \cong \begin{cases} \mathbb{Z} & k = 0, 2, \\ \mathbb{Z}^2 & k = 1, \\ 0 & k \geq 3. \end{cases}$$

7.5.2 Orientable surfaces of genus g

Theorem 7.7 (Homology of Σ_g). *Let Σ_g denote the closed orientable surface of genus $g \geq 0$. Then*

$$H_k(\Sigma_g) \cong \begin{cases} \mathbb{Z} & k = 0, 2, \\ \mathbb{Z}^{2g} & k = 1, \\ 0 & k \geq 3. \end{cases}$$

Proof. We use induction on g . The case $g = 0$ is $\Sigma_0 = S^2$, handled by Theorem 7.5. The case $g = 1$ is the torus, computed above. For $g \geq 2$, decompose Σ_g along a separating simple closed curve into a punctured Σ_{g-1} (call it A) and a punctured torus (call it B), with $A \cap B \simeq S^1$. Both A and B deformation retract onto wedges of circles: $A \simeq \vee^{2(g-1)} S^1$ and $B \simeq S^1 \vee S^1$. Applying Mayer–Vietoris and chasing the maps yields the result. Alternatively, one can use the CW structure and cellular homology to give a direct proof. \square

7.5.3 The real projective plane

Example 7.8 (Homology of \mathbb{RP}^2). Decompose \mathbb{RP}^2 into a Möbius band A and a disk B , glued along their boundaries. We have:

- A deformation retracts onto S^1 , so $\tilde{H}_1(A) \cong \mathbb{Z}$ and $\tilde{H}_k(A) = 0$ for $k \neq 1$.
- B is contractible: $\tilde{H}_k(B) = 0$ for all k .
- $A \cap B$ is an annulus, hence $A \cap B \simeq S^1$, so $\tilde{H}_1(A \cap B) \cong \mathbb{Z}$ and $\tilde{H}_0(A \cap B) = 0$.

The relevant piece of the reduced Mayer–Vietoris sequence is:

$$0 \longrightarrow \tilde{H}_2(\mathbb{RP}^2) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\Phi_1} \mathbb{Z} \oplus 0 \xrightarrow{\Psi_1} \tilde{H}_1(\mathbb{RP}^2) \longrightarrow 0$$

The inclusion of the boundary circle of the Möbius band into the Möbius band wraps around twice (the boundary of a Möbius band goes around the core circle twice). Thus Φ_1 is multiplication by 2. Therefore:

- $\tilde{H}_2(\mathbb{RP}^2) = \ker(\times 2) = 0$,
- $\tilde{H}_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$,
- $\tilde{H}_k(\mathbb{RP}^2) = 0$ for $k \geq 3$.

In summary:

$$H_k(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2\mathbb{Z} & k = 1, \\ 0 & k \geq 2. \end{cases}$$

7.5.4 The Klein bottle

Example 7.9 (Homology of the Klein bottle K). The Klein bottle can be decomposed as the union of two Möbius bands A and B glued along their common boundary circle, so $A \cap B \simeq S^1$. Each Möbius band deformation retracts onto S^1 , so $\tilde{H}_1(A) \cong \mathbb{Z} \cong \tilde{H}_1(B)$. The boundary circle of a Möbius band wraps twice around the core circle, so the inclusion-induced maps $\tilde{H}_1(A \cap B) \rightarrow \tilde{H}_1(A)$ and $\tilde{H}_1(A \cap B) \rightarrow \tilde{H}_1(B)$ are both multiplication by 2. The map $\Phi_1: \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ is therefore $\Phi_1(1) = (2, -2)$. The reduced Mayer–Vietoris sequence gives:

$$0 \rightarrow \tilde{H}_2(K) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\Phi_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Psi_1} \tilde{H}_1(K) \rightarrow 0$$

Since Φ_1 is injective ($\ker \Phi_1 = 0$), we get $\tilde{H}_2(K) = 0$. For $\tilde{H}_1(K)$, we compute $\text{coker } \Phi_1 = (\mathbb{Z} \oplus \mathbb{Z}) / \langle (2, -2) \rangle \cong \mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z}$. (One can verify this by the change of basis $(a, b) \mapsto (a + b, a - b)$.) Thus:

$$H_k(K) \cong \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z} & k = 1, \\ 0 & k \geq 2. \end{cases}$$

Note that $H_2(K) = 0$, reflecting the fact that the Klein bottle is non-orientable.

7.6 Homology of wedge sums

The Mayer–Vietoris sequence provides a quick proof of the following useful result.

Proposition 7.10 (Homology of a wedge sum). *Let X and Y be path-connected spaces with basepoints $x_0 \in X$ and $y_0 \in Y$. Then for all $n \geq 1$,*

$$\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y).$$

Proof. Choose open neighborhoods U of x_0 in X and V of y_0 in Y , both contractible. Let $A = X \cup V \subseteq X \vee Y$ and $B = U \cup Y \subseteq X \vee Y$ (where we identify the basepoints). Then $A \simeq X$, $B \simeq Y$, and $A \cap B = U \cup V \simeq *$ (contractible). The reduced Mayer–Vietoris sequence yields

$$\cdots \rightarrow \underbrace{\tilde{H}_n(A \cap B)}_{=0} \rightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \xrightarrow{\Psi} \tilde{H}_n(X \vee Y) \xrightarrow{\partial} \underbrace{\tilde{H}_{n-1}(A \cap B)}_{=0} \rightarrow \cdots$$

so Ψ is an isomorphism, giving $\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$. \square

Corollary 7.11. *For a finite wedge sum $\bigvee_{i=1}^k X_i$ of path-connected spaces,*

$$\tilde{H}_n\left(\bigvee_{i=1}^k X_i\right) \cong \bigoplus_{i=1}^k \tilde{H}_n(X_i) \quad \text{for all } n \geq 1.$$

In particular, $\tilde{H}_n(\bigvee^k S^m) \cong \mathbb{Z}^k$ for $n = m$ and vanishes for $n \neq m$.

7.7 The Künneth formula (simplified)

The Mayer–Vietoris sequence, combined with induction, leads to the computation of the homology of product spaces. The Künneth formula expresses the homology of a product $X \times Y$ in terms of the homology of the factors.

Theorem 7.12 (Künneth formula, free case). *Let X and Y be topological spaces. If $H_n(Y; \mathbb{Z})$ is a free abelian group for all n , then there is a natural isomorphism*

$$H_n(X \times Y; \mathbb{Z}) \cong \bigoplus_{p+q=n} H_p(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H_q(Y; \mathbb{Z}).$$

The isomorphism is realized by the *cross product* (or exterior product):

$$\times : H_p(X; \mathbb{Z}) \otimes H_q(Y; \mathbb{Z}) \longrightarrow H_{p+q}(X \times Y; \mathbb{Z}), \quad [\alpha] \otimes [\beta] \longmapsto [\alpha \times \beta],$$

constructed via the Eilenberg–Zilber map.

Example 7.13 (Homology of T^2 via Künneth). Using the Künneth formula, the homology of $T^2 = S^1 \times S^1$ is

$$\begin{aligned} H_0(T^2) &\cong H_0(S^1) \otimes H_0(S^1) \cong \mathbb{Z}, \\ H_1(T^2) &\cong (H_0(S^1) \otimes H_1(S^1)) \oplus (H_1(S^1) \otimes H_0(S^1)) \cong \mathbb{Z} \oplus \mathbb{Z}, \\ H_2(T^2) &\cong H_1(S^1) \otimes H_1(S^1) \cong \mathbb{Z}, \end{aligned}$$

agreeing with our Mayer–Vietoris computation.

Example 7.14 (Homology of $S^m \times S^n$). For $m, n \geq 1$ with $m \neq n$, the Künneth formula gives

$$H_k(S^m \times S^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, m, n, m+n, \\ 0 & \text{otherwise.} \end{cases}$$

When $m = n$, $H_m(S^m \times S^m; \mathbb{Z}) \cong \mathbb{Z}^2$. This shows, for example, that $S^m \times S^n \not\cong S^{m+n}$ for $m, n \geq 1$, since their homology groups differ.

Remark 7.15 (The general Künneth formula). When the homology groups have torsion, the full Künneth formula includes a Tor correction term:

$$0 \rightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \rightarrow H_n(X \times Y) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)) \rightarrow 0.$$

This short exact sequence splits (non-naturally). Key properties of Tor that are useful in computations:

- (i) $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\text{gcd}(m, n)\mathbb{Z}$.
- (ii) $\text{Tor}_1^{\mathbb{Z}}(A, B) = 0$ if A or B is free.
- (iii) Tor commutes with finite direct sums.

Example 7.16 (Homology of $\mathbb{R}P^2 \times \mathbb{R}P^2$). Since $H_0(\mathbb{R}P^2) = \mathbb{Z}$, $H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$, and $H_k(\mathbb{R}P^2) = 0$ for $k \geq 2$, the Künneth formula gives for $X = \mathbb{R}P^2 \times \mathbb{R}P^2$:

$$\begin{aligned} H_0(X) &\cong \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}, \\ H_1(X) &\cong (\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2, \\ H_2(X) &\cong (\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}) \oplus \text{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \oplus \text{Tor}_1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus 0 \oplus 0 = \mathbb{Z}/2\mathbb{Z}, \\ H_3(X) &\cong \text{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \\ H_k(X) &= 0 \quad \text{for } k \geq 4. \end{aligned}$$

7.8 The Euler characteristic

Definition 7.17 (Euler characteristic). Let X be a topological space with finitely generated homology groups, all but finitely many of which vanish. The *Euler characteristic* of X is

$$\chi(X) = \sum_{n \geq 0} (-1)^n \text{rank } H_n(X; \mathbb{Z}).$$

Proposition 7.18 (Additivity of χ under Mayer–Vietoris). *If $X = A \cup B$ with $A \cap B \neq \emptyset$ and all spaces have well-defined Euler characteristics, then*

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Proof. This follows from the exactness of the Mayer–Vietoris sequence and the general fact that in any long exact sequence of finitely generated abelian groups $\cdots \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow A_{n-1} \rightarrow \cdots$, the alternating sum of ranks vanishes. \square

Example 7.19 (Euler characteristics of familiar spaces). (i) $\chi(S^n) = 1 + (-1)^n$, so $\chi(S^{2k}) = 2$ and $\chi(S^{2k+1}) = 0$.

(ii) $\chi(T^2) = 1 - 2 + 1 = 0$.

(iii) $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$.

(iv) $\chi(\mathbb{R}P^2) = 1 - 0 + 0 = 1$ (using ranks of the free parts only).

(v) $\chi(K) = 1 - 1 + 0 = 0$ for the Klein bottle K .

Proposition 7.20 (Multiplicativity of χ). *If X and Y have well-defined Euler characteristics, then*

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

Proof. This follows from the Künneth formula. If $h_k = \text{rank } H_k(X)$ and $h'_k = \text{rank } H_k(Y)$,

then

$$\chi(X \times Y) = \sum_n (-1)^n \sum_{p+q=n} h_p h'_q = \left(\sum_p (-1)^p h_p \right) \left(\sum_q (-1)^q h'_q \right) = \chi(X) \cdot \chi(Y). \quad \square$$

Example 7.21. $\chi(T^n) = \chi(S^1)^n = 0^n = 0$ for all $n \geq 1$. Also $\chi(S^2 \times S^2) = \chi(S^2)^2 = 4$.

7.9 Summary of homology computations

We collect the results of this chapter in a single table for reference.

Space	H_0	H_1	H_2	H_3	χ
S^n ($n \geq 1$)	\mathbb{Z}	0 ($n > 1$)	\cdots	\mathbb{Z} in deg. n	$1 + (-1)^n$
T^2	\mathbb{Z}	\mathbb{Z}^2	\mathbb{Z}	0	0
Σ_g	\mathbb{Z}	\mathbb{Z}^{2g}	\mathbb{Z}	0	$2 - 2g$
\mathbb{RP}^2	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$	0	0	1
Klein bottle	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	0	0	0
$S^1 \vee S^1$	\mathbb{Z}	\mathbb{Z}^2	0	0	-1

Remark 7.22. Observe that $H_2(\Sigma_g) \cong \mathbb{Z}$ for all orientable closed surfaces, reflecting their orientability. The non-orientable surfaces \mathbb{RP}^2 and the Klein bottle have $H_2 = 0$. More generally, a closed connected n -manifold M satisfies $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$ if and only if M is orientable.

7.10 Exercises

Exercise 7.1. Use the Mayer–Vietoris sequence to compute the homology groups of $S^1 \vee S^1$ (the wedge of two circles). *Hint:* Take A and B to be open neighborhoods of each circle, with $A \cap B$ contractible.

Exercise 7.2. Compute the homology of $S^1 \vee S^2$ using Mayer–Vietoris.

Exercise 7.3. Let $X = S^2 \vee S^2$. Compute $H_*(X; \mathbb{Z})$ using Mayer–Vietoris. Then verify your answer using the general formula for the homology of a wedge sum.

Exercise 7.4. Use the Mayer–Vietoris sequence to show that for any $n \geq 1$, the sphere S^n is not a retract of D^{n+1} . Deduce the Brouwer fixed point theorem in all dimensions.

Exercise 7.5. Let X be the space obtained from S^2 by identifying the north and south poles. Compute $H_*(X; \mathbb{Z})$ by decomposing X appropriately and applying Mayer–Vietoris.

Exercise 7.6. Compute the homology of the genus-2 surface Σ_2 by decomposing it as the connected sum $\Sigma_2 = T^2 \# T^2$ and applying Mayer–Vietoris. Verify that your result agrees with Theorem 7.7.

Exercise 7.7. Let M be the Möbius band and let $\partial M \cong S^1$ be its boundary. Compute the relative homology groups $H_*(M, \partial M; \mathbb{Z})$ using the long exact sequence of the pair and your knowledge of $H_*(M)$ and $H_*(\partial M)$.

Exercise 7.8. Prove that if $X = A \cup B$ with A , B , and $A \cap B$ all path-connected, then the Mayer–Vietoris connecting homomorphism $\partial_1: H_1(X) \rightarrow H_0(A \cap B)$ is trivial.

Exercise 7.9. Show that the n -fold torus $T^n = (S^1)^n$ has homology

$$H_k(T^n; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{k}}.$$

Hint: Use the Künneth formula and induction on n .

Exercise 7.10. Let $X = \mathbb{R}P^2 \times \mathbb{R}P^2$. Use the Künneth formula (with the Tor term) to compute $H_*(X; \mathbb{Z})$.

Chapter 8

Singular Cohomology and the Cup Product

Cohomology is the “dual” of homology and carries richer algebraic structure: the cup product endows the cohomology groups with the structure of a graded ring. This additional structure often distinguishes spaces that have isomorphic homology groups, making cohomology a strictly finer invariant.

8.1 Cochains and the coboundary operator

Definition 8.1 (Singular cochains). Let X be a topological space and G an abelian group. The *group of singular n -cochains* is

$$C^n(X; G) = \text{Hom}_{\mathbb{Z}}(C_n(X; \mathbb{Z}), G),$$

the group of all homomorphisms from the free abelian group of singular n -chains to G . An element $\varphi \in C^n(X; G)$ assigns to each singular n -simplex $\sigma: \Delta^n \rightarrow X$ a value $\varphi(\sigma) \in G$.

Definition 8.2 (Coboundary operator). The *coboundary operator* $\delta^n: C^n(X; G) \rightarrow C^{n+1}(X; G)$ is the dual of the boundary map: for $\varphi \in C^n(X; G)$ and $\sigma: \Delta^{n+1} \rightarrow X$,

$$(\delta^n \varphi)(\sigma) = \varphi(\partial_{n+1} \sigma) = \sum_{i=0}^{n+1} (-1)^i \varphi(\sigma \circ d_i),$$

where $d_i: \Delta^n \rightarrow \Delta^{n+1}$ is the i -th face inclusion.

Lemma 8.3. $\delta^{n+1} \circ \delta^n = 0$ for all $n \geq 0$.

Proof. For any $\varphi \in C^n(X; G)$ and $(n+2)$ -simplex σ ,

$$(\delta^{n+1} \delta^n \varphi)(\sigma) = (\delta^n \varphi)(\partial_{n+2} \sigma) = \varphi(\partial_{n+1} \partial_{n+2} \sigma) = \varphi(0) = 0,$$

since $\partial^2 = 0$. □

Thus $(C^\bullet(X; G), \delta)$ is a cochain complex:

$$C^0(X; G) \xrightarrow{\delta^0} C^1(X; G) \xrightarrow{\delta^1} C^2(X; G) \xrightarrow{\delta^2} \dots$$

8.2 Singular cohomology groups

Definition 8.4 (Cohomology groups). The n -th *singular cohomology group* of X with coefficients in G is

$$H^n(X; G) = \frac{\ker(\delta^n: C^n(X; G) \rightarrow C^{n+1}(X; G))}{\operatorname{im}(\delta^{n-1}: C^{n-1}(X; G) \rightarrow C^n(X; G))} = \frac{Z^n(X; G)}{B^n(X; G)},$$

where $Z^n(X; G) = \ker \delta^n$ is the group of n -cocycles and $B^n(X; G) = \operatorname{im} \delta^{n-1}$ is the group of n -coboundaries.

Remark 8.5. A continuous map $f: X \rightarrow Y$ induces a chain map $f_\#: C_\bullet(X) \rightarrow C_\bullet(Y)$, which dualizes to a cochain map $f^\#: C^\bullet(Y; G) \rightarrow C^\bullet(X; G)$ (note the reversal of direction). This induces $f^*: H^n(Y; G) \rightarrow H^n(X; G)$. Cohomology is thus a *contravariant* functor.

Example 8.6 (Degree zero). $H^0(X; G) \cong \prod_\alpha G$, where the product runs over the path components of X . If X is path-connected, $H^0(X; G) \cong G$. Indeed, a 0-cocycle $\varphi \in Z^0(X; G)$ satisfies $(\delta\varphi)(\sigma) = \varphi(\sigma(1)) - \varphi(\sigma(0)) = 0$ for every path σ , so φ is constant on path components.

Proposition 8.7 (Homotopy invariance of cohomology). *If $f, g: X \rightarrow Y$ are homotopic, then $f^* = g^*: H^n(Y; G) \rightarrow H^n(X; G)$ for all n . In particular, if $X \simeq Y$, then $H^n(X; G) \cong H^n(Y; G)$.*

Proof. If $F: X \times [0, 1] \rightarrow Y$ is a homotopy from f to g , then the prism operator $P: C_n(X) \rightarrow C_{n+1}(Y)$ satisfying $\partial P + P\partial = g_\# - f_\#$ dualizes to a cochain homotopy $P^*: C^{n+1}(Y; G) \rightarrow C^n(X; G)$ satisfying $P^*\delta + \delta P^* = g^\# - f^\#$. Hence $g^\# - f^\#$ sends cocycles to coboundaries, so $f^* = g^*$. \square

8.3 The Universal Coefficient Theorem

The following theorem relates cohomology to homology.

Theorem 8.8 (Universal Coefficient Theorem for cohomology). *Let X be a topological space and G an abelian group. There is a natural short exact sequence*

$$0 \rightarrow \operatorname{Ext}_{\mathbb{Z}}^1(H_{n-1}(X; \mathbb{Z}), G) \rightarrow H^n(X; G) \rightarrow \operatorname{Hom}_{\mathbb{Z}}(H_n(X; \mathbb{Z}), G) \rightarrow 0.$$

This sequence splits (non-naturally), so

$$H^n(X; G) \cong \text{Hom}(H_n(X; \mathbb{Z}), G) \oplus \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(X; \mathbb{Z}), G).$$

Remark 8.9 (Key special cases). (i) If $H_*(X; \mathbb{Z})$ is free abelian in every degree, then $\text{Ext}^1 = 0$ and $H^n(X; G) \cong \text{Hom}(H_n(X; \mathbb{Z}), G)$.

(ii) $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z}, G) \cong G/mG$.

(iii) $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, G) = 0$.

(iv) $\text{Ext}_{\mathbb{Z}}^1$ commutes with finite direct sums.

Proof sketch. Since $C_n(X; \mathbb{Z})$ is free abelian (it is a free abelian group with basis the set of singular n -simplices), one constructs a free resolution of each $H_n(X; \mathbb{Z})$ from the chain complex. The key algebraic input is the following: given a short exact sequence of chain complexes $0 \rightarrow Z_{\bullet} \rightarrow C_{\bullet} \rightarrow B_{\bullet-1} \rightarrow 0$ (where $Z_n = \ker \partial_n$ and $B_n = \text{im } \partial_{n+1}$), applying $\text{Hom}(-, G)$ and taking cohomology yields the desired short exact sequence. The Ext^1 term arises because $\text{Hom}(-, G)$ is only left exact, not exact, so its first right derived functor appears. \square

Example 8.10 (UCT for \mathbb{RP}^2). For \mathbb{RP}^2 with $G = \mathbb{Z}$: since $H_0(\mathbb{RP}^2) = \mathbb{Z}$, $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, $H_k(\mathbb{RP}^2) = 0$ for $k \geq 2$, the UCT gives

$$H^0(\mathbb{RP}^2; \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus 0 = \mathbb{Z},$$

$$H^1(\mathbb{RP}^2; \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \oplus \text{Ext}^1(\mathbb{Z}, \mathbb{Z}) = 0 \oplus 0 = 0,$$

$$H^2(\mathbb{RP}^2; \mathbb{Z}) \cong \text{Hom}(0, \mathbb{Z}) \oplus \text{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0 \oplus \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}.$$

Note the “shift” of torsion from H_1 to H^2 .

Example 8.11 (UCT with $\mathbb{Z}/2\mathbb{Z}$ coefficients). For $X = S^n$ with $G = \mathbb{Z}/2\mathbb{Z}$:

$$\begin{aligned} H^k(S^n; \mathbb{Z}/2\mathbb{Z}) &\cong \text{Hom}(H_k(S^n), \mathbb{Z}/2\mathbb{Z}) \oplus \text{Ext}^1(H_{k-1}(S^n), \mathbb{Z}/2\mathbb{Z}) \\ &\cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & k = 0, n, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since $\text{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\text{Ext}^1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$.

Example 8.12 (UCT for the Klein bottle). The Klein bottle K has $H_0(K) = \mathbb{Z}$, $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H_k(K) = 0$ for $k \geq 2$. The UCT with $G = \mathbb{Z}$ gives:

$$H^0(K; \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z},$$

$$H^1(K; \mathbb{Z}) \cong \text{Hom}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \oplus \text{Ext}^1(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \oplus 0 = \mathbb{Z},$$

$$H^2(K; \mathbb{Z}) \cong \text{Hom}(0, \mathbb{Z}) \oplus \text{Ext}^1(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0 \oplus \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}.$$

Comparing with homology: $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ but $H^1(K; \mathbb{Z}) \cong \mathbb{Z}$. The torsion in

H_1 “migrates” to H^2 via Ext^1 .

8.4 The cup product

The most important additional structure on cohomology is the cup product, which makes $H^*(X; R)$ into a graded ring when R is a commutative ring.

Definition 8.13 (Cup product on cochains). Let R be a commutative ring with unity. For $\varphi \in C^p(X; R)$ and $\psi \in C^q(X; R)$, the *cup product* $\varphi \smile \psi \in C^{p+q}(X; R)$ is defined by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q}])$$

for every singular $(p+q)$ -simplex $\sigma: \Delta^{p+q} \rightarrow X$. Here $[v_0, \dots, v_p]$ denotes the front p -face and $[v_p, \dots, v_{p+q}]$ the back q -face of Δ^{p+q} .

Lemma 8.14 (Leibniz rule). For $\varphi \in C^p(X; R)$ and $\psi \in C^q(X; R)$,

$$\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^p \varphi \smile (\delta\psi).$$

Proof. This is a direct (though somewhat lengthy) computation from the definitions. One expands both sides using the formula for δ and observes that the terms cancel in pairs except for those contributing to the stated formula. The key identity is that the face maps satisfy appropriate composition relations with the front and back face inclusions. \square

Corollary 8.15. *The cup product on cochains descends to cohomology: if $[\varphi] \in H^p(X; R)$ and $[\psi] \in H^q(X; R)$, then $[\varphi] \smile [\psi] = [\varphi \smile \psi] \in H^{p+q}(X; R)$ is well defined.*

Proof. If φ and ψ are cocycles ($\delta\varphi = 0 = \delta\psi$), the Leibniz rule gives $\delta(\varphi \smile \psi) = 0$, so $\varphi \smile \psi$ is a cocycle. If $\varphi = \delta\alpha$ is a coboundary, then $\varphi \smile \psi = \delta(\alpha \smile \psi)$ by the Leibniz rule (since $\delta\psi = 0$), so the product is a coboundary. Similarly if ψ is a coboundary. \square

8.5 Properties of the cup product

Theorem 8.16 (Properties of the cup product). Let R be a commutative ring with unity, and let X be a topological space. The cup product on $H^*(X; R) = \bigoplus_{n \geq 0} H^n(X; R)$ satisfies:

- (i) **Associativity:** $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$ for all $\alpha \in H^p$, $\beta \in H^q$, $\gamma \in H^r$.
- (ii) **Unit:** The element $1 \in H^0(X; R) \cong R$ (the cocycle sending every 0-simplex to 1_R) satisfies $1 \smile \alpha = \alpha = \alpha \smile 1$ for all α .
- (iii) **Graded commutativity:** $\alpha \smile \beta = (-1)^{pq} \beta \smile \alpha$ for $\alpha \in H^p(X; R)$ and $\beta \in H^q(X; R)$.

(iv) **Bilinearity:** The cup product is R -bilinear: $(\alpha + \alpha') \smile \beta = \alpha \smile \beta + \alpha' \smile \beta$ and $\alpha \smile (\beta + \beta') = \alpha \smile \beta + \alpha \smile \beta'$.

(v) **Naturality:** If $f: X \rightarrow Y$ is continuous, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.

Proof sketch. (i) and (ii) follow directly from the cochain-level definitions. (iv) is immediate from linearity. (v) follows from the fact that $f_{\#}$ preserves front and back faces.

The proof of (iii) is the most substantial. It does *not* hold at the cochain level; rather, one constructs an explicit chain homotopy showing that $\varphi \smile \psi - (-1)^{pq} \psi \smile \varphi$ is a coboundary whenever φ and ψ are cocycles. This uses the *Alexander–Whitney* and *Eilenberg–Zilber* maps and the acyclic models theorem. \square

Example 8.17 (Cup product structure on a point). For $X = \{*\}$, $H^0(\{*\}; R) \cong R$ and $H^n(\{*\}; R) = 0$ for $n > 0$. The cohomology ring is simply $H^*(\{*\}; R) \cong R$, concentrated in degree zero.

Example 8.18 (Cup product on $H^*(S^1; \mathbb{Z})$). We have $H^0(S^1; \mathbb{Z}) \cong \mathbb{Z}$ with generator 1, and $H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ with generator α . Since $H^2(S^1; \mathbb{Z}) = 0$, the product $\alpha \smile \alpha = 0$. The full ring is $H^*(S^1; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$ with $|\alpha| = 1$.

8.6 The cohomology ring

Definition 8.19 (Cohomology ring). Let R be a commutative ring. The *cohomology ring* of X is the graded R -algebra

$$H^*(X; R) = \bigoplus_{n \geq 0} H^n(X; R),$$

with multiplication given by the cup product. By Theorem 8.16, $H^*(X; R)$ is a graded-commutative, associative, unital R -algebra.

Remark 8.20. The cohomology ring is a strictly finer invariant than the cohomology groups alone. For example, $\mathbb{C}P^2$ and $S^2 \vee S^4$ have isomorphic integral cohomology groups:

$$H^n(\mathbb{C}P^2; \mathbb{Z}) \cong H^n(S^2 \vee S^4; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, 2, 4, \\ 0 & \text{otherwise.} \end{cases}$$

However, their cohomology rings differ: in $H^*(\mathbb{C}P^2; \mathbb{Z})$, the square of the degree-2 generator is the degree-4 generator, whereas in $H^*(S^2 \vee S^4; \mathbb{Z})$, all cup products in positive degrees vanish.

8.7 Computations of cohomology rings

8.7.1 Cohomology ring of S^n

Proposition 8.21 (Cohomology ring of S^n). For $n \geq 1$,

$$H^*(S^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2), \quad |\alpha| = n,$$

where $|\alpha|$ denotes the degree of α . That is, $H^*(S^n; \mathbb{Z})$ is the exterior algebra on one generator of degree n .

Proof. We have $H^k(S^n; \mathbb{Z}) \cong \mathbb{Z}$ for $k = 0, n$ and vanishes otherwise. Let $1 \in H^0$ be the unit and $\alpha \in H^n$ a generator. The only potentially nontrivial cup product is $\alpha \smile \alpha \in H^{2n}(S^n; \mathbb{Z})$. Since $H^{2n}(S^n; \mathbb{Z}) = 0$ for $n \geq 1$, we must have $\alpha^2 = 0$. \square

8.7.2 Cohomology ring of the torus T^2

Theorem 8.22 (Cohomology ring of T^2).

$$H^*(T^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2, \alpha\beta + \beta\alpha), \quad |\alpha| = |\beta| = 1.$$

Equivalently, $H^*(T^2; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(\alpha, \beta)$, the exterior algebra on two generators of degree 1.

Proof. Since $T^2 = S^1 \times S^1$, we can use the Künneth formula for cohomology (Theorem 8.29 below). The cohomology groups are:

$$H^0(T^2; \mathbb{Z}) \cong \mathbb{Z}, \quad H^1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2, \quad H^2(T^2; \mathbb{Z}) \cong \mathbb{Z}.$$

Let $\alpha, \beta \in H^1(T^2; \mathbb{Z})$ be generators corresponding to the two S^1 factors. By graded commutativity, $\alpha \smile \beta = -\beta \smile \alpha$, and $\alpha \smile \alpha = -\alpha \smile \alpha$ forces $\alpha^2 = 0$ (and similarly $\beta^2 = 0$).

To see that $\alpha \smile \beta$ generates $H^2(T^2; \mathbb{Z}) \cong \mathbb{Z}$: by the Künneth formula for cohomology rings of products, the cross product $\alpha \times \beta$ corresponds to $\alpha \smile \beta$ under the identification $T^2 = S^1 \times S^1$, and this is a generator of H^2 . \square

8.7.3 Cohomology ring of \mathbb{RP}^n

Theorem 8.23 (Cohomology ring of \mathbb{RP}^n with $\mathbb{Z}/2\mathbb{Z}$ coefficients).

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^{n+1}), \quad |\alpha| = 1.$$

That is, $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$ is the truncated polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ on a single generator α of degree 1.

Proof sketch. The cohomology groups are $H^k(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for $0 \leq k \leq n$ and vanish for $k > n$. Let $\alpha \in H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$ be the unique nonzero element.

The key step is to show that $\alpha^k \neq 0$ in $H^k(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ for all $1 \leq k \leq n$. Since $H^k(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, any nonzero element is the unique generator.

This is proved by induction on n using the long exact sequence in cohomology associated to the pair $(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ and naturality of the cup product with respect to the inclusion $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n$. The crucial geometric input is that $\mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n$, which identifies the relevant maps. \square

Remark 8.24. The generator $\alpha \in H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ is the first Stiefel–Whitney class w_1 of the tautological line bundle over $\mathbb{R}P^n$. The ring structure of $H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ plays a fundamental role in the theory of characteristic classes and in the proof of the Borsuk–Ulam theorem via cohomological methods.

Corollary 8.25. *The cohomology ring of $\mathbb{R}P^n$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients places constraints on immersions and embeddings. For instance, one can use it to show that $\mathbb{R}P^n$ cannot be immersed in \mathbb{R}^m for certain small values of m , via characteristic class arguments.*

8.7.4 Cohomology ring of $\mathbb{C}P^n$

Theorem 8.26 (Cohomology ring of $\mathbb{C}P^n$).

$$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^{n+1}), \quad |\beta| = 2.$$

That is, $H^*(\mathbb{C}P^n; \mathbb{Z})$ is the truncated polynomial algebra over \mathbb{Z} on a single generator β of degree 2.

Proof sketch. The integral cohomology groups are $H^{2k}(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$ for $0 \leq k \leq n$ and $H^{\text{odd}}(\mathbb{C}P^n; \mathbb{Z}) = 0$. Let $\beta \in H^2(\mathbb{C}P^n; \mathbb{Z})$ be a generator.

The proof that β^k generates $H^{2k}(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$ for all $1 \leq k \leq n$ proceeds by induction on n . One uses the cofiber sequence $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n \rightarrow \mathbb{C}P^n/\mathbb{C}P^{n-1} \cong S^{2n}$ and the associated long exact sequence in cohomology. The inductive step shows that the restriction map $H^{2k}(\mathbb{C}P^n; \mathbb{Z}) \rightarrow H^{2k}(\mathbb{C}P^{n-1}; \mathbb{Z})$ is an isomorphism for $k < n$, and one verifies that β^n generates H^{2n} by analyzing the Gysin sequence of the Hopf fibration $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$. \square

Remark 8.27. The generator $\beta \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is the first Chern class c_1 of the tautological line bundle over $\mathbb{C}P^n$. Taking the limit $n \rightarrow \infty$, one obtains $H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\beta]$, an honest polynomial ring. The space $\mathbb{C}P^\infty$ is the classifying space $BU(1)$ for complex line bundles.

Remark 8.28 (Comparison of $\mathbb{R}P^n$ and $\mathbb{C}P^n$). The following table highlights the structural analogy:

	$\mathbb{R}P^n$	$\mathbb{C}P^n$
Coefficient ring	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{Z}
Generator degree	1	2
Truncation	$\alpha^{n+1} = 0$	$\beta^{n+1} = 0$
Ring	$(\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^{n+1})$	$\mathbb{Z}[\beta]/(\beta^{n+1})$
Dimension	n	$2n$
Bundle interpretation	$w_1(\gamma^1)$	$c_1(\gamma^1)$

Both are truncated polynomial algebras, but the parity and coefficient choices differ in ways that reflect the real vs. complex geometry.

8.8 The Künneth formula in cohomology

Theorem 8.29 (Künneth formula for cohomology). *Let R be a principal ideal domain and let X, Y be topological spaces such that $H^*(Y; R)$ is a finitely generated free R -module in each degree. Then there is a natural isomorphism of graded R -algebras*

$$H^*(X \times Y; R) \cong H^*(X; R) \otimes_R H^*(Y; R),$$

where the right-hand side has the product

$$(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') = (-1)^{|\beta||\alpha'|} (\alpha \smile \alpha') \otimes (\beta \smile \beta').$$

Remark 8.30. The sign $(-1)^{|\beta||\alpha'|}$ is the Koszul sign rule: whenever two elements of degrees p and q are transposed, a sign $(-1)^{pq}$ is introduced.

Example 8.31 (Cohomology ring of T^n). Applying the Künneth formula iteratively to $T^n = (S^1)^n$:

$$H^*(T^n; R) \cong H^*(S^1; R)^{\otimes n} \cong \bigwedge_R(\alpha_1, \dots, \alpha_n), \quad |\alpha_i| = 1,$$

the exterior algebra on n generators of degree 1. In particular, $H^k(T^n; R) \cong R^{\binom{n}{k}}$.

Example 8.32 (Cohomology ring of $S^m \times S^n$). For $m, n \geq 1$,

$$H^*(S^m \times S^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2), \quad |\alpha| = m, |\beta| = n,$$

with the relation $\alpha \smile \beta = (-1)^{mn} \beta \smile \alpha$.

8.9 Applications of the cohomology ring

8.9.1 Distinguishing spaces with the same cohomology groups

Example 8.33 ($\mathbb{C}P^2$ vs. $S^2 \vee S^4$). As mentioned in Section 8.6, both spaces have

$$H^k \cong \begin{cases} \mathbb{Z} & k = 0, 2, 4, \\ 0 & \text{otherwise.} \end{cases}$$

In $H^*(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^3)$, the generator $\beta \in H^2$ satisfies $\beta^2 \neq 0$ (it generates H^4). In $H^*(S^2 \vee S^4; \mathbb{Z})$, let $\gamma \in H^2$ and $\mu \in H^4$ be generators. Since the inclusion $S^2 \hookrightarrow S^2 \vee S^4$ factors through the wedge, the cup product γ^2 is computed in $H^4(S^2; \mathbb{Z}) = 0$ and then pushed forward, giving $\gamma^2 = 0$. Therefore $H^*(\mathbb{C}P^2; \mathbb{Z}) \not\cong H^*(S^2 \vee S^4; \mathbb{Z})$ as graded rings, so $\mathbb{C}P^2 \not\cong S^2 \vee S^4$.

8.9.2 The Hopf invariant

Definition 8.34 (Hopf invariant). Let $f: S^{2n-1} \rightarrow S^n$ be a continuous map, $n \geq 2$. Form the mapping cone $C_f = S^n \cup_f D^{2n}$. Then $H^k(C_f; \mathbb{Z}) \cong \mathbb{Z}$ for $k = 0, n, 2n$ and vanishes otherwise. Let $\alpha \in H^n(C_f; \mathbb{Z})$ and $\beta \in H^{2n}(C_f; \mathbb{Z})$ be generators. The *Hopf invariant* of f is the integer $h(f)$ defined by

$$\alpha \smile \alpha = h(f) \cdot \beta.$$

Remark 8.35. The Hopf invariant is a homotopy invariant of f . The classical Hopf map $\eta: S^3 \rightarrow S^2$ has Hopf invariant $h(\eta) = 1$. The theorem of Adams states that a map $S^{2n-1} \rightarrow S^n$ of Hopf invariant ± 1 exists if and only if $n \in \{1, 2, 4, 8\}$, corresponding to the real numbers, complex numbers, quaternions, and octonions.

8.10 Connection to de Rham cohomology

When $X = M$ is a smooth manifold, there is an alternative definition of cohomology using differential forms.

Definition 8.36 (de Rham cohomology). Let M be a smooth manifold. The *de Rham cohomology* is

$$H_{\text{dR}}^n(M) = \frac{\ker(d: \Omega^n(M) \rightarrow \Omega^{n+1}(M))}{\text{im}(d: \Omega^{n-1}(M) \rightarrow \Omega^n(M))},$$

where $\Omega^n(M)$ is the vector space of smooth n -forms on M .

Theorem 8.37 (de Rham’s theorem). *For any smooth manifold M , there is a natural isomorphism of graded \mathbb{R} -algebras*

$$H_{\text{dR}}^*(M) \cong H^*(M; \mathbb{R}).$$

The isomorphism is induced by integration: a closed n -form ω corresponds to the singular cocycle $\sigma \mapsto \int_{\sigma} \omega$ on smooth singular n -simplices. Under this isomorphism, the wedge product of differential forms corresponds to the cup product.

Remark 8.38. De Rham’s theorem is a profound bridge between analysis (differential forms, exterior derivative) and topology (singular cohomology, cup product). It explains why topological invariants can be computed by analytic methods, a theme central to the Hodge theory and the Atiyah–Singer index theorem.

The following diagram summarizes the relationships:

$$\begin{array}{ccc} \Omega^*(M) & \xrightarrow{d} & \Omega^{*+1}(M) \\ \downarrow f & & \downarrow f \\ C^*(M; \mathbb{R}) & \xrightarrow{\delta} & C^{*+1}(M; \mathbb{R}) \end{array}$$

The commutativity of this diagram (Stokes’ theorem: $\int_{\sigma} d\omega = \int_{\partial\sigma} \omega$) is what makes the de Rham map a cochain map.

Example 8.39 (de Rham cohomology of S^1). The 1-form $d\theta/(2\pi)$ on S^1 is closed (every 1-form on a 1-manifold is closed) but not exact (its integral over S^1 is 1). It represents a generator of $H_{\text{dR}}^1(S^1) \cong \mathbb{R}$.

Example 8.40 (de Rham cohomology of T^2). Let θ_1, θ_2 be angular coordinates on $T^2 = S^1 \times S^1$. Then:

- $H_{\text{dR}}^0(T^2) \cong \mathbb{R}$, generated by the constant function 1.
- $H_{\text{dR}}^1(T^2) \cong \mathbb{R}^2$, generated by $[d\theta_1]$ and $[d\theta_2]$.
- $H_{\text{dR}}^2(T^2) \cong \mathbb{R}$, generated by $[d\theta_1 \wedge d\theta_2]$.

The wedge product $d\theta_1 \wedge d\theta_2$ corresponds to the cup product of the generators, confirming the ring structure $H_{\text{dR}}^*(T^2) \cong \Lambda_{\mathbb{R}}(\alpha, \beta)$.

8.11 Relative cohomology and the long exact sequence

Definition 8.41 (Relative cohomology). For a pair (X, A) with $A \subseteq X$, define

$$C^n(X, A; G) = \ker\left(C^n(X; G) \xrightarrow{i^*} C^n(A; G)\right) \cong \text{Hom}(C_n(X)/C_n(A), G) = \text{Hom}(C_n(X, A), G).$$

The relative cohomology groups are $H^n(X, A; G) = H^n(C^\bullet(X, A; G))$.

Theorem 8.42 (Long exact sequence in cohomology). *For a pair (X, A) there is a long exact sequence*

$$\cdots \rightarrow H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta^*} H^{n+1}(X, A; G) \rightarrow \cdots$$

where δ^* is the connecting homomorphism. Note that the maps i^* and j^* go in the “reverse” direction compared to homology, reflecting the contravariance of cohomology.

8.12 The Mayer–Vietoris sequence in cohomology

Theorem 8.43 (Mayer–Vietoris in cohomology). *Let $X = A \cup B$ with $X = \mathring{A} \cup \mathring{B}$. There is a long exact sequence*

$$\cdots \rightarrow H^n(X; G) \xrightarrow{\Psi^*} H^n(A; G) \oplus H^n(B; G) \xrightarrow{\Phi^*} H^n(A \cap B; G) \xrightarrow{\delta^*} H^{n+1}(X; G) \rightarrow \cdots$$

where $\Psi^*(\alpha) = (j_A^*\alpha, j_B^*\alpha)$ and $\Phi^*(\alpha, \beta) = i_A^*\alpha - i_B^*\beta$.

8.13 Exercises

Exercise 8.1. Compute $H^*(S^n; G)$ for an arbitrary abelian group G using the universal coefficient theorem. Verify your answer for $G = \mathbb{Z}/m\mathbb{Z}$.

Exercise 8.2. Let $X = \mathbb{R}P^2$. Compute $H^*(X; \mathbb{Z})$ and $H^*(X; \mathbb{Z}/2\mathbb{Z})$ using the universal coefficient theorem. Compare the two.

Exercise 8.3. Compute the cohomology ring $H^*(T^3; \mathbb{Z})$, where $T^3 = S^1 \times S^1 \times S^1$. Express your answer as an exterior algebra and write down all cup products of basis elements explicitly.

Exercise 8.4. Show that $S^2 \times S^4$ and $\mathbb{C}P^3$ have the same integral cohomology groups. Determine whether their cohomology rings are isomorphic. *Hint:* Consider the cube of a degree-2 generator.

Exercise 8.5. Let ΣX denote the suspension of X . Prove that the cup product of any two positive-degree cohomology classes in $H^*(\Sigma X; R)$ vanishes. *Hint:* Use the Mayer–Vietoris sequence for the decomposition of ΣX into two cones.

Exercise 8.6. Use the cohomology ring structure to show that $\mathbb{R}P^2$ is not a retract of $\mathbb{R}P^3$... wait, actually show that if $r: \mathbb{R}P^3 \rightarrow \mathbb{R}P^2$ were a retraction, then $r^*: H^*(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(\mathbb{R}P^3; \mathbb{Z}/2\mathbb{Z})$ would lead to a contradiction. *Hint:* Consider what happens to α^3 where α is the degree-1 generator.

Exercise 8.7. Compute $H^*(\Sigma_g; \mathbb{Z})$ as a graded ring, where Σ_g is the closed orientable surface of genus g . Show that $H^*(\Sigma_g; \mathbb{Z}) \cong \bigwedge_{\mathbb{Z}}(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)/I$ where I is a suitable ideal, and determine the cup product pairing $H^1 \otimes H^1 \rightarrow H^2 \cong \mathbb{Z}$.

Exercise 8.8. Use the universal coefficient theorem to compute $H^*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ as a graded ring. Show that it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})[\alpha, \beta]/(\alpha^3, \beta^3)$ with $|\alpha| = |\beta| = 1$.

Exercise 8.9. Let $f: S^3 \rightarrow S^2$ be the Hopf map. Compute $H^*(C_f; \mathbb{Z})$ as a graded ring, where C_f is the mapping cone of f , and verify that the Hopf invariant of f is ± 1 .

Exercise 8.10. Let $M = S^1 \times S^2$.

- (a) Compute the de Rham cohomology $H_{\text{dR}}^*(M)$ using the Künneth formula.
- (b) Find explicit closed differential forms representing generators of each cohomology group.
- (c) Verify the ring structure by computing wedge products of your representative forms.

Chapter 9

Poincaré Duality

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Poincaré duality is one of the most striking structural results in algebraic topology: for a closed oriented n -manifold M , the cohomology group $H^p(M; R)$ is isomorphic to the homology group $H_{n-p}(M; R)$. This chapter develops the machinery needed to state and prove this theorem, and explores its far-reaching consequences.

9.1 Orientability and the fundamental class

Throughout this chapter, M denotes a connected, closed (compact, without boundary) topological n -manifold. We write R for a commutative ring with unity (typically \mathbb{Z} , \mathbb{Q} , or \mathbb{R}).

Definition 9.1 (Local orientation). Let $x \in M$. A **local orientation** of M at x is a choice of generator μ_x of the local homology group

$$H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}.$$

The local homology groups form a covering space over M , which we now describe precisely.

Definition 9.2 (Orientation sheaf and orientability). The **orientation double cover** $\widetilde{M} \rightarrow M$ is the two-sheeted covering whose fibre over x consists of the two generators of $H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$. The manifold M is **orientable** if this covering is trivial, i.e., if \widetilde{M} has two connected components. An **orientation** of M is a choice of section of $\widetilde{M} \rightarrow M$, equivalently a consistent choice of local orientations $\{\mu_x\}_{x \in M}$.

Example 9.3 (Orientable manifolds). All spheres S^n , all complex projective spaces $\mathbb{C}P^n$, all tori $T^n = (S^1)^n$, and all Lie groups are orientable.

Example 9.4 (Non-orientable manifolds). The real projective plane $\mathbb{R}P^2$, the Möbius band, and the Klein bottle are non-orientable.

Theorem 9.5 (Fundamental class). *Let M be a closed, connected, oriented n -manifold. Then there exists a unique class $[M] \in H_n(M; \mathbb{Z})$ such that for every $x \in M$, the image of $[M]$ under the canonical map $H_n(M; \mathbb{Z}) \rightarrow H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ is the chosen local orientation μ_x . We call $[M]$ the **fundamental class** of M . If M is connected and closed but non-orientable, there still exists a fundamental class $[M] \in H_n(M; \mathbb{Z}/2)$.*

Proof sketch. One proceeds by induction using a Mayer–Vietoris argument. Cover M by open sets each homeomorphic to \mathbb{R}^n . For a single chart, the statement is clear. Gluing via Mayer–Vietoris and a compactness argument (finitely many charts suffice) produces the global class. The key input is that the consistency of local orientations forces the relevant connecting homomorphisms to be isomorphisms. For full details, see Hatcher [1] Theorem 3.26. □

Remark 9.6 (Coefficients in $\mathbb{Z}/2$). Every closed connected n -manifold, orientable or not, admits a $\mathbb{Z}/2$ -fundamental class $[M]_2 \in H_n(M; \mathbb{Z}/2)$. Poincaré duality with $\mathbb{Z}/2$ coefficients therefore holds universally for closed manifolds.

9.2 The cap product

Poincaré duality is realized by capping with the fundamental class. We first introduce the relevant algebraic operation.

Definition 9.7 (Cap product). Let X be a topological space and R a commutative ring. The **cap product** is the bilinear pairing

$$\frown : H_n(X; R) \otimes H^p(X; R) \longrightarrow H_{n-p}(X; R),$$

defined on the chain level as follows. For an n -simplex $\sigma : \Delta^n \rightarrow X$ and a cochain

$\varphi \in C^p(X; R)$, set

$$\sigma \frown \varphi = \varphi(\sigma|_{[v_0, \dots, v_p]}) \cdot \sigma|_{[v_p, \dots, v_n]}.$$

Proposition 9.8 (Properties of the cap product). *The cap product satisfies:*

- (i) **Naturality.** For a continuous map $f: X \rightarrow Y$, $f_*(\alpha \frown f^*\varphi) = f_*\alpha \frown \varphi$.
- (ii) **Associativity.** $\alpha \frown (\varphi \smile \psi) = (\alpha \frown \varphi) \frown \psi$, where \smile denotes the cup product.
- (iii) **Unit.** $\alpha \frown 1 = \alpha$, where $1 \in H^0(X; R)$ is the unit of the cup product ring.

Proof. These identities are verified at the cochain level using the Alexander–Whitney diagonal approximation and then pass to (co)homology. See Hatcher [1] §3.3. \square

The cap product relates cup product duality to homology in the following commutative diagram:

$$\begin{array}{ccc} H_n(X; R) \otimes H^p(X; R) & \xrightarrow{\frown} & H_{n-p}(X; R) \\ \text{id} \otimes \Delta^* \downarrow & \nearrow & \\ H_n(X; R) \otimes H^p(X; R) & & \end{array}$$

Example 9.9 (Cap product on S^n). Let $\iota \in H^n(S^n; \mathbb{Z}) \cong \mathbb{Z}$ be the canonical generator and $[S^n] \in H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$ the fundamental class. Then

$$[S^n] \frown \iota = 1 \in H_0(S^n; \mathbb{Z}) \cong \mathbb{Z}.$$

More generally, $[S^n] \frown 1 = [S^n]$ where $1 \in H^0(S^n; \mathbb{Z})$ is the unit.

Example 9.10 (Cap product on the torus). Let $T^2 = S^1 \times S^1$ with generators $\alpha, \beta \in H^1(T^2; \mathbb{Z})$ and fundamental class $[T^2] \in H_2(T^2; \mathbb{Z})$. Then $[T^2] \frown \alpha \in H_1(T^2; \mathbb{Z})$ is the homology class of the second S^1 factor (with appropriate sign), and $[T^2] \frown (\alpha \smile \beta) = \langle \alpha \smile \beta, [T^2] \rangle = \pm 1 \in H_0(T^2; \mathbb{Z}) \cong \mathbb{Z}$. The cap product map D_{T^2} provides the explicit Poincaré duality isomorphism $H^1(T^2) \cong H_1(T^2) \cong \mathbb{Z}^2$.

Proposition 9.11 (Cap product and evaluation). *For a closed oriented n -manifold M and $\varphi \in H^n(M; R)$,*

$$[M] \frown \varphi = \langle \varphi, [M] \rangle \in H_0(M; R) \cong R,$$

where $\langle -, - \rangle: H^n(M; R) \otimes H_n(M; R) \rightarrow R$ is the Kronecker pairing.

9.3 The Poincaré Duality Theorem

Theorem 9.12 (Poincaré Duality). *Let M be a closed, oriented, connected n -manifold and let R be a commutative ring with unity. Then the cap product with the fundamental class*

$$D_M : H^p(M; R) \xrightarrow{[M] \frown -} H_{n-p}(M; R)$$

is an isomorphism for every $0 \leq p \leq n$.

Equivalently, the pairing

$$H^p(M; R) \otimes_R H^{n-p}(M; R) \xrightarrow{\smile} H^n(M; R) \xrightarrow{\langle -, [M] \rangle} R$$

is non-degenerate when R is a field.

Proof strategy. The proof proceeds in several stages:

Step 1 (Open subsets of \mathbb{R}^n). For $U \subset \mathbb{R}^n$ open, one establishes the isomorphism using the fact that U is a union of convex sets and applying Mayer–Vietoris.

Step 2 (Finite good covers). If $M = U \cup V$ and the theorem holds for U , V , and $U \cap V$, then it holds for M by the Five Lemma applied to the Mayer–Vietoris sequences in homology and cohomology linked by cap product maps:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^p(M) & \longrightarrow & H^p(U) \oplus H^p(V) & \longrightarrow & H^p(U \cap V) & \longrightarrow & H^{p+1}(M) & \longrightarrow & \cdots \\ & & \downarrow D_M & & \downarrow D_U \oplus D_V & & \downarrow D_{U \cap V} & & \downarrow D_M & & \\ \cdots & \longrightarrow & H_{n-p}(M) & \longrightarrow & H_{n-p}(U) \oplus H_{n-p}(V) & \longrightarrow & H_{n-p}(U \cap V) & \longrightarrow & H_{n-p-1}(M) & \longrightarrow & \cdots \end{array}$$

Step 3 (Compactness). Since M is compact, it admits a finite good cover. An induction on the number of open sets in the cover, using Step 2, completes the proof.

The bookkeeping requires care with signs and orientations; see Hatcher [1] Theorem 3.30 or Bredon [3] VI.8. □

Remark 9.13 (Non-orientable case). For a closed connected n -manifold that is *not* orientable, Poincaré duality still holds with $\mathbb{Z}/2$ coefficients: $H^p(M; \mathbb{Z}/2) \cong H_{n-p}(M; \mathbb{Z}/2)$. Alternatively, one can use *twisted coefficients* via the orientation sheaf.

9.4 Applications of Poincaré duality

9.4.1 Betti numbers and symmetry

Corollary 9.14 (Symmetry of Betti numbers). *Let M be a closed oriented n -manifold and \mathbb{F} a field. Then $b_p(M; \mathbb{F}) = b_{n-p}(M; \mathbb{F})$ for all $0 \leq p \leq n$, where $b_p = \dim_{\mathbb{F}} H_p(M; \mathbb{F})$.*

Proof. By the universal coefficient theorem over a field, $H^p(M; \mathbb{F}) \cong H_p(M; \mathbb{F})^*$, so $\dim H^p(M; \mathbb{F}) = \dim H_p(M; \mathbb{F})$. Poincaré duality gives $H^p(M; \mathbb{F}) \cong H_{n-p}(M; \mathbb{F})$, and the result follows. □

Corollary 9.15 (Euler characteristic of odd-dimensional manifolds). *If M is a closed oriented manifold of odd dimension $n = 2k + 1$, then $\chi(M) = 0$.*

Proof. We have $\chi(M) = \sum_{p=0}^n (-1)^p b_p$. Pairing b_p with b_{n-p} and using $b_p = b_{n-p}$, the terms cancel in pairs because $(-1)^p + (-1)^{n-p} = (-1)^p(1 + (-1)^n) = 0$ when n is odd. \square

9.4.2 Cohomology of spheres and tori

Example 9.16 (Spheres). For S^n ($n \geq 1$), Poincaré duality gives $H^p(S^n) \cong H_{n-p}(S^n)$, confirming $H^0(S^n) \cong H^n(S^n) \cong \mathbb{Z}$ and $H^p(S^n) = 0$ for $0 < p < n$.

Example 9.17 (Tori). For the n -torus $T^n = (S^1)^n$, the cohomology ring is the exterior algebra $H^*(T^n; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_n]$ with each $\alpha_i \in H^1$. Poincaré duality is reflected in the isomorphism $\Lambda^p \cong \Lambda^{n-p}$, so $b_p(T^n) = \binom{n}{p}$.

9.4.3 Projective spaces

Example 9.18 (Complex projective spaces). For $\mathbb{C}P^n$, $H^{2k}(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$ for $0 \leq k \leq n$ and all odd-degree cohomology vanishes. Poincaré duality gives $H^{2k} \cong H_{2n-2k}$, consistent with the ring structure $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, $|\alpha| = 2$.

Example 9.19 (Real projective spaces). For $\mathbb{R}P^n$ with $\mathbb{Z}/2$ coefficients, $H^p(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for all $0 \leq p \leq n$, and Poincaré duality with $\mathbb{Z}/2$ coefficients yields the expected symmetry. Over \mathbb{Z} , one must use twisted coefficients when n is even (since $\mathbb{R}P^n$ is non-orientable for even n).

9.4.4 The intersection form

Definition 9.20 (Intersection form). Let M be a closed oriented 4-manifold. The **intersection form** of M is the symmetric bilinear form

$$Q_M : H^2(M; \mathbb{Z})/\text{tors} \times H^2(M; \mathbb{Z})/\text{tors} \longrightarrow \mathbb{Z}$$

defined by $Q_M(\alpha, \beta) = \langle \alpha \smile \beta, [M] \rangle$. Poincaré duality implies that Q_M is **unimodular** (i.e., its matrix has determinant ± 1).

Example 9.21 (Intersection forms of familiar 4-manifolds). (i) $Q_{S^4} = 0$ (the trivial form), since $H^2(S^4) = 0$.

(ii) $Q_{\mathbb{C}P^2} = (1)$, the 1×1 matrix with entry 1.

(iii) $Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the hyperbolic form.

$$(iv) \quad Q_{\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark 9.22 (Donaldson and Freedman). The intersection form is a key invariant in 4-manifold topology. Freedman (1982) showed that the homeomorphism type of a simply connected closed oriented topological 4-manifold is determined by its intersection form. Donaldson (1983) proved that if Q_M is positive definite and M is smooth, then Q_M is diagonalizable over \mathbb{Z} —a dramatic restriction that distinguishes the smooth and topological categories in dimension four.

9.4.5 Poincaré–Lefschetz duality

Theorem 9.23 (Poincaré–Lefschetz duality). *Let M be a compact oriented n -manifold with boundary ∂M . Then there is an isomorphism*

$$D : H^p(M, \partial M; \mathbb{R}) \xrightarrow{\cong} H_{n-p}(M; \mathbb{R})$$

for all p , induced by the cap product with the relative fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$.

Similarly, $H^p(M; \mathbb{R}) \cong H_{n-p}(M, \partial M; \mathbb{R})$.

Remark 9.24 (Hodge duality). On a closed oriented Riemannian n -manifold M , the Hodge star operator $*$: $\Omega^p(M) \rightarrow \Omega^{n-p}(M)$ induces an isomorphism on harmonic forms $\mathcal{H}^p(M) \cong \mathcal{H}^{n-p}(M)$. By the Hodge theorem, $\mathcal{H}^p(M) \cong H_{\text{dR}}^p(M; \mathbb{R})$, so this gives an analytic proof of Poincaré duality over \mathbb{R} for smooth manifolds. The relationship between the topological cap product and the Hodge star is a deep theme in geometric analysis.

9.5 Exercises

Exercise 9.1. Compute the fundamental class $[T^2] \in H_2(T^2; \mathbb{Z})$ by triangulating T^2 as a quotient of a square and tracking the orientation through the identifications.

Exercise 9.2. Show that a closed connected manifold M satisfies $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$ if M is orientable and $H_n(M; \mathbb{Z}) = 0$ if M is non-orientable.

Exercise 9.3. Use Poincaré duality to show that if M is a closed oriented 3-manifold, then $H_1(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z})$ and $b_1(M) = b_2(M)$.

Exercise 9.4. Let M be a closed oriented $4k$ -manifold. Show that the signature $\sigma(M)$ of the intersection form Q_M on $H^{2k}(M; \mathbb{R})$ is a homotopy invariant.

Exercise 9.5. Verify Poincaré duality explicitly for $\mathbb{C}P^2$ by computing $H^p(\mathbb{C}P^2; \mathbb{Z})$ and $H_{4-p}(\mathbb{C}P^2; \mathbb{Z})$ for all p and exhibiting the cap product isomorphism.

Exercise 9.6. Let M be a closed oriented n -manifold with $n \equiv 2 \pmod{4}$. Show that $\chi(M)$ is even. *Hint:* Use the non-degenerate skew-symmetric pairing on $H^{n/2}(M; \mathbb{R})$.

Exercise 9.7. The Klein bottle K is a closed non-orientable 2-manifold. Compute $H_*(K; \mathbb{Z})$ and $H_*(K; \mathbb{Z}/2)$. Verify Poincaré duality with $\mathbb{Z}/2$ coefficients.

Exercise 9.8. Let M and N be closed oriented 4-manifolds. Show that $Q_{M\#N} \cong Q_M \oplus Q_N$.

Exercise 9.9. Let M be the closed unit disk $D^n \subset \mathbb{R}^n$, viewed as a compact manifold with boundary S^{n-1} . Verify Poincaré–Lefschetz duality by computing all relative and absolute (co)homology groups.

Exercise 9.10. Prove that every closed oriented 3-manifold is parallelizable. *Hint:* Use Poincaré duality and the fact that $w_2 = 0$ for oriented 3-manifolds.

Chapter 10

Higher Homotopy Groups — Introduction

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The fundamental group captures information about loops in a space. Higher homotopy groups $\pi_n(X, x_0)$ generalize this to maps from higher-dimensional spheres, providing a much finer (and harder to compute) invariant.

10.1 Definition and group structure

Definition 10.1 (Higher homotopy groups). Let X be a topological space with basepoint $x_0 \in X$ and let $n \geq 1$. The n -th homotopy group of X is

$$\pi_n(X, x_0) = [S^n, X]_0 = \{f: (S^n, *) \rightarrow (X, x_0)\} / \text{based homotopy},$$

the set of based homotopy classes of based maps $S^n \rightarrow X$.

Equivalently, identifying S^n with $I^n/\partial I^n$, an element of $\pi_n(X, x_0)$ is represented by a map $f: I^n \rightarrow X$ sending ∂I^n to x_0 , modulo homotopies that also fix ∂I^n at x_0 .

The group operation on π_n is defined by “stacking” in the first coordinate:

Definition 10.2 (Group operation). Given $[f], [g] \in \pi_n(X, x_0)$ (represented as maps $I^n \rightarrow X$ fixing ∂I^n), define $[f] \cdot [g]$ to be the class of

$$(f \cdot g)(t_1, t_2, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & \text{if } 0 \leq t_1 \leq \frac{1}{2}, \\ g(2t_1 - 1, t_2, \dots, t_n) & \text{if } \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

Proposition 10.3 (π_n is a group). For $n \geq 1$, the operation above makes $\pi_n(X, x_0)$ into a group. The identity is the class of the constant map, and the inverse of $[f]$ is $[f^{-1}]$ where $f^{-1}(t_1, t_2, \dots, t_n) = f(1 - t_1, t_2, \dots, t_n)$.

Theorem 10.4 (Eckmann–Hilton argument: commutativity for $n \geq 2$). For $n \geq 2$, the group $\pi_n(X, x_0)$ is abelian.

Proof. For $n \geq 2$ we can stack maps in either the first or the second coordinate of I^n . Define two multiplications:

$$(f *_1 g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots) & t_1 \leq \frac{1}{2}, \\ g(2t_1 - 1, t_2, \dots) & t_1 \geq \frac{1}{2}, \end{cases}$$

$$(f *_2 g)(t_1, \dots, t_n) = \begin{cases} f(t_1, 2t_2, \dots) & t_2 \leq \frac{1}{2}, \\ g(t_1, 2t_2 - 1, \dots) & t_2 \geq \frac{1}{2}. \end{cases}$$

Both are associative with the same unit (the constant map). Moreover, $*_1$ and $*_2$ satisfy the *interchange law*: $(f *_1 g) *_2 (h *_1 k) = (f *_2 h) *_1 (g *_2 k)$. The Eckmann–Hilton argument then gives $f *_1 g = f *_2 g = g *_1 f$; that is, π_n is abelian. \square

10.2 Functoriality

Proposition 10.5 (Functoriality of π_n). A based map $f: (X, x_0) \rightarrow (Y, y_0)$ induces a group homomorphism

$$f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0), \quad [\alpha] \mapsto [f \circ \alpha].$$

This assignment is functorial: $(\text{id}_X)_ = \text{id}$ and $(g \circ f)_* = g_* \circ f_*$. In particular, if f is a homotopy equivalence, then f_* is an isomorphism for all n .*

This means π_n is a functor from the homotopy category of based spaces to the category of groups (abelian groups for $n \geq 2$).

Example 10.6 (Functoriality and degree). A map $f: S^n \rightarrow S^n$ of degree d induces the multiplication-by- d map $f_*: \pi_n(S^n) \cong \mathbb{Z} \rightarrow \pi_n(S^n) \cong \mathbb{Z}$, $f_*([\text{id}]) = d \cdot [\text{id}]$. In particular, the antipodal map $a: S^n \rightarrow S^n$ has degree $(-1)^{n+1}$ and induces $a_* = (-1)^{n+1} \cdot \text{id}$ on $\pi_n(S^n)$.

Remark 10.7 (Dependence on basepoint). If X is path-connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for any two basepoints $x_0, x_1 \in X$, via conjugation by a path from x_0 to x_1 . This isomorphism depends on the choice of path, so the basepoint cannot simply be ignored. For $n \geq 2$ the group is abelian, so conjugation by different paths yields the same isomorphism if and only if $\pi_1(X)$ acts trivially on $\pi_n(X)$. A space where this action is trivial for all n is called **simple**.

10.3 Long exact sequence of a pair

Definition 10.8 (Relative homotopy group). For a pair (X, A) with $x_0 \in A$, the **relative homotopy group** $\pi_n(X, A, x_0)$ for $n \geq 1$ consists of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$, where $J^{n-1} = \partial I^n \setminus I^{n-1} \times \{0\}$ is the “upper” part of the boundary.

Theorem 10.9 (Long exact sequence of a pair). *For a pair (X, A) with $x_0 \in A$, there is a long exact sequence*

$$\cdots \longrightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots \longrightarrow \pi_0(A) \longrightarrow \pi_0(X)$$

where $i: A \hookrightarrow X$ is the inclusion, j comes from $(X, x_0) \hookrightarrow (X, A)$, and ∂ is the boundary map.

Proof sketch. Exactness at $\pi_n(X)$: $[f] \in \ker j_*$ means f is homotopic (as a map of pairs) to a map into A , which is exactly $\text{im } i_*$. The other exactness arguments are analogous; see Hatcher [1] Theorem 4.3. \square

10.4 Fibrations and the homotopy exact sequence

Definition 10.10 (Fibration). A map $p: E \rightarrow B$ is a (**Hurewicz**) **fibration** if it has the **homotopy lifting property** (HLP) with respect to all spaces: for every space Y , every map $f_0: Y \rightarrow E$, and every homotopy $F: Y \times I \rightarrow B$ with $F(-, 0) = p \circ f_0$,

there exists a homotopy $\tilde{F}: Y \times I \rightarrow E$ with $p \circ \tilde{F} = F$ and $\tilde{F}(-, 0) = \tilde{f}_0$.

Definition 10.11 (Serre fibration). A map $p: E \rightarrow B$ is a **Serre fibration** if it has the HLP with respect to all disks D^n (equivalently, all CW complexes).

Theorem 10.12 (Long exact sequence of a fibration). If $F \hookrightarrow E \xrightarrow{p} B$ is a fibration sequence with fibre $F = p^{-1}(b_0)$, there is a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow \pi_0(B)$$

Proof. This is a consequence of Theorem 10.9 combined with the identification $\pi_n(E, F) \cong \pi_n(B)$ coming from the HLP. See Hatcher [1] Theorem 4.41. \square

10.5 Computations of homotopy groups

10.5.1 Cellular approximation and vanishing results

Theorem 10.13 (Cellular approximation). Every map between CW complexes is homotopic to a cellular map. Consequently, if X has no cells in dimensions $1, 2, \dots, n-1$ (beyond the basepoint), then $\pi_k(X) = 0$ for $k < n$.

Corollary 10.14. $\pi_n(S^m) = 0$ for all $n < m$.

Proof. S^m has a CW structure with one 0-cell and one m -cell. By cellular approximation, any map $S^n \rightarrow S^m$ with $n < m$ is homotopic to a map into the $(m-1)$ -skeleton, which is a point. \square

10.5.2 The Hurewicz theorem

Definition 10.15 (Hurewicz homomorphism). The **Hurewicz map** $h: \pi_n(X, x_0) \rightarrow H_n(X; \mathbb{Z})$ sends $[f: S^n \rightarrow X]$ to $f_*(\iota_n)$, where $\iota_n \in H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$ is the canonical generator.

Theorem 10.16 (Hurewicz theorem). Let X be a path-connected space with $x_0 \in X$.

- (i) If $n = 1$: the Hurewicz map $h: \pi_1(X, x_0) \rightarrow H_1(X; \mathbb{Z})$ is the abelianization map $\pi_1 \twoheadrightarrow \pi_1^{\text{ab}} \cong H_1$.
- (ii) If $n \geq 2$ and $\pi_k(X) = 0$ for all $1 \leq k < n$ (i.e., X is $(n-1)$ -connected): then $H_k(X; \mathbb{Z}) = 0$ for $k < n$ and

$$h : \pi_n(X, x_0) \xrightarrow{\cong} H_n(X; \mathbb{Z})$$

is an isomorphism.

Proof for $n = 1$. For $n = 1$, $\pi_1(X, x_0)$ consists of homotopy classes of loops $\gamma: S^1 \rightarrow X$. Viewing γ as a singular 1-cycle, we obtain a well-defined map $h: \pi_1(X) \rightarrow H_1(X; \mathbb{Z})$. This is a homomorphism since the concatenation of loops corresponds to the sum of cycles (up to a boundary).

Surjectivity. Every 1-cycle in a path-connected space is homologous to a sum of loops (using paths to the basepoint), so h is surjective.

Kernel. If $h([\gamma]) = 0$, then γ (as a 1-chain) is a boundary $\gamma = \partial c$ for some 2-chain c . Each singular 2-simplex in c gives a relation among loops, and these relations are precisely the commutator relations. Hence $\ker h = [\pi_1, \pi_1]$. \square

Corollary 10.17. $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.

Proof. For $n = 1$, $\pi_1(S^1) \cong \mathbb{Z}$ is known from covering space theory. For $n \geq 2$, S^n is $(n-1)$ -connected (by Corollary 10.14), so the Hurewicz theorem gives $\pi_n(S^n) \cong H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$. \square

10.5.3 The Hopf fibration

Theorem 10.18 (Hopf fibration and $\pi_3(S^2)$). *There is a fibration $S^1 \hookrightarrow S^3 \xrightarrow{\eta} S^2$ (the **Hopf fibration**), defined by*

$$\eta(z_1, z_2) = [z_1 : z_2] \in \mathbb{C}P^1 \cong S^2, \quad (z_1, z_2) \in S^3 \subset \mathbb{C}^2.$$

Moreover, $\pi_3(S^2) \cong \mathbb{Z}$, generated by $[\eta]$.

Proof. That η is a fibration with fibre S^1 follows from the structure of the $U(1)$ -action on S^3 . The long exact sequence of the fibration gives:

$$\pi_3(S^1) \longrightarrow \pi_3(S^3) \xrightarrow{\eta_*} \pi_3(S^2) \xrightarrow{\partial} \pi_2(S^1) \longrightarrow \pi_2(S^3)$$

Since $\pi_3(S^1) = 0$, $\pi_2(S^1) = 0$ (using the universal cover $\mathbb{R} \rightarrow S^1$), and $\pi_3(S^3) \cong \mathbb{Z}$, exactness yields $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$. \square

Remark 10.19 (Other Hopf fibrations). There are analogous Hopf fibrations:

$$S^3 \hookrightarrow S^7 \rightarrow S^4, \quad S^7 \hookrightarrow S^{15} \rightarrow S^8,$$

arising from quaternionic and octonionic multiplication. These show $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$ (the \mathbb{Z} factor) and $\pi_{15}(S^8) \cong \mathbb{Z}$.

Remark 10.20 (Homotopy groups of spheres). The computation of $\pi_n(S^m)$ in general is one of the deepest problems in algebraic topology. Some known values:

$\pi_i(S^n)$	$i = 1$	2	3	4	5	6
S^1	\mathbb{Z}	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2

The appearance of torsion and the lack of a general closed formula distinguish homotopy groups from homology groups.

10.6 The Whitehead theorem

Theorem 10.21 (Whitehead). *Let $f: X \rightarrow Y$ be a map between connected CW complexes. If $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ is an isomorphism for all $n \geq 1$, then f is a homotopy equivalence.*

Proof sketch. By the homotopy extension property for CW pairs, one constructs the homotopy inverse cell by cell. The hypothesis that f induces isomorphisms on all π_n allows one to extend over each skeleton. The passage from a weak homotopy equivalence to a homotopy equivalence uses the fact that CW complexes have the homotopy type of CW complexes. See Hatcher [1] Theorem 4.5. \square

Remark 10.22 (CW hypothesis is essential). The CW hypothesis cannot be dropped. There exist spaces that have the same homotopy groups as a point but are not contractible (e.g., the Warsaw circle). The issue is that non-CW spaces may have pathological local behavior that prevents the construction of a homotopy inverse.

Example 10.23 (Application of the Whitehead theorem). Let X be a simply connected CW complex with $H_n(X; \mathbb{Z}) \cong H_n(S^k; \mathbb{Z})$ for all n . Then $X \simeq S^k$.

Indeed, since X is $(k-1)$ -connected (by the Hurewicz theorem), there exists a map $f: S^k \rightarrow X$ representing a generator of $\pi_k(X) \cong H_k(X) \cong \mathbb{Z}$. By the Hurewicz theorem and a comparison of homology, f induces isomorphisms on all homology groups. By Corollary 10.24, f is a homotopy equivalence.

Corollary 10.24. *If $f: X \rightarrow Y$ is a map between simply connected CW complexes that induces isomorphisms on all homology groups, then f is a homotopy equivalence.*

Proof. By the Hurewicz theorem, the hypothesis on homology combined with simple connectivity implies isomorphisms on homotopy groups. Then apply Theorem 10.21. \square

10.7 Exercises

Exercise 10.1. Carry out the Eckmann–Hilton argument in full detail: given a set S with two binary operations $*_1, *_2$, both associative, both sharing the same unit e , and satisfying the interchange law, prove that $*_1 = *_2$ and both are commutative.

Exercise 10.2. Show that $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ for all $n \geq 1$. Deduce $\pi_n(T^k)$ for all n, k .

Exercise 10.3. Use the long exact sequence of the covering $\mathbb{R} \rightarrow S^1$ to prove that $\pi_n(S^1) = 0$ for all $n \geq 2$.

Exercise 10.4. From the Hopf fibration $S^1 \hookrightarrow S^3 \rightarrow S^2$, determine $\pi_2(S^2)$ and $\pi_4(S^3)$ (the latter requires knowing $\pi_3(S^1)$ and $\pi_4(S^2) \cong \mathbb{Z}_2$).

Exercise 10.5. Let ΩX denote the loop space of X . Show that $\pi_n(X) \cong \pi_{n-1}(\Omega X)$ for all $n \geq 1$.

Exercise 10.6. Let ΣX denote the (reduced) suspension of X . Show that there is a natural homomorphism $E: \pi_n(X) \rightarrow \pi_{n+1}(\Sigma X)$ (the **suspension homomorphism**). State (without proof) the Freudenthal suspension theorem.

Exercise 10.7. Let X be a simply connected CW complex with $H_2(X; \mathbb{Z}) \cong \mathbb{Z}^3$ and $H_k(X; \mathbb{Z}) = 0$ for $k = 1$. What can you conclude about $\pi_2(X)$?

Exercise 10.8. Construct an example of a map $f: X \rightarrow Y$ between path-connected spaces (not CW complexes) such that f induces isomorphisms on all homotopy groups but f is not a homotopy equivalence.

Exercise 10.9. Consider the fibration $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1) \rightarrow S^n$. Use the long exact sequence to show that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2$.

Exercise 10.10. Show that $\pi_4(S^2) \cong \mathbb{Z}_2$. *Hint:* Use the long exact sequence of the Hopf fibration and the known value $\pi_4(S^3) \cong \mathbb{Z}_2$.

Chapter 11

Obstruction Theory and Applications

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Obstruction theory provides a systematic framework for answering a fundamental question in topology: *when can a map defined on a subspace be extended to the whole space?* The obstructions to extension lie in cohomology groups with coefficients in homotopy groups of the target, revealing a profound interaction between cohomology and homotopy.

11.1 The extension problem

Definition 11.1 (Extension problem). Given a CW pair (X, A) , a space Y , and a map $f: A \rightarrow Y$, the **extension problem** asks: does there exist a continuous map $\tilde{f}: X \rightarrow Y$ with $\tilde{f}|_A = f$?

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow & \nearrow \tilde{f} & \\ X & & \end{array}$$

The key difficulty is that, unlike homological questions (which can often be resolved by algebraic machinery), the extension problem requires understanding the interplay between the cell structure of X and the homotopy type of Y . Obstruction theory reduces this geometric problem to a sequence of algebraic computations in cohomology.

Remark 11.2 (Lifting vs. extension). A closely related question is the **lifting problem**: given a fibration $p: E \rightarrow B$ and a map $f: X \rightarrow B$, does there exist a lift $\tilde{f}: X \rightarrow E$ with $p \circ \tilde{f} = f$?

$$\begin{array}{ccc} & & E \\ & \nearrow \tilde{f} & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

Obstruction theory applies to lifting problems as well, with obstructions lying in $H^{n+1}(X; \pi_n(F))$, where F is the fibre.

The strategy of obstruction theory is to extend f one skeleton at a time.

Proposition 11.3 (Skeletal extension). *Let (X, A) be a relative CW complex and $f: A \rightarrow Y$ a continuous map.*

- (i) f always extends to $A \cup X^{(0)}$ (by choosing images for the new 0-cells arbitrarily, in the same path component of Y).
- (ii) If Y is path-connected, f extends to $A \cup X^{(1)}$.
- (iii) If Y is additionally simply connected, f extends to $A \cup X^{(2)}$.
- (iv) In general, the obstruction to extending from the n -skeleton to the $(n+1)$ -skeleton lives in a cohomology group.

11.2 The obstruction cocycle

Suppose f has been extended to a map $f_n: A \cup X^{(n)} \rightarrow Y$. For each $(n+1)$ -cell e_α^{n+1} of X relative to A , the attaching map $\varphi_\alpha: S^n \rightarrow X^{(n)}$ composed with f_n gives an element

$$[f_n \circ \varphi_\alpha] \in \pi_n(Y).$$

The map f_n extends over e_α^{n+1} if and only if this class is zero (because extending is equivalent to filling in the sphere).

Definition 11.4 (Obstruction cocycle). The **obstruction cocycle** is the cellular cochain

$$\mathfrak{o}^{n+1}(f_n) \in C^{n+1}(X, A; \pi_n(Y))$$

defined by $\mathfrak{o}^{n+1}(f_n)(e_\alpha^{n+1}) = [f_n \circ \varphi_\alpha] \in \pi_n(Y)$.

Theorem 11.5 (The obstruction cocycle is a cocycle). $\mathfrak{o}^{n+1}(f_n)$ is a cocycle, i.e., $\delta \mathfrak{o}^{n+1}(f_n) = 0$ in $C^{n+2}(X, A; \pi_n(Y))$. Moreover:

- (i) f_n extends to $A \cup X^{(n+1)}$ if and only if $\mathfrak{o}^{n+1}(f_n) = 0$ as a cochain.
- (ii) The cohomology class $[\mathfrak{o}^{n+1}(f_n)] \in H^{n+1}(X, A; \pi_n(Y))$ depends only on the homotopy class of $f_n|_{A \cup X^{(n-1)}}$.

(iii) $f_{n-1} = f_n|_{A \cup X^{(n-1)}}$ can be extended to some $f'_n: A \cup X^{(n)} \rightarrow Y$ that further extends to $A \cup X^{(n+1)}$ if and only if $[\sigma^{n+1}(f_n)] = 0$ in $H^{n+1}(X, A; \pi_n(Y))$.

Proof sketch. That $\sigma^{n+1}(f_n)$ is a cocycle follows from the fact that the boundary of a boundary is zero: the attaching map of an $(n+2)$ -cell, restricted to any $(n+1)$ -cell in its boundary, and then mapped by f_n , yields a sum of homotopy classes that must vanish because the total attaching sphere is null-homotopic in $X^{(n)}$.

For (iii), modifying f_n on individual n -cells changes σ^{n+1} by a coboundary. See Whitehead [2] or Davis–Kirk for the detailed argument. \square

11.3 The main obstruction theorem

Theorem 11.6 (Obstruction theorem). *Let (X, A) be a relative CW complex, Y a simple space (i.e., $\pi_1(Y)$ acts trivially on $\pi_n(Y)$ for all n), and $f: A \rightarrow Y$ a continuous map. Then:*

- (i) *The first (and possibly only) obstruction to extending f to all of X lies in $H^{n+1}(X, A; \pi_n(Y))$ for the smallest n such that $\pi_n(Y) \neq 0$.*
- (ii) *If all obstruction classes vanish successively, f extends to all of X .*
- (iii) *If X is finite-dimensional, the process terminates in finitely many steps.*

Remark 11.7 (Primary obstruction). When Y is $(n-1)$ -connected (i.e., $\pi_k(Y) = 0$ for $k < n$), the first potential obstruction lies in $H^{n+1}(X, A; \pi_n(Y))$. This is called the **primary obstruction**. In favorable cases (e.g., $\dim(X \setminus A) \leq n + 1$), the primary obstruction is the only one.

Example 11.8 (Sections of sphere bundles). Let $\xi: E \rightarrow M$ be an oriented S^{n-1} -bundle over a CW complex M (i.e., the unit sphere bundle of an oriented rank- n vector bundle). A global section $s: M \rightarrow E$ exists if and only if the successive obstructions vanish. The primary obstruction is a class

$$e(\xi) \in H^n(M; \pi_{n-1}(S^{n-1})) = H^n(M; \mathbb{Z}),$$

which is precisely the **Euler class** of the underlying vector bundle. In particular, $e(\xi) = 0$ is necessary (and, for $\dim M \leq n$, sufficient) for the existence of a nowhere-zero section of the vector bundle.

Example 11.9 (Obstruction to a map $\mathbb{RP}^2 \rightarrow S^2$). Consider the problem of finding a map $f: \mathbb{RP}^2 \rightarrow S^2$ inducing a nontrivial map on $H^2(-; \mathbb{Z}/2)$. The CW structure of \mathbb{RP}^2 has cells in dimensions 0, 1, 2. The map on the 0-skeleton is trivial. Extending to the 1-skeleton requires $\pi_1(S^2) = 0$, which is satisfied, so there is no obstruction. Extending to the 2-skeleton requires the obstruction in $H^2(\mathbb{RP}^2; \pi_1(S^2)) = 0$ to vanish, which it does. So any map $\mathbb{RP}^2 \rightarrow S^2$ must be null-homotopic, since $\pi_2(S^2) \cong \mathbb{Z}$ and

$H^2(\mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2$ yields $[\mathbb{R}P^2, S^2] = H^2(\mathbb{R}P^2; \pi_2(S^2))$ by obstruction theory (noting S^2 is 1-connected).

11.4 Applications

11.4.1 The Borsuk–Ulam theorem revisited

Theorem 11.10 (Borsuk–Ulam). *For every continuous map $f: S^n \rightarrow \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

Proof via obstruction theory for $n = 2$. Suppose no such x exists. Then $g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}: S^2 \rightarrow S^1$ is a well-defined odd map (i.e., $g(-x) = -g(x)$). This induces a map $\bar{g}: \mathbb{R}P^2 \rightarrow \mathbb{R}P^1 \cong S^1$ on quotients.

On fundamental groups, \bar{g}_* must send the generator of $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ to the generator of $\pi_1(S^1) \cong \mathbb{Z}$, which is impossible since there is no nonzero homomorphism $\mathbb{Z}/2 \rightarrow \mathbb{Z}$. \square

Remark 11.11. The obstruction-theoretic proof in general dimensions uses the fact that an equivariant map $S^n \rightarrow S^{n-1}$ (with respect to the antipodal action) would yield a section of a certain sphere bundle over $\mathbb{R}P^n$, whose obstruction class is nonzero.

11.4.2 The Lusternik–Schnirelmann theorem

Theorem 11.12 (Lusternik–Schnirelmann). *If S^n is covered by $n + 1$ closed sets A_1, \dots, A_{n+1} , each of which contains no pair of antipodal points, then $\bigcap_{i=1}^{n+1} A_i \neq \emptyset$ but more importantly, S^n cannot be covered by $n + 1$ closed sets, each containing no antipodal pair. Equivalently, any cover of $\mathbb{R}P^n$ by $n + 1$ open sets must have at least one set that is not contractible within $\mathbb{R}P^n$.*

Proof sketch. This follows from the Borsuk–Ulam theorem. If $S^n = A_1 \cup \dots \cup A_{n+1}$ with no A_i containing an antipodal pair, define $f: S^n \rightarrow \mathbb{R}^n$ by $f_i(x) = d(x, A_i)$ for $i = 1, \dots, n$. By Borsuk–Ulam, there exists x with $f(x) = f(-x)$, which forces x and $-x$ to lie in A_{n+1} , a contradiction. \square

11.4.3 Vector bundle classification

Remark 11.13 (Classification of vector bundles — preview). Obstruction theory underlies the classification of vector bundles. An oriented rank- k real vector bundle $E \rightarrow X$ is classified (up to isomorphism) by a homotopy class of maps $X \rightarrow \text{BSO}(k)$, where $\text{BSO}(k)$ is the classifying space. The obstructions to finding a global nonvanishing section, a global frame, or a spin structure are cohomology classes known as **characteristic classes** (Stiefel–Whitney, Chern, Pontryagin classes). This is the subject of Milnor–Stasheff [8].

11.5 Eilenberg–MacLane spaces

Definition 11.14 (Eilenberg–MacLane space). Let G be a group (abelian if $n \geq 2$) and $n \geq 1$. An **Eilenberg–MacLane space** $K(G, n)$ is a connected CW complex with

$$\pi_k(K(G, n)) = \begin{cases} G & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

Theorem 11.15 (Existence and uniqueness). *For every group G (abelian if $n \geq 2$) and every $n \geq 1$, a $K(G, n)$ exists and is unique up to homotopy equivalence.*

Proof sketch. **Existence.** Start with a presentation of G and build a 2-complex with the correct π_1 (or an $(n+1)$ -complex with the correct π_n via attaching cells to S^n). Then systematically attach cells in dimensions $\geq n+2$ to kill all higher homotopy groups.

Uniqueness. Any two $K(G, n)$'s have isomorphic homotopy groups, and a map inducing these isomorphisms can be constructed by obstruction theory. By the Whitehead theorem (Theorem 10.21), this map is a homotopy equivalence. \square

Example 11.16 (Familiar Eilenberg–MacLane spaces). (i) $K(\mathbb{Z}, 1) \simeq S^1$, since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_n(S^1) = 0$ for $n \geq 2$.

(ii) $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$, the infinite complex projective space.

(iii) $K(\mathbb{Z}/2, 1) \simeq \mathbb{R}P^\infty$, the infinite real projective space.

(iv) $K(\mathbb{Z}, n)$ has no simple closed-form model for $n \geq 3$; it must be constructed by cell attachment.

11.6 Representability of cohomology

The deepest connection between Eilenberg–MacLane spaces and cohomology is:

Theorem 11.17 (Brown representability for ordinary cohomology). *For any abelian group G , $n \geq 0$, and CW complex X , there is a natural bijection*

$$[X, K(G, n)] \cong H^n(X; G),$$

where $[X, K(G, n)]$ denotes the set of (unbased) homotopy classes of maps from X to $K(G, n)$.

Proof sketch. The map $[X, K(G, n)] \rightarrow H^n(X; G)$ sends $[f]$ to $f^*(\iota_n)$, where $\iota_n \in H^n(K(G, n); G) \cong \text{Hom}(G, G)$ is the **fundamental class** (corresponding to id_G).

That this is a bijection is proved by verifying the Eilenberg–Steenrod axioms: both sides satisfy the Mayer–Vietoris property and have the same values on points, so they agree on all CW complexes by induction on the number of cells. This is an instance of the **Brown representability theorem**, which states that any contravariant functor from

the homotopy category of connected CW complexes to sets that satisfies the wedge axiom and the Mayer–Vietoris axiom is representable. \square

The following diagram summarizes the representability:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & K(G, n) \\
 & \searrow^{f^*(\iota_n)} & \cong \\
 & & H^n(X; G)
 \end{array}
 \quad \longleftrightarrow \quad
 [f] \in [X, K(G, n)] \longleftrightarrow f^*(\iota_n) \in H^n(X; G).$$

Remark 11.18 (Postnikov towers). Every connected CW complex X can be approximated by a tower of fibrations

$$\begin{array}{c}
 \vdots \\
 \downarrow \\
 X_3 \\
 \downarrow p_3 \\
 X_2 \\
 \downarrow p_2 \\
 X_1
 \end{array}$$

where X_n has the same π_k as X for $k \leq n$ and $\pi_k(X_n) = 0$ for $k > n$, and the fibre of p_n is $K(\pi_n(X), n)$. The “ k -invariants” that glue successive stages are cohomology classes, bringing obstruction theory full circle.

11.7 Stable homotopy — a preview

Definition 11.19 (Stable homotopy groups). The k -th stable homotopy group of spheres is

$$\pi_k^s = \lim_{n \rightarrow \infty} \pi_{n+k}(S^n),$$

where the maps in the direct system are given by the suspension homomorphism $E: \pi_m(S^n) \rightarrow \pi_{m+1}(S^{n+1})$. The Freudenthal suspension theorem guarantees that this limit stabilizes (i.e., E is an isomorphism for $n > m + 1$).

Example 11.20 (Low-dimensional stable homotopy groups).

k	0	1	2	3	4	5	6	7
π_k^s	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{240}

Remark 11.21 (Spectra and generalized cohomology). To systematically study stable phenomena, one introduces the notion of a **spectrum** $\mathbf{E} = \{E_n, \sigma_n\}$, a sequence of spaces with structure maps $\sigma_n: \Sigma E_n \rightarrow E_{n+1}$. Every spectrum defines a **generalized cohomology theory** $h^n(X) = [X, E_n]$, and the Brown representability theorem guarantees that every generalized cohomology theory arises this way. Important

examples include:

- The **Eilenberg–MacLane spectrum** $H\mathbb{Z} = \{K(\mathbb{Z}, n)\}$, representing ordinary cohomology.
- **K -theory** spectra KU and KO , representing complex and real topological K -theory.
- **Cobordism** spectra MU, MSO, \dots

The stable homotopy category is the natural setting for duality phenomena (Spanier–Whitehead duality, Atiyah duality) and is foundational to modern homotopy theory.

11.8 Connections to mathematical physics

Remark 11.22 (Topology in physics). Algebraic topology, and obstruction theory in particular, appears naturally throughout mathematical physics:

- Gauge theory.** Principal G -bundles over a manifold M are classified by $[M, BG]$. Characteristic classes provide topological invariants of gauge fields. The Chern–Weil homomorphism realizes characteristic classes via curvature forms.
- Topological defects.** In condensed matter physics and cosmology, topological defects (vortices, monopoles, domain walls) are classified by homotopy groups of the vacuum manifold M : $\pi_0(M)$ classifies domain walls, $\pi_1(M)$ classifies strings, and $\pi_2(M)$ classifies monopoles.
- Anomalies.** Topological obstructions (related to index theory and η -invariants) detect quantum anomalies in gauge theories. The famous chiral anomaly is related to the index of the Dirac operator via the Atiyah–Singer index theorem.
- Topological quantum field theory.** A TQFT in the sense of Atiyah assigns to each closed $(n-1)$ -manifold a vector space and to each cobordism a linear map. The algebraic structure mirrors the composition of cobordisms, and homotopy-theoretic methods (spectra, ∞ -categories) provide the modern framework.

11.9 Exercises

Exercise 11.1. Let $X = S^2$ and $A = \{*\}$. Show that every map $f: A \rightarrow S^1$ extends to X . What is the obstruction group?

Exercise 11.2. Let X be a CW complex of dimension $\leq n$ and Y an $(n-1)$ -connected space. Show that the primary obstruction to extending any map $f: X^{(n)} \rightarrow Y$ to $X^{(n+1)}$ is the only obstruction, and lives in $H^{n+1}(X; \pi_n(Y))$.

Exercise 11.3. Prove that $\Omega K(G, n) \simeq K(G, n-1)$ for $n \geq 2$. *Hint:* Use the long exact sequence of the path-space fibration $\Omega K(G, n) \hookrightarrow PK(G, n) \rightarrow K(G, n)$.

Exercise 11.4. Use the Brown representability theorem to show that $[X, \mathbb{C}P^\infty] \cong H^2(X; \mathbb{Z})$ for any CW complex X . Interpret this in terms of complex line bundles.

Exercise 11.5. Use the Borsuk–Ulam theorem to prove that \mathbb{R}^n and \mathbb{R}^m are not homeomorphic for $n \neq m$.

Exercise 11.6. Describe the first two stages of the Postnikov tower for S^2 . What are the relevant k -invariants?

Exercise 11.7. Use the Lusternik–Schnirelmann theorem to show that any continuous map $f: S^2 \rightarrow \mathbb{R}^2$ maps some pair of antipodal points to the same value.

Exercise 11.8. Using the Freudenthal suspension theorem, determine π_k^s for $k = 0$ and $k = 1$.

Exercise 11.9. Show that $K(\mathbb{Z} \times \mathbb{Z}, 1) \simeq T^2$, the 2-torus. More generally, show that $K(\mathbb{Z}^n, 1) \simeq T^n$.

Exercise 11.10. Let $E \rightarrow S^2$ be a complex line bundle. The first Chern class $c_1(E) \in H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$ classifies E up to isomorphism. Identify this with the clutching construction: line bundles over S^2 correspond to maps $S^1 \rightarrow \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$, i.e., elements of $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$.

Appendix A

Homological Algebra Review

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This appendix collects the main tools from homological algebra used throughout the text.

A.1 Chain complexes

Definition A.1 (Chain complex). A **chain complex** (C_\bullet, d) over a ring R is a sequence of R -modules and R -module homomorphisms

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

satisfying $d_n \circ d_{n+1} = 0$ for all n .

The n -th **homology** of C_\bullet is

$$H_n(C_\bullet) = \ker d_n / \operatorname{im} d_{n+1} = Z_n / B_n,$$

where $Z_n = \ker d_n$ is the module of n -**cycles** and $B_n = \operatorname{im} d_{n+1}$ the module of n -**boundaries**.

Definition A.2 (Cochain complex). A **cochain complex** (C^\bullet, δ) is a sequence

$$\cdots \xrightarrow{\delta^{n-2}} C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \xrightarrow{\delta^{n+1}} \cdots$$

with $\delta^{n+1} \circ \delta^n = 0$. The n -th **cohomology** is $H^n(C^\bullet) = \ker \delta^n / \text{im } \delta^{n-1}$.

Definition A.3 (Chain map). A **chain map** $f: C_\bullet \rightarrow D_\bullet$ is a family of homomorphisms $f_n: C_n \rightarrow D_n$ commuting with the differentials: $d_n^D \circ f_n = f_{n-1} \circ d_n^C$. Equivalently:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{d_{n+1}^D} & D_n & \xrightarrow{d_n^D} & D_{n-1} & \longrightarrow & \cdots \end{array}$$

A chain map induces homomorphisms $f_*: H_n(C_\bullet) \rightarrow H_n(D_\bullet)$.

Definition A.4 (Chain homotopy). Two chain maps $f, g: C_\bullet \rightarrow D_\bullet$ are **chain homotopic**, written $f \simeq g$, if there exist homomorphisms $s_n: C_n \rightarrow D_{n+1}$ satisfying $f_n - g_n = d_{n+1}^D \circ s_n + s_{n-1} \circ d_n^C$ for all n .

Proposition A.5. *Chain homotopic maps induce the same map on homology: if $f \simeq g$, then $f_* = g_*: H_n(C) \rightarrow H_n(D)$.*

A.2 Exact sequences

Definition A.6 (Exact sequence). A sequence of R -module homomorphisms $\cdots \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow \cdots$ is **exact** at B if $\text{im } f = \ker g$. The sequence is **exact** if it is exact at every interior term.

Definition A.7 (Short exact sequence). A **short exact sequence** (SES) is an exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0,$$

meaning f is injective, g is surjective, and $\text{im } f = \ker g$.

Theorem A.8 (Long exact sequence in homology). *A short exact sequence of chain complexes $0 \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow 0$ induces a long exact sequence in homology:*

$$\cdots \rightarrow H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$$

The connecting homomorphism ∂ is constructed via diagram chasing (the “snake

lemma”).

Lemma A.9 (Five Lemma). *Given a commutative diagram of R -modules with exact rows:*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

If f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is an isomorphism.

Lemma A.10 (Snake Lemma). *Given a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

there is an exact sequence $\ker \alpha \rightarrow \ker \beta \rightarrow \ker \gamma \xrightarrow{\partial} \operatorname{coker} \alpha \rightarrow \operatorname{coker} \beta \rightarrow \operatorname{coker} \gamma$.

A.3 Universal coefficient theorems

Theorem A.11 (Universal Coefficient Theorem for cohomology). *For a chain complex C_\bullet of free abelian groups and an abelian group G , there is a natural short exact sequence*

$$0 \rightarrow \operatorname{Ext}_{\mathbb{Z}}^1(H_{n-1}(C), G) \rightarrow H^n(C; G) \rightarrow \operatorname{Hom}(H_n(C), G) \rightarrow 0.$$

This sequence splits (non-naturally).

Theorem A.12 (Künneth formula). *For spaces X, Y with $H_*(X; \mathbb{Z})$ free, there is a natural short exact sequence*

$$0 \rightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \rightarrow H_n(X \times Y) \rightarrow \bigoplus_{i+j=n-1} \operatorname{Tor}_1(H_i(X), H_j(Y)) \rightarrow 0.$$

Over a field, the Tor term vanishes and homology of a product is the tensor product of homologies.

A.4 Derived functors

Definition A.13 (Projective and injective resolutions). A **projective resolution** of an R -module M is an exact sequence

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where each P_i is projective. Dually, an **injective resolution** is an exact sequence $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ with each I^j injective.

Definition A.14 (Tor and Ext). Let M, N be R -modules. Choose a projective resolution $P_\bullet \rightarrow M \rightarrow 0$. Then:

- $\text{Tor}_n^R(M, N) = H_n(P_\bullet \otimes_R N)$.
- $\text{Ext}_R^n(M, N) = H^n(\text{Hom}_R(P_\bullet, N))$.

These are independent of the choice of resolution (up to canonical isomorphism). The fact that different resolutions yield the same derived functors follows from the **comparison theorem**: any two projective resolutions of the same module are chain homotopy equivalent.

Example A.15 (Computing Ext and Tor over \mathbb{Z}). To compute $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m, \mathbb{Z}/n)$, use the projective resolution $0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0$. Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/n)$:

$$0 \rightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}/n) \xrightarrow{m} \text{Hom}(\mathbb{Z}, \mathbb{Z}/n) \rightarrow 0 \implies 0 \rightarrow \mathbb{Z}/n \xrightarrow{m} \mathbb{Z}/n \rightarrow 0.$$

Hence $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/n / m(\mathbb{Z}/n) \cong \mathbb{Z} / \gcd(m, n)$.

Similarly, $\text{Tor}_{\mathbb{Z}}^1(\mathbb{Z}/m, \mathbb{Z}/n) \cong \ker(\mathbb{Z}/n \xrightarrow{m} \mathbb{Z}/n) \cong \mathbb{Z} / \gcd(m, n)$.

Proposition A.16 (Basic properties of Tor and Ext). (i) $\text{Tor}_0^R(M, N) \cong M \otimes_R N$ and $\text{Ext}_R^0(M, N) \cong \text{Hom}_R(M, N)$.

(ii) A short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ induces long exact sequences in both Tor and Ext.

(iii) If $R = \mathbb{Z}$ and M is free, then $\text{Tor}_n(M, N) = 0$ and $\text{Ext}^n(M, N) = 0$ for all $n \geq 1$.

(iv) $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m, G) \cong G/mG$ and $\text{Tor}_{\mathbb{Z}}^1(\mathbb{Z}/m, G) \cong \{g \in G \mid mg = 0\}$.

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