

Category Theory

Lecture Notes

M1-PhD — 2025–2026

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“I thought the notion of category was needed to axiomatise natural transformations.” — Saunders Mac Lane

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Preface

Category theory provides a unifying language for mathematics. Originally introduced by Eilenberg and MacLane in the 1940s to formalise natural transformations in algebraic topology, it has since become an indispensable tool in algebra, geometry, logic, computer science, and mathematical physics.

This text is designed for a one-semester graduate course. We assume familiarity with basic algebra (groups, rings, modules) and point-set topology, but no prior knowledge of category theory itself. The exposition emphasises both the abstract framework and its concrete manifestations: every definition is illustrated by examples drawn from algebra, topology, and analysis.

Organisation. Chapters 1 and 2 lay the foundations: categories, functors, natural transformations, and the duality principle. Chapters 3 and 4 develop limits, colimits, and adjunctions. Later chapters treat representability, the Yoneda lemma, Kan extensions, abelian categories, and monoidal categories.

Exercises. Each chapter ends with a graded exercise set. Working through these is essential for mastering the material.

Notation and Conventions

Symbol	Meaning
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Natural numbers (including 0), integers, rationals, reals, complex numbers
\mathbb{K}	An arbitrary field
Set, Grp, Ab	Categories of sets, groups, abelian groups
Ring, Top, Vect$_{\mathbb{K}}$	Categories of rings, topological spaces, \mathbb{K} -vector spaces
Mod$_R$	Category of left R -modules
Cat	Category of small categories
$\text{Hom}_{\mathcal{C}}(A, B)$	Set of morphisms $A \rightarrow B$ in \mathcal{C}
\mathcal{C}^{op}	Opposite (dual) category of \mathcal{C}
$F: \mathcal{C} \rightarrow \mathcal{D}$	Functor from \mathcal{C} to \mathcal{D}
$\alpha: F \Rightarrow G$	Natural transformation from F to G
id_A	Identity morphism on A
$g \circ f$	Composition: first f , then g
$[\mathcal{C}, \mathcal{D}]$ or $\text{Fun}(\mathcal{C}, \mathcal{D})$	Functor category

Throughout, “category” means a locally small category unless otherwise stated. We use the terms “morphism”, “arrow”, and “map” interchangeably. All rings are assumed to have a multiplicative identity.

Chapter 1

Categories, Functors, and Natural Transformations

Categories, functors, and natural transformations constitute the three fundamental notions of category theory. A category axiomatises the idea of “objects with structure-preserving maps between them”. A functor is a morphism of categories, and a natural transformation is a morphism of functors. Already at this first level of abstraction one finds a striking pattern: the passage from objects to morphisms between objects repeats at every stage.

In this chapter we introduce these three notions carefully, give copious examples, and establish basic terminology—monomorphisms, epimorphisms, isomorphisms, initial and terminal objects—that will be used throughout the course.

1.1 Categories

Definition 1.1.1 (Category). A **category** \mathcal{C} consists of the following data:

- (i) A collection $\text{Ob}(\mathcal{C})$ of *objects*.
- (ii) For every ordered pair of objects $A, B \in \text{Ob}(\mathcal{C})$, a set $\text{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* (or *arrows*) from A to B . We write $f: A \rightarrow B$ to indicate $f \in \text{Hom}_{\mathcal{C}}(A, B)$.
- (iii) For every triple of objects A, B, C , a *composition law*
$$\circ: \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \longrightarrow \text{Hom}_{\mathcal{C}}(A, C), \quad (g, f) \longmapsto g \circ f.$$

(iv) For every object A , an *identity morphism* $\text{id}_A \in \text{Hom}_{\mathcal{C}}(A, A)$.

These data are subject to two axioms:

(C1) Associativity. For all $f: A \rightarrow B$, $g: B \rightarrow C$, $h: C \rightarrow D$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(C2) Identity. For all $f: A \rightarrow B$,

$$f \circ \text{id}_A = f = \text{id}_B \circ f.$$

Remark 1.1.2. We require the hom-sets to be pairwise disjoint: every morphism f has a uniquely determined *domain* (or *source*) and *codomain* (or *target*). Some authors encode this by defining a morphism as a triple (f, A, B) .

Notation 1.1.3. We write $\text{Hom}_{\mathcal{C}}(A, B)$, or $\mathcal{C}(A, B)$, or $\text{Mor}_{\mathcal{C}}(A, B)$ interchangeably. When the ambient category is clear we simply write $\text{Hom}(A, B)$.

1.2 First examples of categories

Example 1.2.1 (Algebraic categories). The following are categories:

- (i) **Set**: objects are sets, morphisms are functions.
- (ii) **Grp**: objects are groups, morphisms are group homomorphisms.
- (iii) **Ab**: objects are abelian groups, morphisms are group homomorphisms.
- (iv) **Ring**: objects are (unital) rings, morphisms are ring homomorphisms (preserving 1).
- (v) **Mod_R**: for a ring R , objects are left R -modules, morphisms are R -linear maps.
- (vi) **Vect_ℚ**: objects are \mathbb{K} -vector spaces, morphisms are \mathbb{K} -linear maps. This is the special case **Mod_ℚ**.

In each case, composition is the usual composition of functions and the identity morphism on an object A is the identity function $\text{id}_A: A \rightarrow A$.

Example 1.2.2 (Topological category). **Top**: objects are topological spaces, morphisms are continuous maps.

Example 1.2.3 (Ordered sets as categories). Let (P, \leq) be a preorder (a set equipped with a reflexive, transitive relation). Define a category \mathcal{C}_P by:

- $\text{Ob}(\mathcal{C}_P) = P$.
- For $a, b \in P$:

$$\text{Hom}(a, b) = \begin{cases} \{*\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise.} \end{cases}$$

Transitivity gives composition; reflexivity gives identities. If (P, \leq) is a partial order, distinct objects are never isomorphic.

Example 1.2.4 (Monoids as categories). A monoid (M, \cdot, e) may be viewed as a category with a single object $*$ and $\text{Hom}(*, *) = M$. Composition is the monoid operation and the identity morphism is e . Conversely, every category with exactly one object is a monoid. A group is a one-object category in which every morphism is invertible.

Example 1.2.5 (The category **Cat**). The category **Cat** has small categories as objects and functors (defined in section 1.6) as morphisms.

1.3 Small and locally small categories

Definition 1.3.1 (Size conditions). A category \mathcal{C} is called:

- (i) **small** if $\text{Ob}(\mathcal{C})$ is a set (not a proper class);
- (ii) **locally small** if for every pair of objects A, B , the collection $\text{Hom}_{\mathcal{C}}(A, B)$ is a set.

A category that is not small is called **large**.

Remark 1.3.2. Every small category is locally small. The categories **Set**, **Grp**, **Top**, etc. are locally small but not small (their collection of objects is a proper class). Every preorder viewed as a category is small provided the underlying set is a set. Throughout this text, “category” means “locally small category” unless otherwise stated.

1.4 Monomorphisms, epimorphisms, and isomorphisms

Definition 1.4.1 (Monomorphism). A morphism $f: A \rightarrow B$ in a category \mathcal{C} is a **monomorphism** (or is *monic*) if for every pair of morphisms $g_1, g_2: C \rightarrow A$,

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

We denote a monomorphism by $f: A \rightarrowtail B$.

Definition 1.4.2 (Epimorphism). A morphism $f: A \rightarrow B$ is an **epimorphism** (or is *epic*) if for every pair $h_1, h_2: B \rightarrow D$,

$$h_1 \circ f = h_2 \circ f \implies h_1 = h_2.$$

We denote an epimorphism by $f: A \twoheadrightarrow B$.

Definition 1.4.3 (Isomorphism). A morphism $f: A \rightarrow B$ is an **isomorphism** if there exists a morphism $g: B \rightarrow A$ such that

$$g \circ f = \text{id}_A \quad \text{and} \quad f \circ g = \text{id}_B.$$

The morphism g is unique and is called the *inverse* of f , written f^{-1} . Two objects are *isomorphic*, $A \cong B$, if there exists an isomorphism between them.

Proposition 1.4.4. *Every isomorphism is both a monomorphism and an epimorphism.*

Proof. Let $f: A \rightarrow B$ be an isomorphism with inverse g . If $f \circ g_1 = f \circ g_2$, apply g on the left: $g_1 = g \circ f \circ g_1 = g \circ f \circ g_2 = g_2$, so f is monic. Dually, f is epic. \square

Remark 1.4.5. The converse fails in general. In **Ring**, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both monic and epic, but it is not an isomorphism.

Example 1.4.6 (Mono and epi in concrete categories). (i) In **Set**: monomorphisms are injections, epimorphisms are surjections, isomorphisms are bijections.

(ii) In **Grp**: monomorphisms are injective homomorphisms, epimorphisms are surjective homomorphisms.

(iii) In **Top**: monomorphisms are injective continuous maps; epimorphisms are surjective continuous maps (but *not* necessarily quotient maps).

Lemma 1.4.7. *Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be morphisms.*

(i) *If $g \circ f$ is monic, then f is monic.*

(ii) *If $g \circ f$ is epic, then g is epic.*

(iii) *If both f and g are monic (resp. epic), then so is $g \circ f$.*

Proof. (i) Suppose $g \circ f$ is monic and $f \circ h_1 = f \circ h_2$. Then $g \circ f \circ h_1 = g \circ f \circ h_2$, hence $h_1 = h_2$. Parts (ii) and (iii) are similar. \square

1.5 Initial and terminal objects

Definition 1.5.1 (Initial object). An object $I \in \mathcal{C}$ is **initial** if for every object $A \in \mathcal{C}$ there exists a unique morphism $I \rightarrow A$.

Definition 1.5.2 (Terminal object). An object $T \in \mathcal{C}$ is **terminal** if for every object $A \in \mathcal{C}$ there exists a unique morphism $A \rightarrow T$.

Definition 1.5.3 (Zero object). An object that is both initial and terminal is called a **zero object**.

Proposition 1.5.4. *If an initial (resp. terminal, resp. zero) object exists, it is unique up to unique isomorphism.*

Proof. Let I and I' be initial. By the universal property there exist unique morphisms $f: I \rightarrow I'$ and $g: I' \rightarrow I$. Then $g \circ f: I \rightarrow I$ must equal id_I (uniqueness of the morphism $I \rightarrow I$), and similarly $f \circ g = \text{id}_{I'}$. The argument for terminal objects is dual. \square

Example 1.5.5 (Initial and terminal objects). (i) In **Set**: the empty set \emptyset is initial; any singleton $\{*\}$ is terminal.

(ii) In **Grp** and **Ab**: the trivial group $\{e\}$ is a zero object.

(iii) In **Ring**: \mathbb{Z} is initial (unique ring homomorphism $\mathbb{Z} \rightarrow R$ sending $1 \mapsto 1_R$); the zero ring $\{0\}$ is terminal.

(iv) In **Top**: the empty space is initial; any one-point space is terminal.

(v) In **Vect $_{\mathbb{K}}$** : the zero space $\{0\}$ is a zero object.

Remark 1.5.6. If \mathcal{C} has a zero object 0 , then for any objects A, B there is a unique morphism $A \rightarrow 0 \rightarrow B$, called the *zero morphism* and denoted 0_{AB} (or simply 0).

1.6 Functors

Definition 1.6.1 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A **(covariant) functor** $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of:

(i) A map on objects: $A \mapsto F(A)$ for each $A \in \text{Ob}(\mathcal{C})$.

(ii) A map on morphisms: for each $f: A \rightarrow B$ in \mathcal{C} , a morphism $F(f): F(A) \rightarrow F(B)$ in \mathcal{D} .

These must satisfy:

(F1) $F(\text{id}_A) = \text{id}_{F(A)}$ for every object A .

(F2) $F(g \circ f) = F(g) \circ F(f)$ for every composable pair f, g .

Definition 1.6.2 (Contravariant functor). A **contravariant functor** $F: \mathcal{C} \rightarrow \mathcal{D}$ is a (covariant) functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$. Equivalently, F reverses the direction of morphisms: $F(f): F(B) \rightarrow F(A)$ when $f: A \rightarrow B$, and $F(g \circ f) = F(f) \circ F(g)$.

Example 1.6.3 (Forgetful functors). (i) $U: \mathbf{Grp} \rightarrow \mathbf{Set}$: sends a group to its underlying set and a homomorphism to the underlying function.

(ii) $U: \mathbf{Top} \rightarrow \mathbf{Set}$: forgets the topology.

(iii) $U: \mathbf{Vect}_{\mathbb{K}} \rightarrow \mathbf{Ab}$: forgets scalar multiplication, retaining only the additive group.

(iv) $U: \mathbf{Ring} \rightarrow \mathbf{Ab}$: sends a ring to its underlying additive group.

Example 1.6.4 (Free functors). (i) $F: \mathbf{Set} \rightarrow \mathbf{Grp}$: sends a set S to the free group on S .

(ii) $F: \mathbf{Set} \rightarrow \mathbf{Vect}_{\mathbb{K}}$: sends a set S to the vector space with basis S .

(iii) $F: \mathbf{Set} \rightarrow \mathbf{Ab}$: sends a set S to the free abelian group $\mathbb{Z}^{(S)} = \bigoplus_{s \in S} \mathbb{Z}$.

Example 1.6.5 (Power-set functor). There are two natural functors $\mathbf{Set} \rightarrow \mathbf{Set}$ associated with the power set:

(i) **Covariant**: $\mathcal{P}: \mathbf{Set} \rightarrow \mathbf{Set}$ sends a set S to $\mathcal{P}(S)$ and a function $f: S \rightarrow T$ to the direct image $f_*: \mathcal{P}(S) \rightarrow \mathcal{P}(T)$, $A \mapsto f(A)$.

(ii) **Contravariant**: $\mathcal{P}^{\text{op}}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ sends $f: S \rightarrow T$ to the pre-image $f^*: \mathcal{P}(T) \rightarrow \mathcal{P}(S)$, $B \mapsto f^{-1}(B)$.

Example 1.6.6 (Hom-functors). For any locally small category \mathcal{C} and object $A \in \mathcal{C}$:

(i) The **covariant hom-functor** $\text{Hom}_{\mathcal{C}}(A, -): \mathcal{C} \rightarrow \mathbf{Set}$ sends $B \mapsto \text{Hom}(A, B)$ and $f: B \rightarrow C$ to $f_*: \text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$, $g \mapsto f \circ g$.

(ii) The **contravariant hom-functor** $\text{Hom}_{\mathcal{C}}(-, B): \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ sends $A \mapsto \text{Hom}(A, B)$ and $f: A' \rightarrow A$ to $f^*: \text{Hom}(A, B) \rightarrow \text{Hom}(A', B)$, $g \mapsto g \circ f$.

Example 1.6.7 (Fundamental group functor). Let \mathbf{Top}_* denote the category of pointed topological spaces and base-point-preserving continuous

maps. The fundamental group construction defines a functor

$$\pi_1: \mathbf{Top}_* \longrightarrow \mathbf{Grp}, \quad (X, x_0) \longmapsto \pi_1(X, x_0).$$

A continuous map $f: (X, x_0) \rightarrow (Y, y_0)$ induces the group homomorphism $f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ given by $[\gamma] \mapsto [f \circ \gamma]$.

1.7 Faithful, full, and essentially surjective functors

Definition 1.7.1 (Faithful, full, fully faithful). A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called:

- (i) **faithful** if for every pair $A, B \in \mathcal{C}$ the map

$$F_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

is injective;

- (ii) **full** if every $F_{A,B}$ is surjective;
 (iii) **fully faithful** if every $F_{A,B}$ is bijective.

Definition 1.7.2 (Essentially surjective). A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is **essentially surjective** (or *essentially surjective on objects*) if for every $D \in \mathcal{D}$ there exists $C \in \mathcal{C}$ with $F(C) \cong D$.

Example 1.7.3. (i) Every forgetful functor (e.g. $U: \mathbf{Grp} \rightarrow \mathbf{Set}$) is faithful. It is not full in general: not every function between the underlying sets of two groups is a homomorphism.

- (ii) The inclusion functor $\mathbf{Ab} \hookrightarrow \mathbf{Grp}$ is fully faithful: a homomorphism between abelian groups is the same thing whether viewed in \mathbf{Ab} or \mathbf{Grp} .

- (iii) A functor is an *equivalence of categories* if and only if it is fully faithful and essentially surjective (assuming the axiom of choice).

Proposition 1.7.4. *A fully faithful functor reflects isomorphisms: if $F(f)$ is an isomorphism in \mathcal{D} , then f is an isomorphism in \mathcal{C} .*

Proof. Let $F(f): F(A) \rightarrow F(B)$ be an isomorphism with inverse h . Since F is full, $h = F(g)$ for some $g: B \rightarrow A$. Then $F(g \circ f) = F(g) \circ F(f) = \text{id}_{F(A)} = F(\text{id}_A)$. Since F is faithful, $g \circ f = \text{id}_A$. Similarly $f \circ g = \text{id}_B$. \square

1.8 Natural transformations

Definition 1.8.1 (Natural transformation). Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors. A **natural transformation** $\alpha: F \Rightarrow G$ is a family of morphisms in \mathcal{D} ,

$$\{\alpha_A: F(A) \rightarrow G(A)\}_{A \in \text{Ob}(\mathcal{C})},$$

called the *components* of α , such that for every morphism $f: A \rightarrow B$ in \mathcal{C} the following *naturality square* commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

That is, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ for all $f: A \rightarrow B$.

Definition 1.8.2 (Natural isomorphism). A natural transformation $\alpha: F \Rightarrow G$ is a **natural isomorphism** if every component α_A is an isomorphism. We then write $F \cong G$ and say F and G are *naturally isomorphic*.

Example 1.8.3 (Double dual). Let $\mathbf{Vect}_{\mathbb{K}}^{\text{fd}}$ denote the category of finite-dimensional \mathbb{K} -vector spaces. Define $\eta: \text{Id} \Rightarrow (-)^{**}$ by

$$\eta_V: V \longrightarrow V^{**}, \quad v \longmapsto (\varphi \mapsto \varphi(v)).$$

This is a natural isomorphism (each η_V is an isomorphism by dimension counting). Crucially, the isomorphism $V \cong V^{**}$ is *natural* in V , whereas the isomorphism $V \cong V^*$ (which requires choosing a basis) is not.

Example 1.8.4 (Determinant). Let **Ring** be the category of commutative rings. Consider the functors $\mathrm{GL}_n, (-)^\times: \mathbf{Ring} \rightarrow \mathbf{Grp}$ sending a commutative ring R to the group of invertible $n \times n$ matrices over R and to the group of units R^\times , respectively. The determinant gives a natural transformation

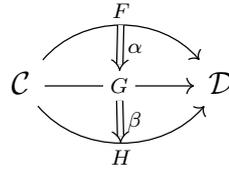
$$\det: \mathrm{GL}_n \Longrightarrow (-)^\times.$$

Naturality says: for every ring homomorphism $\varphi: R \rightarrow S$ and every $M \in \mathrm{GL}_n(R)$, $\varphi(\det M) = \det(\varphi(M))$.

Definition 1.8.5 (Vertical composition). Let $\alpha: F \Rightarrow G$ and $\beta: G \Rightarrow H$ be natural transformations between functors $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$. Their **vertical composition** $\beta \circ \alpha: F \Rightarrow H$ has components

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A.$$

This is depicted schematically as:



Lemma 1.8.6. *Vertical composition is associative, and the identity natural transformation $\mathrm{id}_F: F \Rightarrow F$ (with components $(\mathrm{id}_F)_A = \mathrm{id}_{F(A)}$) serves as identity.*

Proof. Both statements follow immediately from the corresponding properties of morphism composition in \mathcal{D} . \square

Definition 1.8.7 (Horizontal composition). Let $\alpha: F \Rightarrow G$ between functors $\mathcal{C} \rightarrow \mathcal{D}$ and $\beta: H \Rightarrow K$ between functors $\mathcal{D} \rightarrow \mathcal{E}$. The **horizontal composition** (or *Godement product*) $\beta * \alpha: H \circ F \Rightarrow K \circ G$ has components

$$(\beta * \alpha)_A = \beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}.$$

The two expressions are equal by naturality of β :

$$\begin{array}{ccc} HF(A) & \xrightarrow{H(\alpha_A)} & HG(A) \\ \beta_{F(A)} \downarrow & & \downarrow \beta_{G(A)} \\ KF(A) & \xrightarrow{K(\alpha_A)} & KG(A) \end{array}$$

1.9 Functor categories

Definition 1.9.1 (Functor category). Let \mathcal{C} and \mathcal{D} be categories with \mathcal{C} small. The **functor category** $[\mathcal{C}, \mathcal{D}]$ (also written $\text{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$) is the category whose:

- objects are functors $F: \mathcal{C} \rightarrow \mathcal{D}$;
- morphisms are natural transformations;
- composition is vertical composition of natural transformations.

Remark 1.9.2. If \mathcal{C} is small and \mathcal{D} is locally small, then $[\mathcal{C}, \mathcal{D}]$ is locally small.

Example 1.9.3 (Presheaf categories). A particularly important special case: let \mathcal{C} be a small category. The functor category

$$\widehat{\mathcal{C}} = [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$

is called the **category of presheaves** on \mathcal{C} . Its objects are contravariant functors $\mathcal{C} \rightarrow \mathbf{Set}$, called *presheaves*. This category has remarkable properties (it is complete, cocomplete, and cartesian closed) that we shall explore later.

Proposition 1.9.4. *A natural transformation $\alpha: F \Rightarrow G$ is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ if and only if each component α_A is an isomorphism in \mathcal{D} .*

Proof. If α is a natural isomorphism, its inverse α^{-1} has components $(\alpha^{-1})_A = (\alpha_A)^{-1}$. One checks naturality of α^{-1} by applying $(-)^{-1}$ to the naturality squares of α . Conversely, if each α_A is an isomorphism, define $\beta_A = (\alpha_A)^{-1}$.

The naturality of β follows from that of α : for $f: A \rightarrow B$,

$$\beta_B \circ G(f) = \beta_B \circ G(f) \circ \alpha_A \circ \beta_A = \beta_B \circ \alpha_B \circ F(f) \circ \beta_A = F(f) \circ \beta_A. \quad \square$$

Definition 1.9.5 (Equivalence of categories). An **equivalence of categories** between \mathcal{C} and \mathcal{D} consists of functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ together with natural isomorphisms

$$\eta: \text{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F \quad \text{and} \quad \varepsilon: F \circ G \xrightarrow{\sim} \text{Id}_{\mathcal{D}}.$$

We write $\mathcal{C} \simeq \mathcal{D}$ and say \mathcal{C} and \mathcal{D} are *equivalent*.

Theorem 1.9.6. *A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is part of an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Proof. (\Rightarrow) Suppose F is an equivalence with quasi-inverse G and natural isomorphisms $\eta: \text{Id} \xrightarrow{\sim} GF$ and $\varepsilon: FG \xrightarrow{\sim} \text{Id}$.

Essentially surjective: For $D \in \mathcal{D}$, $F(G(D)) \cong D$ via ε_D .

Faithful: Suppose $F(f) = F(g)$ for $f, g: A \rightarrow B$. Then $GF(f) = GF(g)$. Naturality of η gives $\eta_B \circ f = GF(f) \circ \eta_A = GF(g) \circ \eta_A = \eta_B \circ g$, and since η_B is an isomorphism, $f = g$.

Full: Let $h: F(A) \rightarrow F(B)$. Consider $f = \eta_B^{-1} \circ G(h) \circ \eta_A: A \rightarrow B$. Then $F(f) = \varepsilon_{F(B)} \circ FG(h) \circ (\varepsilon_{F(A)})^{-1}$. By naturality of ε , $\varepsilon_{F(B)} \circ FG(h) = h \circ \varepsilon_{F(A)}$, so $F(f) = h$.

(\Leftarrow) Assume F is fully faithful and essentially surjective. For each $D \in \mathcal{D}$, choose $G(D) \in \mathcal{C}$ with an isomorphism $\varepsilon_D: FG(D) \xrightarrow{\sim} D$. For $h: D \rightarrow D'$, define $G(h)$ to be the unique morphism (existing by full faithfulness) satisfying $F(G(h)) = \varepsilon_{D'}^{-1} \circ h \circ \varepsilon_D$. One verifies G is a functor and that ε and the induced η are natural isomorphisms. \square

1.10 Exercises for Chapter 1

Exercise 1.10.1. Define a category **Rel** whose objects are sets and whose morphisms $A \rightarrow B$ are relations $R \subseteq A \times B$. What is composition? What are the identity morphisms? Show that **Rel** is indeed a category.

Exercise 1.10.2. A *groupoid* is a category in which every morphism is an isomorphism. Show that the following are groupoids:

- (i) A group, viewed as a one-object category.

- (ii) The *fundamental groupoid* $\Pi_1(X)$ of a topological space X : objects are points of X , morphisms $x \rightarrow y$ are homotopy classes of paths from x to y .
- (iii) The *core* $\text{core}(\mathcal{C})$ of any category \mathcal{C} : same objects, but only isomorphisms.

Exercise 1.10.3. Let \mathcal{C} be a category and $A \in \mathcal{C}$ an object. The *slice category* (or *over category*) \mathcal{C}/A has:

- Objects: morphisms $f: X \rightarrow A$ in \mathcal{C} .
- Morphisms: from $(f: X \rightarrow A)$ to $(g: Y \rightarrow A)$, a morphism $h: X \rightarrow Y$ in \mathcal{C} such that $g \circ h = f$:

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & & A \end{array}$$

Verify that \mathcal{C}/A is a category. What is the terminal object? Describe $\mathbf{Set}/\{0, 1\}$ explicitly.

Exercise 1.10.4. Let $F: \mathcal{A} \rightarrow \mathcal{C}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors. Define the *comma category* $(F \downarrow G)$ and show that slice categories and coslice categories are special cases.

- Exercise 1.10.5.**
- (i) Show that in any category, the composition of two monomorphisms is a monomorphism.
 - (ii) Show that in \mathbf{Set} , a morphism is an epimorphism if and only if it is surjective.
 - (iii) Find an example of a morphism in a concrete category that is both monic and epic but not an isomorphism (beyond the example of $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in \mathbf{Ring}).

Exercise 1.10.6. For any object A in a category \mathcal{C} , show that $\text{End}_{\mathcal{C}}(A) = \text{Hom}_{\mathcal{C}}(A, A)$ forms a monoid under composition. When is this monoid a group?

Exercise 1.10.7. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors. Show that the composition $G \circ F: \mathcal{A} \rightarrow \mathcal{C}$ is again a functor. Verify that functor composition is associative and that the identity functor $\text{Id}_{\mathcal{C}}$ is a two-sided identity.

Exercise 1.10.8. Let $F: \mathcal{C} \rightarrow \mathcal{C}$ be a functor. Show that the following are equivalent:

- (i) $\alpha: \text{Id}_{\mathcal{C}} \Rightarrow F$ is a natural transformation;
- (ii) For every morphism $f: A \rightarrow B$, one has $F(f) \circ \alpha_A = \alpha_B \circ f$.

Exercise 1.10.9. Prove the *interchange law* for horizontal and vertical composition of natural transformations. Specifically, given

$$\begin{array}{ccc} \mathcal{A} & & \mathcal{B} \\ \begin{array}{c} \xrightarrow{F_1} \\ \downarrow \alpha \\ \xrightarrow{F_2} \end{array} & & \begin{array}{c} \xrightarrow{G_1} \\ \downarrow \beta \\ \xrightarrow{G_2} \end{array} \\ \mathcal{B} & & \mathcal{C} \end{array}$$

and natural transformations $\alpha': F_2 \Rightarrow F_3$, $\beta': G_2 \Rightarrow G_3$, show that

$$(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha).$$

Exercise 1.10.10. A category \mathcal{C} is *skeletal* if isomorphic objects are equal. A *skeleton* of \mathcal{C} is a full subcategory that is skeletal and contains exactly one object from each isomorphism class. Show that every category has a skeleton and that inclusion of the skeleton is an equivalence of categories.

Exercise 1.10.11. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors, $\alpha: F \Rightarrow G$ a natural transformation, and $H: \mathcal{B} \rightarrow \mathcal{C}$ a functor. Define the *whiskering* $\alpha H: FH \Rightarrow GH$ by $(\alpha H)_B = \alpha_{H(B)}$. Show that αH is a natural transformation. Similarly define whiskering on the other side.

Chapter 2

Duality and the Dual Principle

One of the most powerful features of category theory is the *duality principle*: every categorical statement has a dual, obtained by reversing all morphisms. This single observation doubles the theorems one can prove “for free”. In this chapter we make this precise by introducing opposite categories, formulating the duality principle, and examining several concrete dualities. We also study the two-variable hom-functor, which will be essential in later chapters.

2.1 The opposite category

Definition 2.1.1 (Opposite category). Let \mathcal{C} be a category. The **opposite category** (or *dual category*) \mathcal{C}^{op} is defined by:

- (i) $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$.
- (ii) For objects A, B : $\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A)$.
- (iii) Composition in \mathcal{C}^{op} : given $f^{\text{op}}: A \rightarrow B$ and $g^{\text{op}}: B \rightarrow C$ in \mathcal{C}^{op} (corresponding to $f: B \rightarrow A$ and $g: C \rightarrow B$ in \mathcal{C}), we set $g^{\text{op}} \circ f^{\text{op}} = (f \circ g)^{\text{op}}$.
- (iv) The identity on A in \mathcal{C}^{op} is $(\text{id}_A)^{\text{op}} = \text{id}_A$.

Proposition 2.1.2. For any category \mathcal{C} , $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$.

Proof. The objects coincide. For morphisms, $\text{Hom}_{(\mathcal{C}^{\text{op}})^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}^{\text{op}}}(B, A) = \text{Hom}_{\mathcal{C}}(A, B)$. Composition: $(f^{\text{op}})^{\text{op}} \circ (g^{\text{op}})^{\text{op}} = ((g^{\text{op}} \circ f^{\text{op}})^{\text{op}})^{\text{op}} = ((f \circ g)^{\text{op}})^{\text{op}} = f \circ g$. Everything reduces to the original data of \mathcal{C} . \square

- Example 2.1.3.** (i) If (P, \leq) is a poset viewed as a category, then P^{op} is the poset (P, \geq) .
- (ii) If G is a group viewed as a one-object category, then G^{op} is the *opposite group*: same elements, with multiplication $a \cdot^{\text{op}} b = b \cdot a$. For abelian groups, $G^{\text{op}} \cong G$.
- (iii) \mathbf{Set}^{op} is a perfectly valid category, but it is not equivalent to any “familiar” category of structured sets (it is, however, equivalent to the category of complete atomic Boolean algebras, by a non-trivial theorem).

2.2 The duality principle

Definition 2.2.1 (Dual statement). Let Σ be a statement formulated purely in the language of category theory (objects, morphisms, composition, identities). The **dual statement** Σ^{op} is obtained from Σ by:

- reversing the direction of every morphism;
- replacing each composition $g \circ f$ by $f \circ g$;
- interchanging domain and codomain.

Theorem 2.2.2 (Duality principle). *If a statement Σ holds in every category, then so does its dual Σ^{op} . More generally, if Σ holds in a category \mathcal{C} , then Σ^{op} holds in \mathcal{C}^{op} .*

Proof. A statement about \mathcal{C}^{op} is, by definition, a statement about \mathcal{C} with all arrows reversed. Since \mathcal{C}^{op} is a legitimate category, any theorem valid for all categories applies to it. Unwinding the definitions in \mathcal{C}^{op} recovers the dual statement in \mathcal{C} . □

Remark 2.2.3. The duality principle immediately gives us dual pairs of concepts:

Concept	Dual concept
monomorphism	epimorphism
initial object	terminal object
product	coproduct
limit	colimit
kernel	cokernel
left adjoint	right adjoint

Once we prove a theorem about monomorphisms, the dual theorem about epimorphisms follows automatically.

Example 2.2.4. We showed in definition 1.4.7 that if $g \circ f$ is monic then f is monic. By duality: if $g \circ f$ is epic, then g is epic. No additional proof is needed—it follows from the proof of the original statement applied to \mathcal{C}^{op} .

2.3 Functors and duality

Proposition 2.3.1. *A contravariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is the same thing as a covariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$, and also the same as a covariant functor $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$.*

Proof. A covariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ assigns to each morphism $f^{\text{op}}: B \rightarrow A$ in \mathcal{C}^{op} (i.e. $f: A \rightarrow B$ in \mathcal{C}) a morphism $F(f^{\text{op}}): F(B) \rightarrow F(A)$ in \mathcal{D} , which is precisely a contravariant assignment. The preservation of composition and identities translates directly. The argument for $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$ is analogous. \square

Remark 2.3.2. Every functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces a functor $F^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ defined by $F^{\text{op}}(A) = F(A)$ on objects and $F^{\text{op}}(f^{\text{op}}) = (F(f))^{\text{op}}$ on morphisms. In fact, $(-)^{\text{op}}$ is a functor $\mathbf{Cat} \rightarrow \mathbf{Cat}$.

2.4 Concrete dualities

The abstract duality principle tells us about dual *concepts*, but there also exist concrete *equivalences* $\mathcal{C}^{\text{op}} \simeq \mathcal{D}$ for specific categories. These are called *concrete dualities* or *dualities of categories*.

Example 2.4.1 (Stone duality). The category **Stone** of Stone spaces (compact, Hausdorff, totally disconnected topological spaces) and continuous maps is equivalent to $\mathbf{Bool}^{\text{op}}$, where **Bool** is the category of Boolean algebras and their homomorphisms:

$$\mathbf{Stone} \simeq \mathbf{Bool}^{\text{op}}.$$

The equivalence is implemented by the functor sending a Stone space to its Boolean algebra of clopen subsets, and conversely sending a Boolean algebra to its space of ultrafilters.

Example 2.4.2 (Pontryagin duality). Let **LCA** be the category of locally compact abelian groups and continuous homomorphisms. Pontryagin duality provides a contravariant equivalence

$$(-)^{\wedge}: \mathbf{LCA}^{\text{op}} \xrightarrow{\sim} \mathbf{LCA},$$

where $G^{\wedge} = \text{Hom}_{\text{cts}}(G, \mathbb{R}/\mathbb{Z})$ is the *Pontryagin dual*. The natural map $G \rightarrow G^{\wedge\wedge}$, $g \mapsto (\chi \mapsto \chi(g))$, is an isomorphism.

Example 2.4.3 (Gelfand duality). There is an equivalence

$$\mathbf{CHaus}^{\text{op}} \simeq \mathbf{cC}^*\mathbf{Alg}_1,$$

where **CHaus** is the category of compact Hausdorff spaces and $\mathbf{cC}^*\mathbf{Alg}_1$ is the category of commutative unital C^* -algebras. The functor sends $X \mapsto C(X, \mathbb{C})$ (continuous functions) and the quasi-inverse sends a C^* -algebra to its maximal ideal space (Gelfand spectrum).

Example 2.4.4 (Finite-dimensional vector space duality). The dual space functor $(-)^*: \mathbf{Vect}_{\mathbb{K}}^{\text{fd}} \rightarrow (\mathbf{Vect}_{\mathbb{K}}^{\text{fd}})^{\text{op}}$ is an equivalence of categories. Combined with $(\mathbf{Vect}_{\mathbb{K}}^{\text{fd}})^{\text{op}} \simeq \mathbf{Vect}_{\mathbb{K}}^{\text{fd}}$ (via the double dual), we see that finite-dimensional vector spaces are self-dual.

2.5 The hom-functor in both variables

Definition 2.5.1 (Bifunctor). Let \mathcal{C} , \mathcal{D} , \mathcal{E} be categories. A **bifunctor** is a functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$.

Remark 2.5.2. A bifunctor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ determines, for each $C \in \mathcal{C}$, a functor $F(C, -): \mathcal{D} \rightarrow \mathcal{E}$ and, for each $D \in \mathcal{D}$, a functor $F(-, D): \mathcal{C} \rightarrow \mathcal{E}$. Conversely, a family of functors in each variable that is “jointly functorial” determines a bifunctor.

Proposition 2.5.3. For any locally small category \mathcal{C} , the assignment

$$\mathrm{Hom}_{\mathcal{C}}(-, -): \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$$

defined on objects by $(A, B) \mapsto \mathrm{Hom}_{\mathcal{C}}(A, B)$ and on morphisms by

$$(f: A' \rightarrow A, g: B \rightarrow B') \longmapsto (h \mapsto g \circ h \circ f): \mathrm{Hom}(A, B) \rightarrow \mathrm{Hom}(A', B'),$$

is a bifunctor (i.e. a functor from the product category $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to \mathbf{Set}).

Proof. We verify the functor axioms. On identities:

$$\mathrm{Hom}(\mathrm{id}_A, \mathrm{id}_B)(h) = \mathrm{id}_B \circ h \circ \mathrm{id}_A = h,$$

so $\mathrm{Hom}(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{\mathrm{Hom}(A, B)}$.

On composition: let $f': A'' \rightarrow A'$, $f: A' \rightarrow A$, $g: B \rightarrow B'$, $g': B' \rightarrow B''$. We need $\mathrm{Hom}(f \circ f', g' \circ g) = \mathrm{Hom}(f', g') \circ \mathrm{Hom}(f, g)$. The left side maps $h \mapsto (g' \circ g) \circ h \circ (f \circ f')$. The right side maps $h \mapsto g' \circ (g \circ h \circ f) \circ f'$, which is the same by associativity. \square

Remark 2.5.4. The bifunctor $\mathrm{Hom}(-, -)$ encodes the two “partial” hom-functors from definition 1.6.6:

- Fixing the first argument: $\mathrm{Hom}(A, -)$ is covariant.
- Fixing the second argument: $\mathrm{Hom}(-, B)$ is contravariant (i.e. covariant from $\mathcal{C}^{\mathrm{op}}$).

The two partial functors determine the bifunctor (they agree on pairs of identities and satisfy a compatibility condition that is automatic from the bifunctor structure).

Proposition 2.5.5. *For any object A in a locally small category \mathcal{C} , the covariant hom-functor $\text{Hom}(A, -): \mathcal{C} \rightarrow \mathbf{Set}$ preserves all limits that exist in \mathcal{C} .*

Proof. This is a consequence of the universal property of limits and will be proved rigorously in chapter 3. The key idea is that a natural bijection $\text{Hom}(A, \lim F) \cong \lim \text{Hom}(A, F-)$ follows directly from the defining universal property of the limit. \square

Definition 2.5.6 (Representable functor (preview)). A functor $F: \mathcal{C} \rightarrow \mathbf{Set}$ is **representable** if it is naturally isomorphic to $\text{Hom}_{\mathcal{C}}(A, -)$ for some object $A \in \mathcal{C}$. The object A is called a *representing object*. A contravariant functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ is representable if $F \cong \text{Hom}_{\mathcal{C}}(-, B)$ for some B . Representable functors are central to category theory; we shall study them in depth in connection with the Yoneda lemma.

2.6 Exercises for Chapter 2

Exercise 2.6.1. Let \mathcal{C} be a category.

- (i) Verify carefully that \mathcal{C}^{op} as defined in definition 2.1.1 satisfies the axioms of a category.
- (ii) Show that $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$ (equality, not merely isomorphism).

Exercise 2.6.2. Show that A is an initial object in \mathcal{C} if and only if A is a terminal object in \mathcal{C}^{op} . Deduce the uniqueness (up to unique isomorphism) of terminal objects from the corresponding result for initial objects via duality.

Exercise 2.6.3. Show that f is a monomorphism in \mathcal{C} if and only if f^{op} is an epimorphism in \mathcal{C}^{op} .

Exercise 2.6.4. Let \mathcal{C} be small and \mathcal{D} any category. Construct a canonical isomorphism of categories

$$[\mathcal{C}, \mathcal{D}]^{\text{op}} \cong [\mathcal{C}, \mathcal{D}^{\text{op}}].$$

(Hint: what happens to naturality squares when you reverse arrows in

the target?)

Exercise 2.6.5. A category \mathcal{C} is called *self-dual* if $\mathcal{C} \simeq \mathcal{C}^{\text{op}}$.

- (i) Show that $\mathbf{Vect}_{\mathbb{K}}^{\text{fd}}$ is self-dual.
- (ii) Show that any groupoid is self-dual.
- (iii) Is \mathbf{Set} self-dual? Justify.

Exercise 2.6.6. Let \mathcal{C} and \mathcal{D} be categories. Define the product category $\mathcal{C} \times \mathcal{D}$ and show that

$$(\mathcal{C} \times \mathcal{D})^{\text{op}} \cong \mathcal{C}^{\text{op}} \times \mathcal{D}^{\text{op}}.$$

Exercise 2.6.7. Let \mathcal{C} be a locally small category. Give a detailed verification that $\text{Hom}_{\mathcal{C}}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$ is a functor by checking the functor axioms (preservation of identities and composition) explicitly.

Exercise 2.6.8. Let $F, G: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be contravariant functors. Spell out explicitly what a natural transformation $\alpha: F \Rightarrow G$ looks like: what are the components, and what does the naturality condition say? Draw the naturality square and compare it with the covariant case.

Exercise 2.6.9. Show that if $\mathcal{C} \simeq \mathcal{D}$, then $\mathcal{C}^{\text{op}} \simeq \mathcal{D}^{\text{op}}$.

Exercise 2.6.10. Let L be a bounded lattice, viewed as a category (poset). Show that L^{op} is again a bounded lattice, with joins and meets interchanged. Describe the dual of the lattice of open sets of a topological space X .

Exercise 2.6.11. Let \mathcal{C} be a category and $A \in \mathcal{C}$ an object. The *coslice category* (or *under category*) A/\mathcal{C} has objects $f: A \rightarrow X$ and morphisms the evident commutative triangles. Show that $(A/\mathcal{C})^{\text{op}} \cong \mathcal{C}^{\text{op}}/A$.

Exercise 2.6.12. Let \mathcal{C} be a locally small category and $A \in \mathcal{C}$. Show that the contravariant hom-functor $\text{Hom}(-, A)$ is representable as a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$, with representing object A (viewed as an object of \mathcal{C}^{op}).

Chapter 3

Limits and Colimits

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Limits and colimits are the categorical generalisations of constructions that pervade all of mathematics: products, intersections, kernels, pullbacks, inverse limits, and their duals. Understanding them in full generality is one of the main rewards of learning category theory. In this chapter we develop the theory systematically, beginning with the language of diagrams and cones, proceeding to the universal property that defines a limit, and then examining the most important special cases. We prove the fundamental existence theorem—that products and equalisers suffice for all finite limits—and establish the crucial relationship between limits and representable functors.

3.1 Diagrams and index categories

Definition 3.1.1 (Diagram). Let \mathcal{C} be a category and let \mathcal{J} be a small category (the *index category* or *shape category*). A **diagram of shape \mathcal{J} in \mathcal{C}** is a functor

$$F: \mathcal{J} \longrightarrow \mathcal{C}.$$

We write F_j for the image of an object $j \in \text{Ob}(\mathcal{J})$ and $F_\alpha: F_j \rightarrow F_k$ for the image of a morphism $\alpha: j \rightarrow k$ in \mathcal{J} .

Example 3.1.2. The following small categories appear constantly as index categories.

- (i) **Discrete categories.** Let \mathcal{J} be a discrete category on a set I (objects I , only identity morphisms). A diagram $F: \mathcal{J} \rightarrow \mathcal{C}$ is simply a family of objects $\{F_i\}_{i \in I}$.
- (ii) **The parallel pair.** Let $\mathcal{J} = (\bullet \rightrightarrows \bullet)$, i.e. two objects $0, 1$ with two non-identity morphisms $\alpha, \beta: 0 \rightarrow 1$. A diagram of this shape is a pair of parallel morphisms $f, g: A \rightarrow B$ in \mathcal{C} .
- (iii) **The span.** $\mathcal{J} = (\bullet \leftarrow \bullet \rightarrow \bullet)$. A diagram is a span $A \xleftarrow{f} C \xrightarrow{g} B$.
- (iv) **The cospan.** $\mathcal{J} = (\bullet \rightarrow \bullet \leftarrow \bullet)$. A diagram is a cospan $A \xrightarrow{f} C \xleftarrow{g} B$.

Remark 3.1.3. We shall sometimes write a diagram simply as $\{F_j, F_\alpha\}_{j \in \mathcal{J}}$ when the index category is understood. A diagram is nothing more than a functor; the terminology “diagram” merely signals that we are about to take its limit or colimit.

3.2 Cones and cocones

Definition 3.2.1 (Cone). Let $F: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram. A **cone over F** is a pair $(N, \{\pi_j\}_{j \in \mathcal{J}})$ consisting of an object $N \in \mathcal{C}$ (the *vertex* of the cone) and a family of morphisms $\pi_j: N \rightarrow F_j$ (the *legs*), one for each object $j \in \mathcal{J}$, such that for every morphism $\alpha: j \rightarrow k$ in \mathcal{J} the following

triangle commutes:

$$\begin{array}{ccc}
 & N & \\
 \pi_j \swarrow & & \searrow \pi_k \\
 F_j & \xrightarrow{F_\alpha} & F_k
 \end{array}$$

That is, $F_\alpha \circ \pi_j = \pi_k$ for every $\alpha: j \rightarrow k$.

Definition 3.2.2 (Morphism of cones). Let (N, π_j) and (N', π'_j) be two cones over F . A **morphism of cones** from (N, π_j) to (N', π'_j) is a morphism $u: N \rightarrow N'$ in \mathcal{C} such that $\pi'_j \circ u = \pi_j$ for every $j \in \mathcal{J}$:

$$\begin{array}{ccc}
 N & \xrightarrow{u} & N' \\
 \pi_j \searrow & & \swarrow \pi'_j \\
 & F_j &
 \end{array}$$

The cones over F and their morphisms form a category, which we denote $\text{Cone}(F)$.

Definition 3.2.3 (Cocone). Dually, a **cocone under F** is a pair $(N, \{\iota_j\}_{j \in \mathcal{J}})$ where $\iota_j: F_j \rightarrow N$ and $\iota_k \circ F_\alpha = \iota_j$ for every $\alpha: j \rightarrow k$:

$$\begin{array}{ccc}
 F_j & \xrightarrow{F_\alpha} & F_k \\
 \iota_j \searrow & & \swarrow \iota_k \\
 & N &
 \end{array}$$

Cocones under F form a category $\text{Cocone}(F)$.

Remark 3.2.4. Let $\Delta_N: \mathcal{J} \rightarrow \mathcal{C}$ denote the constant functor sending every object to N and every morphism to id_N . Then a cone (N, π_j) over F is exactly a natural transformation $\pi: \Delta_N \Rightarrow F$. Dually, a cocone is a natural transformation $\iota: F \Rightarrow \Delta_N$. This reformulation will be important when we study the relationship between limits and the functor category $\text{Fun}(\mathcal{J}, \mathcal{C})$.

3.3 Limits

Definition 3.3.1 (Limit). Let $F: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram. A **limit** of F is a cone $(\varprojlim F, \{\pi_j\}_j)$ that is *terminal* in $\mathbf{Cone}(F)$. Explicitly, this means:

- (a) $(\varprojlim F, \pi_j)$ is a cone over F ; and
- (b) for every cone (N, ψ_j) over F , there exists a *unique* morphism $u: N \rightarrow \varprojlim F$ such that $\pi_j \circ u = \psi_j$ for every $j \in \mathcal{J}$:

$$\begin{array}{ccc}
 N & \overset{\exists! u}{\dashrightarrow} & \varprojlim F \\
 \psi_j \searrow & & \swarrow \pi_j \\
 & F_j &
 \end{array}$$

We also write $\lim_{j \in \mathcal{J}} F_j$ or simply $\lim F$. The morphisms π_j are called the **projection morphisms** (or **canonical projections**).

Proposition 3.3.2 (Uniqueness of limits). *If $F: \mathcal{J} \rightarrow \mathcal{C}$ admits a limit, then the limit is unique up to unique isomorphism.*

Proof. This is a standard argument for terminal objects. Suppose (L, π_j) and (L', π'_j) are both limits of F . By the universal property of L , there is a unique $u: L' \rightarrow L$ with $\pi_j \circ u = \pi'_j$ for all j . By the universal property of L' , there is a unique $v: L \rightarrow L'$ with $\pi'_j \circ v = \pi_j$ for all j . Then $\pi_j \circ (u \circ v) = \pi'_j \circ v = \pi_j$ for all j , and by uniqueness applied to L with the cone (L, π_j) , we get $u \circ v = \text{id}_L$. Similarly $v \circ u = \text{id}_{L'}$. Hence u is an isomorphism, and it is the unique isomorphism of cones.

$$\begin{array}{ccc}
 & v & \\
 L & \xrightarrow{\quad} & L' \\
 & \text{u} & \\
 \pi_j \searrow & & \swarrow \pi'_j \\
 & F_j &
 \end{array}$$

□

3.4 Colimits

Definition 3.4.1 (Colimit). Let $F: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram. A **colimit** of F is a cocone $(\varinjlim F, \{\iota_j\}_j)$ that is *initial* in $\text{Cocone}(F)$. That is, for every cocone (N, ψ_j) under F , there exists a unique morphism $u: \varinjlim F \rightarrow N$ with $u \circ \iota_j = \psi_j$ for all j :

$$\begin{array}{ccc}
 F_j & \xrightarrow{\iota_j} & \varinjlim F \\
 & \searrow \psi_j & \swarrow \exists! u \\
 & & N
 \end{array}$$

The morphisms ι_j are the **coprojection morphisms** (or **canonical injections**).

Remark 3.4.2. A colimit of $F: \mathcal{J} \rightarrow \mathcal{C}$ is the same as a limit of $F^{\text{op}}: \mathcal{J}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$. Hence every theorem about limits has a dual statement about colimits. We will exploit this duality systematically.

3.5 Products and coproducts

Products and coproducts are limits and colimits indexed by a discrete category.

Definition 3.5.1 (Product). Let $\{A_i\}_{i \in I}$ be a family of objects in \mathcal{C} , viewed as a diagram $F: I_{\text{disc}} \rightarrow \mathcal{C}$. The **product** $\prod_{i \in I} A_i$ is the limit of this diagram. Concretely, it is an object $\prod_i A_i$ equipped with projections $\pi_i: \prod_j A_j \rightarrow A_i$ such that for every object N with morphisms $f_i: N \rightarrow A_i$, there exists a unique $\langle f_i \rangle: N \rightarrow \prod_j A_j$ with $\pi_i \circ \langle f_i \rangle = f_i$:

$$\begin{array}{ccccc}
 & & N & & \\
 & f_1 \swarrow & \downarrow f_2 & \searrow f_3 & \\
 A_1 & & \langle f_i \rangle & & A_3 \\
 & \swarrow \pi_1 & \uparrow \pi_2 & \searrow \pi_3 & \\
 & & \prod_j A_j & &
 \end{array}$$

For a binary product we write $A \times B$ with projections $\pi_1: A \times B \rightarrow A$ and $\pi_2: A \times B \rightarrow B$.

Definition 3.5.2 (Coproduct). Dually, the **coproduct** $\coprod_{i \in I} A_i$ is the colimit of a family indexed by a discrete category. It is an object with injections $\iota_i: A_i \rightarrow \coprod_j A_j$ such that for every object N with morphisms $g_i: A_i \rightarrow N$, there is a unique $[g_i]: \coprod_j A_j \rightarrow N$ with $[g_i] \circ \iota_i = g_i$:

$$\begin{array}{ccc}
 A_1 & & A_2 \\
 \searrow^{\iota_1} & & \swarrow^{\iota_2} \\
 & \coprod_j A_j & \\
 \swarrow_{g_1} & \downarrow [g_j] & \searrow_{g_2} \\
 & N &
 \end{array}$$

For a binary coproduct we write $A \amalg B$ or $A + B$.

Example 3.5.3 (Products and coproducts in concrete categories). (i)

Set. The product is the Cartesian product $\prod_i A_i = \{(a_i)_{i \in I} : a_i \in A_i\}$ with the usual projections. The coproduct is the disjoint union $\coprod_i A_i$.

- (ii) **Grp.** The product is the direct product of groups (Cartesian product with component-wise operations). The coproduct is the *free product* $*_i G_i$, which is considerably more complicated than the direct product.
- (iii) **Ab.** Both the product and the finite coproduct coincide with the direct sum $\bigoplus_i A_i$ (for finite families). For infinite families the product is the full direct product and the coproduct is the direct sum (elements with finitely many nonzero components).
- (iv) **Top.** The product is the Cartesian product with the product topology (the coarsest topology making all projections continuous). The coproduct is the disjoint union with the coproduct (disjoint union) topology.
- (v) **Vect $_{\mathbb{K}}$.** The product is the direct product of vector spaces and the finite coproduct is the direct sum. For infinite families, $\prod_i V_i \neq \bigoplus_i V_i$ in general.

below:

$$\begin{array}{ccccc}
 A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B & \xrightarrow{q} & Q \\
 & & & \searrow & \dashrightarrow \exists! u \\
 & & & & N \\
 & & & \searrow h & \\
 & & & & N
 \end{array}$$

We write $Q = \text{coeq}(f, g)$.

Example 3.6.5. In **Set**, the coequaliser of $f, g: A \rightrightarrows B$ is the quotient B/\sim where \sim is the equivalence relation generated by $f(a) \sim g(a)$ for all $a \in A$.

Proposition 3.6.6. *Every coequaliser is an epimorphism.*

Proof. Dual to the proof that equalisers are monic (Proposition 3.6.3). \square

3.7 Kernels and cokernels

Definition 3.7.1 (Kernel). Let \mathcal{C} be a category with a zero object 0 . For a morphism $f: A \rightarrow B$, the **kernel** of f is the equaliser of f and the zero morphism $0_{AB}: A \rightarrow 0 \rightarrow B$:

$$\text{Ker}(f) = \text{eq}(f, 0_{AB}).$$

Concretely, it is a morphism $k: K \rightarrow A$ such that $f \circ k = 0$ and every $h: N \rightarrow A$ with $f \circ h = 0$ factors uniquely through k :

$$\begin{array}{ccccc}
 N & \xrightarrow{h} & A & \xrightarrow{f} & B \\
 & \dashrightarrow \exists! u & \downarrow k & & \\
 & & K & \xrightarrow{k} & A
 \end{array}$$

Definition 3.7.2 (Cokernel). The **cokernel** of $f: A \rightarrow B$ is the coequaliser of f and 0_{AB} :

$$\text{coker}(f) = \text{coeq}(f, 0_{AB}).$$

It is a morphism $c: B \rightarrow Q$ with $c \circ f = 0$, universal with this property.

Example 3.7.3. In \mathbf{Ab} , the kernel of a homomorphism $f: A \rightarrow B$ is $\text{Ker}(f) = \{a \in A : f(a) = 0\}$ with the inclusion into A , and the cokernel is $\text{coker}(f) = B/\text{im}(f)$ with the projection.

Exercise 3.7.4. Show that in any category with a zero object, a kernel is a monomorphism. (This follows from the fact that equalisers are monic.)

3.8 Pullbacks and pushouts

Definition 3.8.1 (Pullback). Given a cospan $A \xrightarrow{f} C \xleftarrow{g} B$, the **pullback** (or **fibre product**) is the limit of this diagram. It is an object $A \times_C B$ equipped with morphisms $p_1: A \times_C B \rightarrow A$ and $p_2: A \times_C B \rightarrow B$ such that $f \circ p_1 = g \circ p_2$, and universal with this property:

$$\begin{array}{ccccc}
 N & & & & \\
 \downarrow \psi_1 & \xrightarrow{\psi_2} & & & \\
 & & A \times_C B & \xrightarrow{p_2} & B \\
 & & \downarrow p_1 & \lrcorner & \downarrow g \\
 & & A & \xrightarrow{f} & C
 \end{array}$$

The symbol \lrcorner in the square indicates that it is a pullback square.

Example 3.8.2 (Pullback in \mathbf{Set}). Given $f: A \rightarrow C$ and $g: B \rightarrow C$ in \mathbf{Set} ,

$$A \times_C B = \{(a, b) \in A \times B : f(a) = g(b)\}.$$

The projections are $(a, b) \mapsto a$ and $(a, b) \mapsto b$.

Example 3.8.3 (Pullback in \mathbf{Top}). In \mathbf{Top} , the pullback is the set-theoretic fibre product $\{(a, b) \in A \times B : f(a) = g(b)\}$ equipped with the subspace topology inherited from $A \times B$.

Example 3.8.4 (Pullback as inverse image). In \mathbf{Set} , consider $f: A \rightarrow C$ and the inclusion $\iota: S \hookrightarrow C$ of a subset. The pullback $A \times_C S$ is the

preimage $f^{-1}(S) = \{a \in A : f(a) \in S\}$.

$$\begin{array}{ccc} f^{-1}(S) & \hookrightarrow & A \\ \downarrow & \lrcorner & \downarrow f \\ S & \xrightarrow{\iota} & C \end{array}$$

Proposition 3.8.5. *If $g: B \rightarrow C$ is a monomorphism, then $p_1: A \times_C B \rightarrow A$ is also a monomorphism. That is, monomorphisms are stable under pullback.*

Proof. Suppose $p_1 \circ u = p_1 \circ v$ for $u, v: N \rightarrow A \times_C B$. Then $f \circ p_1 \circ u = f \circ p_1 \circ v$, hence $g \circ p_2 \circ u = g \circ p_2 \circ v$, hence $p_2 \circ u = p_2 \circ v$ since g is monic. Now u and v are both morphisms $N \rightarrow A \times_C B$ satisfying $p_1 \circ u = p_1 \circ v$ and $p_2 \circ u = p_2 \circ v$. By the uniqueness part of the pullback universal property, $u = v$. \square

Definition 3.8.6 (Pushout). Given a span $A \xleftarrow{f} C \xrightarrow{g} B$, the **pushout** (or **amalgamated sum**) is the colimit of this diagram. It is an object $A \amalg_C B$ with morphisms $q_1: A \rightarrow A \amalg_C B$ and $q_2: B \rightarrow A \amalg_C B$ such that $q_1 \circ f = q_2 \circ g$, universal with this property:

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ f \downarrow & \lrcorner & \downarrow q_2 \\ A & \xrightarrow{q_1} & A \amalg_C B \end{array} \quad \begin{array}{c} \searrow \psi_2 \\ \downarrow \\ \exists! u \\ \downarrow \\ N \end{array}$$

ψ_1

Example 3.8.7 (Pushout in **Set**). Given a span $A \xleftarrow{f} C \xrightarrow{g} B$ in **Set**,

$$A \amalg_C B = (A \amalg B) / \sim$$

where \sim is generated by $f(c) \sim g(c)$ for all $c \in C$.

Example 3.8.8 (Pushout in **Top**). In **Top**, pushouts correspond to gluing constructions. If $f: A \hookrightarrow X$ is a subspace inclusion and $g: A \rightarrow Y$ is

continuous, then $X \amalg_A Y$ is the space obtained by gluing Y to X along A via g . A fundamental instance: if D^n is the n -disc, S^{n-1} its boundary, and $\varphi: S^{n-1} \rightarrow X$ an attaching map, then

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \varphi \downarrow & & \downarrow \\ X & \longrightarrow & X \cup_{\varphi} D^n \end{array}$$

is a pushout square giving the space X with an n -cell attached.

Exercise 3.8.9. Show that epimorphisms are stable under pushout. (This is dual to Proposition 3.8.5.)

Exercise 3.8.10. (Pullback pasting lemma.) Consider a diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

Show that if both squares are pullbacks, then the outer rectangle is a pullback. Show also that if the right square and the outer rectangle are pullbacks, then the left square is a pullback.

3.9 General existence theorem

The following fundamental theorem reduces the existence of all finite limits to two basic building blocks.

Theorem 3.9.1 (Products and equalisers give all finite limits). *A category \mathcal{C} has all finite limits if and only if it has all finite products and all equalisers.*

Proof. The “only if” direction is clear, since products and equalisers are special cases of finite limits.

For the “if” direction, let $F: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram where \mathcal{J} is a finite category. We construct $\lim F$ as an equaliser of two maps between products.

Step 1. Form the products

$$P = \prod_{j \in \text{Ob}(\mathcal{J})} F_j, \quad Q = \prod_{\alpha \in \text{Mor}(\mathcal{J})} F_{\text{cod}(\alpha)},$$

where for each morphism $\alpha: j \rightarrow k$ in \mathcal{J} , the factor of Q indexed by α is $F_k = F_{\text{cod}(\alpha)}$.

Step 2. Define two morphisms $s, t: P \rightarrow Q$ as follows. For each morphism $\alpha: j \rightarrow k$ in \mathcal{J} , the α -component of s is the composite

$$s_\alpha: P \xrightarrow{\pi_j} F_j \xrightarrow{F_\alpha} F_k,$$

and the α -component of t is

$$t_\alpha: P \xrightarrow{\pi_k} F_k.$$

$$\text{eq}(s, t) \xrightarrow{e} P \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} Q$$

Step 3. Let $e: E \rightarrow P$ be the equaliser of s and t . We claim $E = \lim F$ with projections $\pi_j \circ e: E \rightarrow F_j$.

The condition $s \circ e = t \circ e$ means that for every $\alpha: j \rightarrow k$,

$$F_\alpha \circ \pi_j \circ e = \pi_k \circ e,$$

which is exactly the cone condition.

Conversely, if (N, ψ_j) is any cone over F , the universal property of P gives a unique $h: N \rightarrow P$ with $\pi_j \circ h = \psi_j$. The cone condition ensures $s \circ h = t \circ h$, so the universal property of the equaliser gives a unique $u: N \rightarrow E$ with $e \circ u = h$. Then $\pi_j \circ e \circ u = \pi_j \circ h = \psi_j$, as required. \square

Corollary 3.9.2. *A category \mathcal{C} is **(finitely) complete** if and only if it has (finite) products and equalisers. Dually, \mathcal{C} is **(finitely) cocomplete** if and only if it has (finite) coproducts and coequalisers.*

Example 3.9.3. The categories **Set**, **Grp**, **Ab**, **Ring**, **Top**, **Vect $_{\mathbb{K}}$** , and **Mod $_R$** (for any ring R) are all complete and cocomplete.

Remark 3.9.4. The same argument works for arbitrary (small) limits: a category with all small products and equalisers has all small limits. A category with all small limits is called **complete**.

3.10 Limits and the Hom functor

Theorem 3.10.1 (Limits commute with Hom). *Let $F: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram admitting a limit $L = \lim F$. For any object $X \in \mathcal{C}$,*

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim_j F_j) \cong \lim_j \mathrm{Hom}_{\mathcal{C}}(X, F_j),$$

*naturally in X . Here the limit on the right is computed in **Set**.*

Proof. Let (L, π_j) be the limit cone. Define

$$\Phi: \mathrm{Hom}(X, L) \longrightarrow \lim_j \mathrm{Hom}(X, F_j)$$

by $\Phi(h) = (\pi_j \circ h)_{j \in \mathcal{J}}$. We verify that $(\pi_j \circ h)_j$ lies in the limit: for $\alpha: j \rightarrow k$, $F_\alpha \circ \pi_j \circ h = \pi_k \circ h$ since (L, π_j) is a cone.

The map Φ is **injective**: if $\Phi(h) = \Phi(h')$, then $\pi_j \circ h = \pi_j \circ h'$ for all j , so $h = h'$ by the uniqueness in the universal property of L .

The map Φ is **surjective**: given $(f_j)_j \in \lim_j \mathrm{Hom}(X, F_j)$, the family $(f_j: X \rightarrow F_j)_j$ is a cone over F with vertex X . By the universal property of L , there exists a unique $h: X \rightarrow L$ with $\pi_j \circ h = f_j$ for all j . Then $\Phi(h) = (f_j)_j$.

Naturality in X is straightforward: for $\varphi: X' \rightarrow X$, $\Phi(h \circ \varphi) = (\pi_j \circ h \circ \varphi)_j$, which is the image of $\Phi(h)$ under pre-composition with φ . \square

Corollary 3.10.2. *Every representable functor $\mathrm{Hom}(X, -)$ preserves all limits that exist in \mathcal{C} .*

Remark 3.10.3. Dually, for colimits:

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_j F_j, X) \cong \lim_j \mathrm{Hom}_{\mathcal{C}}(F_j, X).$$

Note the *contravariance*: the colimit on the left becomes a limit on the right.

3.11 Functors and limits

Definition 3.11.1 (Preservation of limits). A functor $G: \mathcal{C} \rightarrow \mathcal{D}$ **preserves the limit** of a diagram $F: \mathcal{J} \rightarrow \mathcal{C}$ if whenever (L, π_j) is a limit cone for F , then $(GL, G\pi_j)$ is a limit cone for $G \circ F$ in \mathcal{D} .

We say G **preserves all limits** (or is **continuous**) if it preserves the limit of every diagram that admits a limit.

Definition 3.11.2 (Reflection and creation of limits). A functor $G: \mathcal{C} \rightarrow \mathcal{D}$ **reflects the limit** of $F: \mathcal{J} \rightarrow \mathcal{C}$ if whenever (L, π_j) is a cone for F such that $(GL, G\pi_j)$ is a limit cone for $G \circ F$, then (L, π_j) is already a limit cone for F .

G **creates the limit** of F if for every limit cone (M, ψ_j) for $G \circ F$ in \mathcal{D} , there exists a unique cone (L, π_j) for F with $GL = M$ and $G\pi_j = \psi_j$, and moreover this cone is a limit of F .

Example 3.11.3. The forgetful functor $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ creates all limits. Indeed, given a diagram of groups, one first forms the limit in \mathbf{Set} (using the Cartesian product construction), then shows this set carries a unique group structure making all projections group homomorphisms. The same is true for $\mathbf{Ab} \rightarrow \mathbf{Set}$, $\mathbf{Ring} \rightarrow \mathbf{Set}$, $\mathbf{Mod}_R \rightarrow \mathbf{Set}$, etc.

3.12 Filtered colimits

Definition 3.12.1 (Filtered category). A small category \mathcal{J} is **filtered** if:

- (i) \mathcal{J} is non-empty;
- (ii) for every pair of objects $j, k \in \mathcal{J}$, there exists an object ℓ and morphisms $j \rightarrow \ell$ and $k \rightarrow \ell$;
- (iii) for every pair of parallel morphisms $\alpha, \beta: j \rightrightarrows k$, there exists a morphism $\gamma: k \rightarrow \ell$ such that $\gamma \circ \alpha = \gamma \circ \beta$.

Definition 3.12.2 (Filtered colimit). A **filtered colimit** (or **direct limit**) is a colimit of a diagram $F: \mathcal{J} \rightarrow \mathcal{C}$ where \mathcal{J} is filtered. We also write $\varinjlim F$.

Example 3.12.3. A partially ordered set (I, \leq) that is directed (every pair has an upper bound) is a filtered category. A functor $F: I \rightarrow \mathcal{C}$ is a directed system, and its colimit is the classical direct limit.

Theorem 3.12.4. *In \mathbf{Set} , filtered colimits commute with finite limits. More precisely, if \mathcal{J} is filtered and \mathcal{K} is finite, then for any functor $F: \mathcal{J} \times \mathcal{K} \rightarrow \mathcal{C}$,*

$\mathcal{K} \rightarrow \mathbf{Set}$,

$$\lim_{\rightarrow j} \lim_{\leftarrow k} F(j, k) \cong \lim_{\leftarrow k} \lim_{\rightarrow j} F(j, k).$$

Proof. We sketch the proof for the key cases. The filtered colimit $\lim_{\rightarrow j} F_j$ in \mathbf{Set} is computed as $(\coprod_j F_j)/\sim$ where $x \in F_j$ and $y \in F_k$ satisfy $x \sim y$ iff there exist morphisms $\alpha: j \rightarrow \ell$ and $\beta: k \rightarrow \ell$ with $F_\alpha(x) = F_\beta(y)$.

Commutation with finite products: An element of $\lim_{\rightarrow j} (F_j \times G_j)$ is an equivalence class $[(x, y)]$ with $x \in F_j, y \in G_j$. The filteredness condition (ii) ensures that any pair of elements from $\lim_{\rightarrow j} F_j$ and $\lim_{\rightarrow j} G_j$ can be represented at a common index, establishing a bijection with $(\lim_{\rightarrow j} F_j) \times (\lim_{\rightarrow j} G_j)$.

Commutation with equalisers: Given parallel natural transformations $f, g: F \rightrightarrows G$, the filteredness condition (iii) ensures that $\lim_{\rightarrow j} \text{eq}(f_j, g_j) \cong \text{eq}(\lim_{\rightarrow j} f_j, \lim_{\rightarrow j} g_j)$.

Since finite products and equalisers generate all finite limits (Theorem 3.9.1), the general statement follows. \square

Example 3.12.5. A filtered colimit of groups is a group, a filtered colimit of rings is a ring, etc. For instance, $\mathbb{Q} = \lim_{\rightarrow n \geq 1} \frac{1}{n}\mathbb{Z}$ as abelian groups, and algebraic closures can be constructed as filtered colimits of finite extensions.

3.13 Exercises

Exercise 3.13.1. Show that a terminal object is the limit of the empty diagram (i.e. $\mathcal{J} = \emptyset$), and an initial object is the colimit of the empty diagram.

Exercise 3.13.2. Show that if \mathcal{C} has binary products and pullbacks, then \mathcal{C} has equalisers. *Hint:* Given $f, g: A \rightrightarrows B$, consider the pullback of $\langle f, g \rangle: A \rightarrow B \times B$ along the diagonal $\Delta: B \rightarrow B \times B$.

Exercise 3.13.3. Show that if \mathcal{C} has a terminal object and pullbacks, then \mathcal{C} has all finite limits.

Exercise 3.13.4. Show that for any small diagram $F: \mathcal{J} \rightarrow \mathbf{Set}$, the

colimit is

$$\varinjlim F = \left(\prod_{j \in \mathcal{J}} F_j \right) / \sim$$

where \sim is the equivalence relation generated by: $x \in F_j$ is equivalent to $F_\alpha(x) \in F_k$ for every $\alpha: j \rightarrow k$ in \mathcal{J} .

Exercise 3.13.5. Let \mathcal{C} be a complete category and \mathcal{J} a small category. Show that $\text{Fun}(\mathcal{J}, \mathcal{C})$ is also complete, and that limits are computed “pointwise”: for a diagram $\Phi: \mathcal{K} \rightarrow \text{Fun}(\mathcal{J}, \mathcal{C})$,

$$(\lim_k \Phi_k)(j) \cong \lim_k (\Phi_k(j))$$

for each $j \in \mathcal{J}$.

Exercise 3.13.6. Let \mathcal{C} be a category with equalisers. Show that if $\alpha: F \Rightarrow G$ is a natural transformation between diagrams $F, G: \mathcal{J} \rightarrow \mathcal{C}$ such that each α_j is a monomorphism, and if $\lim F$ and $\lim G$ both exist, then the induced morphism $\lim F \rightarrow \lim G$ is also a monomorphism.

Exercise 3.13.7. Let p be a prime. Consider the inverse system $\cdots \rightarrow \mathbb{Z}/p^3\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ in **Ab**. Show that the inverse limit is the ring of p -adic integers \mathbb{Z}_p .

Exercise 3.13.8. Show that filtered colimits in \mathbf{Mod}_R (for R a ring) are exact. That is, if $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ is a short exact sequence of directed systems, then $0 \rightarrow \varinjlim F \rightarrow \varinjlim G \rightarrow \varinjlim H \rightarrow 0$ is exact.

Chapter 4

Adjunctions

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Adjunctions are arguably the most important concept in category theory. Saunders Mac Lane famously wrote that “adjoint functors arise everywhere.” An adjunction captures a universal relationship between two functors going in opposite directions, generalising at a stroke the notions of free construction, tensor–hom duality, and a host of other fundamental correspondences throughout mathematics.

In this chapter we give three equivalent definitions of adjunctions, prove their equivalence in detail, explore the fundamental examples, and establish the key theorems—most notably, that left adjoints preserve colimits and right adjoints preserve limits. We conclude with Freyd’s adjoint functor theorems.

4.1 Definition via natural bijection

Definition 4.1.1 (Adjunction). Let \mathcal{C} and \mathcal{D} be categories and $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$ be functors. We say that F is **left adjoint** to G (equivalently, G is **right adjoint** to F), written $F \dashv G$, if there is a natural bijection

$$\Phi_{A,B}: \text{Hom}_{\mathcal{D}}(FA, B) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(A, GB)$$

for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$, natural in both variables. That is, Φ is a natural isomorphism

$$\Phi: \text{Hom}_{\mathcal{D}}(F-, -) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(-, G-)$$

of bifunctors $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$.

Remark 4.1.2. We depict an adjunction as:

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \end{array}$$

The symbol \perp (or \dashv) indicates $F \dashv G$.

Let us spell out what naturality means in this context.

Proposition 4.1.3 (Naturality conditions). *The bijection $\Phi_{A,B}$ is natural in A and B if and only if:*

(i) **Naturality in B :** For every $g: B \rightarrow B'$ and $f: FA \rightarrow B$,

$$\Phi_{A,B'}(g \circ f) = Gg \circ \Phi_{A,B}(f).$$

(ii) **Naturality in A :** For every $h: A' \rightarrow A$ and $f: FA \rightarrow B$,

$$\Phi_{A',B}(f \circ Fh) = \Phi_{A,B}(f) \circ h.$$

Proof. This is a direct translation of the naturality squares for the bifunctor

$\text{Hom}_{\mathcal{D}}(F-, -) \Rightarrow \text{Hom}_{\mathcal{C}}(-, G-)$. For (i), the naturality square in B is:

$$\begin{array}{ccc} \text{Hom}(FA, B) & \xrightarrow{\Phi_{A,B}} & \text{Hom}(A, GB) \\ g \circ - \downarrow & & \downarrow Gg \circ - \\ \text{Hom}(FA, B') & \xrightarrow{\Phi_{A,B'}} & \text{Hom}(A, GB') \end{array}$$

Commutativity gives $Gg \circ \Phi_{A,B}(f) = \Phi_{A,B'}(g \circ f)$. The argument for (ii) is analogous. \square

4.2 Unit and counit

Definition 4.2.1 (Unit and counit). Given an adjunction $F \dashv G$ with bijection Φ , define:

- The **unit** $\eta: \text{Id}_{\mathcal{C}} \Rightarrow GF$ by $\eta_A = \Phi_{A,FA}(\text{id}_{FA})$ for each $A \in \mathcal{C}$.
- The **counit** $\varepsilon: FG \Rightarrow \text{Id}_{\mathcal{D}}$ by $\varepsilon_B = \Phi_{GB,B}^{-1}(\text{id}_{GB})$ for each $B \in \mathcal{D}$.

Proposition 4.2.2. *The unit η and counit ε are natural transformations. Moreover, the bijection Φ can be recovered from them:*

$$\Phi_{A,B}(f: FA \rightarrow B) = Gf \circ \eta_A, \quad \Phi_{A,B}^{-1}(g: A \rightarrow GB) = \varepsilon_B \circ Fg.$$

Proof. **Recovery formula.** For $f: FA \rightarrow B$, naturality of Φ in B gives

$$\Phi_{A,B}(f) = \Phi_{A,B}(f \circ \text{id}_{FA}) = Gf \circ \Phi_{A,FA}(\text{id}_{FA}) = Gf \circ \eta_A.$$

Similarly, for $g: A \rightarrow GB$, naturality in A gives

$$\Phi_{A,B}^{-1}(g) = \Phi_{A,B}^{-1}(\text{id}_{GB} \circ g) = \Phi_{GB,B}^{-1}(\text{id}_{GB}) \circ Fg = \varepsilon_B \circ Fg.$$

Naturality of η . We must show that for $h: A \rightarrow A'$, $\eta_{A'} \circ h = GFh \circ \eta_A$:

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ \eta_A \downarrow & & \downarrow \eta_{A'} \\ GFA & \xrightarrow{GFh} & GFA' \end{array}$$

We have $\eta_{A'} \circ h = \Phi_{A',FA'}(\text{id}_{FA'}) \circ h = \Phi_{A,FA'}(\text{id}_{FA'} \circ Fh) = \Phi_{A,FA'}(Fh) = GFh \circ \eta_A$, using naturality in A for the second equality and the recovery formula for the last.

Naturality of ε is proved dually. \square

4.3 Triangle identities

Theorem 4.3.1 (Triangle identities). *Let $F \dashv G$ with unit η and counit ε . Then the following two composites are identities:*

(a) $\varepsilon F \circ F\eta = \text{id}_F$, i.e. for every A :

$$\begin{array}{ccc} FA & \xrightarrow{F\eta_A} & FGFA \\ & \searrow & \downarrow \varepsilon_{FA} \\ & & FA \end{array}$$

(b) $G\varepsilon \circ \eta G = \text{id}_G$, i.e. for every B :

$$\begin{array}{ccc} GB & \xrightarrow{\eta_{GB}} & GFGB \\ & \searrow & \downarrow G\varepsilon_B \\ & & GB \end{array}$$

These are called the **triangle identities** (or **zig-zag identities**) because of their shape in the following diagram:

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow & \downarrow \varepsilon_F \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow & \downarrow G\varepsilon \\ & & G \end{array}$$

Proof. (a): We must show $\varepsilon_{FA} \circ F\eta_A = \text{id}_{FA}$. By the recovery formula (Proposition 4.2.2), $\Phi_{A,FA}^{-1}(\eta_A) = \varepsilon_{FA} \circ F\eta_A$. But $\eta_A = \Phi_{A,FA}(\text{id}_{FA})$, so $\Phi_{A,FA}^{-1}(\eta_A) = \text{id}_{FA}$. Hence $\varepsilon_{FA} \circ F\eta_A = \text{id}_{FA}$.

(b): We must show $G\varepsilon_B \circ \eta_{GB} = \text{id}_{GB}$. By the recovery formula, $\Phi_{GB,B}(\varepsilon_B) = G\varepsilon_B \circ \eta_{GB}$. But $\varepsilon_B = \Phi_{GB,B}^{-1}(\text{id}_{GB})$, so $\Phi_{GB,B}(\varepsilon_B) = \text{id}_{GB}$. Hence $G\varepsilon_B \circ \eta_{GB} = \text{id}_{GB}$. \square

4.4 Equivalence of definitions

We now state and prove the equivalence of three different ways of specifying an adjunction.

Theorem 4.4.1 (Three equivalent definitions of adjunction). *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be functors. The following data are in bijective correspondence:*

(A) A natural isomorphism $\Phi: \text{Hom}_{\mathcal{D}}(F-, -) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(-, G-)$.

(B) Natural transformations $\eta: \text{Id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon: FG \Rightarrow \text{Id}_{\mathcal{D}}$ satisfying the triangle identities:

$$\varepsilon F \circ F\eta = \text{id}_F, \quad G\varepsilon \circ \eta G = \text{id}_G.$$

(C) For each $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$ and a morphism $\eta_A: A \rightarrow GFA$ that is **universal**: for every $B \in \mathcal{D}$ and $g: A \rightarrow GB$, there exists a unique $f: FA \rightarrow B$ with $Gf \circ \eta_A = g$:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & \searrow g & \downarrow Gf \\ & & GB \end{array}$$

Proof. (A) \Rightarrow (B): Given Φ , define $\eta_A = \Phi_{A,FA}(\text{id}_{FA})$ and $\varepsilon_B = \Phi_{GB,B}^{-1}(\text{id}_{GB})$. By Proposition 4.2.2, these are natural transformations. The triangle identities are proved in Theorem 4.3.1.

(B) \Rightarrow (A): Given (η, ε) satisfying the triangle identities, define

$$\Phi_{A,B}(f) = Gf \circ \eta_A, \quad \Psi_{A,B}(g) = \varepsilon_B \circ Fg.$$

We verify $\Psi \circ \Phi = \text{id}$ and $\Phi \circ \Psi = \text{id}$.

For $f: FA \rightarrow B$:

$$\begin{aligned} \Psi(\Phi(f)) &= \Psi(Gf \circ \eta_A) = \varepsilon_B \circ F(Gf \circ \eta_A) \\ &= \varepsilon_B \circ FGf \circ F\eta_A = f \circ \varepsilon_{FA} \circ F\eta_A \quad (\text{naturality of } \varepsilon) \\ &= f \circ \text{id}_{FA} = f \quad (\text{triangle identity (a)}). \end{aligned}$$

For $g: A \rightarrow GB$:

$$\begin{aligned} \Phi(\Psi(g)) &= \Phi(\varepsilon_B \circ Fg) = G(\varepsilon_B \circ Fg) \circ \eta_A \\ &= G\varepsilon_B \circ GFg \circ \eta_A = G\varepsilon_B \circ \eta_{GB} \circ g \quad (\text{naturality of } \eta) \\ &= \text{id}_{GB} \circ g = g \quad (\text{triangle identity (b)}). \end{aligned}$$

Naturality of Φ in both variables follows from the naturality of η and the functoriality of G .

(A) \Leftrightarrow (C): Given Φ , setting $\eta_A = \Phi_{A,FA}(\text{id}_{FA})$ provides the universal morphism. The universality statement is precisely the bijectivity of $\Phi_{A,B}$: given $g: A \rightarrow GB$, the unique f with $Gf \circ \eta_A = g$ is $f = \Phi_{A,B}^{-1}(g)$.

Conversely, given universal arrows (η_A) , define $\Phi_{A,B}(f) = Gf \circ \eta_A$. The universality ensures this is a bijection. Naturality follows from the naturality of the collection $(\eta_A)_A$ (which itself follows from universality applied to the composites $GFh \circ \eta_A$). \square

4.5 Fundamental examples

4.5.1 Free–forgetful adjunctions

Example 4.5.1 (Free group). Let $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ be the forgetful functor and $F: \mathbf{Set} \rightarrow \mathbf{Grp}$ the free group functor. Then $F \dashv U$:

$$\begin{array}{ccc} & F & \\ \text{Set} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \text{Grp} \\ & U & \end{array}$$

The adjunction bijection is

$$\text{Hom}_{\mathbf{Grp}}(F(S), G) \cong \text{Hom}_{\mathbf{Set}}(S, U(G)) :$$

a group homomorphism from the free group on S is determined by where the generators go, i.e. by a function $S \rightarrow U(G)$.

The unit $\eta_S: S \rightarrow UF(S)$ sends each element to the corresponding generator in the free group. The counit $\varepsilon_G: FU(G) \rightarrow G$ sends a word in the “generators” $U(G)$ to its evaluation in G .

Example 4.5.2 (Free abelian group). The free abelian group functor $\mathbb{Z}[-]: \mathbf{Set} \rightarrow \mathbf{Ab}$ is left adjoint to the forgetful functor $U: \mathbf{Ab} \rightarrow \mathbf{Set}$:

$$\text{Hom}_{\mathbf{Ab}}(\mathbb{Z}[S], A) \cong \text{Hom}_{\mathbf{Set}}(S, U(A)).$$

Here $\mathbb{Z}[S] = \bigoplus_{s \in S} \mathbb{Z}$ is the free abelian group on S .

Example 4.5.3 (Free R -module). For a ring R , the free module functor $R[-]: \mathbf{Set} \rightarrow \mathbf{Mod}_R$ is left adjoint to the forgetful functor.

Example 4.5.4 (Free–forgetful between **Ab** and **Grp**). The inclusion functor $\iota: \mathbf{Ab} \hookrightarrow \mathbf{Grp}$ has a left adjoint: the abelianisation $\text{ab}: \mathbf{Grp} \rightarrow \mathbf{Ab}$ sending G to $G/[G, G]$.

$$\text{Hom}_{\mathbf{Ab}}(G/[G, G], A) \cong \text{Hom}_{\mathbf{Grp}}(G, \iota A).$$

4.5.2 Tensor–Hom adjunction

Example 4.5.5 (Tensor–Hom). Let R be a commutative ring and M an R -module. Then $(- \otimes_R M) \dashv \text{Hom}_R(M, -)$:

$$\text{Hom}_R(N \otimes_R M, P) \cong \text{Hom}_R(N, \text{Hom}_R(M, P))$$

naturally in N and P . This is the universal property of the tensor product.

4.5.3 Product–Internal Hom (Cartesian closed categories)

Example 4.5.6 (Cartesian closed). In a Cartesian closed category (e.g. **Set**), for each object B the product functor $(- \times B)$ has a right adjoint, the **internal Hom** (or **exponential**) $B^{(-)}$ or $[B, -]$:

$$\text{Hom}(A \times B, C) \cong \text{Hom}(A, C^B).$$

In **Set**, $C^B = \{f: B \rightarrow C\}$ is the set of functions. This is the **currying** correspondence in computer science.

4.5.4 Direct and inverse image (sheaves)

Example 4.5.7 (Direct and inverse image). Let $f: X \rightarrow Y$ be a continuous map of topological spaces. There is an adjunction

$$f^* \dashv f_*$$

between the inverse image functor $f^*: \mathbf{Sh}(Y) \rightarrow \mathbf{Sh}(X)$ and the direct image functor $f_*: \mathbf{Sh}(X) \rightarrow \mathbf{Sh}(Y)$ on categories of sheaves. This adjunction is fundamental in algebraic geometry and sheaf theory.

4.5.5 Suspension–loops adjunction

Example 4.5.8 (Suspension and loops). In the category \mathbf{Top}_* of pointed topological spaces, the (reduced) suspension functor Σ is left adjoint to the loops functor Ω :

$$\Sigma \dashv \Omega, \quad [\Sigma X, Y]_* \cong [X, \Omega Y]_*$$

where $[-, -]_*$ denotes pointed homotopy classes. This adjunction underlies the theory of stable homotopy.

Example 4.5.9 (Galois connections). If \mathcal{C} and \mathcal{D} are posets (viewed as categories), then an adjunction $F \dashv G$ is the same as a **Galois connection**: $F(a) \leq b$ if and only if $a \leq G(b)$.

4.6 Left adjoints preserve colimits

Theorem 4.6.1 (Left adjoints preserve colimits). *If $F \dashv G$ with $F: \mathcal{C} \rightarrow \mathcal{D}$, then F preserves all colimits that exist in \mathcal{C} .*

Proof. Let $D: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram with colimit $(\varinjlim D, \iota_j)$. We must show that $(F(\varinjlim D), F\iota_j)$ is a colimit of $F \circ D$.

Let (N, ψ_j) be a cocone under $F \circ D$ in \mathcal{D} , so $\psi_j: FD_j \rightarrow N$ with compatibility. For each j , applying the adjunction bijection gives

$$\tilde{\psi}_j = \Phi_{D_j, N}(\psi_j): D_j \rightarrow GN.$$

Claim: $(GN, \tilde{\psi}_j)$ is a cocone under D in \mathcal{C} . Indeed, for $\alpha: j \rightarrow k$ in \mathcal{J} :

$$\begin{aligned} \tilde{\psi}_k \circ D_\alpha &= \Phi_{D_k, N}(\psi_k) \circ D_\alpha \\ &= \Phi_{D_j, N}(\psi_k \circ FD_\alpha) \quad (\text{naturality of } \Phi \text{ in } A) \\ &= \Phi_{D_j, N}(\psi_j) = \tilde{\psi}_j \quad (\text{cocone condition for } \psi). \end{aligned}$$

By the universal property of $\varinjlim D$, there is a unique $\tilde{u}: \varinjlim D \rightarrow GN$ with $\tilde{u} \circ \iota_j = \tilde{\psi}_j$ for all j .

Now set $u = \Phi_{\varinjlim D, N}^{-1}(\tilde{u}): F(\varinjlim D) \rightarrow N$. Then for each j , using naturality of Φ^{-1} in A :

$$u \circ F\iota_j = \Phi_{D_j, N}^{-1}(\tilde{u} \circ \iota_j) = \Phi_{D_j, N}^{-1}(\tilde{\psi}_j) = \psi_j.$$

This gives existence.

Uniqueness: If $u': F(\varinjlim D) \rightarrow N$ also satisfies $u' \circ F\iota_j = \psi_j$, then $\Phi_{\varinjlim D, N}(u') \circ \iota_j = \tilde{\psi}_j$ for all j , so by uniqueness for $\varinjlim D$, $\Phi(u') = \tilde{u} = \Phi(u)$, hence $u' = u$.

$$\begin{array}{ccc} FD_j & \xrightarrow{F\iota_j} & F(\varinjlim D) \\ & \searrow \psi_j & \downarrow \exists! u \\ & & N \end{array}$$

□

4.7 Right adjoints preserve limits

Theorem 4.7.1 (Right adjoints preserve limits). *If $F \dashv G$ with $G: \mathcal{D} \rightarrow \mathcal{C}$, then G preserves all limits that exist in \mathcal{D} .*

Proof. Let $D: \mathcal{J} \rightarrow \mathcal{D}$ be a diagram with limit (L, π_j) , where $\pi_j: L \rightarrow D_j$. We show $(GL, G\pi_j)$ is a limit of $G \circ D$.

Let (N, ψ_j) be a cone over $G \circ D$ in \mathcal{C} , so $\psi_j: N \rightarrow GD_j$. By the adjunction, for each j let $\tilde{\psi}_j = \Phi_{N, D_j}^{-1}(\psi_j): FN \rightarrow D_j$.

Claim: $(FN, \tilde{\psi}_j)$ is a cone over D . For $\alpha: j \rightarrow k$:

$$\begin{aligned} D_\alpha \circ \tilde{\psi}_j &= D_\alpha \circ \Phi_{N, D_j}^{-1}(\psi_j) \\ &= \Phi_{N, D_k}^{-1}(GD_\alpha \circ \psi_j) \quad (\text{naturality of } \Phi^{-1} \text{ in } B) \\ &= \Phi_{N, D_k}^{-1}(\psi_k) = \tilde{\psi}_k \quad (\text{cone condition for } \psi). \end{aligned}$$

By the universal property of L , there is a unique $\tilde{v}: FN \rightarrow L$ with $\pi_j \circ \tilde{v} = \tilde{\psi}_j$. Set $v = \Phi_{N, L}(\tilde{v}): N \rightarrow GL$. Then

$$G\pi_j \circ v = G\pi_j \circ \Phi_{N, L}(\tilde{v}) = \Phi_{N, D_j}(\pi_j \circ \tilde{v}) = \Phi_{N, D_j}(\tilde{\psi}_j) = \psi_j,$$

using naturality of Φ in B .

Uniqueness follows as before from the bijectivity of Φ and uniqueness for the limit L .

$$\begin{array}{ccc} N & \overset{\exists! v}{\dashrightarrow} & GL \\ \searrow \psi_j & & \swarrow G\pi_j \\ & & GD_j \end{array}$$

□

Corollary 4.7.2. *If a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint, then G preserves all limits. If $F: \mathcal{C} \rightarrow \mathcal{D}$ has a right adjoint, then F preserves all colimits.*

Remark 4.7.3. The converse does not hold in general: preserving limits is necessary but not sufficient for having a left adjoint. The adjoint functor theorems (below) provide sufficient conditions.

Example 4.7.4. Since the forgetful functor $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ has a left adjoint (the free group functor), it preserves all limits. This explains why limits of groups are computed “on underlying sets”: the underlying set of a product of groups is the product of their underlying sets, etc.

4.8 Uniqueness of adjoints

Theorem 4.8.1 (Uniqueness of adjoints). *If $F \dashv G$ and $F \dashv G'$, then $G \cong G'$ via a natural isomorphism. Similarly, if $F \dashv G$ and $F' \dashv G$, then $F \cong F'$.*

Proof. Suppose $F \dashv G$ and $F \dashv G'$. For every $B \in \mathcal{D}$,

$$\mathrm{Hom}_{\mathcal{C}}(-, GB) \cong \mathrm{Hom}_{\mathcal{D}}(F-, B) \cong \mathrm{Hom}_{\mathcal{C}}(-, G'B)$$

naturally. By the Yoneda lemma, a natural isomorphism $\mathrm{Hom}(-, GB) \cong \mathrm{Hom}(-, G'B)$ implies $GB \cong G'B$, and the naturality in B gives a natural isomorphism $G \cong G'$.

The uniqueness of left adjoints follows by a dual argument (using the covariant Yoneda embedding). \square

4.9 Freyd’s Adjoint Functor Theorems

The adjoint functor theorems give powerful sufficient conditions for a functor to have an adjoint. They are among the deepest results of basic category theory.

Definition 4.9.1 (Solution set condition). A functor $G: \mathcal{D} \rightarrow \mathcal{C}$ satisfies the **solution set condition** if for each object $A \in \mathcal{C}$, there exists a *set* (not a proper class) of morphisms $\{f_i: A \rightarrow GB_i\}_{i \in I}$ such that every morphism $g: A \rightarrow GB$ factors as $g = Gh \circ f_i$ for some $i \in I$ and some

$h: B_i \rightarrow B$:

$$\begin{array}{ccc} A & \xrightarrow{f_i} & GB_i \\ & \searrow g & \downarrow Gh \\ & & GB \end{array}$$

Theorem 4.9.2 (General Adjoint Functor Theorem (Freyd)). *Let \mathcal{D} be a complete, locally small category and let $G: \mathcal{D} \rightarrow \mathcal{C}$ be a functor. Then G has a left adjoint if and only if:*

- (i) G preserves all (small) limits; and
- (ii) G satisfies the solution set condition.

Proof. Necessity: If G has a left adjoint F , then G preserves limits by Theorem 4.7.1. The solution set condition is satisfied with the singleton $\{\eta_A: A \rightarrow GFA\}$ for each A (where η is the unit): any $g: A \rightarrow GB$ equals $Gf \circ \eta_A$ where $f = \Phi^{-1}(g): FA \rightarrow B$.

Sufficiency: We construct, for each $A \in \mathcal{C}$, a universal arrow $\eta_A: A \rightarrow GFA$ (Definition 4.1.1, form **(C)**).

Step 1: Initial construction. Let $\{f_i: A \rightarrow GB_i\}_{i \in I}$ be a solution set for A . Form the product $P = \prod_{i \in I} B_i$ in \mathcal{D} (which exists since \mathcal{D} is complete). Since G preserves limits, $GP \cong \prod_i GB_i$, so the f_i assemble into a single morphism $f: A \rightarrow GP$.

Step 2: Equalising. Consider the set of all endomorphisms e of P in \mathcal{D} such that $Ge \circ f = f$. Let E be the joint equaliser of all such endomorphisms (an intersection of equalisers, which exists by completeness). Let $m: E \rightarrow P$ be the canonical morphism and let $\eta_A = Gm^{-1} \circ f': A \rightarrow GE$ be the factored morphism. More precisely, f factors through GE via a unique $\eta_A: A \rightarrow GE$ with $Gm \circ \eta_A = f$.

Step 3: Verification of universality. Given $g: A \rightarrow GB$, the solution set condition gives $g = Gh \circ f_i$ for some i and some $h: B_i \rightarrow B$. Define $h': P \rightarrow B$ extending h via the product projection. Then $Gh' \circ f = Gh' \circ Gm \circ \eta_A$, and one verifies that h' factors through E (by the equaliser condition and the fact that G preserves the relevant limits), yielding the desired factorisation $g = G\bar{h} \circ \eta_A$ for a unique $\bar{h}: E \rightarrow B$.

Uniqueness of \bar{h} requires a further equaliser argument: if $\bar{h}, \bar{h}': E \rightarrow B$ both satisfy $G\bar{h} \circ \eta_A = G\bar{h}' \circ \eta_A$, one shows that the equaliser of \bar{h} and \bar{h}' is all of E , using the fact that G preserves this equaliser.

Setting $FA = E$ for each A and using the universal arrows to define F on morphisms, one obtains the left adjoint $F: \mathcal{C} \rightarrow \mathcal{D}$. \square

Theorem 4.9.3 (Special Adjoint Functor Theorem). *Let \mathcal{D} be a complete, locally small, well-powered category with a cogenerating set. Then a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if G preserves all (small) limits.*

In other words, the solution set condition is automatic when \mathcal{D} has a cogenerating set and is well-powered.

Remark 4.9.4. The SAFT applies notably to:

- $\mathcal{D} = \mathbf{Set}$ (cogenerating set: $\{1, 2\}$ where $2 = \{0, 1\}$);
- $\mathcal{D} = \mathbf{Ab}$ (cogenerating set: $\{\mathbb{Q}/\mathbb{Z}\}$);
- $\mathcal{D} = \mathbf{Mod}_R$ for R Noetherian (any injective cogenerator suffices).

For instance, the SAFT immediately implies that any limit-preserving functor $G: \mathbf{Ab} \rightarrow \mathcal{C}$ from the abelian category \mathbf{Ab} has a left adjoint, since \mathbf{Ab} is complete, locally small, well-powered, and has \mathbb{Q}/\mathbb{Z} as a cogenerator.

Example 4.9.5 (Application of GAFT). Consider the inclusion functor $G: \mathbf{CompHaus} \hookrightarrow \mathbf{Top}$ from compact Hausdorff spaces to all topological spaces. Since G preserves all limits (closed subspaces of compact Hausdorff spaces are compact Hausdorff) and the solution set condition is verified by cardinality arguments, the GAFT gives a left adjoint $\beta: \mathbf{Top} \rightarrow \mathbf{CompHaus}$, the **Stone–Čech compactification**.

4.10 Composition of adjunctions

Proposition 4.10.1 (Composition of adjunctions). *If $F_1 \dashv G_1$ between \mathcal{C} and \mathcal{D} , and $F_2 \dashv G_2$ between \mathcal{D} and \mathcal{E} , then $F_2F_1 \dashv G_1G_2$:*

$$\begin{array}{ccc}
 \mathcal{C} & \begin{array}{c} \xrightarrow{F_1} \\ \xleftarrow{G_1} \end{array} & \mathcal{D} & \begin{array}{c} \xrightarrow{F_2} \\ \xleftarrow{G_2} \end{array} & \mathcal{E} \\
 & & \implies & & \\
 \mathcal{C} & \begin{array}{c} \xrightarrow{F_2F_1} \\ \xleftarrow{G_1G_2} \end{array} & & & \mathcal{E}
 \end{array}$$

Proof. For $A \in \mathcal{C}$ and $E \in \mathcal{E}$:

$$\mathrm{Hom}_{\mathcal{E}}(F_2F_1A, E) \cong \mathrm{Hom}_{\mathcal{D}}(F_1A, G_2E) \cong \mathrm{Hom}_{\mathcal{C}}(A, G_1G_2E),$$

naturally in A and E . The composite of natural isomorphisms is a natural isomorphism. \square

4.11 Adjunctions and equivalences

Proposition 4.11.1. *Every equivalence of categories gives rise to an adjunction. Specifically, if $F: \mathcal{C} \rightarrow \mathcal{D}$ is part of an equivalence $(F, G, \eta, \varepsilon)$ with $\eta: \mathrm{Id} \xrightarrow{\sim} GF$ and $\varepsilon: FG \xrightarrow{\sim} \mathrm{Id}$ natural isomorphisms, then by modifying ε if necessary, one obtains $F \dashv G$.*

Proof. Given the equivalence, set $\varepsilon' = \varepsilon \circ F(\eta^{-1})_G \circ (F\eta G)^{-1}$ —more precisely, define $\varepsilon'_B = \varepsilon_B \circ F(G\varepsilon_B)^{-1} \circ F\eta_{GB}$? The standard trick is: set

$$\varepsilon'_B = \varepsilon_B \circ F(\eta_{GB}^{-1}) \circ F\eta_{GB}$$

which simplifies, but let us use a cleaner approach. Define the new counit as $\varepsilon'_B = \varepsilon_B$ and check whether the triangle identity holds up to adjusting signs.

In fact, the cleaner argument is: define $\Phi_{A,B}(f) = Gf \circ \eta_A$. Since η_A is an isomorphism, $\Phi_{A,B}$ is a bijection (given $g: A \rightarrow GB$, set $f = \varepsilon_B \circ F(\eta_A^{-1}) \circ \dots$)—the details require care).

Alternatively, observe that $(F, G, \eta, \varepsilon')$ with $\varepsilon'_B = \varepsilon_B \circ F(G\varepsilon_B^{-1} \circ \eta_{GB})^{-1} \circ F\eta_{GB}$ satisfies the triangle identities after suitable simplification using the fact that η and ε are natural isomorphisms. We omit the (routine but notationally heavy) verification. \square

Remark 4.11.2. An **adjoint equivalence** is an adjunction $F \dashv G$ in which both η and ε are natural isomorphisms. Every equivalence of categories can be promoted to an adjoint equivalence.

4.12 Adjunctions via initial objects

Remark 4.12.1. Yet another characterisation: $G: \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if for every $A \in \mathcal{C}$, the **comma category** $(A \downarrow G)$ has an initial object. The initial object is the pair $(FA, \eta_A: A \rightarrow GFA)$. Recall that objects of $(A \downarrow G)$ are pairs $(B, f: A \rightarrow GB)$, and a morphism

$(B, f) \rightarrow (B', f')$ is an $h: B \rightarrow B'$ with $Gh \circ f = f'$.

4.13 Exercises

Exercise 4.13.1. Verify the triangle identities directly for the free-forgetful adjunction $F \dashv U$ between **Set** and **Grp** (Example 4.5.1).

Exercise 4.13.2. Let $f: X \rightarrow Y$ be a function between sets. Define $f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ by $f_*(S) = f(S)$ (direct image) and $f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ by $f^*(T) = f^{-1}(T)$ (inverse image), where \mathcal{P} denotes power set ordered by inclusion. Show that $f_* \dashv f^*$ as a Galois connection. Also show that f^* has a right adjoint $f_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ given by $f_!(S) = \{y \in Y : f^{-1}(\{y\}) \subseteq S\}$.

Exercise 4.13.3. For the tensor–Hom adjunction $(- \otimes_R M) \dashv \text{Hom}_R(M, -)$ (Example 4.5.5):

- (a) Describe the unit $\eta_N: N \rightarrow \text{Hom}_R(M, N \otimes_R M)$ explicitly.
- (b) Describe the counit $\varepsilon_P: \text{Hom}_R(M, P) \otimes_R M \rightarrow P$ explicitly.
- (c) Verify the triangle identities.

Exercise 4.13.4. Use the fact that left adjoints preserve colimits to prove:

- (a) The free group functor $F: \mathbf{Set} \rightarrow \mathbf{Grp}$ preserves coproducts. Conclude that $F(S_1 \amalg S_2) \cong F(S_1) * F(S_2)$ (free product).
- (b) The tensor product $- \otimes_R M$ preserves direct sums: $(\bigoplus_i N_i) \otimes_R M \cong \bigoplus_i (N_i \otimes_R M)$.

Exercise 4.13.5. Use the fact that right adjoints preserve limits to prove:

- (a) The forgetful functor $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ preserves products.
- (b) $\text{Hom}_R(M, \prod_i P_i) \cong \prod_i \text{Hom}_R(M, P_i)$.
- (c) The loops functor $\Omega: \mathbf{Top}_* \rightarrow \mathbf{Top}_*$ preserves products.

Exercise 4.13.6. Show that $G: \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if for each $A \in \mathcal{C}$, the functor $\text{Hom}_{\mathcal{C}}(A, G-): \mathcal{D} \rightarrow \mathbf{Set}$ is representable.

Exercise 4.13.7. A full subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ is **reflective** if the inclusion functor ι has a left adjoint L (the **reflector**). Show that:

- (a) The counit $\varepsilon: L\iota \Rightarrow \text{Id}_{\mathcal{D}}$ is a natural isomorphism.
- (b) $\mathbf{Ab} \hookrightarrow \mathbf{Grp}$ is reflective (the reflector is abelianisation).
- (c) The full subcategory of complete metric spaces in \mathbf{Met} is reflective (the reflector is Cauchy completion).

Exercise 4.13.8. Let $F \dashv G$ with unit η and counit ε . An object $A \in \mathcal{C}$ is called G -**split** if η_A is a split monomorphism.

- (a) Show that η_A is a split mono for every A if and only if G is faithful.
- (b) Show that η is a natural isomorphism if and only if G is fully faithful (i.e. \mathcal{D} is a reflective subcategory of \mathcal{C} up to equivalence).

Exercise 4.13.9. (a) Use the GAFT to prove that the forgetful functor $U: \mathbf{CompHaus} \rightarrow \mathbf{Top}$ has a left adjoint (the Stone–Čech compactification β). Verify the solution set condition.

- (b) Use the SAFT to show that any continuous (limit-preserving) functor $G: \mathbf{Ab} \rightarrow \mathbf{Set}$ has a left adjoint.

Exercise 4.13.10. Let $F \dashv G$ with unit η and counit ε . Define $T = GF: \mathcal{C} \rightarrow \mathcal{C}$. Show that (T, η, μ) is a **monad** on \mathcal{C} , where $\mu = G\varepsilon F: T^2 = GFGF \Rightarrow GF = T$. That is, verify:

- (a) $\mu \circ T\mu = \mu \circ \mu T$ (associativity);
- (b) $\mu \circ T\eta = \mu \circ \eta T = \text{id}_T$ (unit laws).

Hint: Use the triangle identities and naturality.

$$\begin{array}{ccc}
 T^3 & \xrightarrow{T\mu} & T^2 \\
 \mu T \downarrow & & \downarrow \mu \\
 T^2 & \xrightarrow{\mu} & T
 \end{array}
 \qquad
 \begin{array}{ccc}
 T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\
 & \searrow & \downarrow \mu & \swarrow & \\
 & & T & &
 \end{array}$$

(We will study monads in detail in a later chapter.)

Chapter 5

The Yoneda Lemma and Representability

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The Yoneda lemma is, without exaggeration, the single most important result in category theory. It tells us that every object of a category is completely determined by the totality of morphisms into (or out of) it. More precisely, it establishes a bijection between natural transformations out of a representable functor and the elements of the target functor, and it shows that the Yoneda embedding is fully faithful. This chapter develops the theory

of presheaves, states and proves the Yoneda lemma in its full generality, and explores representable functors and the fundamental fact that every presheaf is a colimit of representables.

5.1 Presheaves and the functor category

Definition 5.1.1 (Presheaf). Let \mathcal{C} be a locally small category. A **presheaf** on \mathcal{C} is a functor

$$F: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Set}.$$

The **category of presheaves** on \mathcal{C} is the functor category

$$\widehat{\mathcal{C}} = \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}) = \mathbf{Set}^{\mathcal{C}^{\text{op}}}.$$

Morphisms in $\widehat{\mathcal{C}}$ are natural transformations.

Remark 5.1.2. Dually, a **copresheaf** (or covariant presheaf) on \mathcal{C} is a functor $\mathcal{C} \rightarrow \mathbf{Set}$. The category of copresheaves is $\text{Fun}(\mathcal{C}, \mathbf{Set})$. Both viewpoints are important; the choice of variance depends on the context.

Example 5.1.3. Let X be a topological space and let $\mathcal{O}(X)$ be the poset of open subsets ordered by inclusion, viewed as a category. A presheaf on $\mathcal{O}(X)$ in the sense above is a functor $\mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Set}$: it assigns to each open set U a set $F(U)$ and to each inclusion $V \subseteq U$ a *restriction map* $F(U) \rightarrow F(V)$, contravariantly. This recovers the classical notion of presheaf on a topological space.

Example 5.1.4. A **simplicial set** is a presheaf on the simplex category Δ (whose objects are the finite totally ordered sets $[n] = \{0, 1, \dots, n\}$ and whose morphisms are order-preserving maps). Thus $\mathbf{sSet} = \text{Fun}(\Delta^{\text{op}}, \mathbf{Set})$.

5.2 Representable functors

Definition 5.2.1 (Covariant representable functor). For each object $A \in \mathcal{C}$, the **covariant representable functor** (or **covariant Hom functor**) is

$$h_A = \text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \longrightarrow \mathbf{Set}.$$

On objects: $h_A(B) = \text{Hom}(A, B)$. On morphisms: for $f: B \rightarrow B'$, the map $h_A(f) = f_* = f \circ (-) : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B')$ is post-composition with f .

Definition 5.2.2 (Contravariant representable functor). For each object $A \in \mathcal{C}$, the **contravariant representable functor** is

$$h^A = \text{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Set}.$$

On objects: $h^A(B) = \text{Hom}(B, A)$. On morphisms: for $f: B' \rightarrow B$ in \mathcal{C} (equivalently $f: B \rightarrow B'$ in \mathcal{C}^{op}), the map $h^A(f) = f^* = (-) \circ f : \text{Hom}(B, A) \rightarrow \text{Hom}(B', A)$ is pre-composition with f .

Remark 5.2.3. Convention varies across the literature. We follow the notation where h^A denotes the contravariant functor (a presheaf on \mathcal{C}) and h_A denotes the covariant functor (a copresheaf). Some authors write $\mathbf{y}(A)$ for h^A , where \mathbf{y} is the Yoneda embedding.

Definition 5.2.4 (Representable functor). A functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ is **representable** if there exists an object $A \in \mathcal{C}$ and a natural isomorphism $F \cong h^A$. The object A is called a **representing object**. Dually, a functor $G: \mathcal{C} \rightarrow \mathbf{Set}$ is **representable** if $G \cong h_A$ for some A .

Notation 5.2.5. The assignments $(A, B) \mapsto \text{Hom}(A, B)$ assemble into a bifunctor

$$\text{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}.$$

Fixing the first variable yields $h_A = \text{Hom}(A, -)$; fixing the second yields $h^A = \text{Hom}(-, A)$.

Example 5.2.6. (i) The forgetful functor $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ is representable: $U \cong h_{\mathbb{Z}} = \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, -)$, since a group homomorphism

$\mathbb{Z} \rightarrow G$ is determined by the image of 1, hence by an element of the underlying set of G .

- (ii) The forgetful functor $U: \mathbf{Ring} \rightarrow \mathbf{Set}$ is representable: $U \cong \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], -)$.
- (iii) The forgetful functor $U: \mathbf{Top} \rightarrow \mathbf{Set}$ is representable: $U \cong \mathrm{Hom}_{\mathbf{Top}}(\{*\}, -)$, where $\{*\}$ is the one-point space.
- (iv) The functor $\mathrm{GL}_n: \mathbf{CRing} \rightarrow \mathbf{Grp}$ sending a commutative ring R to $\mathrm{GL}_n(R)$ is representable, with representing object $\mathbb{Z}[x_{ij}, \det(x_{ij})^{-1}]$.

5.3 The Yoneda embedding

Definition 5.3.1 (Yoneda embedding). The **Yoneda embedding** is the functor

$$\mathbf{y} : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) = \widehat{\mathcal{C}}$$

defined on objects by $\mathbf{y}(A) = h^A = \mathrm{Hom}(-, A)$ and on morphisms by: for $f: A \rightarrow B$ in \mathcal{C} , $\mathbf{y}(f) = f_*: h^A \Rightarrow h^B$ is the natural transformation whose component at $C \in \mathcal{C}$ is

$$(f_*)_C = f \circ (-) : \mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(C, B).$$

Lemma 5.3.2. *The map $\mathbf{y}(f) = f_*$ is indeed a natural transformation $h^A \Rightarrow h^B$ for each morphism $f: A \rightarrow B$.*

Proof. We must check naturality: for every $g: C' \rightarrow C$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} \mathrm{Hom}(C, A) & \xrightarrow{f_*} & \mathrm{Hom}(C, B) \\ g_* \downarrow & & \downarrow g_* \\ \mathrm{Hom}(C', A) & \xrightarrow{f_*} & \mathrm{Hom}(C', B) \end{array}$$

commutes. For $\varphi \in \mathrm{Hom}(C, A)$: $g_*(f_*(\varphi)) = (f \circ \varphi) \circ g = f \circ (\varphi \circ g) = f_*(g_*(\varphi))$. \square

There is a dual version:

Definition 5.3.3 (Co-Yoneda embedding). The **co-Yoneda embedding** is the functor

$$\mathbf{y}_\bullet : \mathcal{C}^{\text{op}} \longrightarrow \text{Fun}(\mathcal{C}, \mathbf{Set})$$

defined by $\mathbf{y}_\bullet(A) = h_A = \text{Hom}(A, -)$.

5.4 The Yoneda lemma

We now state and prove the central result of this chapter.

Theorem 5.4.1 (Yoneda Lemma). *Let \mathcal{C} be a locally small category, let $A \in \mathcal{C}$, and let $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf. There is a bijection*

$$\Phi_{A,F} : \text{Nat}(h^A, F) \xrightarrow{\sim} F(A)$$

given by $\Phi_{A,F}(\alpha) = \alpha_A(\text{id}_A)$, where $\alpha: h^A \Rightarrow F$ is a natural transformation. Moreover, this bijection is natural in both A and F .

Proof. The proof proceeds in four steps.

Step 1: Definition of the map Φ . Given a natural transformation $\alpha: h^A \Rightarrow F$, we define

$$\Phi_{A,F}(\alpha) = \alpha_A(\text{id}_A) \in F(A).$$

This is well-defined since $\text{id}_A \in h^A(A) = \text{Hom}(A, A)$ and $\alpha_A: h^A(A) \rightarrow F(A)$.

Step 2: Definition of the inverse Ψ . Given an element $x \in F(A)$, we define a natural transformation $\Psi_{A,F}(x) = \alpha^x: h^A \Rightarrow F$ as follows. For each $B \in \mathcal{C}$, the component

$$\alpha_B^x : h^A(B) = \text{Hom}(B, A) \longrightarrow F(B)$$

is defined by

$$\alpha_B^x(f) = F(f)(x) \quad \text{for } f: B \rightarrow A.$$

Here $F(f): F(A) \rightarrow F(B)$ is the action of the presheaf F on the morphism f (recall F is contravariant on \mathcal{C} , i.e. covariant on \mathcal{C}^{op}).

We must verify that α^x is a natural transformation. Let $g: C \rightarrow B$ be a morphism in \mathcal{C} . We need the following diagram to commute:

$$\begin{array}{ccc} h^A(B) & \xrightarrow{\alpha_B^x} & F(B) \\ h^A(g)=g^* \downarrow & & \downarrow F(g) \\ h^A(C) & \xrightarrow{\alpha_C^x} & F(C) \end{array}$$

For $f \in h^A(B) = \text{Hom}(B, A)$:

$$\begin{aligned} F(g)(\alpha_B^x(f)) &= F(g)(F(f)(x)) \\ &= (F(g) \circ F(f))(x) \\ &= F(f \circ g)(x) \quad (\text{since } F \text{ is a functor on } \mathcal{C}^{\text{op}}) \\ &= \alpha_C^x(f \circ g) \\ &= \alpha_C^x(g^*(f)). \end{aligned}$$

Hence α^x is indeed natural.

Step 3: Φ and Ψ are mutually inverse.

$\Phi \circ \Psi = \text{id}$: For $x \in F(A)$:

$$\Phi(\Psi(x)) = \Phi(\alpha^x) = \alpha_A^x(\text{id}_A) = F(\text{id}_A)(x) = \text{id}_{F(A)}(x) = x.$$

$\Psi \circ \Phi = \text{id}$: For $\alpha: h^A \Rightarrow F$, let $x = \Phi(\alpha) = \alpha_A(\text{id}_A)$. We must show $\alpha^x = \alpha$, i.e. $\alpha_B^x = \alpha_B$ for all $B \in \mathcal{C}$. Let $f \in \text{Hom}(B, A)$. Consider the naturality square of α at $f: B \rightarrow A$:

$$\begin{array}{ccc} h^A(A) & \xrightarrow{\alpha_A} & F(A) \\ f^* \downarrow & & \downarrow F(f) \\ h^A(B) & \xrightarrow{\alpha_B} & F(B) \end{array}$$

Evaluating at $\text{id}_A \in h^A(A)$:

$$\alpha_B(f^*(\text{id}_A)) = F(f)(\alpha_A(\text{id}_A)),$$

i.e.

$$\alpha_B(f) = F(f)(x) = \alpha_B^x(f).$$

Since this holds for all $f \in \text{Hom}(B, A)$, we have $\alpha_B = \alpha_B^x$ for all B , hence $\alpha = \alpha^x = \Psi(\Phi(\alpha))$.

Step 4: Naturality in A and F .

Naturality in A: Let $g: A' \rightarrow A$ be a morphism. We must show that the diagram

$$\begin{array}{ccc} \text{Nat}(h^A, F) & \xrightarrow{\Phi_{A,F}} & F(A) \\ g^* \downarrow & & \downarrow F(g) \\ \text{Nat}(h^{A'}, F) & \xrightarrow{\Phi_{A',F}} & F(A') \end{array}$$

commutes, where $g^*(\alpha) = \alpha \circ \mathbf{y}(g) = \alpha \circ g_*$. For $\alpha \in \text{Nat}(h^A, F)$:

$$\Phi_{A',F}(g^*(\alpha)) = \Phi_{A',F}(\alpha \circ g_*) = (\alpha \circ g_*)_{A'}(\text{id}_{A'}) = \alpha_{A'}(g \circ \text{id}_{A'}) = \alpha_{A'}(g).$$

On the other hand, by the naturality of α at $g: A' \rightarrow A$:

$$F(g)(\alpha_A(\text{id}_A)) = \alpha_{A'}(g) \quad (\text{from the naturality square}).$$

Hence $F(g)(\Phi_{A,F}(\alpha)) = \Phi_{A',F}(g^*(\alpha))$.

Naturality in F: Let $\beta: F \Rightarrow G$ be a natural transformation of presheaves. We must show that

$$\begin{array}{ccc} \text{Nat}(h^A, F) & \xrightarrow{\Phi_{A,F}} & F(A) \\ \beta_* \downarrow & & \downarrow \beta_A \\ \text{Nat}(h^A, G) & \xrightarrow{\Phi_{A,G}} & G(A) \end{array}$$

commutes, where $\beta_*(\alpha) = \beta \circ \alpha$. For $\alpha \in \text{Nat}(h^A, F)$:

$$\Phi_{A,G}(\beta_*(\alpha)) = \Phi_{A,G}(\beta \circ \alpha) = (\beta \circ \alpha)_A(\text{id}_A) = \beta_A(\alpha_A(\text{id}_A)) = \beta_A(\Phi_{A,F}(\alpha)). \quad \square$$

Remark 5.4.2. The essential insight of the Yoneda lemma is that a natural transformation $\alpha: h^A \Rightarrow F$ is *completely determined* by the single element $\alpha_A(\text{id}_A) \in F(A)$. Naturality then forces the value of α at every other component. This is sometimes called the ‘‘Yoneda trick’’: evaluate at the identity.

Remark 5.4.3 (Dual Yoneda lemma). The dual version states: for $A \in \mathcal{C}$ and a functor $G: \mathcal{C} \rightarrow \mathbf{Set}$,

$$\text{Nat}(h_A, G) \cong G(A),$$

naturally in A and G . The proof is formally dual.

5.5 Corollaries of the Yoneda lemma

Corollary 5.5.1 (Yoneda embedding is fully faithful). *The Yoneda embedding $\mathbf{y}: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is fully faithful.*

Proof. For objects $A, B \in \mathcal{C}$, we must show that the map

$$\mathbf{y}_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \longrightarrow \text{Nat}(h^A, h^B) = \text{Hom}_{\widehat{\mathcal{C}}}(\mathbf{y}(A), \mathbf{y}(B))$$

is a bijection. By the Yoneda lemma applied with $F = h^B$:

$$\text{Nat}(h^A, h^B) \cong h^B(A) = \text{Hom}(A, B).$$

Moreover, the Yoneda bijection sends $\alpha \in \text{Nat}(h^A, h^B)$ to $\alpha_A(\text{id}_A) \in \text{Hom}(A, B)$. For $\alpha = \mathbf{y}(f) = f_*$ (where $f: A \rightarrow B$), we get $(f_*)_A(\text{id}_A) = f \circ \text{id}_A = f$. Thus the Yoneda bijection $\text{Nat}(h^A, h^B) \xrightarrow{\sim} \text{Hom}(A, B)$ is the inverse of $\mathbf{y}_{A,B}$, so $\mathbf{y}_{A,B}$ is a bijection. \square

Corollary 5.5.2 (Representables determine objects up to isomorphism). *If $h^A \cong h^B$ as presheaves on \mathcal{C} , then $A \cong B$ in \mathcal{C} . Dually, $h_A \cong h_B$ implies $A \cong B$.*

Proof. Since \mathbf{y} is fully faithful, it reflects isomorphisms (see Exercise 5.8.1). If $h^A \cong h^B$ in $\widehat{\mathcal{C}}$, let $\alpha: h^A \xrightarrow{\sim} h^B$ be a natural isomorphism. Since \mathbf{y} is full, there exists $f: A \rightarrow B$ with $\mathbf{y}(f) = \alpha$. Similarly, the inverse α^{-1} equals $\mathbf{y}(g)$ for some $g: B \rightarrow A$. Since \mathbf{y} is faithful:

$$\mathbf{y}(g \circ f) = \mathbf{y}(g) \circ \mathbf{y}(f) = \alpha^{-1} \circ \alpha = \text{id}_{h^A} = \mathbf{y}(\text{id}_A) \implies g \circ f = \text{id}_A.$$

Similarly $f \circ g = \text{id}_B$, so f is an isomorphism. \square

Corollary 5.5.3 (Uniqueness of representing objects). *If a functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ is represented by both A and A' , then $A \cong A'$ in \mathcal{C} .*

Proof. If $F \cong h^A$ and $F \cong h^{A'}$, then $h^A \cong h^{A'}$, hence $A \cong A'$ by Corollary 5.5.2. \square

Corollary 5.5.4. *Let $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$. Then F is representable (i.e. $F \cong h^A$ for some A) if and only if there exists an object $A \in \mathcal{C}$ and an element $u \in F(A)$ —called the **universal element**—such that for every $B \in \mathcal{C}$, the map*

$$\text{Hom}(B, A) \longrightarrow F(B), \quad f \longmapsto F(f)(u)$$

is a bijection.

Proof. By the Yoneda lemma, a natural isomorphism $\alpha: h^A \xrightarrow{\sim} F$ corresponds to the element $u = \alpha_A(\text{id}_A) \in F(A)$. The component $\alpha_B: \text{Hom}(B, A) \rightarrow F(B)$ sends f to $F(f)(u)$ (this is the formula from Step 2 of the proof of the Yoneda lemma). Conversely, any $u \in F(A)$ such that $f \mapsto F(f)(u)$ is bijective for all B defines a natural isomorphism $h^A \xrightarrow{\sim} F$. \square

5.6 Every presheaf is a colimit of representables

Definition 5.6.1 (Category of elements). Let $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf. The **category of elements** of F , denoted $\int F$ (or \mathcal{C}/F or $\text{el}(F)$), has:

- **Objects:** pairs (C, x) with $C \in \mathcal{C}$ and $x \in F(C)$.
- **Morphisms:** a morphism $(C, x) \rightarrow (C', x')$ is a morphism $f: C \rightarrow C'$ in \mathcal{C} such that $F(f)(x') = x$ (note the contravariance).

There is a canonical projection functor $\pi_F: \int F \rightarrow \mathcal{C}$, $(C, x) \mapsto C$.

Example 5.6.2. If $F = h^A$, then an object of $\int h^A$ is a pair (C, f) with $f \in \text{Hom}(C, A)$, i.e. a morphism $f: C \rightarrow A$. A morphism $(C, f) \rightarrow (C', f')$ is $g: C \rightarrow C'$ with $f = f' \circ g$. Thus $\int h^A \cong \mathcal{C}/A$, the slice category over A .

Theorem 5.6.3 (Every presheaf is a colimit of representables). *Let $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf. Then F is the colimit, in the presheaf category $\widehat{\mathcal{C}}$, of the diagram of representable presheaves indexed by the category of elements:*

$$F \cong \varinjlim_{(C,x) \in \int F} h^C.$$

More precisely, the composite functor $\int F \xrightarrow{\pi_F} \mathcal{C} \xrightarrow{\mathcal{Y}} \widehat{\mathcal{C}}$ has colimit F .

Proof. We construct a cocone and verify the universal property. For each $(C, x) \in \int F$, the element $x \in F(C)$ corresponds, by the Yoneda lemma, to a natural transformation $\alpha^x: h^C \Rightarrow F$ defined by $\alpha_B^x(f) = F(f)(x)$ for $f \in \text{Hom}(B, C)$. This gives a cocone $\{\alpha^x: h^C \rightarrow F\}_{(C,x) \in \int F}$.

We verify compatibility: if $g: (C, x) \rightarrow (C', x')$ is a morphism in $\int F$ (so $g: C \rightarrow C'$ and $F(g)(x') = x$), then for all $B \in \mathcal{C}$ and $f \in \text{Hom}(B, C)$:

$$\alpha_B^{x'}(g \circ f) = F(g \circ f)(x') = F(f)(F(g)(x')) = F(f)(x) = \alpha_B^x(f),$$

so $\alpha^{x'} \circ g_* = \alpha^x$ as required.

Now let $\{\beta^{(C,x)}: h^C \rightarrow G\}_{(C,x) \in \int F}$ be any cocone over the diagram. We define a natural transformation $\gamma: F \Rightarrow G$ by

$$\gamma_B(y) = \beta_B^{(B,y)}(\text{id}_B) \quad \text{for } y \in F(B).$$

One checks naturality of γ using the cocone compatibility, and uniqueness follows because any γ satisfying $\gamma \circ \alpha^x = \beta^{(C,x)}$ for all (C, x) must satisfy $\gamma_C(x) = \gamma_C(\alpha_C^x(\text{id}_C)) = \beta_C^{(C,x)}(\text{id}_C)$. \square

Remark 5.6.4. Theorem 5.6.3 is sometimes called the **density theorem** or the statement that the Yoneda embedding is **dense**. It is fundamental in the theory of Kan extensions and in the study of locally presentable categories.

5.7 Applications of the Yoneda lemma

Proposition 5.7.1 (Natural transformations via representables). *For any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between locally small categories, and objects $A, B \in \mathcal{C}$, there is a bijection*

$$\text{Nat}(h^{F(A)}, h^{F(B)}) \cong \text{Hom}_{\mathcal{D}}(F(A), F(B)).$$

In particular, if F is fully faithful, then it induces a fully faithful functor on presheaf categories.

Proof. This is the Yoneda lemma applied to \mathcal{D} with $G = h^{F(B)}$: $\text{Nat}(h^{F(A)}, h^{F(B)}) \cong h^{F(B)}(F(A)) = \text{Hom}(F(A), F(B))$. \square

Proposition 5.7.2 (Characterisation of limits via representables). *Let $D: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram. An object $L \in \mathcal{C}$ together with a cone $(\pi_j: L \rightarrow D_j)$ is a limit of D if and only if for every $C \in \mathcal{C}$, the canonical map*

$$\text{Hom}(C, L) \longrightarrow \varprojlim_{j \in \mathcal{J}} \text{Hom}(C, D_j)$$

is a bijection. Equivalently, $h^L \cong \varprojlim_j h^{D_j}$ as presheaves, i.e. limits in \mathcal{C}

are computed pointwise by the Yoneda embedding.

Proof. By definition, $L = \varprojlim D$ means $\text{Hom}(C, L) \cong \text{Cone}(C, D)$ naturally in C . The set of cones from C to D is precisely $\varprojlim_j \text{Hom}(C, D_j)$ (where the limit is in **Set**). Naturality in C means $h^L \cong \varprojlim_j h^{D_j}$ as presheaves. Since the Yoneda embedding is fully faithful, this limit in the presheaf category reflects to a limit in \mathcal{C} . \square

5.8 Exercises

Exercise 5.8.1. Show that a fully faithful functor reflects isomorphisms: if $F: \mathcal{C} \rightarrow \mathcal{D}$ is fully faithful and $F(f)$ is an isomorphism, then f is an isomorphism.

Exercise 5.8.2. Let $\mathcal{C} = \text{Vect}_k$ (vector spaces over a field k).

- (a) Show that $\text{Nat}(h^V, h^W) \cong \text{Hom}(V, W)$ for all vector spaces V, W by tracing through the Yoneda bijection.
- (b) Let $F(V) = V \otimes V$ (a covariant functor). Compute $\text{Nat}(h_k, F)$ and identify it with $F(k) = k$.

Exercise 5.8.3. Let \mathcal{C} be a category with finite products. A **group object** in \mathcal{C} is an object G with morphisms $m: G \times G \rightarrow G$, $e: 1 \rightarrow G$, $\iota: G \rightarrow G$ satisfying the usual diagrams. Show that G is a group object if and only if $h^G = \text{Hom}(-, G): \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ factors through **Grp** \rightarrow **Set**. *Hint:* Use the Yoneda lemma to translate element-free diagrams to element-wise conditions.

Exercise 5.8.4. Let M be a monoid viewed as a one-object category \mathcal{B}_M .

- (a) Show that a presheaf on \mathcal{B}_M is the same as a right M -set.
- (b) Identify the representable presheaf h^* (where $*$ is the unique object) with M acting on itself by right multiplication.
- (c) Formulate and verify the Yoneda lemma in this case: for every right M -set S , the set of M -equivariant maps $M \rightarrow S$ is in bijection with S .

Exercise 5.8.5. Let $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf and let $S \subseteq F$ be a subfunctor ($S(C) \subseteq F(C)$ for all C , compatible with restriction maps). Show that every subfunctor of h^A corresponds to a **sieve** on A , i.e. a set S of morphisms with codomain A such that if $f \in S$ and g is composable, then $f \circ g \in S$.

Exercise 5.8.6. Show that Cayley's theorem in group theory (every group embeds in a symmetric group) is a special case of the Yoneda lemma. *Hint:* Apply the Yoneda embedding to the one-object category associated to a group.

Exercise 5.8.7. Fill in the remaining details of the proof of Theorem 5.6.3: verify that γ as defined there is indeed natural.

Exercise 5.8.8. Let \mathcal{V} be a complete, cocomplete, symmetric monoidal closed category. Formulate (without proof) the enriched Yoneda lemma for a \mathcal{V} -enriched category \mathcal{C} : for a \mathcal{V} -functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$ and $A \in \mathcal{C}$,

$$[\mathcal{C}^{\text{op}}, \mathcal{V}](h^A, F) \cong F(A)$$

in \mathcal{V} , where the left-hand side is the \mathcal{V} -object of \mathcal{V} -natural transformations.

Chapter 6

Monads and Algebras

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Monads are one of the most versatile concepts in category theory, appearing in algebra, topology, logic, and computer science. A monad on a

category captures the essence of algebraic structure: it packages together a “free construction” and the “equations” that the resulting algebraic objects must satisfy. In this chapter we define monads, show how they arise from adjunctions, study the categories of algebras over a monad (the Eilenberg–Moore and Kleisli categories), and prove Beck’s monadicity theorem, which characterises those adjunctions that arise from monads.

6.1 Monads

Definition 6.1.1 (Monad). A **monad** on a category \mathcal{C} is a triple (T, η, μ) where:

- $T: \mathcal{C} \rightarrow \mathcal{C}$ is an endofunctor,
- $\eta: \text{Id}_{\mathcal{C}} \Rightarrow T$ is a natural transformation called the **unit**,
- $\mu: T^2 \Rightarrow T$ is a natural transformation called the **multiplication**,

subject to the following diagrams commuting for every object $A \in \mathcal{C}$:

Associativity:

$$\begin{array}{ccc} T^3 A & \xrightarrow{T\mu_A} & T^2 A \\ \mu_{TA} \downarrow & & \downarrow \mu_A \\ T^2 A & \xrightarrow{\mu_A} & TA \end{array}$$

Unit laws:

$$\begin{array}{ccc} TA & \xrightarrow{T\eta_A} & T^2 A & \xleftarrow{\eta_{TA}} & TA \\ & \searrow & \downarrow \mu_A & \swarrow & \\ & & TA & & \end{array}$$

Equivalently, $\mu \circ T\mu = \mu \circ \mu T$ and $\mu \circ T\eta = \mu \circ \eta T = \text{id}_T$.

Remark 6.1.2. A monad is a **monoid** in the monoidal category $(\text{End}(\mathcal{C}), \circ, \text{Id})$ of endofunctors with composition as the monoidal product. The unit η is the monoid unit and the multiplication μ is the monoid multiplication. This perspective is extremely useful.

6.2 Monads from adjunctions

Proposition 6.2.1 (Every adjunction gives a monad). *Let $F \dashv G$ be an adjunction with $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$, unit $\eta: \text{Id}_{\mathcal{C}} \Rightarrow GF$, and counit $\varepsilon: FG \Rightarrow \text{Id}_{\mathcal{D}}$. Then (T, η, μ) is a monad on \mathcal{C} , where $T = GF$ and $\mu = G\varepsilon F: GF GF \Rightarrow GF$.*

Proof. Associativity: We must show $\mu \circ T\mu = \mu \circ \mu T$, i.e. $G\varepsilon F \circ GF \cdot G\varepsilon F = G\varepsilon F \circ G\varepsilon F \cdot GF$. Equivalently, for each A :

$$\begin{aligned}\mu_A \circ T\mu_A &= G\varepsilon_{FA} \circ GF G\varepsilon_{FA} = G(\varepsilon_{FA} \circ FG\varepsilon_{FA}), \\ \mu_A \circ \mu_{TA} &= G\varepsilon_{FA} \circ G\varepsilon_{FGFA} = G(\varepsilon_{FA} \circ \varepsilon_{FGFA}).\end{aligned}$$

By naturality of ε at $\varepsilon_{FA}: FGFA \rightarrow FA$: $\varepsilon_{FA} \circ FG\varepsilon_{FA} = \varepsilon_{FA} \circ \varepsilon_{FGFA}$. Hence the two expressions agree.

Left unit law: We show $\mu \circ \eta T = \text{id}_T$. For each A : $\mu_A \circ \eta_{TA} = G\varepsilon_{FA} \circ \eta_{GFA} = \text{id}_{GFA} = \text{id}_{TA}$, by one of the triangle identities ($G\varepsilon \circ \eta G = \text{id}_G$ applied to FA).

Right unit law: We show $\mu \circ T\eta = \text{id}_T$. For each A : $\mu_A \circ T\eta_A = G\varepsilon_{FA} \circ GF\eta_A = G(\varepsilon_{FA} \circ F\eta_A) = G(\text{id}_{FA}) = \text{id}_{TA}$, by the other triangle identity ($\varepsilon F \circ F\eta = \text{id}_F$ applied to A). \square

Remark 6.2.2. Dually, the adjunction $F \dashv G$ also gives a **comonad** (S, ε, δ) on \mathcal{D} , where $S = FG$, the counit is $\varepsilon: FG \Rightarrow \text{Id}_{\mathcal{D}}$, and the comultiplication is $\delta = F\eta G: FG \Rightarrow FGFG$.

6.3 Examples of monads

Example 6.3.1 (Free monoid monad). The free-forgetful adjunction $F \dashv U: \mathbf{Mon} \rightarrow \mathbf{Set}$ gives rise to the monad $T = UF$ on \mathbf{Set} . Explicitly:

$$T(X) = \coprod_{n \geq 0} X^n = \{(x_1, \dots, x_n) : n \geq 0, x_i \in X\},$$

the set of finite words on the alphabet X . The unit $\eta_X: X \rightarrow T(X)$ sends x to the one-letter word (x) . The multiplication $\mu_X: T^2(X) \rightarrow T(X)$ is “flattening” (concatenation): it sends a word of words to a single word by removing parentheses. The monad axioms correspond to the associativity of concatenation and the fact that the empty word is a unit.

Example 6.3.2 (Free group monad). The adjunction $F \dashv U: \mathbf{Grp} \rightarrow \mathbf{Set}$ gives a monad T where $T(X)$ is the underlying set of the free group on X , i.e. the set of reduced words in $X \cup X^{-1}$.

Example 6.3.3 (Free R -module monad). For a ring R , the adjunction $F \dashv U: R\text{-Mod} \rightarrow \mathbf{Set}$ gives a monad with

$$T(X) = \bigoplus_{x \in X} R = \left\{ \sum_{x \in X} r_x \cdot x : r_x \in R, \text{ finitely many nonzero} \right\}.$$

Example 6.3.4 (Power set monad). The **power set monad** \mathcal{P} on \mathbf{Set} is defined by $\mathcal{P}(X) = 2^X$ (the power set), with $\eta_X(x) = \{x\}$ and $\mu_X(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$ for $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$. This monad arises from the adjunction between \mathbf{Set} and the category of complete sup-lattices.

Example 6.3.5 (Maybe monad). The functor $T(X) = X \sqcup \{*\}$ on \mathbf{Set} , with η_X the inclusion $X \hookrightarrow X \sqcup \{*\}$ and μ_X collapsing the two copies of $*$, is a monad. In computer science, this is the **Maybe** (or **option**) monad, modelling computations that may fail.

Example 6.3.6 (Continuation monad). Fix a set R . The functor $T(X) = R^{R^X}$ on \mathbf{Set} carries a monad structure called the **continuation monad**. In functional programming, it arises from the self-adjunction $(-)^R \dashv (-)^R: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$.

Example 6.3.7 (Monad from a closure operator). Let (P, \leq) be a poset viewed as a category. A monad on P is a **closure operator**: a monotone map $c: P \rightarrow P$ with $x \leq c(x)$ (unit) and $c(c(x)) \leq c(x)$ (multiplication), which combined with monotonicity gives $c(c(x)) = c(x)$ (idempotence).

6.4 Eilenberg–Moore algebras

Definition 6.4.1 (Algebra over a monad). Let (T, η, μ) be a monad on \mathcal{C} . A T -**algebra** (or **Eilenberg–Moore algebra**) is a pair (A, a) where $A \in \mathcal{C}$ is an object and $a: TA \rightarrow A$ is a morphism (the **structure map**) such that:

Unit: $a \circ \eta_A = \text{id}_A$.

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow & \downarrow a \\ & & A \end{array}$$

Associativity: $a \circ \mu_A = a \circ Ta$.

$$\begin{array}{ccc} T^2A & \xrightarrow{\mu_A} & TA \\ Ta \downarrow & & \downarrow a \\ TA & \xrightarrow{a} & A \end{array}$$

Definition 6.4.2 (Morphism of algebras). A **morphism of T -algebras** $f: (A, a) \rightarrow (B, b)$ is a morphism $f: A \rightarrow B$ in \mathcal{C} such that $f \circ a = b \circ Tf$:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

Definition 6.4.3 (Eilenberg–Moore category). The **Eilenberg–Moore category** \mathcal{C}^T has:

- **Objects:** T -algebras (A, a) ,
- **Morphisms:** morphisms of T -algebras,
- **Composition and identities:** inherited from \mathcal{C} .

There is a forgetful functor $U^T: \mathcal{C}^T \rightarrow \mathcal{C}$, $(A, a) \mapsto A$, which has a left adjoint $F^T: \mathcal{C} \rightarrow \mathcal{C}^T$ defined by $F^T(A) = (TA, \mu_A)$.

Proposition 6.4.4. *The pair $F^T \dashv U^T$ is an adjunction, and the monad arising from this adjunction (via Proposition 6.2.1) is precisely (T, η, μ) .*

Proof. $F^T(A) = (TA, \mu_A)$ is a T -algebra: The unit axiom is $\mu_A \circ \eta_{TA} = \text{id}_{TA}$ (left unit of the monad). The associativity axiom is $\mu_A \circ \mu_{TA} = \mu_A \circ T\mu_A$ (monad associativity).

Adjunction: For $(A, a) \in \mathcal{C}^T$ and $B \in \mathcal{C}$, define

$$\Phi: \text{Hom}_{\mathcal{C}^T}(F^T B, (A, a)) \longrightarrow \text{Hom}_{\mathcal{C}}(B, U^T(A, a) = A)$$

by $\Phi(f) = f \circ \eta_B$. The inverse is $\Psi(g) = a \circ Tg$ for $g: B \rightarrow A$. One verifies that $\Psi(g)$ is a T -algebra morphism and that Φ, Ψ are mutually inverse using the algebra axioms and monad axioms.

Recovery of the monad: $U^T F^T(A) = U^T(TA, \mu_A) = TA = T(A)$, so $U^T F^T = T$. The unit of the adjunction $F^T \dashv U^T$ is η . The counit at (A, a) is the algebra map $\varepsilon_{(A,a)} = a: (TA, \mu_A) \rightarrow (A, a)$, which is a T -algebra morphism by the associativity axiom for (A, a) . The induced multiplication is $U^T \varepsilon_{F^T} \cdot \eta = \mu$, as desired. \square

- Example 6.4.5.** (i) **Free monoid monad** (Example 6.3.1): T -algebras are monoids. A structure map $a: T(X) \rightarrow X$ takes a word (x_1, \dots, x_n) and produces a single element: this is the multiplication. The algebra axioms force associativity and unitality.
- (ii) **Free group monad** (Example 6.3.2): T -algebras are groups. More precisely, $\mathbf{Set}^T \simeq \mathbf{Grp}$.
- (iii) **Free module monad** (Example 6.3.3): T -algebras are R -modules, i.e. $\mathbf{Set}^T \simeq R\text{-Mod}$.
- (iv) **Power set monad** (Example 6.3.4): T -algebras are complete sup-lattices with sup-preserving maps as morphisms.
- (v) **Closure operators** (Example 6.3.7): the algebras for a closure operator c on a poset P are exactly the **closed elements**, i.e. those x with $c(x) = x$.

6.5 Free algebras

Definition 6.5.1 (Free T -algebra). A T -algebra (A, a) is **free** if it is isomorphic to $F^T(X) = (TX, \mu_X)$ for some $X \in \mathcal{C}$.

Proposition 6.5.2. *The free T -algebra on X satisfies the universal property: for every T -algebra (A, a) and every morphism $f: X \rightarrow A$ in \mathcal{C} , there is a unique T -algebra morphism $\bar{f}: (TX, \mu_X) \rightarrow (A, a)$ with $\bar{f} \circ \eta_X = f$. Explicitly, $\bar{f} = a \circ Tf$.*

Proof. This is simply the adjunction $F^T \dashv U^T$ applied to the pair $(X, (A, a))$: $\text{Hom}_{\mathcal{C}^T}(F^T X, (A, a)) \cong \text{Hom}_{\mathcal{C}}(X, A)$. \square

6.6 The Kleisli category

Definition 6.6.1 (Kleisli category). Let (T, η, μ) be a monad on \mathcal{C} . The Kleisli category \mathcal{C}_T has:

- **Objects:** the same as \mathcal{C} ,
- **Morphisms:** $\text{Hom}_{\mathcal{C}_T}(A, B) = \text{Hom}_{\mathcal{C}}(A, TB)$,
- **Identity:** $\text{id}_A = \eta_A: A \rightarrow TA$,
- **Composition:** given $f: A \rightarrow TB$ and $g: B \rightarrow TC$ (in \mathcal{C}), their Kleisli composite is

$$g \circ_T f = \mu_C \circ Tg \circ f : A \rightarrow TC.$$

Proposition 6.6.2. *The Kleisli category \mathcal{C}_T is well-defined, i.e. composition is associative and η_A is a two-sided identity.*

Proof. *Left identity:* $\text{id}_B \circ_T f = \mu_B \circ T\eta_B \circ f = \text{id}_{TB} \circ f = f$ (right unit law of the monad).

Right identity: $f \circ_T \text{id}_A = \mu_B \circ Tf \circ \eta_A = f$ (naturality of η : $Tf \circ \eta_A = \eta_{TB} \circ f$, then $\mu_B \circ \eta_{TB} = \text{id}_{TB}$).

Associativity: Let $f: A \rightarrow TB$, $g: B \rightarrow TC$, $h: C \rightarrow TD$. Then:

$$\begin{aligned} h \circ_T (g \circ_T f) &= \mu_D \circ Th \circ \mu_C \circ Tg \circ f, \\ (h \circ_T g) \circ_T f &= \mu_D \circ T(\mu_D \circ Th \circ g) \circ f = \mu_D \circ T\mu_D \circ T^2h \circ Tg \circ f. \end{aligned}$$

By naturality of μ at h : $\mu_D \circ Th = \mu_D \circ Th$. More precisely, we use $\mu_D \circ T\mu_D = \mu_D \circ \mu_{TD}$ (associativity of the monad) and naturality of μ at h : $\mu_D \circ Th = Th \circ \mu_C$ is not quite right; let us be more careful. By naturality of μ at $h: C \rightarrow TD$, $\mu_D \circ T^2h = Th \circ \mu_C \dots$ actually this requires $h: TC \rightarrow TD$. Instead we use:

$$\mu_D \circ Th \circ \mu_C = \mu_D \circ \mu_{TD} \circ T^2h \quad (\text{naturality of } \mu \text{ at } h: \text{ applied as } \mu_D \circ Th = \dots)$$

Let us argue differently. By naturality of μ applied to $Th: TC \rightarrow T^2D$:

$$\mu_D \circ T\mu_D = \mu_D \circ \mu_{TD} \quad (\text{monad associativity}).$$

And by naturality of μ at the morphism $h: C \rightarrow TD$: $\mu_{TD} \circ T^2h = Th \circ \mu_C \dots$ No, $h: C \rightarrow TD$, so $Th: TC \rightarrow T^2D$ and $T^2h: T^2C \rightarrow T^3D$. Naturality of μ gives $\mu_D \circ Th = h^* \circ \mu_C \dots$ this is getting confused.

The cleanest argument: define $f^\sharp = \mu_B \circ Tf$ for $f: A \rightarrow TB$, so $g \circ_T f = g^\sharp \circ f$. Then associativity $(h \circ_T g) \circ_T f = h \circ_T (g \circ_T f)$ is equivalent to $(h^\sharp \circ g)^\sharp = h^\sharp \circ g^\sharp$, i.e. $\mu \circ T(h^\sharp \circ g) = h^\sharp \circ \mu \circ Tg$, which is $\mu \circ Th^\sharp \circ Tg = h^\sharp \circ \mu \circ Tg$. This reduces to $\mu \circ Th^\sharp = h^\sharp \circ \mu$, i.e. $\mu \circ T(\mu \circ Th) = \mu \circ Th \circ \mu$, which follows from monad associativity and naturality of μ . \square

Proposition 6.6.3. *There is an adjunction $F_T \dashv G_T$ with $F_T: \mathcal{C} \rightarrow \mathcal{C}_T$ and $G_T: \mathcal{C}_T \rightarrow \mathcal{C}$ such that $G_T F_T = T$ and the monad arising from $F_T \dashv G_T$ is (T, η, μ) . Explicitly:*

- $F_T(A) = A$ on objects, $F_T(f) = \eta_B \circ f$ on morphisms $f: A \rightarrow B$,
- $G_T(A) = TA$ on objects, $G_T(g) = \mu_B \circ Tg$ for $g: A \rightarrow TB$ (a Kleisli morphism $A \rightarrow B$).

Proof. The adjunction is verified directly: $\text{Hom}_{\mathcal{C}_T}(F_T A, B) = \text{Hom}_{\mathcal{C}}(A, TB) = \text{Hom}_{\mathcal{C}}(A, G_T B)$. The unit is η , the counit at B is $\text{id}_{TB}: TB \rightarrow TB$ viewed as a Kleisli morphism $F_T G_T B = F_T(TB) \rightarrow B$. One checks that $G_T F_T = T$ and the induced monad is (T, η, μ) . \square

Remark 6.6.4. The Kleisli category is the **initial** object and the Eilenberg–Moore category is the **terminal** object in the category of adjunctions giving rise to the monad T . More precisely, if $F \dashv G: \mathcal{D} \rightarrow \mathcal{C}$ is any adjunction with associated monad T , then there exist unique (up to isomorphism) comparison functors

$$\mathcal{C}_T \longrightarrow \mathcal{D} \longrightarrow \mathcal{C}^T$$

compatible with the adjunctions. The comparison functor $K: \mathcal{D} \rightarrow \mathcal{C}^T$ sends D to $(GD, G\varepsilon_D)$.

Example 6.6.5 (Kleisli for the power set monad). The Kleisli category for the power set monad \mathcal{P} on **Set** has sets as objects and $\text{Hom}(A, B) = \text{Hom}(A, \mathcal{P}(B)) \cong \{R \subseteq A \times B\}$ as morphisms, i.e. morphisms are **relations**. Kleisli composition corresponds to composition of relations.

Example 6.6.6 (Kleisli for the Maybe monad). The Kleisli category for the Maybe monad $T(X) = X \sqcup \{*\}$ has morphisms $A \rightarrow B \sqcup \{*\}$, i.e. partial functions. This captures the idea of computations that may fail.

6.7 Beck's monadicity theorem

The central question is: given an adjunction $F \dashv G$, when is the comparison functor $K: \mathcal{D} \rightarrow \mathcal{C}^T$ an equivalence?

Definition 6.7.1 (Comparison functor). Let $F \dashv G: \mathcal{D} \rightarrow \mathcal{C}$ be an adjunction with associated monad (T, η, μ) on \mathcal{C} (where $T = GF$). The **comparison functor** $K: \mathcal{D} \rightarrow \mathcal{C}^T$ is defined by:

- On objects: $K(D) = (GD, G\varepsilon_D)$, where $\varepsilon_D: FGD \rightarrow D$ is the counit. (This is a T -algebra: the unit and associativity axioms follow from the triangle identities and naturality of ε .)
- On morphisms: $K(f) = Gf$ for $f: D \rightarrow D'$. (This is a T -algebra morphism by naturality of ε .)

Definition 6.7.2 (Monadic functor). An adjunction $F \dashv G$ (equivalently, the right adjoint G) is **monadic** if the comparison functor K is an equivalence of categories. It is **strictly monadic** if K is an isomorphism of categories.

Definition 6.7.3 (G -split coequaliser). Let $G: \mathcal{D} \rightarrow \mathcal{C}$ be a functor. A parallel pair $f, g: D \rightrightarrows D'$ in \mathcal{D} is called a **G -split pair** if the image $Gf, Gg: GD \rightrightarrows GD'$ admits a **split coequaliser** in \mathcal{C} : a diagram

$$GD \begin{array}{c} \xrightarrow{Gf} \\ \xrightarrow{Gg} \end{array} GD' \begin{array}{c} \xrightarrow{e} \\ \xleftarrow{t} \end{array} C$$

with $e \circ Gf = e \circ Gg$, $e \circ s = \text{id}_C$, $Gf \circ t = s \circ e$, and $Gg \circ t = \text{id}_{GD'}$. A **G -split coequaliser** of f, g is a coequaliser of f, g in \mathcal{D} whose image under G is this split coequaliser.

Remark 6.7.4. Split coequalisers are **absolute**: they are preserved by any functor whatsoever. Indeed, if (e, s, t) is a split coequaliser as above, then applying any functor H preserves all the relevant equations, so He is the coequaliser of HGf, HGg . This crucial property is what makes Beck's theorem work.

Lemma 6.7.5. *A split coequaliser is a coequaliser.*

Proof. Let $Gf, Gg: A \rightrightarrows B$ with split coequaliser (e, s, t) as in Definition 6.7.3

(with $A = GD$, $B = GD'$). First, e coequalises: $e \circ Gf = e \circ Gg$ by assumption. For universality, let $h: B \rightarrow Z$ with $h \circ Gf = h \circ Gg$. Define $u = h \circ t: C \rightarrow Z$. Then $u \circ e = h \circ t \circ e = h \circ Gf \circ s = h \circ Gg \circ s \dots$ wait, we need: $u \circ e = h \circ t \circ e$. Using $Gf \circ t = s \circ e$: $h \circ s \circ e = h \circ Gf \circ t \dots$ let us recompute.

We have $e \circ s = \text{id}_C$, so $u \circ e = h \circ t \circ e$. Now $t \circ e = Gf \circ t \dots$ no, $Gf \circ t = s \circ e$. Actually: from $Gg \circ t = \text{id}_B$, we get

$$u \circ e = h \circ t \circ e = h \circ \text{id}_B = h?$$

No. Let us be careful. We need: From $Gg \circ t = \text{id}_B$: this gives $t: C \rightarrow A$ is a section of $Gg \dots$ no, $t: C \rightarrow B$ and $Gg: A \rightarrow B$, so this does not type-check. Let me reconsider.

In the standard formulation, a split coequaliser of $f_1, f_2: A \rightrightarrows B$ is a diagram $A \rightrightarrows B \xrightarrow{e} C$ with $s: C \rightarrow B$ and $t: B \rightarrow A$ such that $e \circ f_1 = e \circ f_2$, $e \circ s = \text{id}_C$, $f_1 \circ t = s \circ e$, $f_2 \circ t = \text{id}_B$.

Then for universality: let $h: B \rightarrow Z$ with $h \circ f_1 = h \circ f_2$. Define $u = h \circ s: C \rightarrow Z$. Then: $u \circ e = h \circ s \circ e = h \circ f_1 \circ t = h \circ f_2 \circ t = h \circ \text{id}_B = h \dots$ wait: $f_2 \circ t = \text{id}_B$, so $h \circ f_2 \circ t = h$. And $h \circ f_1 \circ t = h \circ f_2 \circ t = h$ since $h \circ f_1 = h \circ f_2$. So $u \circ e = h \circ s \circ e = h \circ f_1 \circ t = h$.

Uniqueness: if $u' \circ e = h$, then $u' = u' \circ e \circ s = h \circ s = u$. \square

Now we can state Beck's theorem.

Theorem 6.7.6 (Beck's Monadicity Theorem). *Let $F \dashv G$ with $G: \mathcal{D} \rightarrow \mathcal{C}$. The following are equivalent:*

- (i) G is monadic (the comparison functor K is an equivalence).
- (ii) G reflects isomorphisms and \mathcal{D} has coequalisers of all G -split pairs, and G preserves them.

Proof. The proof is substantial. We break it into parts.

(i) \Rightarrow (ii):

Assume $K: \mathcal{D} \rightarrow \mathcal{C}^T$ is an equivalence. Since $U^T \circ K = G$ and K is an equivalence, it suffices to show that $U^T: \mathcal{C}^T \rightarrow \mathcal{C}$ reflects isomorphisms and that \mathcal{C}^T has coequalisers of U^T -split pairs.

U^T reflects isomorphisms: Let $f: (A, a) \rightarrow (B, b)$ be a T -algebra morphism such that $f: A \rightarrow B$ is an isomorphism in \mathcal{C} . We claim $f^{-1}: (B, b) \rightarrow (A, a)$ is a T -algebra morphism. We need $f^{-1} \circ b = a \circ Tf^{-1}$. Since $f \circ a = b \circ Tf$ (as f is a T -algebra morphism), we get $a = f^{-1} \circ b \circ Tf$, hence $a \circ Tf^{-1} = f^{-1} \circ b \circ Tf \circ Tf^{-1} = f^{-1} \circ b \circ T(f \circ f^{-1}) = f^{-1} \circ b$.

\mathcal{C}^T has coequalisers of U^T -split pairs: Let $f, g: (A, a) \rightrightarrows (B, b)$ be T -algebra morphisms such that $f, g: A \rightrightarrows B$ admits a split coequaliser $(e: B \rightarrow C, s: C \rightarrow B, t: B \rightarrow A)$ in \mathcal{C} . We must endow C with a T -algebra structure.

Define $c: TC \rightarrow C$ by

$$c = e \circ b \circ Ts : TC \rightarrow C.$$

c is well-defined as a structure map: We verify the algebra axioms. Unit: $c \circ \eta_C = e \circ b \circ Ts \circ \eta_C = e \circ b \circ \eta_B \circ s = e \circ s = \text{id}_C$ (using naturality of η and the algebra unit for (B, b)). Associativity: $c \circ \mu_C = e \circ b \circ Ts \circ \mu_C = e \circ b \circ T(b \circ Ts) = e \circ b \circ Tb \circ T^2s \dots$ We use the algebra associativity for (B, b) : $b \circ Tb = b \circ \mu_B$. So $c \circ \mu_C = e \circ b \circ \mu_B \circ T^2s = e \circ b \circ Ts \circ T(e \circ b \circ Ts) \dots$ let us use $\mu_C = T(s) \circ \dots$ no. Actually:

$$\begin{aligned} c \circ Tc &= (e \circ b \circ Ts) \circ T(e \circ b \circ Ts) \\ &= e \circ b \circ Ts \circ Te \circ Tb \circ T^2s \\ &= e \circ b \circ T(s \circ e) \circ Tb \circ T^2s \\ &= e \circ b \circ T(f \circ t) \circ Tb \circ T^2s \quad (\text{since } s \circ e = f \circ t \dots \text{ no, } f_1 \circ t = s \circ e) \end{aligned}$$

We have $f \circ t = s \circ e$ from the split coequaliser. So $T(s \circ e) = Tf \circ Tt$. Then:

$$\begin{aligned} c \circ Tc &= e \circ b \circ Tf \circ Tt \circ Tb \circ T^2s \\ &= e \circ f \circ a \circ Tt \circ Tb \circ T^2s \quad (b \circ Tf = f \circ a \text{ since } f \text{ is a } T\text{-morphism}) \\ &= e \circ g \circ a \circ Tt \circ Tb \circ T^2s \quad (e \circ f = e \circ g) \\ &= e \circ b \circ Tg \circ Tt \circ Tb \circ T^2s \quad (b \circ Tg = g \circ a \text{ since } g \text{ is a } T\text{-morphism}) \\ &= e \circ b \circ T(g \circ t) \circ Tb \circ T^2s \\ &= e \circ b \circ T(\text{id}_B) \circ Tb \circ T^2s \quad (g \circ t = \text{id}_B) \\ &= e \circ b \circ Tb \circ T^2s \\ &= e \circ b \circ \mu_B \circ T^2s \quad (\text{associativity for } (B, b)) \\ &= e \circ b \circ Ts \circ \mu_C \quad (\text{naturality of } \mu \text{ at } s: \mu_C = Ts \dots \text{ no}) \end{aligned}$$

Naturality of μ at $s: C \rightarrow B$ gives $\mu_B \circ T^2s = Ts \circ \mu_C$. Hence:

$$c \circ Tc = e \circ b \circ Ts \circ \mu_C = c \circ \mu_C.$$

So (C, c) is indeed a T -algebra.

e is a T -algebra morphism: We check $e \circ b = c \circ Te$. $c \circ Te = e \circ b \circ Ts \circ Te = e \circ b \circ T(s \circ e) = e \circ b \circ Tf \circ Tt = e \circ f \circ a \circ Tt$. But also $e \circ b = e \circ b \circ T(g \circ t) = e \circ g \circ a \circ Tt = e \circ f \circ a \circ Tt \dots$ this is not right either. We need $c \circ Te = e \circ b$ directly. $c \circ Te = e \circ b \circ T(s \circ e) = e \circ b \circ T(f \circ t) = e \circ b \circ Tf \circ Tt = e \circ f \circ a \circ Tt$.

And $e \circ b = e \circ b \circ \text{id}_{TB} = e \circ b \circ T(g \circ t) = e \circ g \circ a \circ Tt = e \circ f \circ a \circ Tt$. Here we used $g \circ t = \text{id}_B$ and $e \circ g = e \circ f$. So indeed $c \circ Te = e \circ b$.

(C, c) is the coequaliser: Suppose $h: (B, b) \rightarrow (Z, z)$ is a T -algebra morphism with $h \circ f = h \circ g$. Since the underlying diagram is a split coequaliser, there is a unique $u: C \rightarrow Z$ in \mathcal{C} with $u \circ e = h$. We must show u is a T -algebra morphism, i.e. $u \circ c = z \circ Tu$.

$$u \circ c = u \circ e \circ b \circ Ts = h \circ b \circ Ts = z \circ Th \circ Ts = z \circ T(h \circ s) = z \circ T(u \circ e \circ s) = z \circ Tu.$$

(Using $h \circ b = z \circ Th$ since h is a T -morphism, and $e \circ s = \text{id}_C$.)

(ii) \Rightarrow (i):

Assume G reflects isomorphisms and \mathcal{D} has coequalisers of G -split pairs, preserved by G . We show K is an equivalence by constructing a quasi-inverse.

K is essentially surjective: Let $(A, a) \in \mathcal{C}^T$. Consider the pair $FTA \rightrightarrows FA$ in \mathcal{D} given by

$$Fa \text{ and } F\mu_A \quad (\text{no: } \varepsilon_{FA}: FGFA = FTA \rightarrow FA).$$

More precisely, consider the canonical pair: $F\mu_A, \varepsilon_{FTA}: FT^2A = FGFTA \rightrightarrows FTA$...

Let us use the standard construction. For $(A, a) \in \mathcal{C}^T$, consider the parallel pair in \mathcal{D} :

$$FTGA = FTA \begin{array}{c} \xrightarrow{Fa} \\ \xrightarrow{\varepsilon_{FA}} \end{array} FA.$$

Wait— $a: TA \rightarrow A$ is a morphism in \mathcal{C} , so $Fa: FTA \rightarrow FA$. And $\varepsilon_{FA}: FGFA = FTA \rightarrow FA$. So we have $Fa, \varepsilon_{FA}: FTA \rightrightarrows FA$.

Applying G : $GFa, G\varepsilon_{FA}: GFTA = T^2A \rightrightarrows TA = GFA$. That is, $Ta, \mu_A: T^2A \rightrightarrows TA$. This pair has a split coequaliser in \mathcal{C} with $e = a: TA \rightarrow A$, $s = \eta_A: A \rightarrow TA$, $t = T\eta_A$... let us verify:

- $e \circ Ta = a \circ Ta$ and $e \circ \mu_A = a \circ \mu_A$. The algebra associativity says $a \circ Ta = a \circ \mu_A$. \checkmark
- $e \circ s = a \circ \eta_A = \text{id}_A$. \checkmark (algebra unit)
- $Ta \circ t = s \circ e$: We need $t: TA \rightarrow T^2A$ with $Ta \circ t = \eta_A \circ a$ and $\mu_A \circ t = \text{id}_{TA}$. Take $t = T\eta_A$. Then $\mu_A \circ T\eta_A = \text{id}_{TA}$ (monad right unit). \checkmark
- $Ta \circ T\eta_A = T(a \circ \eta_A) = T(\text{id}_A) = \text{id}_{TA}$. But we need $Ta \circ t = s \circ e = \eta_A \circ a$. We have $Ta \circ T\eta_A = \text{id}_{TA} \neq \eta_A \circ a$ in general.

So let us take $t = \eta_{TA}: TA \rightarrow T^2A$ instead.

- $\mu_A \circ \eta_{TA} = \text{id}_{TA}$. ✓ (monad left unit)
- $Ta \circ \eta_{TA} = \eta_A \circ a$: this is naturality of η at a . ✓

So the split coequaliser data is: $e = a$, $s = \eta_A$, $t = \eta_{TA}$, with the coequalised pair being (Ta, μ_A) .

So the pair (Fa, ε_{FA}) is a G -split pair. By hypothesis, \mathcal{D} has a coequaliser $q: FA \rightarrow D$ of (Fa, ε_{FA}) , and G preserves it. Since G preserves this coequaliser and the split coequaliser of (Ta, μ_A) is $a: TA \rightarrow A$, we get $Gq = a$ (up to the canonical isomorphism). More precisely, $GD \cong A$ via the isomorphism coming from the fact that $Gq: GFA = TA \rightarrow GD$ is the coequaliser of (Ta, μ_A) in \mathcal{C} , which is a with target A . So $GD \cong A$.

Now $K(D) = (GD, G\varepsilon_D)$. Under the identification $GD \cong A$, we get a T -algebra structure on A that agrees with a . Hence $K(D) \cong (A, a)$, and K is essentially surjective.

K is fully faithful: For $D, D' \in \mathcal{D}$, the map $K_{D,D'}: \text{Hom}_{\mathcal{D}}(D, D') \rightarrow \text{Hom}_{\mathcal{C}T}(K(D), K(D'))$ sends $f \mapsto Gf$. Since morphisms of T -algebras are just morphisms in \mathcal{C} satisfying a compatibility condition, $\text{Hom}_{\mathcal{C}T}(K(D), K(D')) \subseteq \text{Hom}_{\mathcal{C}}(GD, GD')$.

Faithful: If $Gf = Gf'$, we want $f = f'$. Consider the coequalisers $q: FA \rightarrow D$ from the essential surjectivity argument. Since $Gf = Gf'$ and G reflects the relevant coequalisers, one shows $f = f'$. Alternatively: for any $D \in \mathcal{D}$, the counit $\varepsilon_D: FGD \rightarrow D$ is a (regular) epimorphism (it is the coequaliser of a G -split pair). If $Gf = Gf'$, then $FGf = FGf'$, so $f \circ \varepsilon_D = \varepsilon_{D'} \circ FGf = \varepsilon_{D'} \circ FGf' = f' \circ \varepsilon_D$. Since ε_D is an epimorphism, $f = f'$.

Full: Let $\varphi: GD \rightarrow GD'$ be a T -algebra morphism $K(D) \rightarrow K(D')$. Consider the diagram in \mathcal{D} :

$$FGFGD \xrightarrow[\varepsilon_{FGD'}]{F(G\varepsilon_D)} FGD \xrightarrow{\varepsilon_D} D$$

and the corresponding one for D' . The map $F\varphi: FGD \rightarrow FGD'$ satisfies the required compatibility (since φ is a T -algebra morphism). Since ε_D is the coequaliser and $\varepsilon_{D'} \circ F\varphi$ coequalises the pair for D , there exists a unique $f: D \rightarrow D'$ with $f \circ \varepsilon_D = \varepsilon_{D'} \circ F\varphi$. Applying G : $Gf \circ G\varepsilon_D = G\varepsilon_{D'} \circ GF\varphi = G\varepsilon_{D'} \circ T\varphi$. But for a T -algebra morphism, $\varphi \circ G\varepsilon_D = G\varepsilon_{D'} \circ T\varphi$, so $Gf \circ G\varepsilon_D = \varphi \circ G\varepsilon_D$. Since $G\varepsilon_D = a$ is an epimorphism (it is a split epimorphism), $Gf = \varphi$.

Hence K is fully faithful and essentially surjective, so it is an equivalence. \square

Remark 6.7.7. There is a **crude monadicity theorem** (also called the **tripleability theorem** in older terminology): G is monadic if and only if G has a left adjoint, G reflects isomorphisms, and \mathcal{D} has and G preserves coequalisers of *reflexive* G -split pairs (a reflexive pair is one admitting a common section). This is sometimes easier to verify in practice.

Corollary 6.7.8. *A monadic functor reflects isomorphisms and creates coequalisers of G -split pairs.*

Proof. This follows immediately from the (i) \Rightarrow (ii) direction of Beck's theorem. \square

6.8 Applications of monadicity

Example 6.8.1 (Algebraic categories are monadic over **Set**). The forgetful functors from **Grp**, **Ab**, **Ring**, $R\text{-Mod}$, and more generally any variety of algebras in the sense of universal algebra, to **Set** are monadic. We verify the conditions of Beck's theorem:

- (a) The forgetful functor U has a left adjoint (the free functor).
- (b) U reflects isomorphisms: a bijective homomorphism is an isomorphism.
- (c) **Set** has all coequalisers, and the free-forgetful adjunction creates them via the algebra structure.

Example 6.8.2 (Compact Hausdorff spaces). The forgetful functor $U: \text{CompHaus} \rightarrow \text{Set}$ is monadic. The associated monad is the ultrafilter monad β , whose algebras are compact Hausdorff spaces. This is a deep theorem of Manes (1969).

Example 6.8.3 (Non-monadic functors). (i) The forgetful functor $U: \text{Top} \rightarrow \text{Set}$ is *not* monadic. It has a left adjoint (discrete topology) but does not reflect isomorphisms: a continuous bijection need not be a homeomorphism.

- (ii) The inclusion $\text{Ab} \hookrightarrow \text{Grp}$ is not monadic: it does not reflect isomorphisms in general, and the associated monad (abelianisation)

does not recover \mathbf{Ab} as the category of algebras (the Eilenberg–Moore category is equivalent to \mathbf{Ab} , but the inclusion is not the comparison functor from \mathbf{Grp}). Actually, the functor $\mathbf{Ab} \rightarrow \mathbf{Grp}$ does reflect isomorphisms (it is fully faithful), so the issue is that coequalisers of split pairs in \mathbf{Grp} need not land in \mathbf{Ab} .

Proposition 6.8.4. *A monadic functor creates all limits that exist in the base category. In particular, if $G: \mathcal{D} \rightarrow \mathcal{C}$ is monadic and \mathcal{C} is complete, then so is \mathcal{D} .*

Proof. Since G is monadic, $\mathcal{D} \simeq \mathcal{C}^T$ for a monad T on \mathcal{C} . It suffices to show U^T creates limits. Let $D: \mathcal{J} \rightarrow \mathcal{C}^T$ be a diagram with $D(j) = (A_j, a_j)$, and suppose $L = \varprojlim_j A_j$ exists in \mathcal{C} with projections $\pi_j: L \rightarrow A_j$. Then $TL = T(\varprojlim_j A_j)$ and we need a T -algebra structure on L . Since T (being a right adjoint composite $G \circ F \dots$ no, T need not preserve limits in general).

For each j , the maps $a_j \circ T\pi_j: TL \rightarrow A_j$ form a cone over the diagram $\{A_j\}$ (using compatibility of the a_j with the algebra morphisms). By the universal property of L , there is a unique $\ell: TL \rightarrow L$ with $\pi_j \circ \ell = a_j \circ T\pi_j$ for all j . One verifies that (L, ℓ) is a T -algebra and is the limit of D in \mathcal{C}^T . The algebra axioms for (L, ℓ) follow from those for each (A_j, a_j) and the fact that the π_j jointly detect equality (since L is their limit). \square

6.9 Exercises

Exercise 6.9.1. Complete the details of Proposition 6.2.1: verify each monad axiom carefully using the triangle identities.

Exercise 6.9.2. Let $T: \mathbf{Set} \rightarrow \mathbf{Set}$ be the “list” functor with $T(X) = \coprod_{n \geq 0} X^n$. Define η and μ explicitly and verify the monad axioms. Show that T -algebras are monoids.

Exercise 6.9.3. Verify all details of Proposition 6.4.4: show that $F^T \dashv U^T$ is an adjunction and that the associated monad is (T, η, μ) .

Exercise 6.9.4. For the free group monad T on \mathbf{Set} , describe Kleisli composition explicitly. Show that a Kleisli morphism $A \rightarrow B$ is a function $A \rightarrow F(B)$ (where $F(B)$ is the free group on B) and that composition corresponds to substitution followed by reduction.

Exercise 6.9.5. A monad (T, η, μ) is **idempotent** if $\mu: T^2 \Rightarrow T$ is a natural isomorphism.

- (a) Show that the following are equivalent: (i) T is idempotent; (ii) $T\eta = \eta T$; (iii) every T -algebra is free.
- (b) Show that the monad arising from a reflective subcategory (i.e. a full subcategory whose inclusion has a left adjoint) is idempotent.
- (c) Deduce that the category of algebras for an idempotent monad is equivalent to the full subcategory of \mathcal{C} consisting of objects of the form TA .

Exercise 6.9.6. Let (S, η^S, μ^S) and (T, η^T, μ^T) be monads on \mathcal{C} . A **distributive law** of S over T is a natural transformation $\lambda: TS \Rightarrow ST$ satisfying four axioms (compatibility with units and multiplications of both monads).

- (a) Write down the four axiom diagrams.
- (b) Show that given a distributive law, the composite ST carries a monad structure with unit $\eta^S \eta^T$ and multiplication $\mu^S \cdot S\lambda T \cdot \mu^T$.
- (c) Show that the ring axiom “multiplication distributes over addition” is a distributive law for the free monoid monad over the free abelian group monad.

Exercise 6.9.7. Use Beck’s monadicity theorem to give an alternative proof that the forgetful functor $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ is monadic. *Hint:* Verify that U reflects isomorphisms and that \mathbf{Grp} has coequalisers of U -split pairs.

Exercise 6.9.8. Show that the forgetful functor $U: \mathbf{Top} \rightarrow \mathbf{Set}$ is not monadic by exhibiting a continuous bijection that is not a homeomorphism. What is the monad $T = UF$ (where F is the discrete topology functor)?

Exercise 6.9.9. Dualise the theory of this chapter: define a **coalgebra** for a comonad (S, ε, δ) on \mathcal{C} and formulate the dual of Beck’s theorem for comonadic functors.

Exercise 6.9.10. A morphism of monads $\varphi: (S, \eta^S, \mu^S) \rightarrow (T, \eta^T, \mu^T)$ on the same category \mathcal{C} is a natural transformation $\varphi: S \Rightarrow T$ compatible with units and multiplications: $\varphi \circ \eta^S = \eta^T$ and $\varphi \circ \mu^S = \mu^T \circ \varphi\varphi$ (where $\varphi\varphi = \varphi T \circ S\varphi = T\varphi \circ \varphi S$).

- (a) Show that a monad morphism $\varphi: S \rightarrow T$ induces a functor $\varphi^*: \mathcal{C}^T \rightarrow \mathcal{C}^S$ (restriction of scalars).
- (b) Show that φ^* has a left adjoint (extension of scalars).

Exercise 6.9.11. In the proof of Beck's theorem (direction (i) \Rightarrow (ii)), verify directly that the candidate T -algebra structure $c = e \circ b \circ Ts$ on C satisfies the associativity axiom, filling in any steps that were left implicit.

Chapter 7

Abelian Categories and Exact Sequences

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7.1 Preadditive and Additive Categories

The passage from general category theory to homological algebra requires enriching the categorical framework with algebraic structure on morphism sets. This section introduces the hierarchy of additive structures on categories.

Definition 7.1.1 (Preadditive category). A **preadditive category** (or **Ab-category**) is a category \mathcal{A} such that:

1. For every pair of objects $A, B \in \text{Ob}(\mathcal{A})$, the set $\text{Hom}_{\mathcal{A}}(A, B)$ carries the structure of an abelian group, with addition denoted $+$ and zero element denoted $0_{A,B}$.

2. Composition is bilinear: for all morphisms $f, f': A \rightarrow B$ and $g, g': B \rightarrow C$,

$$g \circ (f + f') = g \circ f + g \circ f', \quad (g + g') \circ f = g \circ f + g' \circ f.$$

Equivalently, \mathcal{A} is a category enriched over $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

Remark 7.1.2. In a preadditive category, for any objects A, B , the zero element $0_{A,B} \in \text{Hom}(A, B)$ satisfies $0_{B,C} \circ f = 0_{A,C}$ and $g \circ 0_{A,B} = 0_{A,C}$ for all $f: A \rightarrow B$ and $g: B \rightarrow C$. Thus the zero morphisms form a compatible system. If \mathcal{A} has a zero object 0 , then $0_{A,B}$ is the composite $A \rightarrow 0 \rightarrow B$.

Example 7.1.3. 1. The category \mathbf{Ab} of abelian groups is preadditive, with pointwise addition of homomorphisms.

2. For any ring R , the category \mathbf{Mod}_R of right R -modules is preadditive.
3. The category \mathbf{Set} is *not* preadditive: there is no natural abelian group structure on $\text{Hom}_{\mathbf{Set}}(X, Y)$ for general sets X, Y .
4. Any ring R can be viewed as a preadditive category with a single object $*$, where $\text{Hom}(*, *) = R$.

Definition 7.1.4 (Additive functor). Let \mathcal{A}, \mathcal{B} be preadditive categories. A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is **additive** if for all objects A, B the map

$$F_{A,B}: \text{Hom}_{\mathcal{A}}(A, B) \longrightarrow \text{Hom}_{\mathcal{B}}(FA, FB)$$

is a group homomorphism: $F(f + g) = F(f) + F(g)$.

Definition 7.1.5 (Biproduct). Let \mathcal{A} be a preadditive category and let A_1, \dots, A_n be objects. A **biproduct** of A_1, \dots, A_n is an object $A_1 \oplus \dots \oplus A_n$ together with morphisms

$$\iota_k: A_k \rightarrow A_1 \oplus \dots \oplus A_n, \quad \pi_k: A_1 \oplus \dots \oplus A_n \rightarrow A_k, \quad 1 \leq k \leq n,$$

satisfying:

1. $\pi_k \circ \iota_j = \delta_{jk} \cdot \text{id}_{A_k}$ for all j, k ,

$$2. \sum_{k=1}^n \iota_k \circ \pi_k = \text{id}_{A_1 \oplus \cdots \oplus A_n}.$$

Proposition 7.1.6. *A biproduct $(A_1 \oplus \cdots \oplus A_n, \iota_k, \pi_k)$ is simultaneously a product and a coproduct. More precisely:*

1. $(A_1 \oplus \cdots \oplus A_n, (\pi_k))$ is a product of A_1, \dots, A_n .
2. $(A_1 \oplus \cdots \oplus A_n, (\iota_k))$ is a coproduct of A_1, \dots, A_n .

Proof. We prove the product property; the coproduct property is dual. Given morphisms $f_k: B \rightarrow A_k$ for $1 \leq k \leq n$, define $f = \sum_{k=1}^n \iota_k \circ f_k: B \rightarrow A_1 \oplus \cdots \oplus A_n$. Then

$$\pi_j \circ f = \sum_{k=1}^n \pi_j \circ \iota_k \circ f_k = \sum_{k=1}^n \delta_{jk} f_k = f_j.$$

For uniqueness, if $g: B \rightarrow A_1 \oplus \cdots \oplus A_n$ satisfies $\pi_k \circ g = f_k$ for all k , then

$$g = \left(\sum_{k=1}^n \iota_k \circ \pi_k \right) \circ g = \sum_{k=1}^n \iota_k \circ f_k = f. \quad \square$$

Definition 7.1.7 (Additive category). An **additive category** is a preadditive category \mathcal{A} that:

1. has a zero object 0 ,
2. has finite biproducts: for any objects $A, B \in \text{Ob}(\mathcal{A})$, the biproduct $A \oplus B$ exists.

Proposition 7.1.8. *Let \mathcal{A} be a preadditive category with a zero object and finite products (or finite coproducts). Then \mathcal{A} is additive.*

Proof. Suppose \mathcal{A} has finite products. Let $A_1, A_2 \in \text{Ob}(\mathcal{A})$, and let $(A_1 \times A_2, \pi_1, \pi_2)$ be their product. Define $\iota_1: A_1 \rightarrow A_1 \times A_2$ as the unique morphism with $\pi_1 \circ \iota_1 = \text{id}_{A_1}$ and $\pi_2 \circ \iota_1 = 0$, and similarly for ι_2 . One checks that $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \text{id}$ using the universal property and the fact that

$$\pi_j \circ (\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) = \delta_{1j} \pi_1 + \delta_{2j} \pi_2 = \pi_j$$

for $j = 1, 2$. Thus $(A_1 \times A_2, \iota_k, \pi_k)$ is a biproduct. \square

Example 7.1.9. In \mathbf{Mod}_R , the biproduct $M \oplus N$ is the usual direct sum. The morphism set $\text{Hom}(M_1 \oplus M_2, N_1 \oplus N_2)$ can be identified with 2×2 matrices:

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, \quad f_{ij}: M_j \rightarrow N_i,$$

with composition given by matrix multiplication. This matrix calculus is a key feature of additive categories.

7.2 Kernels, Cokernels, and Abelian Categories

Definition 7.2.1 (Kernel and cokernel). Let $f: A \rightarrow B$ be a morphism in a preadditive category \mathcal{A} .

1. A **kernel** of f is an equalizer of f and $0_{A,B}$: it is a morphism $k: K \rightarrow A$ such that $f \circ k = 0$ and for every $g: X \rightarrow A$ with $f \circ g = 0$, there exists a unique $\bar{g}: X \rightarrow K$ with $k \circ \bar{g} = g$.

$$\begin{array}{ccccc} X & & & & \\ \downarrow \exists! \bar{g} & \searrow g & & \searrow 0 & \\ K & \xrightarrow{k} & A & \xrightarrow{f} & B \end{array}$$

We write $\text{Ker}(f) = (K, k)$ or simply $\text{Ker}(f) = K$.

2. A **cokernel** of f is a coequalizer of f and $0_{A,B}$: it is a morphism $c: B \rightarrow C$ such that $c \circ f = 0$ and for every $h: B \rightarrow Y$ with $h \circ f = 0$, there exists a unique $\bar{h}: C \rightarrow Y$ with $\bar{h} \circ c = h$. We write $\text{coker}(f) = (C, c)$.

Definition 7.2.2 (Image and coimage). Let $f: A \rightarrow B$ be a morphism in a preadditive category.

1. The **image** of f is $\text{im}(f) = \text{Ker}(\text{coker}(f))$.
2. The **coimage** of f is $\text{coim}(f) = \text{coker}(\text{Ker}(f))$.

There is a canonical morphism $\bar{f}: \text{coim}(f) \rightarrow \text{im}(f)$ induced by the universal properties.

Definition 7.2.3 (Abelian category). An **abelian category** is an additive category \mathcal{A} satisfying:

- (AB1) Every morphism has a kernel and a cokernel.
- (AB2) For every morphism f , the canonical morphism $\bar{f}: \text{coim}(f) \rightarrow \text{im}(f)$ is an isomorphism.

Remark 7.2.4. Condition (AB2) can equivalently be stated: every monomorphism is a kernel and every epimorphism is a cokernel. That is, every mono $m: K \hookrightarrow A$ is the kernel of some morphism out of A , and dually for epimorphisms.

Example 7.2.5. 1. **Ab** is an abelian category. Kernels and cokernels are the usual ones.

- 2. \mathbf{Mod}_R is abelian for any ring R . This is the prototypical example.
- 3. For a topological space X , the category $\text{Sh}(X, \mathbf{Ab})$ of sheaves of abelian groups on X is abelian.
- 4. The category of finitely generated abelian groups is abelian.
- 5. The category of free abelian groups is additive but *not* abelian: the cokernel of $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ would be $\mathbb{Z}/2\mathbb{Z}$, which is not free.

Definition 7.2.6 (Grothendieck axioms). Let \mathcal{A} be an abelian category. Grothendieck introduced additional axioms:

- (AB3) \mathcal{A} has arbitrary (set-indexed) coproducts.
- (AB3*) \mathcal{A} has arbitrary (set-indexed) products.
- (AB4) (AB3) holds and coproducts are exact: if $f_i: A_i \rightarrow B_i$ is a family of monomorphisms, then $\bigoplus f_i: \bigoplus A_i \rightarrow \bigoplus B_i$ is a monomorphism.
- (AB4*) (AB3*) holds and products are exact.
- (AB5) (AB3) holds and filtered colimits are exact: filtered colimits commute with finite limits.
- (AB5*) (AB3*) holds and filtered limits (cofiltered limits) are exact.

A **Grothendieck category** is an (AB5) abelian category with a generator.

Example 7.2.7. 1. \mathbf{Mod}_R satisfies (AB3)–(AB5) and (AB3*)–(AB4*). It is a Grothendieck category with generator R . It does *not* satisfy (AB5*) in general.

2. $\mathbf{Sh}(X, \mathbf{Ab})$ is a Grothendieck category.

3. The category \mathbf{Ab}^f of finite abelian groups satisfies (AB1)–(AB2) but not (AB3): infinite coproducts do not exist.

Theorem 7.2.8 (Freyd–Mitchell embedding). *Let \mathcal{A} be a small abelian category. Then there exists a ring R and an exact, fully faithful functor $F: \mathcal{A} \hookrightarrow \mathbf{Mod}_R$.*

Remark 7.2.9. The Freyd–Mitchell theorem justifies “diagram chasing” in abstract abelian categories: any statement about finite diagrams involving kernels, cokernels, exactness, etc., that holds in \mathbf{Mod}_R for all rings R automatically holds in any abelian category. We shall freely use element-wise arguments in proofs below, understood via this embedding.

7.3 Exact Sequences

Definition 7.3.1 (Exact sequence). Let \mathcal{A} be an abelian category. A sequence of morphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots$$

is **exact at** A_n if $\text{im}(f_{n+1}) = \text{Ker}(f_n)$ (as subobjects of A_n). Equivalently, $f_n \circ f_{n+1} = 0$ and the induced morphism $\text{coim}(f_{n+1}) \rightarrow \text{Ker}(f_n)$ is an isomorphism. The sequence is **exact** if it is exact at every term.

Definition 7.3.2 (Short exact sequence). A **short exact sequence** (SES) in \mathcal{A} is an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

This means:

- f is a monomorphism (exactness at A),
- g is an epimorphism (exactness at C),
- $\text{im}(f) = \text{Ker}(g)$ (exactness at B).

- Example 7.3.3.**
1. In \mathbf{Ab} : $0 \rightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$ for any $n \geq 1$.
 2. In \mathbf{Mod}_R : for any submodule $N \subseteq M$, we have $0 \rightarrow N \hookrightarrow M \rightarrow M/N \rightarrow 0$.
 3. A SES $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ **splits** if there exists $s: C \rightarrow B$ with $g \circ s = \text{id}_C$, or equivalently $r: B \rightarrow A$ with $r \circ f = \text{id}_A$. In this case $B \cong A \oplus C$.

Proposition 7.3.4 (Splitting lemma). *For a short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ in an abelian category, the following are equivalent:*

1. *There exists $s: C \rightarrow B$ with $g \circ s = \text{id}_C$ (right splitting).*
2. *There exists $r: B \rightarrow A$ with $r \circ f = \text{id}_A$ (left splitting).*
3. *$B \cong A \oplus C$ via an isomorphism compatible with f and g .*

Proof. (3) \Rightarrow (1) and (3) \Rightarrow (2) are clear from the biproduct structure.

(1) \Rightarrow (3): Given $s: C \rightarrow B$ with $g \circ s = \text{id}_C$, define $\varphi: A \oplus C \rightarrow B$ by $\varphi = f \circ \pi_A + s \circ \pi_C$. Then $g \circ \varphi \circ \iota_C = g \circ s = \text{id}_C$ and $g \circ \varphi \circ \iota_A = g \circ f = 0$. We claim φ is an isomorphism. By the five lemma (Theorem 7.6.1 below), or by direct argument: for $b \in B$, write $b = f(a) + s(g(b))$ where a is the unique element with $f(a) = b - s(g(b))$ (which lies in $\text{Ker}(g) = \text{im}(f)$).

(2) \Rightarrow (3): Dual argument. □

7.4 Exact Functors

Definition 7.4.1 (Exact functor). Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories.

1. F is **left exact** if for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the sequence $0 \rightarrow FA \rightarrow FB \rightarrow FC$ is exact.
2. F is **right exact** if for every SES, $FA \rightarrow FB \rightarrow FC \rightarrow 0$ is exact.

3. F is **exact** if it is both left and right exact, i.e., it preserves short exact sequences.

Example 7.4.2. 1. For any R -module M , the functor $\text{Hom}_R(M, -)$ is left exact (covariant). The functor $\text{Hom}_R(-, M)$ is left exact (contravariant, i.e., exact as a functor $\mathbf{Mod}_R^{\text{op}} \rightarrow \mathbf{Ab}$).

2. The tensor product functor $M \otimes_R -$ is right exact for any right R -module M .
3. The forgetful functor $\mathbf{Mod}_R \rightarrow \mathbf{Ab}$ is exact.
4. An R -module M is **flat** if and only if $M \otimes_R -$ is exact.

Proposition 7.4.3. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. Then:*

1. F is left exact if and only if F preserves kernels.
2. F is right exact if and only if F preserves cokernels.
3. F is exact if and only if F preserves short exact sequences.

7.5 The Snake Lemma

The snake lemma is one of the fundamental tools of homological algebra. It produces long exact sequences from short ones.

Theorem 7.5.1 (Snake lemma). *Consider a commutative diagram in an abelian category \mathcal{A} with exact rows:*

$$\begin{array}{ccccccc} A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

Then there exists an exact sequence

$$\text{Ker}(a) \xrightarrow{\text{Ker}(f')} \text{Ker}(b) \xrightarrow{\text{Ker}(g')} \text{Ker}(c) \xrightarrow{\delta} \text{coker}(a) \xrightarrow{\text{coker}(f)} \text{coker}(b) \xrightarrow{\text{coker}(g)} \text{coker}(c),$$

*where δ is the **connecting morphism**. Moreover:*

- *If f' is a monomorphism, so is $\text{Ker}(a) \rightarrow \text{Ker}(b)$.*

- If g is an epimorphism, so is $\text{coker}(b) \rightarrow \text{coker}(c)$.

Proof. We give a complete proof using element-chasing (justified by the Freyd–Mitchell theorem).

Step 1: The induced morphisms. The morphisms $\text{Ker}(a) \rightarrow \text{Ker}(b)$ and $\text{Ker}(b) \rightarrow \text{Ker}(c)$ are induced by f' and g' respectively. Indeed, if $x \in \text{Ker}(a)$, then $b(f'(x)) = f(a(x)) = f(0) = 0$, so $f'(x) \in \text{Ker}(b)$. Similarly for g' . The morphisms $\text{coker}(a) \rightarrow \text{coker}(b) \rightarrow \text{coker}(c)$ are induced dually.

Step 2: Construction of δ . Let $x \in \text{Ker}(c) \subseteq C'$. Since g' is surjective (onto C'), choose $y \in B'$ with $g'(y) = x$. Consider $b(y) \in B$. Then

$$g(b(y)) = c(g'(y)) = c(x) = 0,$$

so $b(y) \in \text{Ker}(g) = \text{im}(f)$. Choose $z \in A$ with $f(z) = b(y)$ (unique since f is injective). Define

$$\delta(x) = [z] \in \text{coker}(a) = A/\text{im}(a).$$

Step 3: Well-definedness of δ . If y' is another choice with $g'(y') = x$, then $g'(y - y') = 0$, so $y - y' = f'(w)$ for some $w \in A'$. Then $b(y) - b(y') = b(f'(w)) = f(a(w))$, so $z - z' = a(w)$ since f is injective. Thus $[z] = [z']$ in $\text{coker}(a)$.

Step 4: Exactness at $\text{Ker}(b)$. The composition $\text{Ker}(a) \rightarrow \text{Ker}(b) \rightarrow \text{Ker}(c)$ is $g' \circ f' = 0$ (since the top row is a complex). Conversely, let $y \in \text{Ker}(b)$ with $g'(y) \in \text{Ker}(c)$ mapping to 0 in $\text{Ker}(c)$ —that is, $g'(y) = 0$. Then $y \in \text{Ker}(g') = \text{im}(f')$, so $y = f'(x)$ for some $x \in A'$. Now $f(a(x)) = b(f'(x)) = b(y) = 0$, and since f is injective, $a(x) = 0$, so $x \in \text{Ker}(a)$.

Step 5: Exactness at $\text{Ker}(c)$. The composition $\text{Ker}(b) \rightarrow \text{Ker}(c) \xrightarrow{\delta} \text{coker}(a)$ is zero: if $y \in \text{Ker}(b)$ and $x = g'(y) \in \text{Ker}(c)$, then in the construction of δ , we may choose y itself, and $b(y) = 0$, so $z = 0$, giving $\delta(x) = 0$.

Conversely, let $x \in \text{Ker}(c)$ with $\delta(x) = 0$. With notation as in Step 2, $[z] = 0$ means $z = a(w)$ for some $w \in A'$. Then $b(y) = f(z) = f(a(w)) = b(f'(w))$, so $y - f'(w) \in \text{Ker}(b)$ and $g'(y - f'(w)) = g'(y) - 0 = x$.

Step 6: Exactness at $\text{coker}(a)$. The composition δ followed by $\text{coker}(a) \rightarrow \text{coker}(b)$ is zero: in the notation above, $\delta(x) = [z]$ and the image in $\text{coker}(b)$ is $[f(z)] = [b(y)] = 0$.

Conversely, let $[z] \in \text{coker}(a)$ with $[f(z)] = 0$ in $\text{coker}(b)$. Then $f(z) = b(y)$ for some $y \in B'$. Now $c(g'(y)) = g(b(y)) = g(f(z)) = 0$, so $g'(y) \in \text{Ker}(c)$, and $\delta(g'(y)) = [z]$ by construction.

Step 7: Exactness at $\text{coker}(b)$. Similar argument, left as an exercise.

Step 8: Additional exactness. If f' is mono, then $\text{Ker}(a) \rightarrow \text{Ker}(b)$ (given by $f'|_{\text{Ker}(a)}$) is mono. If g is epi, then $\text{coker}(b) \rightarrow \text{coker}(c)$ is epi by a dual argument. \square

The snake lemma is often remembered via the following diagram, where the “snake” traces the path through the kernels and cokernels:

$$\begin{array}{ccccccc}
 & & \text{Ker}(a) & \longrightarrow & \text{Ker}(b) & \longrightarrow & \text{Ker}(c) & \dashrightarrow & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
 & & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 & \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & & \\
 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
 & & \text{coker}(a) & \longrightarrow & \text{coker}(b) & \longrightarrow & \text{coker}(c) & & &
 \end{array}$$

(Note: Dashed arrows form a snake path: $\text{Ker}(a) \rightarrow \text{Ker}(b) \rightarrow \text{Ker}(c) \rightarrow \text{Coker}(a) \rightarrow \text{Coker}(b) \rightarrow \text{Coker}(c)$)

7.6 The Five Lemma

Theorem 7.6.1 (Five lemma). *Consider a commutative diagram with exact rows in an abelian category:*

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
 \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \alpha_4 \downarrow & & \alpha_5 \downarrow \\
 B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
 \end{array}$$

1. If α_1 is an epimorphism and α_2, α_4 are monomorphisms, then α_3 is a monomorphism.
2. If α_5 is a monomorphism and α_2, α_4 are epimorphisms, then α_3 is an epimorphism.
3. If α_1 is an epimorphism, α_5 is a monomorphism, and α_2, α_4 are isomorphisms, then α_3 is an isomorphism.

Proof. We use element chasing (justified by the Freyd–Mitchell embedding).

Part (1): Suppose $\alpha_3(a_3) = 0$ for some $a_3 \in A_3$. Then $\alpha_4(f_3(a_3)) = g_3(\alpha_3(a_3)) = 0$. Since α_4 is mono, $f_3(a_3) = 0$. By exactness at A_3 , there exists $a_2 \in A_2$ with $f_2(a_2) = a_3$. Now $g_2(\alpha_2(a_2)) = \alpha_3(f_2(a_2)) = \alpha_3(a_3) = 0$. By exactness at B_2 , there exists $b_1 \in B_1$ with $g_1(b_1) = \alpha_2(a_2)$. Since α_1 is epi, $b_1 = \alpha_1(a_1)$ for some $a_1 \in A_1$. Then

$$\alpha_2(f_1(a_1)) = g_1(\alpha_1(a_1)) = g_1(b_1) = \alpha_2(a_2).$$

Since α_2 is mono, $f_1(a_1) = a_2$, so $a_3 = f_2(a_2) = f_2(f_1(a_1)) = 0$ by exactness at A_2 (since $\text{im}(f_1) = \text{Ker}(f_2)$). Thus α_3 is mono.

Part (2): Let $b_3 \in B_3$. Consider $g_3(b_3) \in B_4$. Since α_4 is epi, there exists $a_4 \in A_4$ with $\alpha_4(a_4) = g_3(b_3)$. Now $\alpha_5(f_4(a_4)) = g_4(\alpha_4(a_4)) = g_4(g_3(b_3)) = 0$ by exactness at B_4 . Since α_5 is mono, $f_4(a_4) = 0$. By exactness at A_4 , there exists $a_3 \in A_3$ with $f_3(a_3) = a_4$. Consider $b_3 - \alpha_3(a_3)$:

$$g_3(b_3 - \alpha_3(a_3)) = g_3(b_3) - g_3(\alpha_3(a_3)) = g_3(b_3) - \alpha_4(f_3(a_3)) = g_3(b_3) - \alpha_4(a_4) = 0.$$

By exactness at B_3 , there exists $b_2 \in B_2$ with $g_2(b_2) = b_3 - \alpha_3(a_3)$. Since α_2 is epi, $b_2 = \alpha_2(a_2)$ for some $a_2 \in A_2$. Then

$$b_3 = \alpha_3(a_3) + g_2(\alpha_2(a_2)) = \alpha_3(a_3) + \alpha_3(f_2(a_2)) = \alpha_3(a_3 + f_2(a_2)).$$

Thus α_3 is epi.

Part (3): Immediate from (1) and (2). □

Corollary 7.6.2 (Short five lemma). *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

if α and γ are isomorphisms, then β is an isomorphism.

Proof. Apply the five lemma with $A_1 = B_1 = 0$ and $A_5 = B_5 = 0$, taking $\alpha_1 = \alpha_5 = 0$, $\alpha_2 = \alpha$, $\alpha_3 = \beta$, $\alpha_4 = \gamma$. □

7.7 Injective and Projective Objects

Definition 7.7.1 (Projective object). An object P in an abelian category \mathcal{A} is **projective** if the functor $\text{Hom}_{\mathcal{A}}(P, -)$ is exact. Equivalently, for

every epimorphism $p: B \twoheadrightarrow C$ and every morphism $f: P \rightarrow C$, there exists a morphism $\tilde{f}: P \rightarrow B$ with $p \circ \tilde{f} = f$:

$$\begin{array}{ccc} & P & \\ \tilde{f} \swarrow & \downarrow f & \\ B & \xrightarrow{p} & C \longrightarrow 0 \end{array}$$

Definition 7.7.2 (Injective object). An object I in \mathcal{A} is **injective** if $\text{Hom}_{\mathcal{A}}(-, I)$ is exact. Equivalently, for every monomorphism $i: A \hookrightarrow B$ and every $f: A \rightarrow I$, there exists $\tilde{f}: B \rightarrow I$ with $\tilde{f} \circ i = f$:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xhookrightarrow{i} & B \\ & & \downarrow f & \swarrow \tilde{f} & \\ & & I & & \end{array}$$

- Example 7.7.3.**
1. In Mod_R , free modules are projective. A module is projective if and only if it is a direct summand of a free module.
 2. In \mathbf{Ab} , the group \mathbb{Q} is injective. More generally, \mathbb{Q}/\mathbb{Z} is injective.
 3. In a semisimple abelian category (every SES splits), every object is both projective and injective.

Definition 7.7.4 (Enough projectives/injectives). An abelian category \mathcal{A} has **enough projectives** if for every object A , there exists an epimorphism $P \twoheadrightarrow A$ with P projective. It has **enough injectives** if for every A , there exists a monomorphism $A \hookrightarrow I$ with I injective.

Theorem 7.7.5 (Baer's criterion). *Let R be a ring and I an R -module. Then I is injective if and only if for every left ideal $J \subseteq R$ and every homomorphism $f: J \rightarrow I$, there exists an extension $\tilde{f}: R \rightarrow I$ with $\tilde{f}|_J = f$.*

Theorem 7.7.6. *Every Grothendieck category has enough injectives. In particular, Mod_R and $\text{Sh}(X, \mathbf{Ab})$ have enough injectives.*

7.8 Resolutions

Definition 7.8.1 (Projective resolution). A **projective resolution** of an object A in an abelian category \mathcal{A} is an exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

where each P_n is projective. We write this as $P_\bullet \xrightarrow{\varepsilon} A$.

Definition 7.8.2 (Injective resolution). An **injective resolution** of A is an exact sequence

$$0 \longrightarrow A \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

where each I^n is injective. We write this as $A \xrightarrow{\eta} I^\bullet$.

Proposition 7.8.3. *If \mathcal{A} has enough projectives, then every object admits a projective resolution. If \mathcal{A} has enough injectives, then every object admits an injective resolution.*

Proof. We construct a projective resolution inductively. Choose an epi $P_0 \twoheadrightarrow A$ with P_0 projective and let $K_0 = \text{Ker}(P_0 \rightarrow A)$. Then choose $P_1 \twoheadrightarrow K_0$ with P_1 projective, set $d_1: P_1 \twoheadrightarrow K_0 \hookrightarrow P_0$, let $K_1 = \text{Ker}(P_1 \rightarrow K_0)$, and continue. \square

Theorem 7.8.4 (Comparison theorem). *Let $P_\bullet \xrightarrow{\varepsilon} A$ be a projective resolution and $Q_\bullet \xrightarrow{\varepsilon'} B$ be any resolution (not necessarily projective). Then any morphism $f: A \rightarrow B$ lifts to a chain map $f_\bullet: P_\bullet \rightarrow Q_\bullet$:*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ \cdots & \longrightarrow & Q_2 & \xrightarrow{d_2} & Q_1 & \xrightarrow{d_1} & Q_0 & \xrightarrow{\varepsilon'} & B & \longrightarrow & 0 \end{array}$$

Moreover, any two such lifts are chain homotopic.

Proof. Existence. We construct f_n inductively. The composite $f \circ \varepsilon: P_0 \rightarrow B$ factors through $\varepsilon': Q_0 \rightarrow B$ since P_0 is projective and ε' is epi. This gives f_0 . For the inductive step, given f_{n-1} , we need f_n making the square commute. The morphism $f_{n-1} \circ d_n: P_n \rightarrow Q_{n-1}$ satisfies $d'_{n-1} \circ f_{n-1} \circ d_n = f_{n-2} \circ d_{n-1} \circ$

$d_n = 0$. Thus $f_{n-1} \circ d_n$ factors through $\text{Ker}(d'_{n-1}) = \text{im}(d'_n)$, and projectivity of P_n gives f_n .

Uniqueness up to homotopy. Suppose f_\bullet, g_\bullet are two lifts. We construct a chain homotopy $s_n: P_n \rightarrow Q_{n+1}$ with $f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n$ inductively, using projectivity at each step. \square

7.9 Exercises

Exercise 7.9.1. Let \mathcal{A} be a preadditive category. Show that \mathcal{A} has a zero object if and only if it has an object Z such that $\text{id}_Z = 0_{Z,Z}$.

Exercise 7.9.2. Show that an additive functor preserves zero objects, biproducts, kernels, and cokernels (when they exist).

Exercise 7.9.3. Let \mathcal{A} be an additive category and $e: A \rightarrow A$ an idempotent ($e^2 = e$). We say e **splits** if there exist morphisms $p: A \rightarrow B$ and $i: B \rightarrow A$ with $p \circ i = \text{id}_B$ and $i \circ p = e$.

1. Show that in an abelian category, every idempotent splits.
2. Show that $B = \text{Ker}(\text{id}_A - e)$ works.

Exercise 7.9.4. (The 3×3 lemma.) Consider a commutative diagram with exact columns and the middle and bottom rows exact:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Prove that the top row is also exact.

Exercise 7.9.5. (Horseshoe lemma.) Let $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ be a short exact sequence in an abelian category with enough projectives. Given projective resolutions $P'_\bullet \rightarrow A'$ and $P''_\bullet \rightarrow A''$, show there exists a projective resolution $P_\bullet \rightarrow A$ with $P_n = P'_n \oplus P''_n$ fitting into a short exact sequence of chain complexes $0 \rightarrow P'_\bullet \rightarrow P_\bullet \rightarrow P''_\bullet \rightarrow 0$.

Exercise 7.9.6. Show that an abelian group G is injective in \mathbf{Ab} if and only if it is divisible (for every $g \in G$ and $n \geq 1$, there exists $h \in G$ with $nh = g$).

Exercise 7.9.7. Let R be a ring and P an R -module. Prove the equivalence:

1. P is projective.
2. Every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$ splits.
3. P is a direct summand of a free module.

Exercise 7.9.8. Let \mathcal{A} be an abelian category. Show that the pullback of an epimorphism is an epimorphism, and the pushout of a monomorphism is a monomorphism.

Chapter 8

Derived Categories — Introduction

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8.1 Chain Complexes and Their Morphisms

Definition 8.1.1 (Chain complex). Let \mathcal{A} be an abelian category. A **chain complex** (or simply **complex**) in \mathcal{A} is a sequence of objects and morphisms

$$A^\bullet: \quad \cdots \longrightarrow A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \cdots$$

such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$. The morphisms d^n are called **differentials**. We use cohomological (superscript) indexing throughout.

Definition 8.1.2 (Cohomology). The n -th **cohomology** of a complex A^\bullet is

$$H^n(A^\bullet) = \text{Ker}(d^n) / \text{im}(d^{n-1}) = \text{Ker}(d^n : A^n \rightarrow A^{n+1}) / \text{im}(d^{n-1} : A^{n-1} \rightarrow A^n).$$

A complex is **acyclic** (or **exact**) if $H^n(A^\bullet) = 0$ for all n .

Definition 8.1.3 (Category of complexes). The **category of chain complexes** $\text{Ch}(\mathcal{A})$ has:

- Objects: chain complexes in \mathcal{A} .
- Morphisms: a **chain map** $f : A^\bullet \rightarrow B^\bullet$ is a collection of morphisms $f^n : A^n \rightarrow B^n$ commuting with differentials: $d_B^n \circ f^n = f^{n+1} \circ d_A^n$ for all n .

We also define the bounded variants:

$$\begin{aligned} \text{Ch}^+(\mathcal{A}) &= \{A^\bullet \mid A^n = 0 \text{ for } n \ll 0\} && \text{(bounded below),} \\ \text{Ch}^-(\mathcal{A}) &= \{A^\bullet \mid A^n = 0 \text{ for } n \gg 0\} && \text{(bounded above),} \\ \text{Ch}^b(\mathcal{A}) &= \text{Ch}^+(\mathcal{A}) \cap \text{Ch}^-(\mathcal{A}) && \text{(bounded).} \end{aligned}$$

Remark 8.1.4. The category $\text{Ch}(\mathcal{A})$ is abelian when \mathcal{A} is abelian, with kernels, cokernels, and exact sequences computed degreewise.

Definition 8.1.5 (Shift functor). The **shift** (or **translation**) functor $[1] : \text{Ch}(\mathcal{A}) \rightarrow \text{Ch}(\mathcal{A})$ is defined by

$$(A[1])^n = A^{n+1}, \quad d_{A[1]}^n = -d_A^{n+1}.$$

More generally, $A[k]^n = A^{n+k}$ with $d_{A[k]}^n = (-1)^k d_A^{n+k}$. For a chain map $f : A^\bullet \rightarrow B^\bullet$, we set $f[1]^n = f^{n+1}$.

Definition 8.1.6 (Mapping cone). Let $f : A^\bullet \rightarrow B^\bullet$ be a chain map. The **mapping cone** $\text{Cone}(f)$ is the complex defined by

$$\text{Cone}(f)^n = A^{n+1} \oplus B^n, \quad d_{\text{Cone}(f)}^n = \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}.$$

One verifies $d^2 = 0$ using $d_B \circ f = f \circ d_A$. There is a natural short exact

sequence of complexes:

$$0 \longrightarrow B^\bullet \xrightarrow{\iota} \text{Cone}(f) \xrightarrow{\pi} A[1]^\bullet \longrightarrow 0.$$

Proposition 8.1.7. *For any chain map $f: A^\bullet \rightarrow B^\bullet$, the mapping cone gives a long exact sequence in cohomology:*

$$\dots \rightarrow H^n(A^\bullet) \xrightarrow{H^n(f)} H^n(B^\bullet) \rightarrow H^n(\text{Cone}(f)) \rightarrow H^{n+1}(A^\bullet) \xrightarrow{H^{n+1}(f)} \dots$$

In particular, f is a quasi-isomorphism if and only if $\text{Cone}(f)$ is acyclic.

8.2 Chain Homotopies and the Homotopy Category

Definition 8.2.1 (Chain homotopy). Let $f, g: A^\bullet \rightarrow B^\bullet$ be chain maps. A **chain homotopy** from f to g is a collection of morphisms $s^n: A^n \rightarrow B^{n-1}$ such that

$$f^n - g^n = d_B^{n-1} \circ s^n + s^{n+1} \circ d_A^n \quad \text{for all } n.$$

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^{n-1} & \xrightarrow{d^{n-1}} & A^n & \xrightarrow{d^n} & A^{n+1} & \longrightarrow & \dots \\ & & \downarrow g^{n-1} & \swarrow f^n \circ s_1^n & \downarrow g^n & \swarrow f^{n+1} \circ s_1^{n+1} & \downarrow g^{n+1} & \swarrow f^{n+1} & \\ \dots & \longrightarrow & B^{n-1} & \xrightarrow{d^{n-1}} & B^n & \xrightarrow{d^n} & B^{n+1} & \longrightarrow & \dots \end{array}$$

We write $f \sim g$ and say f and g are **homotopic**. If $f \sim 0$, we say f is **null-homotopic**.

Proposition 8.2.2. *Chain homotopy is an equivalence relation on $\text{Hom}_{\text{Ch}(A)}(A^\bullet, B^\bullet)$, compatible with composition: if $f \sim g$ then $h \circ f \sim h \circ g$ and $f \circ k \sim g \circ k$ for any compatible chain maps h, k .*

Proof. Reflexivity: take $s = 0$. Symmetry: negate s . Transitivity: add homotopies. For compatibility, if s is a homotopy from f to g , then $h \circ s$ is a homotopy from $h \circ f$ to $h \circ g$, and $s \circ k$ is a homotopy from $f \circ k$ to $g \circ k$. \square

Proposition 8.2.3. *If $f \sim g: A^\bullet \rightarrow B^\bullet$, then $H^n(f) = H^n(g)$ for all n .*

Proof. For $[a] \in H^n(A^\bullet)$ with $d_A^n(a) = 0$: $f^n(a) - g^n(a) = d_B^{n-1}(s^n(a)) + s^{n+1}(d_A^n(a)) = d_B^{n-1}(s^n(a))$, which is a coboundary. Thus $[f^n(a)] = [g^n(a)]$. \square

Definition 8.2.4 (Homotopy category). The **homotopy category** $K(\mathcal{A})$ has:

- Objects: chain complexes in \mathcal{A} (same as $\text{Ch}(\mathcal{A})$).
- Morphisms: $\text{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \text{Hom}_{\text{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$, i.e., chain maps modulo homotopy.

The bounded variants $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$ are defined similarly.

Remark 8.2.5. The homotopy category $K(\mathcal{A})$ is additive but *not* abelian in general. It does, however, carry a **triangulated structure** (see Section 8.4).

Definition 8.2.6 (Homotopy equivalence). A chain map $f: A^\bullet \rightarrow B^\bullet$ is a **homotopy equivalence** if it becomes an isomorphism in $K(\mathcal{A})$, i.e., there exists $g: B^\bullet \rightarrow A^\bullet$ with $g \circ f \sim \text{id}_A$ and $f \circ g \sim \text{id}_B$.

8.3 Quasi-Isomorphisms and the Derived Category

Definition 8.3.1 (Quasi-isomorphism). A chain map $f: A^\bullet \rightarrow B^\bullet$ is a **quasi-isomorphism** if the induced maps $H^n(f): H^n(A^\bullet) \rightarrow H^n(B^\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$. We denote the class of quasi-isomorphisms by qis .

Remark 8.3.2. Every homotopy equivalence is a quasi-isomorphism (by Proposition 8.2.3), but the converse is false in general. The derived category is obtained by formally inverting all quasi-isomorphisms.

Example 8.3.3. In \mathbf{Ab} , consider the complexes $A^\bullet: 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow 0$ (in degrees 0 and 1) and $B^\bullet: 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ (in degree 1). The natural projection $A^\bullet \rightarrow B^\bullet$ is a quasi-isomorphism but not a homotopy

equivalence (since $\text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$, there is no map backward in degree 1).

8.3.1 Localization of categories

Definition 8.3.4 (Localization). Let \mathcal{C} be a category and S a class of morphisms in \mathcal{C} . The **localization** $\mathcal{C}[S^{-1}]$ is a category equipped with a functor $Q: \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ such that:

1. $Q(s)$ is an isomorphism for every $s \in S$.
2. Q is universal with this property: for any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ sending S to isomorphisms, there exists a unique $\bar{F}: \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$ with $\bar{F} \circ Q = F$.

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{Q} & \mathcal{C}[S^{-1}] \\
 & \searrow F & \downarrow \exists! \bar{F} \\
 & & \mathcal{D}
 \end{array}$$

Remark 8.3.5. The localization always exists (Gabriel–Zisman): objects of $\mathcal{C}[S^{-1}]$ are those of \mathcal{C} , and morphisms are equivalence classes of “zigzag” chains

$$A = C_0 \xleftarrow{s_1} C_1 \xrightarrow{f_1} C_2 \xleftarrow{s_2} C_3 \xrightarrow{f_2} \dots \rightarrow B,$$

where the backward arrows s_i belong to S , subject to appropriate relations. In general, Hom sets may fail to be sets (set-theoretic issues). However, when S admits a **calculus of fractions**, morphisms simplify to “roofs” and the issues become manageable.

Definition 8.3.6 (Right/left roof). When S admits a **calculus of right fractions**, every morphism in $\mathcal{C}[S^{-1}]$ from A to B can be represented by a **right roof**:

$$\begin{array}{ccc}
 & C & \\
 s \swarrow & & \searrow f \\
 A & & B
 \end{array}$$

where $s \in S$. This represents the “fraction” $f \circ s^{-1}$. Two roofs (C, s, f) and (C', s', f') represent the same morphism if and only if they can be

“completed” to a common refinement.

8.3.2 The derived category

Definition 8.3.7 (Derived category). Let \mathcal{A} be an abelian category. The **derived category** $D(\mathcal{A})$ is the localization of $K(\mathcal{A})$ (the homotopy category) at the class of quasi-isomorphisms:

$$D(\mathcal{A}) = K(\mathcal{A})[\text{qis}^{-1}].$$

The bounded variants are:

$$D^+(\mathcal{A}) = K^+(\mathcal{A})[\text{qis}^{-1}],$$

$$D^-(\mathcal{A}) = K^-(\mathcal{A})[\text{qis}^{-1}],$$

$$D^b(\mathcal{A}) = K^b(\mathcal{A})[\text{qis}^{-1}].$$

Remark 8.3.8. The construction proceeds in two steps: $\text{Ch}(\mathcal{A}) \rightarrow K(\mathcal{A}) \rightarrow D(\mathcal{A})$. The first step (modding out homotopies) ensures that quasi-isomorphisms satisfy the Ore conditions (calculus of fractions), making the second localization well-behaved. If one tried to localize $\text{Ch}(\mathcal{A})$ directly at quasi-isomorphisms, the calculus of fractions would fail.

Theorem 8.3.9. *The class of quasi-isomorphisms in $K(\mathcal{A})$ admits both a left and a right calculus of fractions. Therefore, morphisms in $D(\mathcal{A})$ from A^\bullet to B^\bullet are represented by roofs*

$$\begin{array}{ccc} & C^\bullet & \\ s \swarrow & & \searrow f \\ A^\bullet & & B^\bullet \end{array} \quad \sim$$

where s is a quasi-isomorphism in $K(\mathcal{A})$ and f is any morphism in $K(\mathcal{A})$.

Proposition 8.3.10. *There is a fully faithful functor $\mathcal{A} \hookrightarrow D(\mathcal{A})$ sending an object A to the complex concentrated in degree zero:*

$$A \mapsto (\cdots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots).$$

Under this embedding, $\text{Hom}_{D(\mathcal{A})}(A, B) \cong \text{Hom}_{\mathcal{A}}(A, B)$ for objects $A, B \in \mathcal{A}$.

\mathcal{A} .

Theorem 8.3.11. *Let \mathcal{A} be an abelian category with enough injectives, and let $K^+(\mathcal{I})$ be the full subcategory of $K^+(\mathcal{A})$ consisting of complexes of injective objects. Then the natural functor*

$$K^+(\mathcal{I}) \xrightarrow{\sim} D^+(\mathcal{A})$$

is an equivalence of categories. Dually, if \mathcal{A} has enough projectives, $K^-(\mathcal{P}) \xrightarrow{\sim} D^-(\mathcal{A})$.

Proof sketch. The functor is the composition $K^+(\mathcal{I}) \hookrightarrow K^+(\mathcal{A}) \xrightarrow{Q} D^+(\mathcal{A})$.

Essential surjectivity: Every bounded below complex A^\bullet admits a quasi-isomorphism $A^\bullet \xrightarrow{\sim} I^\bullet$ to a complex of injectives (by inductively resolving, using an injective Cartan–Eilenberg resolution).

Fully faithfulness: For complexes of injectives I^\bullet, J^\bullet , a quasi-isomorphism $s: I^\bullet \xrightarrow{\sim} J^\bullet$ is already a homotopy equivalence (a standard argument using injectivity at each degree). Thus roofs reduce to ordinary morphisms in $K^+(\mathcal{I})$. \square

8.4 Triangulated Structure

Definition 8.4.1 (Triangulated category). A **triangulated category** is an additive category \mathcal{T} equipped with:

1. An automorphism $[1]: \mathcal{T} \rightarrow \mathcal{T}$ (the **shift functor**).
2. A class of **distinguished triangles** (or **exact triangles**)

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1],$$

satisfying axioms (TR1)–(TR4):

(TR1) (a) For every A , the triangle $A \xrightarrow{\text{id}} A \rightarrow 0 \rightarrow A[1]$ is distinguished.
 (b) Every morphism $f: A \rightarrow B$ can be completed to a distinguished triangle $A \xrightarrow{f} B \rightarrow C \rightarrow A[1]$. (c) A triangle isomorphic to a distinguished triangle is distinguished.

(TR2) (**Rotation**) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is distinguished if and only if $B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$ is.

(TR3) (**Morphism of triangles**) Given distinguished triangles and morphisms α, β making the left square commute, there exists (not necessarily unique) γ completing the morphism of triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \alpha[1] \downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & A'[1] \end{array}$$

(TR4) (**Octahedron axiom**) Given composable morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ and distinguished triangles completing f , g , and $g \circ f$, there exists a distinguished triangle relating the three cones, making all faces commute.

Theorem 8.4.2. *The homotopy category $K(\mathcal{A})$ is triangulated with:*

- *Shift: $A^\bullet \mapsto A[1]^\bullet$ (Definition 8.1.5).*
- *Distinguished triangles: those isomorphic in $K(\mathcal{A})$ to*

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\iota} \text{Cone}(f) \xrightarrow{\pi} A[1]^\bullet$$

for some chain map f .

Theorem 8.4.3. *The derived category $D(\mathcal{A})$ inherits a triangulated structure from $K(\mathcal{A})$ via the localization functor $Q: K(\mathcal{A}) \rightarrow D(\mathcal{A})$: a triangle in $D(\mathcal{A})$ is distinguished if it is isomorphic to the image of a distinguished triangle in $K(\mathcal{A})$.*

Proposition 8.4.4. *A short exact sequence of complexes $0 \rightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \rightarrow 0$ gives rise to a distinguished triangle*

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{\delta} A[1]^\bullet$$

in $D(\mathcal{A})$, where δ is the connecting morphism. In particular, the long exact sequence in cohomology associated to a short exact sequence of complexes is a consequence of the triangulated structure.

Proposition 8.4.5. *Let \mathcal{T} be a triangulated category and \mathcal{A} an abelian category. An additive functor $H: \mathcal{T} \rightarrow \mathcal{A}$ is **cohomological** if for every distinguished triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$, the sequence*

$$H(A) \longrightarrow H(B) \longrightarrow H(C)$$

is exact. In this case, writing $H^n = H \circ [n]$, every distinguished triangle gives a long exact sequence

$$\cdots \rightarrow H^n(A) \rightarrow H^n(B) \rightarrow H^n(C) \rightarrow H^{n+1}(A) \rightarrow \cdots$$

Example 8.4.6. 1. The cohomology functor $H^0: D(\mathcal{A}) \rightarrow \mathcal{A}$ is cohomological.

2. For any object X , $\text{Hom}_{D(\mathcal{A})}(X, -)$ is cohomological with values in \mathbf{Ab} .

8.5 Derived Functors

Definition 8.5.1 (Right derived functor). Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor between abelian categories, with \mathcal{A} having enough injectives. The **right derived functor** $RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is defined as follows. For $A^\bullet \in D^+(\mathcal{A})$:

1. Choose a quasi-isomorphism $A^\bullet \xrightarrow{\sim} I^\bullet$ with $I^\bullet \in K^+(\mathcal{I})$ (injective resolution).
2. Define $RF(A^\bullet) = F(I^\bullet)$ (apply F degreewise).

By Theorem 8.3.11, this is well-defined (independent of the choice of resolution, up to canonical isomorphism in $D^+(\mathcal{B})$).

The **classical right derived functors** are

$$R^n F(A) = H^n(RF(A)) \quad \text{for } A \in \mathcal{A}, n \geq 0.$$

Note $R^0 F \cong F$ since F is left exact.

Definition 8.5.2 (Left derived functor). Let $G: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor with \mathcal{A} having enough projectives. The **left derived functor** $LG: D^-(\mathcal{A}) \rightarrow D^-(\mathcal{B})$ is defined by:

1. Choose a quasi-isomorphism $P^\bullet \xrightarrow{\sim} A^\bullet$ with $P^\bullet \in K^-(\mathcal{P})$.
2. Define $LG(A^\bullet) = G(P^\bullet)$.

The classical left derived functors are $L_n G(A) = H^{-n}(LG(A))$ for $n \geq 0$, with $L_0 G \cong G$.

Remark 8.5.3. The right derived functor RF is characterized by the universal property: it is the “closest approximation” to F that is an exact functor (i.e., preserves distinguished triangles) on the derived category. More precisely, RF is the right Kan extension of $Q_{\mathcal{B}} \circ F$ along $Q_{\mathcal{A}}$.

Theorem 8.5.4 (Derived functor of a composition). *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$ be left exact functors between abelian categories with enough injectives. If F sends injective objects of \mathcal{A} to G -acyclic objects (i.e., $R^n G(F(I)) = 0$ for $n > 0$ and I injective), then*

$$R(G \circ F) \cong RG \circ RF.$$

*At the level of classical derived functors, there is a **Grothendieck spectral sequence**:*

$$E_2^{p,q} = R^p G(R^q F(A)) \implies R^{p+q}(G \circ F)(A).$$

8.6 Ext and Tor

Definition 8.6.1 (Ext groups). Let \mathcal{A} be an abelian category with enough injectives (or enough projectives). For objects $A, B \in \mathcal{A}$, the **Ext groups** are

$$\mathrm{Ext}_{\mathcal{A}}^n(A, B) = R^n \mathrm{Hom}_{\mathcal{A}}(A, -)(B) = H^n(R \mathrm{Hom}(A, B)).$$

Equivalently, if I^\bullet is an injective resolution of B :

$$\mathrm{Ext}^n(A, B) \cong H^n(\mathrm{Hom}(A, I^\bullet)).$$

If \mathcal{A} has enough projectives, using a projective resolution $P_\bullet \rightarrow A$:

$$\mathrm{Ext}^n(A, B) \cong H^n(\mathrm{Hom}(P_\bullet, B)).$$

Proposition 8.6.2. For a module category \mathbf{Mod}_R :

1. $\text{Ext}_R^0(M, N) \cong \text{Hom}_R(M, N)$.
2. $\text{Ext}_R^n(M, N) = 0$ for all $n > 0$ if M is projective or N is injective.
3. The two definitions (via injective or projective resolutions) agree: this is the **balancing of Ext**.
4. $\text{Ext}_R^n(M, N)$ classifies n -fold extensions of M by N (Yoneda's interpretation).

Example 8.6.3. 1. In \mathbf{Ab} : $\text{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\text{gcd}(m, n)\mathbb{Z}$.
Indeed, the projective resolution $0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0$ gives $\text{Ext}^1 \cong \mathbb{Z}/n\mathbb{Z}/m(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\text{gcd}(m, n)\mathbb{Z}$.

2. $\text{Ext}_{\mathbb{Z}}^n(A, B) = 0$ for all $n \geq 2$ and all abelian groups A, B (since \mathbb{Z} has global dimension 1).

Definition 8.6.4 (Tor groups). For a ring R and modules $M \in \mathbf{Mod}_{R^{\text{op}}}$, $N \in \mathbf{Mod}_R$, the **Tor groups** are

$$\text{Tor}_n^R(M, N) = L_n(M \otimes_R -)(N) = H^{-n}(M \otimes_R^L N).$$

If $P_\bullet \rightarrow N$ is a projective resolution:

$$\text{Tor}_n^R(M, N) \cong H_n(M \otimes_R P_\bullet).$$

Proposition 8.6.5. 1. $\text{Tor}_0^R(M, N) \cong M \otimes_R N$.

2. $\text{Tor}_n^R(M, N) = 0$ for all $n > 0$ if M or N is flat.
3. Tor is balanced: $L_n(M \otimes_R -)(N) \cong L_n(- \otimes_R N)(M)$.
4. M is flat if and only if $\text{Tor}_1^R(M, N) = 0$ for all N .

Example 8.6.6. 1. $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\text{gcd}(m, n)\mathbb{Z}$. Using $0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0$: $\text{Tor}_1 = \text{Ker}(\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\text{gcd}(m, n)\mathbb{Z}$.

2. $\text{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for $n \geq 2$.

8.7 Long Exact Sequences

Theorem 8.7.1. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor and $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ a short exact sequence in \mathcal{A} . Then there is a long exact sequence:*

$$0 \rightarrow FA' \rightarrow FA \rightarrow FA'' \xrightarrow{\delta^0} R^1FA' \rightarrow R^1FA \rightarrow R^1FA'' \xrightarrow{\delta^1} R^2FA' \rightarrow \dots$$

Dually, for a right exact functor G and $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$:

$$\dots \rightarrow L_2GA'' \xrightarrow{\delta_2} L_1GA' \rightarrow L_1GA \rightarrow L_1GA'' \xrightarrow{\delta_1} GA' \rightarrow GA \rightarrow GA'' \rightarrow 0.$$

Proof. By Proposition 8.4.4, the short exact sequence gives a distinguished triangle $A' \rightarrow A \rightarrow A'' \rightarrow A'[1]$ in $D^+(\mathcal{A})$. Applying RF (which preserves distinguished triangles) gives a distinguished triangle

$$RF(A') \rightarrow RF(A) \rightarrow RF(A'') \rightarrow RF(A')[1]$$

in $D^+(\mathcal{B})$. The cohomological functor H^0 (Proposition 8.4.5) then yields the long exact sequence, since $H^n(RF(A)) = R^nF(A)$. \square

Corollary 8.7.2. *For any short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in \mathbf{Mod}_R and any module M :*

1. (Covariant) *There is a long exact sequence*

$$0 \rightarrow \mathrm{Hom}(M, A') \rightarrow \mathrm{Hom}(M, A) \rightarrow \mathrm{Hom}(M, A'') \rightarrow \mathrm{Ext}^1(M, A') \rightarrow \dots$$

2. (Contravariant) *There is a long exact sequence*

$$0 \rightarrow \mathrm{Hom}(A'', M) \rightarrow \mathrm{Hom}(A, M) \rightarrow \mathrm{Hom}(A', M) \rightarrow \mathrm{Ext}^1(A'', M) \rightarrow \dots$$

Corollary 8.7.3. *For any short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in \mathbf{Mod}_R and any module M :*

$$\dots \rightarrow \mathrm{Tor}_1(M, A') \rightarrow \mathrm{Tor}_1(M, A) \rightarrow \mathrm{Tor}_1(M, A'') \rightarrow M \otimes A' \rightarrow M \otimes A \rightarrow M \otimes A'' \rightarrow 0.$$

Example 8.7.4. Consider the exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$ in \mathbf{Ab} . Applying $\mathrm{Hom}(-, A)$ for an abelian group A :

$$0 \rightarrow \mathrm{Hom}(\mathbb{Z}/n, A) \rightarrow \mathrm{Hom}(\mathbb{Z}, A) \xrightarrow{n} \mathrm{Hom}(\mathbb{Z}, A) \rightarrow \mathrm{Ext}^1(\mathbb{Z}/n, A) \rightarrow 0.$$

This gives $\text{Hom}(\mathbb{Z}/n, A) \cong A[n] = \{a \in A : na = 0\}$ and $\text{Ext}^1(\mathbb{Z}/n, A) \cong A/nA$.

8.8 Derived Category in Practice

Definition 8.8.1 (RHom and derived tensor). For $\mathcal{A} = \mathbf{Mod}_R$:

1. $R\text{Hom}_R(M, N) \in D^+(\mathbf{Ab})$ is the right derived functor of $\text{Hom}_R(M, -)$. Its cohomology gives $H^n(R\text{Hom}(M, N)) = \text{Ext}_R^n(M, N)$.
2. $M \otimes_R^L N \in D^-(\mathbf{Ab})$ is the left derived functor of $M \otimes_R -$. Its cohomology gives $H^{-n}(M \otimes_R^L N) = \text{Tor}_n^R(M, N)$.

These extend to functors on derived categories:

$$R\text{Hom}: D^-(\mathbf{Mod}_R)^{\text{op}} \times D^+(\mathbf{Mod}_R) \rightarrow D^+(\mathbf{Ab}),$$

$$\otimes_R^L: D^-(\mathbf{Mod}_{R^{\text{op}}}) \times D^-(\mathbf{Mod}_R) \rightarrow D^-(\mathbf{Ab}).$$

Proposition 8.8.2 (Derived Hom–Tensor adjunction). *There is a natural isomorphism in $D(\mathbf{Ab})$:*

$$R\text{Hom}_R(M \otimes_S^L N, P) \cong R\text{Hom}_S(N, R\text{Hom}_R(M, P))$$

for appropriate bimodule structures.

Proposition 8.8.3 (Derived category of a hereditary category). *Let \mathcal{A} be a hereditary abelian category (i.e., $\text{Ext}^n = 0$ for $n \geq 2$). Then every object of $D^b(\mathcal{A})$ is isomorphic to the direct sum of its cohomology objects (each placed in its own degree):*

$$A^\bullet \cong \bigoplus_{n \in \mathbb{Z}} H^n(A^\bullet)[-n] \quad \text{in } D^b(\mathcal{A}).$$

Remark 8.8.4. The philosophy of derived categories can be summarized as follows:

1. Objects of \mathcal{A} are replaced by complexes, which carry “higher-order” information through their cohomology in all degrees.

2. Functors that are only partially exact (left or right exact) become exact on derived categories, at the cost of working with complexes.
3. The derived category provides a unified framework for cohomological computations: spectral sequences, universal coefficient theorems, Künneth formulas, etc. all arise from the triangulated structure.

Notation 8.8.5. We summarize the key notation:

$\text{Ch}(\mathcal{A})$	Category of chain complexes
$K(\mathcal{A})$	Homotopy category
$D(\mathcal{A})$	Derived category
D^+, D^-, D^b	Bounded variants
RF	Right derived functor
LG	Left derived functor
$R\text{Hom}$	Derived Hom
\otimes^L	Derived tensor product
$\text{Ext}^n(A, B)$	$= H^n(R\text{Hom}(A, B))$
$\text{Tor}_n(M, N)$	$= H^{-n}(M \otimes^L N)$

8.9 Exercises

Exercise 8.9.1. Let $f: A^\bullet \rightarrow B^\bullet$ be a chain map. Show that f is a quasi-isomorphism if and only if $\text{Cone}(f)$ is acyclic.

Exercise 8.9.2. Show that the homotopy category $K(\mathcal{A})$ is additive. Verify that it is not abelian by finding an explicit example (e.g., in $K(\mathbf{Ab})$) of a morphism that has no kernel in $K(\mathbf{Ab})$.

Exercise 8.9.3. Show that the shift functor $[1]: D(\mathcal{A}) \rightarrow D(\mathcal{A})$ is an exact functor of triangulated categories (sends distinguished triangles to distinguished triangles).

Exercise 8.9.4. Let k be a field. Compute $\text{Ext}_{k[x]}^n(k, k)$ for all $n \geq 0$, where k is viewed as a $k[x]$ -module via $x \mapsto 0$. (Use the projective resolution $0 \rightarrow k[x] \xrightarrow{x} k[x] \rightarrow k \rightarrow 0$.)

Exercise 8.9.5. Let R be a commutative ring and M an R -module.

1. Show that M is flat if and only if $\text{Tor}_1^R(M, R/I) = 0$ for every ideal $I \subseteq R$.
2. Compute $\text{Tor}_n^{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z})$ for all n, m .

Exercise 8.9.6. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of R -modules and N any R -module. Write down explicitly the long exact sequences in Ext and Tor obtained by applying $\text{Hom}_R(N, -)$, $\text{Hom}_R(-, N)$, and $N \otimes_R -$.

Exercise 8.9.7. The **projective dimension** $\text{pd}(M)$ of a module M is the infimum of lengths of projective resolutions of M . The **global dimension** of R is $\text{gldim}(R) = \sup_M \text{pd}(M)$.

1. Show that $\text{pd}(M) \leq n$ if and only if $\text{Ext}_R^{n+1}(M, N) = 0$ for all N .
2. Show that $\text{gldim}(\mathbb{Z}) = 1$.
3. Show that $\text{gldim}(k[x_1, \dots, x_n]) = n$ for a field k (this is the Hilbert syzygy theorem).

Exercise 8.9.8. Let \mathcal{T} be a triangulated category and $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ a distinguished triangle. For any object X , show there are long exact sequences:

$$\begin{aligned} \cdots \rightarrow \text{Hom}(X, A) \rightarrow \text{Hom}(X, B) \rightarrow \text{Hom}(X, C) \rightarrow \text{Hom}(X, A[1]) \rightarrow \cdots \\ \cdots \rightarrow \text{Hom}(C, X) \rightarrow \text{Hom}(B, X) \rightarrow \text{Hom}(A, X) \rightarrow \text{Hom}(C[-1], X) \rightarrow \cdots \end{aligned}$$

Exercise 8.9.9. Show that in $D(\mathcal{A})$, for objects $A, B \in \mathcal{A}$ (viewed as complexes concentrated in degree 0):

$$\text{Hom}_{D(\mathcal{A})}(A, B[n]) \cong \text{Ext}_{\mathcal{A}}^n(A, B) \quad \text{for all } n \geq 0.$$

This is one of the key motivations for derived categories: the Ext groups

are simply Hom spaces with shifts.

Exercise 8.9.10. (Verdier quotient.) Let \mathcal{T} be a triangulated category and $\mathcal{N} \subseteq \mathcal{T}$ a **thick subcategory** (a full triangulated subcategory closed under direct summands). Define the Verdier quotient \mathcal{T}/\mathcal{N} as the localization at morphisms whose cone lies in \mathcal{N} .

1. Show that $D(\mathcal{A}) \cong K(\mathcal{A})/\text{Ac}(\mathcal{A})$, where $\text{Ac}(\mathcal{A})$ is the thick subcategory of acyclic complexes.
2. Show that the Verdier quotient inherits a triangulated structure.

Chapter 9

Monoidal and Braided Categories

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In many areas of mathematics and theoretical physics, one encounters categories equipped with a “tensor product”—a bifunctor that combines objects in a way that is associative and unital up to coherent isomorphism. Such categories are called *monoidal*. They provide the natural setting for studying algebras, coalgebras, and their modules in a category-theoretic framework. When the tensor product is commutative up to isomorphism, one obtains *braided* and *symmetric* monoidal categories, which underpin the theory of quantum groups and topological quantum field theories. In this chapter we develop the foundations of monoidal category theory, state Mac Lane’s coherence theorem, study enriched and closed categories, and touch upon the Curry–Howard–Lambek correspondence.

9.1 Monoidal categories

Definition 9.1.1 (Monoidal category). A **monoidal category** is a tuple $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ where:

- (i) \mathcal{C} is a category;
- (ii) $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a bifunctor called the **tensor product**;
- (iii) $I \in \text{Ob}(\mathcal{C})$ is an object called the **unit object** (or **monoidal unit**);
- (iv) $\alpha_{A,B,C}: (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ is a natural isomorphism called the **associator**;
- (v) $\lambda_A: I \otimes A \xrightarrow{\sim} A$ is a natural isomorphism called the **left unitor**;
- (vi) $\rho_A: A \otimes I \xrightarrow{\sim} A$ is a natural isomorphism called the **right unitor**;

subject to the *pentagon axiom* and the *triangle axiom* below.

Definition 9.1.2 (Pentagon axiom). For all objects A, B, C, D of \mathcal{C} , the following diagram commutes:

$$\begin{array}{ccc}
 & (A \otimes B) \otimes (C \otimes D) & \\
 \alpha_{A \otimes B, C, D} \nearrow & & \searrow \alpha_{A, B, C \otimes D} \\
 ((A \otimes B) \otimes C) \otimes D & & A \otimes (B \otimes (C \otimes D)) \\
 \alpha_{A, B, C} \otimes \text{id}_D \downarrow & & \uparrow \text{id}_A \otimes \alpha_{B, C, D} \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D)
 \end{array}$$

Definition 9.1.3 (Triangle axiom). For all objects A, B of \mathcal{C} , the following diagram commutes:

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{\alpha_{A, I, B}} & A \otimes (I \otimes B) \\
 \rho_A \otimes \text{id}_B \searrow & & \swarrow \text{id}_A \otimes \lambda_B \\
 & A \otimes B &
 \end{array}$$

Definition 9.1.4 (Strict monoidal category). A monoidal category is called **strict** if α , λ , and ρ are all identity natural transformations, i.e. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, $I \otimes A = A$, and $A \otimes I = A$ on the nose.

Example 9.1.5. The following are important examples of monoidal categories.

- (i) **(Set, \times , $\{*\}$)**: the category of sets with cartesian product.
- (ii) **(Ab, $\otimes_{\mathbb{Z}}$, \mathbb{Z})**: abelian groups with the tensor product over \mathbb{Z} .
- (iii) **(Vect $_{\mathbb{K}}$, $\otimes_{\mathbb{K}}$, \mathbb{K})**: vector spaces over a field \mathbb{K} with the usual tensor product.
- (iv) **(R-Mod, \otimes_R , R)** for a commutative ring R .
- (v) **(Cat, \times , $\mathbf{1}$)**: the category of small categories with the cartesian product.
- (vi) **(End(\mathcal{C}), \circ , $\text{id}_{\mathcal{C}}$)**: endofunctors of a category \mathcal{C} with composition. This is a strict monoidal category.
- (vii) **(Set, \sqcup , \emptyset)**: sets with disjoint union.
- (viii) The category of chain complexes $\text{Ch}(R)$ with the tensor product of complexes.

Remark 9.1.6. We often write just $(\mathcal{C}, \otimes, I)$ for a monoidal category, leaving the structural isomorphisms implicit.

9.2 Mac Lane's coherence theorem

The pentagon and triangle axioms may appear insufficient to guarantee that *all* diagrams built from α , λ , ρ and their inverses commute. Mac Lane's celebrated coherence theorem asserts that this is indeed the case.

Theorem 9.2.1 (Mac Lane's coherence theorem). *Every monoidal category is monoidally equivalent to a strict monoidal category. Equivalently, every diagram in a monoidal category whose morphisms are composites of instances of α , α^{-1} , λ , λ^{-1} , ρ , ρ^{-1} , identities, and tensor products thereof, commutes.*

Proof sketch. One constructs a strict monoidal category \mathcal{C}_s whose objects are finite words in $\text{Ob}(\mathcal{C})$ (including the empty word, which serves as the strict unit) and whose morphisms are induced by those of \mathcal{C} . The multiplication is word concatenation. An explicit monoidal equivalence $\mathcal{C}_s \simeq \mathcal{C}$ is built by iterating the tensor product. The commutativity of arbitrary constraint diagrams then reduces to the trivially commuting diagrams in \mathcal{C}_s . For the full proof, see Mac Lane [2] or [1]. \square

Remark 9.2.2. The practical consequence of the coherence theorem is that, for calculations, one may pretend that every monoidal category is strict: one simply omits all parentheses and all instances of the associator and unitors. We shall make frequent use of this simplification in what follows.

9.3 Braided and symmetric monoidal categories

Definition 9.3.1 (Braided monoidal category). A **braided monoidal category** is a monoidal category $(\mathcal{C}, \otimes, I)$ equipped with a natural isomorphism

$$\sigma_{A,B}: A \otimes B \xrightarrow{\sim} B \otimes A$$

called the **braiding**, satisfying the two *hexagon axioms*:

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{\alpha} & A \otimes (B \otimes C) \xrightarrow{\sigma_{A,B \otimes C}} (B \otimes C) \otimes A \\ \sigma_{A,B} \otimes \text{id} \downarrow & & \downarrow \alpha \\ (B \otimes A) \otimes C & \xrightarrow{\alpha} & B \otimes (A \otimes C) \xrightarrow{\text{id} \otimes \sigma_{A,C}} B \otimes (C \otimes A) \end{array}$$

and

$$\begin{array}{ccc} A \otimes (B \otimes C) & \xrightarrow{\alpha^{-1}} & (A \otimes B) \otimes C \xrightarrow{\sigma_{A \otimes B, C}} C \otimes (A \otimes B) \\ \text{id} \otimes \sigma_{B,C} \downarrow & & \downarrow \alpha^{-1} \\ A \otimes (C \otimes B) & \xrightarrow{\alpha^{-1}} & (A \otimes C) \otimes B \xrightarrow{\sigma_{A,C} \otimes \text{id}} (C \otimes A) \otimes B \end{array}$$

Definition 9.3.2 (Symmetric monoidal category). A braided monoidal category is **symmetric** if $\sigma_{B,A} \circ \sigma_{A,B} = \text{id}_{A \otimes B}$ for all A, B .

Example 9.3.3. (i) $(\mathbf{Set}, \times, \{*\})$, $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$, and $(\mathbf{Vect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$ are all symmetric monoidal categories with the evident swap maps.

- (ii) The category of R -modules with \otimes_R is symmetric monoidal.
- (iii) The category of graded R -modules with the Koszul sign rule $\sigma(a \otimes b) = (-1)^{|a||b|} b \otimes a$ is symmetric monoidal.
- (iv) The category of representations of a quantum group is braided monoidal but typically not symmetric.

9.4 Monoid objects

Definition 9.4.1 (Monoid object). Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A **monoid object** (or **monoid**) in \mathcal{C} is a triple (M, μ, η) where $M \in \text{Ob}(\mathcal{C})$, $\mu: M \otimes M \rightarrow M$ is the **multiplication**, and $\eta: I \rightarrow M$ is the **unit**, satisfying associativity and unitality:

$$\begin{array}{ccccc}
 (M \otimes M) \otimes M & \xrightarrow{\alpha} & M \otimes (M \otimes M) & \xrightarrow{\text{id} \otimes \mu} & M \otimes M & & I \otimes M & \xrightarrow{\eta \otimes \text{id}} & M \otimes M & \xleftarrow{\text{id} \otimes \eta} & M \otimes I \\
 \mu \otimes \text{id} \downarrow & & & & \downarrow \mu & & & \searrow \lambda & \downarrow \mu & & \swarrow \rho \\
 M \otimes M & \xrightarrow{\mu} & M & & & & & & M & &
 \end{array}$$

A **morphism of monoid objects** $f: (M, \mu, \eta) \rightarrow (M', \mu', \eta')$ is a morphism $f: M \rightarrow M'$ in \mathcal{C} with $f \circ \mu = \mu' \circ (f \otimes f)$ and $f \circ \eta = \eta'$. The category of monoid objects in \mathcal{C} is denoted $\text{Mon}(\mathcal{C})$.

Example 9.4.2. (i) A monoid object in $(\mathbf{Set}, \times, \{*\})$ is a monoid in the usual sense.

- (ii) A monoid object in $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ is a ring (not necessarily commutative).
- (iii) A monoid object in $(\mathbf{Vect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$ is a \mathbb{K} -algebra.
- (iv) A monoid object in $(\text{End}(\mathcal{C}), \circ, \text{id}_{\mathcal{C}})$ is a monad on \mathcal{C} .
- (v) A **commutative monoid object** in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma)$ is a monoid (M, μ, η) with $\mu \circ \sigma_{M,M} = \mu$.

Definition 9.4.3 (Comonoid object). Dually, a **comonoid object** in $(\mathcal{C}, \otimes, I)$ is a triple (C, Δ, ε) where $\Delta: C \rightarrow C \otimes C$ (the *comultiplication*) and $\varepsilon: C \rightarrow I$ (the *counit*) satisfy the duals of the associativity and unitality diagrams.

9.5 Enriched categories

Ordinary category theory is “enriched over **Set**”: for each pair of objects A, B , the hom $\text{Hom}(A, B)$ is a *set*. But in practice these hom-sets often carry additional structure—they may be abelian groups (**Ab**-enriched), topological spaces (**Top**-enriched), or chain complexes (dg-enriched).

Definition 9.5.1 (\mathcal{V} -enriched category). Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. A \mathcal{V} -enriched category (or \mathcal{V} -category) \mathcal{A} consists of:

- (i) a class $\text{Ob}(\mathcal{A})$ of objects;
- (ii) for each pair $A, B \in \text{Ob}(\mathcal{A})$, an object $\mathcal{A}(A, B) \in \text{Ob}(\mathcal{V})$ (the *hom-object*);
- (iii) for each triple A, B, C , a *composition morphism*

$$c_{A,B,C}: \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C) \quad \text{in } \mathcal{V};$$

- (iv) for each object A , an *identity morphism* $j_A: I \rightarrow \mathcal{A}(A, A)$ in \mathcal{V} ;

subject to associativity and unitality axioms expressed as commutative diagrams in \mathcal{V} .

Example 9.5.2. (i) A **Set**-enriched category is an ordinary (locally small) category.

(ii) An **Ab**-enriched category is a *preadditive category*: hom-sets are abelian groups and composition is bilinear.

(iii) A **Top**-enriched category is a *topological category*: hom-sets are topological spaces and composition is continuous.

(iv) A category enriched over the monoidal category $([0, \infty]^{\text{op}}, +, 0)$ (extended non-negative reals, reversed order, addition) is a (gener-

alised) *metric space* in the sense of Lawvere.

(v) A category enriched over chain complexes is a *dg-category*.

Definition 9.5.3 (\mathcal{V} -functor). A \mathcal{V} -functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between \mathcal{V} -categories consists of a map $F: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{B})$ and morphisms $F_{A,B}: \mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$ in \mathcal{V} compatible with composition and identities.

Definition 9.5.4 (\mathcal{V} -natural transformation). A \mathcal{V} -natural transformation $\eta: F \Rightarrow G$ between \mathcal{V} -functors $F, G: \mathcal{A} \rightarrow \mathcal{B}$ consists of a family of morphisms $\eta_A: I \rightarrow \mathcal{B}(FA, GA)$ in \mathcal{V} , one for each $A \in \text{Ob}(\mathcal{A})$, satisfying the enriched naturality condition.

9.6 Closed monoidal and cartesian closed categories

Definition 9.6.1 (Closed monoidal category). A monoidal category $(\mathcal{C}, \otimes, I)$ is **(left) closed** if for each object B the functor $(-) \otimes B: \mathcal{C} \rightarrow \mathcal{C}$ has a right adjoint, denoted $[B, -]$ or $\underline{\text{Hom}}(B, -)$:

$$\text{Hom}_{\mathcal{C}}(A \otimes B, C) \cong \text{Hom}_{\mathcal{C}}(A, [B, C]) \quad \text{naturally in } A \text{ and } C.$$

The object $[B, C]$ is called the **internal hom** (or **exponential object**).

Definition 9.6.2 (Cartesian closed category). A category \mathcal{C} with finite products is **cartesian closed** if it is closed monoidal with respect to the cartesian monoidal structure $(\mathcal{C}, \times, 1)$. That is, for every object B the functor $(-) \times B$ has a right adjoint $(-)^B$:

$$\text{Hom}(A \times B, C) \cong \text{Hom}(A, C^B).$$

Example 9.6.3. (i) **Set** is cartesian closed with $C^B = \text{Hom}_{\text{Set}}(B, C)$.

(ii) The category **Cat** of small categories is cartesian closed with the exponential $\mathcal{D}^{\mathcal{C}} = \text{Fun}(\mathcal{C}, \mathcal{D})$.

- (iii) Any elementary topos is cartesian closed (Theorem 10.3.4).
- (iv) **Top** is *not* cartesian closed in general; one must pass to a convenient subcategory such as compactly generated Hausdorff spaces.
- (v) **Ab** is not cartesian closed (the cartesian product is the direct sum, and $\text{Hom}_{\mathbf{Ab}}(A \oplus B, C) \not\cong \text{Hom}_{\mathbf{Ab}}(A, C^B)$ in general), but it is closed monoidal with $\otimes_{\mathbb{Z}}$ and internal hom $\text{Hom}_{\mathbb{Z}}(B, C)$.

Proposition 9.6.4 (Evaluation and coevaluation). *In a closed monoidal category, there are natural morphisms:*

- (a) the **evaluation**: $\text{ev}_{B,C}: [B, C] \otimes B \rightarrow C$, corresponding to $\text{id}_{[B,C]}$ under the adjunction;
- (b) the **coevaluation**: $\text{coev}_{A,B}: A \rightarrow [B, A \otimes B]$, corresponding to $\text{id}_{A \otimes B}$ under the adjunction.

Proof. Apply the adjunction isomorphism $\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, [B, C])$ to the appropriate identity morphisms. \square

9.7 The Curry–Howard–Lambek correspondence

The Curry–Howard–Lambek correspondence is a remarkable three-way dictionary between logic, computation, and category theory. We give a brief account aimed at illustrating the categorical perspective.

Logic	Type theory	Category theory
Proposition A	Type A	Object A
Proof of A	Term of type A	Morphism $1 \rightarrow A$
$A \Rightarrow B$	Function type $A \rightarrow B$	Exponential B^A
$A \wedge B$	Product type $A \times B$	Product $A \times B$
$A \vee B$	Sum type $A + B$	Coproduct $A + B$
\top (true)	Unit type 1	Terminal object
\perp (false)	Empty type 0	Initial object
Modus ponens	Function application	Evaluation $B^A \times A \rightarrow B$

Theorem 9.7.1 (Lambek). *The simply-typed lambda calculus (with products) is the internal language of cartesian closed categories. Specifically:*

- (a) *Every cartesian closed category gives rise to a simply-typed lambda calculus whose types are the objects and whose terms are the morphisms.*
- (b) *Every simply-typed lambda calculus (modulo $\beta\eta$ -equivalence) gives rise to a cartesian closed category, the syntactic category.*
- (c) *These two constructions are mutually inverse (up to equivalence).*

Remark 9.7.2. The correspondence extends far beyond the simply-typed case:

- Dependent types correspond to locally cartesian closed categories.
- Linear logic corresponds to $*$ -autonomous categories (a form of closed monoidal category).
- Homotopy type theory corresponds to $(\infty, 1)$ -toposes.

9.8 Exercises

Exercise 9.8.1. Show that every strict monoidal category is a monoidal category (the axioms are trivially satisfied with identity natural transformations).

Exercise 9.8.2. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. Show that the unit object I is unique up to isomorphism, in the sense that if (I', λ', ρ') also satisfies the axioms, then $I \cong I'$.

Exercise 9.8.3. Let $(\mathcal{C}, \otimes, I, \sigma)$ be a symmetric monoidal category. Show that the category $\text{CMon}(\mathcal{C})$ of commutative monoid objects in \mathcal{C} admits a monoidal structure inherited from that of \mathcal{C} .

Exercise 9.8.4. Show that every set S has a unique comonoid structure in $(\mathbf{Set}, \times, \{*\})$, given by $\Delta(s) = (s, s)$ and $\varepsilon(s) = *$.

Exercise 9.8.5. Let \mathcal{A} be a \mathcal{V} -enriched category where \mathcal{V} is closed monoidal. Define the *underlying ordinary category* \mathcal{A}_0 of \mathcal{A} and show it is indeed a category. *Hint:* take $\text{Hom}_{\mathcal{A}_0}(A, B) = \text{Hom}_{\mathcal{V}}(I, \mathcal{A}(A, B))$.

Exercise 9.8.6. Let $(\mathcal{C}, \otimes, I)$ be a small monoidal category. Show that the category of presheaves $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ carries a monoidal structure given by *Day convolution*:

$$(F \star G)(C) = \int^{A, B \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(A \otimes B, C) \times F(A) \times G(B).$$

Identify the unit object of this monoidal structure.

Exercise 9.8.7. Verify explicitly that in \mathbf{Set} , the adjunction $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C^B)$ corresponds to currying of functions: $f: A \times B \rightarrow C$ is sent to $\hat{f}: A \rightarrow C^B$ defined by $\hat{f}(a)(b) = f(a, b)$.

Chapter 10

Grothendieck Toposes—Introduction

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Topos theory, introduced by Grothendieck and his school in the 1960s for the purposes of algebraic geometry, has grown into one of the deepest and most far-reaching branches of category theory. A *Grothendieck topos* is a category of sheaves on a site; it simultaneously generalises the category of sheaves on a topological space and the category of presheaves on a small category. Elementary toposes, introduced by Lawvere and Tierney, isolate the key categorical properties that make a category “behave like the category of sets.”

In this chapter we give an introduction to the subject, defining sites, Grothendieck topologies, sheaves, and the resulting toposes. We discuss elementary toposes, subobject classifiers, and the internal logic, then state Giraud's theorem characterising Grothendieck toposes.

10.1 Sites and Grothendieck topologies

Definition 10.1.1 (Sieve). Let \mathcal{C} be a category and $C \in \text{Ob}(\mathcal{C})$. A **sieve on C** is a collection S of morphisms with codomain C such that if $f: D \rightarrow C$ is in S and $g: E \rightarrow D$ is any morphism, then $f \circ g \in S$. Equivalently, a sieve on C is a subfunctor of the representable presheaf $\text{Hom}(-, C)$.

Definition 10.1.2 (Grothendieck topology). A **Grothendieck topology J** on a small category \mathcal{C} assigns to each object C a collection $J(C)$ of sieves on C , called **covering sieves**, satisfying:

- (i) **Maximal sieve:** The maximal sieve $t_C = \{f \mid \text{cod}(f) = C\}$ is in $J(C)$.
- (ii) **Stability:** If $S \in J(C)$ and $h: D \rightarrow C$ is any morphism, then the pullback sieve $h^*S = \{g: E \rightarrow D \mid h \circ g \in S\}$ is in $J(D)$.
- (iii) **Transitivity:** If $S \in J(C)$ and R is a sieve on C such that for every $f: D \rightarrow C$ in S the pullback $f^*R \in J(D)$, then $R \in J(C)$.

Definition 10.1.3 (Site). A **site** is a pair (\mathcal{C}, J) consisting of a small category \mathcal{C} and a Grothendieck topology J on \mathcal{C} .

Definition 10.1.4 (Coverage). In practice, Grothendieck topologies are often specified via **coverages**: for each $C \in \text{Ob}(\mathcal{C})$, one gives a collection of families $\{f_i: C_i \rightarrow C\}_{i \in I}$, called **covering families**, satisfying a stability condition under pullback. The covering families generate sieves, hence a Grothendieck topology.

Example 10.1.5. (i) **Trivial topology.** Only the maximal sieves cover. Then every presheaf is a sheaf.

(ii) **Discrete topology.** Every sieve is a covering sieve. The only sheaf

is the terminal presheaf.

- (iii) **Canonical topology on \mathbf{Top} .** For an open set U , a covering sieve is generated by an open cover $\{U_i\}$ of U .
- (iv) **Zariski topology.** On the category of commutative rings (with the opposite orientation), covering families correspond to collections of localisations $\{R \rightarrow R[f_i^{-1}]\}$ where $(f_1, \dots, f_n) = R$.
- (v) **Étale topology.** On the category of schemes, covering families are jointly surjective families of étale morphisms.
- (vi) **fppf and fpqc topologies.** Finer topologies on schemes used in algebraic geometry.

10.2 Sheaves on a site

Definition 10.2.1 (Presheaf). A **presheaf** on a small category \mathcal{C} is a functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$. The category of presheaves is $\mathbf{PSh}(\mathcal{C}) = [\mathcal{C}^{\text{op}}, \mathbf{Set}]$.

Definition 10.2.2 (Sheaf on a site). Let (\mathcal{C}, J) be a site. A presheaf $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ is a **sheaf** (for the topology J) if for every covering sieve $S \in J(\mathcal{C})$, the natural map

$$F(\mathcal{C}) \longrightarrow \varprojlim_{(f: D \rightarrow \mathcal{C}) \in S} F(D)$$

is a bijection. Equivalently, for every covering family $\{f_i: C_i \rightarrow \mathcal{C}\}$ generating a covering sieve, the diagram

$$F(\mathcal{C}) \longrightarrow \prod_i F(C_i) \rightrightarrows \prod_{i,j} F(C_i \times_{\mathcal{C}} C_j)$$

is an equaliser.

Definition 10.2.3 (Category of sheaves). The full subcategory of $\mathbf{PSh}(\mathcal{C})$ consisting of sheaves for the topology J is denoted $\mathbf{Sh}(\mathcal{C}, J)$. It is called a **Grothendieck topos**.

Theorem 10.2.4 (Sheafification). *The inclusion $i: \text{Sh}(\mathcal{C}, J) \hookrightarrow \text{PSh}(\mathcal{C})$ has a left adjoint $a: \text{PSh}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{C}, J)$ called **sheafification** (or the **associated sheaf functor**). Moreover:*

- (a) *a is left exact (preserves finite limits);*
- (b) *$a \circ i \cong \text{id}$;*
- (c) *the adjunction $a \dashv i$ is a reflection.*

Proof sketch. The sheafification aF is constructed by the “plus construction” applied twice: $(F^+)^+$. Given F , one defines

$$F^+(C) = \varinjlim_{S \in J(C)} \text{Match}(S, F)$$

where $\text{Match}(S, F)$ is the set of matching families for the sieve S . One application of $(-)^+$ makes F separated; a second application yields a sheaf. Left exactness follows from the fact that the colimit over covering sieves is filtered. \square

Example 10.2.5. If X is a topological space and $\mathcal{O}(X)$ is the poset of open subsets (viewed as a category), then the open-cover topology makes $(\mathcal{O}(X), J)$ a site and $\text{Sh}(\mathcal{O}(X), J)$ is the classical category of sheaves on X .

10.3 Elementary toposes

Definition 10.3.1 (Elementary topos). An **elementary topos** is a category \mathcal{E} satisfying:

- (i) \mathcal{E} has all finite limits;
- (ii) \mathcal{E} has all finite colimits;
- (iii) \mathcal{E} is cartesian closed;
- (iv) \mathcal{E} has a **subobject classifier**.

Definition 10.3.2 (Subobject classifier). A **subobject classifier** in a category \mathcal{E} with finite limits is a monomorphism $\text{true}: 1 \rightarrow \Omega$ such that

for every monomorphism $m: S \rightarrow A$ there exists a unique morphism $\chi_m: A \rightarrow \Omega$ making the following square a pullback:

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ m \downarrow & & \downarrow \text{true} \\ A & \xrightarrow{\chi_m} & \Omega \end{array}$$

The morphism χ_m is called the **characteristic morphism** (or **classifying morphism**) of the subobject m .

Example 10.3.3. (i) In **Set**, $\Omega = \{0, 1\}$ with $\text{true}(\ast) = 1$. The classifying morphism of a subset $S \subseteq A$ is the characteristic function χ_S .

(ii) In the topos of sheaves $\text{Sh}(X)$ on a topological space X , Ω is the sheaf of open subsets: $\Omega(U) = \{V \subseteq U \mid V \text{ open}\}$.

(iii) In a presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, $\Omega(C) = \{\text{sieves on } C\}$.

Theorem 10.3.4. *Every elementary topos is cartesian closed. In particular, every Grothendieck topos is cartesian closed.*

Proof sketch. The power object $P(B) = \Omega^B$ exists by assumption in an elementary topos (cartesian closedness is part of the definition). For a Grothendieck topos, cartesian closedness follows from the fact that $\text{Sh}(\mathcal{C}, J)$ is a reflective subcategory of the cartesian closed presheaf category, and the reflection preserves finite products. \square

Proposition 10.3.5. *Every elementary topos \mathcal{E} satisfies:*

(a) \mathcal{E} is balanced (every monic epic is an isomorphism);

(b) Epimorphisms in \mathcal{E} are coequalizers;

(c) Every morphism $f: A \rightarrow B$ factors as a regular epimorphism followed by a monomorphism (image factorisation).

10.4 Internal logic

Every topos has an *internal logic* that allows one to reason about objects and morphisms using a language that resembles set-theoretic reasoning, but is *intuitionistic*: the law of excluded middle need not hold.

Definition 10.4.1 (Internal logic of a topos). The **internal logic** (or **Mitchell–Bénabou language**) of a topos \mathcal{E} interprets:

- *types* as objects of \mathcal{E} ;
- *terms* of type A in context Γ as morphisms $\Gamma \rightarrow A$;
- *propositions* in context Γ as morphisms $\Gamma \rightarrow \Omega$ (subobjects of Γ);
- logical connectives via operations on Ω : $\wedge, \vee, \Rightarrow, \neg, \forall, \exists$.

Remark 10.4.2. The internal logic of a topos is **intuitionistic higher-order logic**. The law of excluded middle $\phi \vee \neg\phi$ holds internally if and only if $\Omega \cong 1 \sqcup 1$, in which case \mathcal{E} is called a **Boolean topos**. The topos **Set** is Boolean; sheaf toposes on non-trivial spaces are typically non-Boolean.

Remark 10.4.3. The precise semantics for interpreting formulae of the internal language in a topos is called **Kripke–Joyal semantics** (or *forcing*). It provides the tool for transferring set-theoretic proofs to arbitrary toposes.

10.5 Geometric morphisms

Definition 10.5.1 (Geometric morphism). A **geometric morphism** between toposes $f: \mathcal{E} \rightarrow \mathcal{F}$ is an adjunction $f^* \dashv f_*$ where:

- $f^*: \mathcal{F} \rightarrow \mathcal{E}$ is called the **inverse image functor**;
- $f_*: \mathcal{E} \rightarrow \mathcal{F}$ is called the **direct image functor**;
- f^* is required to be left exact (preserve finite limits).

Example 10.5.2. (i) A continuous map $f: X \rightarrow Y$ of topological spaces induces a geometric morphism $f: \text{Sh}(X) \rightarrow \text{Sh}(Y)$ with f_* the direct image of sheaves and f^* the inverse image.

(ii) For any topos \mathcal{E} , there is an essentially unique geometric morphism $\gamma: \mathcal{E} \rightarrow \mathbf{Set}$ given by $\gamma^* = \Delta$ (constant sheaf) and $\gamma_* = \Gamma$ (global sections).

(iii) The sheafification adjunction $a \dashv i$ for a site (\mathcal{C}, J) is a geometric morphism $\text{Sh}(\mathcal{C}, J) \rightarrow \text{PSh}(\mathcal{C})$.

Definition 10.5.3 (Categories of geometric morphisms). A **2-cell** between geometric morphisms $f, g: \mathcal{E} \rightarrow \mathcal{F}$ is a natural transformation $\alpha: f^* \Rightarrow g^*$ (equivalently, $\beta: g_* \Rightarrow f_*$ by the mate correspondence). This gives a 2-category \mathfrak{Top} of (Grothendieck) toposes.

10.6 Giraud's theorem

Giraud's theorem gives a purely categorical characterisation of Grothendieck toposes, without reference to sites.

Theorem 10.6.1 (Giraud). *A category \mathcal{E} is a Grothendieck topos (i.e. equivalent to $\text{Sh}(\mathcal{C}, J)$ for some site (\mathcal{C}, J)) if and only if it satisfies the following conditions:*

- (i) \mathcal{E} has all small colimits;
- (ii) colimits in \mathcal{E} are universal (stable under pullback);
- (iii) coproducts in \mathcal{E} are disjoint;
- (iv) every equivalence relation in \mathcal{E} is effective (is the kernel pair of its coequaliser);
- (v) \mathcal{E} has a small set of generators.

Remark 10.6.2. Giraud's theorem implies, in particular, that every Grothendieck topos:

- is an elementary topos;

- has all small limits and colimits;
- is a locally presentable category;
- satisfies the axiom of choice internally if and only if every epimorphism splits.

10.7 Exercises

Exercise 10.7.1. Show that sieves on an object C in a category \mathcal{C} are in bijection with subfunctors of the representable presheaf $\text{Hom}(-, C)$.

Exercise 10.7.2. Let \mathcal{C} be a small category. Show that $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is a Grothendieck topos (for the trivial topology). Identify the subobject classifier explicitly.

Exercise 10.7.3. Let \mathcal{E} be an elementary topos. Show that the subobject classifier Ω carries the structure of an *internal Heyting algebra*: there exist morphisms $\wedge, \vee, \Rightarrow: \Omega \times \Omega \rightarrow \Omega$ satisfying the Heyting algebra axioms internally.

Exercise 10.7.4. Show that geometric morphisms compose: if $f: \mathcal{E} \rightarrow \mathcal{F}$ and $g: \mathcal{F} \rightarrow \mathcal{G}$ are geometric morphisms, then so is $g \circ f$ with $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* = g_* \circ f_*$.

Exercise 10.7.5. Let (\mathcal{C}, J) be a site and $\{f_i: C_i \rightarrow C\}_{i \in I}$ a covering family. Show that a presheaf F satisfies the sheaf condition for this covering if and only if the diagram

$$F(C) \longrightarrow \prod_{i \in I} F(C_i) \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \prod_{(i,j) \in I \times I} F(C_i \times_C C_j)$$

is an equaliser, where d_0 and d_1 are induced by the two projections from the fibre product.

Exercise 10.7.6. Show that a topos \mathcal{E} is Boolean if and only if every subobject has a complement, i.e. for every mono $m: S \rightarrow A$ there exists

$m': S' \rightarrow A$ with $S \sqcup S' \cong A$ and $S \cap S' \cong 0$.

Exercise 10.7.7. Describe the étale site of $\text{Spec}(\mathbb{Z})$ and explain why the resulting topos is related to the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Chapter 11

∞ -Categories—Overview

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Classical category theory considers objects, morphisms between objects, and equalities between morphisms. But in many contexts—homotopy theory, homological algebra, derived algebraic geometry—one needs morphisms between morphisms (2-cells), morphisms between those (3-cells), and so on, ad infinitum, with all compositions associative and unital only up to coherent higher-dimensional cells. This is the domain of *higher category theory* and, in particular, $(\infty, 1)$ -categories.

This chapter provides a survey of the main ideas, definitions, and results. Proofs are mostly omitted; we aim to give the reader a map of the landscape and precise pointers to the literature.

11.1 Motivation

The need for higher categories arises from several independent sources.

- (i) **Homotopy theory.** The homotopy category $\mathrm{Ho}(\mathbf{Top})$ of topological spaces loses too much information: mapping spaces are truncated to sets. One wants a “category” where $\mathrm{Hom}(X, Y)$ is a space (or a simplicial set), composition is associative up to homotopy, and all higher coherences are recorded.
- (ii) **Derived categories.** The derived category $D(\mathcal{A})$ of an abelian category \mathcal{A} is obtained by formally inverting quasi-isomorphisms, but the resulting category is poorly behaved (e.g. it lacks functorial cones). The ∞ -categorical enhancement—the *derived ∞ -category*—remedies these defects.
- (iii) **Stacks and higher stacks.** In algebraic geometry, stacks are “sheaves of groupoids.” Higher stacks are sheaves of ∞ -groupoids, and their natural home is the ∞ -topos.
- (iv) **Topological quantum field theory.** The cobordism hypothesis of Baez–Dolan (proved by Lurie) requires the language of (∞, n) -categories.

11.2 Quasi-categories

The most developed model for $(\infty, 1)$ -categories is that of *quasi-categories*, introduced by Boardman–Vogt and extensively developed by Joyal and Lurie.

Definition 11.2.1 (Simplicial set). Let Δ denote the simplex category (finite non-empty ordinals and order-preserving maps). A **simplicial set** is a functor $X: \Delta^{\mathrm{op}} \rightarrow \mathbf{Set}$. We write $X_n = X([n])$ for the set of n -simplices. The category of simplicial sets is $\mathbf{sSet} = [\Delta^{\mathrm{op}}, \mathbf{Set}]$.

Definition 11.2.2 (Horn). For $0 \leq k \leq n$, the **k -th horn** $\Lambda_k^n \subset \Delta^n$ is the simplicial subset obtained by removing the interior of Δ^n and the face opposite vertex k . The horn is called **inner** if $0 < k < n$ and **outer** otherwise.

Definition 11.2.3 (Quasi-category). A simplicial set X is a **quasi-category** (or **∞ -category**, or **weak Kan complex**) if every inner horn has a filler: for every $0 < k < n$ and every map $\Lambda_k^n \rightarrow X$, there exists an

extension $\Delta^n \rightarrow X$:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^n & & \end{array}$$

Note that the filler is *not required to be unique*.

Remark 11.2.4. If one requires fillers for *all* horns (inner and outer), one obtains a **Kan complex**, which models an ∞ -*groupoid*: every 1-simplex is invertible up to higher homotopy. A quasi-category is to an ∞ -category what a Kan complex is to an ∞ -groupoid.

Example 11.2.5. (i) The nerve $N(\mathcal{C})$ of an ordinary category \mathcal{C} is a quasi-category in which inner horn fillers are *unique*.

(ii) The singular simplicial set $\text{Sing}(X)$ of a topological space X is a Kan complex, hence a quasi-category.

(iii) The **dg-nerve** of a dg-category is a quasi-category.

(iv) The **homotopy coherent nerve** of a simplicial category is a quasi-category.

Definition 11.2.6 (Homotopy category of a quasi-category). Given a quasi-category X , its **homotopy category** $\text{h}X$ is the ordinary category with:

- objects: the vertices (0-simplices) of X ;
- morphisms $x \rightarrow y$: homotopy classes of edges (1-simplices) from x to y , where two edges are homotopic if they are related by a 2-simplex with degenerate third edge;
- composition: given by choosing fillers for inner 2-horns.

11.3 Models for $(\infty, 1)$ -categories

There are several equivalent models for $(\infty, 1)$ -categories.

Definition 11.3.1 ($(\infty, 1)$ -category—informal). An $(\infty, 1)$ -category is a “category enriched in spaces”—a structure with objects, morphisms, homotopies between morphisms, homotopies between homotopies, etc., where all k -morphisms for $k \geq 2$ are invertible (up to higher morphisms).

Remark 11.3.2. The principal models, all Quillen equivalent, include:

- (i) **Quasi-categories** (simplicial sets with inner horn fillers) — Joyal, Lurie.
- (ii) **Complete Segal spaces** — Rezk.
- (iii) **Segal categories** — Hirschowitz–Simpson.
- (iv) **Simplicial categories** (sSet-enriched categories) — Bergner.
- (v) **Relative categories** (categories with weak equivalences) — Barwick–Kan.

The fact that these models are all equivalent is a deep theorem (the “comparison theorem”), proved using Quillen model structures.

11.4 The ∞ -Yoneda lemma

The Yoneda lemma generalises to the ∞ -categorical setting.

Definition 11.4.1 (∞ -presheaf). Let \mathcal{C} be a small $(\infty, 1)$ -category. The ∞ -category of presheaves on \mathcal{C} is $\mathcal{P}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$, where \mathcal{S} denotes the ∞ -category of spaces (i.e. ∞ -groupoids, modelled by Kan complexes).

Theorem 11.4.2 (∞ -Yoneda lemma). *Let \mathcal{C} be a small $(\infty, 1)$ -category. The Yoneda embedding*

$$\mathbf{y}: \mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C}), \quad C \mapsto \text{Map}_{\mathcal{C}}(-, C)$$

is fully faithful. Moreover, for any presheaf $F \in \mathcal{P}(\mathcal{C})$ and any $C \in \mathcal{C}$,

$$\text{Map}_{\mathcal{P}(\mathcal{C})}(\mathbf{y}(C), F) \simeq F(C)$$

as objects of \mathcal{S} (an equivalence of spaces).

Remark 11.4.3. The ∞ -category $\mathcal{P}(\mathcal{C})$ is the **free cocompletion** of \mathcal{C} under small colimits, just as in the 1-categorical case. Every ∞ -presheaf is a colimit of representables.

11.5 ∞ -adjunctions and ∞ -limits

Definition 11.5.1 (∞ -adjunction). An ∞ -adjunction between $(\infty, 1)$ -categories \mathcal{C} and \mathcal{D} is a pair of functors $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$ together with an equivalence of mapping spaces

$$\mathrm{Map}_{\mathcal{D}}(FC, D) \simeq \mathrm{Map}_{\mathcal{C}}(C, GD)$$

natural in C and D .

Remark 11.5.2. The theory of ∞ -adjunctions parallels the 1-categorical theory:

- (i) ∞ -adjoints are unique up to contractible choice;
- (ii) an ∞ -left adjoint preserves ∞ -colimits;
- (iii) an ∞ -right adjoint preserves ∞ -limits;
- (iv) the ∞ -adjoint functor theorem holds (under appropriate presentability hypotheses).

Definition 11.5.3 (∞ -limit). Let $F: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram in an $(\infty, 1)$ -category \mathcal{C} . An ∞ -limit of F is an object $L \in \mathcal{C}$ together with equivalences

$$\mathrm{Map}_{\mathcal{C}}(X, L) \simeq \lim_{j \in \mathcal{J}} \mathrm{Map}_{\mathcal{C}}(X, Fj)$$

natural in X , where the right-hand side is a homotopy limit of spaces. Dually for ∞ -colimits.

11.6 Higher toposes

Definition 11.6.1 (∞ -topos). An ∞ -**topos** is an $(\infty, 1)$ -category that is an accessible left exact localisation of an ∞ -presheaf category $\mathcal{P}(\mathcal{C})$ for some small $(\infty, 1)$ -category \mathcal{C} .

Remark 11.6.2. Lurie proves an ∞ -categorical analogue of Giraud’s theorem: an $(\infty, 1)$ -category \mathcal{E} is an ∞ -topos if and only if:

- (i) \mathcal{E} is presentable (cocomplete and accessible);
- (ii) colimits in \mathcal{E} are universal;
- (iii) groupoid objects in \mathcal{E} are effective.

Note that the disjointness condition for coproducts, present in Giraud’s classical theorem, is automatic in the ∞ -setting.

Example 11.6.3. (i) The ∞ -category \mathcal{S} of spaces (Kan complexes) is the initial ∞ -topos, playing the role of **Set**.
 (ii) For a topological space X , the ∞ -category $\mathrm{Sh}_\infty(X)$ of ∞ -sheaves (sheaves of spaces) on X is an ∞ -topos.
 (iii) Derived algebraic geometry uses ∞ -toposes as the ambient framework for “derived stacks.”

Remark 11.6.4. ∞ -toposes provide the semantic models for **homotopy type theory** (HoTT). The univalence axiom of Voevodsky holds in every ∞ -topos, and the internal language of an ∞ -topos is (conjecturally) homotopy type theory. This is an active area of research connecting category theory, homotopy theory, and the foundations of mathematics.

11.7 Exercises

Exercise 11.7.1. Let \mathcal{C} be an ordinary category. Show that the nerve $N(\mathcal{C})$ is a quasi-category and that inner horn fillers are unique. Deduce that N is fully faithful as a functor $\mathbf{Cat} \rightarrow \mathbf{sSet}$.

Exercise 11.7.2. Show that a quasi-category X is a Kan complex if and only if every morphism (1-simplex) in X is an equivalence (invertible up to a 2-simplex).

Exercise 11.7.3. Let \mathcal{C} be an ordinary category. Show that $\mathbf{h}(N(\mathcal{C})) \cong \mathcal{C}$.

Exercise 11.7.4. Let \mathcal{C} be a small $(\infty, 1)$ -category and $F \in \mathcal{P}(\mathcal{C})$ an ∞ -presheaf. Using the ∞ -Yoneda lemma, show that F is a colimit of representable presheaves.

Exercise 11.7.5. Show that if $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor of $(\infty, 1)$ -categories that admits a right adjoint G , then G is unique up to a contractible space of choices. *Hint:* use the ∞ -Yoneda lemma to express $G(D)$ via the presheaf $C \mapsto \mathrm{Map}_{\mathcal{D}}(FC, D)$.

Exercise 11.7.6. Let \mathcal{E} be an ∞ -topos and $X \in \mathcal{E}$ an object. Show that the slice ∞ -category $\mathcal{E}_{/X}$ is again an ∞ -topos.

Exercise 11.7.7 (Descent). Let \mathcal{E} be an ∞ -topos and $f: X \rightarrow Y$ a morphism in \mathcal{E} . Show that the pullback functor $f^*: \mathcal{E}_{/Y} \rightarrow \mathcal{E}_{/X}$ preserves all colimits. This is the ∞ -categorical formulation of **descent**.

Appendix A

Review of Algebra and Topology

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We collect here the basic definitions from algebra and topology that are used throughout the text. The reader familiar with these notions may safely skip this appendix and refer back to it as needed.

A.1 Groups, rings, and modules

Definition A.1.1 (Group). A **group** is a set G equipped with a binary operation $\cdot : G \times G \rightarrow G$, an identity element $e \in G$, and an inverse map $(-)^{-1} : G \rightarrow G$, satisfying associativity, identity, and inverse laws. A group is **abelian** if $a \cdot b = b \cdot a$ for all $a, b \in G$.

Definition A.1.2 (Ring). A **ring** $(R, +, \cdot, 0, 1)$ is an abelian group $(R, +, 0)$ equipped with an associative multiplication with unit 1, such that multiplication distributes over addition. A ring is **commutative** if

$ab = ba$ for all $a, b \in R$.

Definition A.1.3 (Module). Let R be a ring. A **left R -module** is an abelian group $(M, +)$ equipped with a scalar multiplication $R \times M \rightarrow M$ satisfying the usual axioms: $r(m + n) = rm + rn$, $(r + s)m = rm + sm$, $(rs)m = r(sm)$, $1m = m$.

Definition A.1.4 (Field). A **field** is a commutative ring \mathbb{K} in which every non-zero element is invertible, i.e. $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$.

Definition A.1.5 (Algebra over a field). A **\mathbb{K} -algebra** is a ring A that is also a \mathbb{K} -vector space, with multiplication \mathbb{K} -bilinear. Equivalently, it is a monoid object in $(\mathbf{Vect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$.

A.2 Topological spaces

Definition A.2.1 (Topological space). A **topological space** is a pair (X, τ) where X is a set and $\tau \subseteq \mathcal{P}(X)$ is a collection of subsets (called *open sets*) satisfying: (i) $\emptyset, X \in \tau$; (ii) τ is closed under arbitrary unions; (iii) τ is closed under finite intersections.

Definition A.2.2 (Continuous map). A map $f: X \rightarrow Y$ between topological spaces is **continuous** if $f^{-1}(U) \in \tau_X$ for every $U \in \tau_Y$.

Definition A.2.3 (Hausdorff space). A topological space X is **Hausdorff** (or T_2) if for every pair of distinct points $x \neq y$ there exist disjoint open sets $U \ni x$ and $V \ni y$.

Definition A.2.4 (Compact space). A topological space X is **compact** if every open cover has a finite subcover.

Definition A.2.5 (Connected space). A topological space X is **connected** if it cannot be written as a disjoint union of two non-empty open sets.

A.3 Simplicial sets

Definition A.3.1 (Simplex category). The **simplex category** Δ has as objects the finite ordinals $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$, and as morphisms the order-preserving maps.

Definition A.3.2 (Simplicial set). A **simplicial set** is a functor $X: \Delta^{\text{op}} \rightarrow \mathbf{Set}$. It is specified by sets X_n (the n -simplices) together with face maps $d_i: X_n \rightarrow X_{n-1}$ and degeneracy maps $s_i: X_n \rightarrow X_{n+1}$ satisfying the simplicial identities.

Definition A.3.3 (Geometric realisation). The **geometric realisation** of a simplicial set X is

$$|X| = \left(\coprod_{n \geq 0} X_n \times \Delta_{\text{top}}^n \right) / \sim$$

where Δ_{top}^n is the topological n -simplex and the equivalence relation is generated by the face and degeneracy maps.

A.4 Basic homological algebra

Definition A.4.1 (Chain complex). A **chain complex** over a ring R is a sequence of R -modules and homomorphisms

$$\dots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

with $d_n \circ d_{n+1} = 0$ for all n . The **homology** is $H_n(C_\bullet) = \ker d_n / \text{im } d_{n+1}$.

Definition A.4.2 (Exact sequence). A chain complex is **exact** at C_n if $H_n = 0$, i.e. $\ker d_n = \text{im } d_{n+1}$. A **short exact sequence** is an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

Definition A.4.3 (Tensor product of modules). For a commutative ring R and R -modules M, N , the **tensor product** $M \otimes_R N$ is the R -module generated by symbols $m \otimes n$ ($m \in M, n \in N$) subject

to bilinearity relations. It is characterised by the universal property: $\text{Hom}_R(M \otimes_R N, P) \cong \text{Bilin}_R(M \times N, P)$.

Definition A.4.4 (Projective and injective modules). An R -module P is **projective** if $\text{Hom}_R(P, -)$ is exact. An R -module Q is **injective** if $\text{Hom}_R(-, Q)$ is exact.

A.5 Posets and lattices

Definition A.5.1 (Partially ordered set). A **partially ordered set** (or **poset**) is a set P equipped with a relation \leq that is reflexive, antisymmetric, and transitive. A poset may be viewed as a category with objects P and a unique morphism $x \rightarrow y$ whenever $x \leq y$.

Definition A.5.2 (Lattice). A **lattice** is a poset in which every pair of elements $\{a, b\}$ has a meet $a \wedge b$ (greatest lower bound) and a join $a \vee b$ (least upper bound). A **complete lattice** has meets and joins for all subsets.

Definition A.5.3 (Heyting algebra). A **Heyting algebra** is a lattice H with a binary operation \Rightarrow (implication) satisfying: $a \wedge b \leq c$ if and only if $a \leq (b \Rightarrow c)$. Every Boolean algebra is a Heyting algebra, but not conversely.

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