Category Theory

A Graduate Course

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Preface

Category theory provides a unifying language for mathematics. Originally introduced by Eilenberg and Mac Lane in the 1940s to formalise natural transformations in algebraic topology, it has since become an indispensable tool in algebra, geometry, logic, computer science, and mathematical physics.

This text is designed for a one-semester graduate course. We assume familiarity with basic algebra (groups, rings, modules) and point-set topology, but no prior knowledge of category theory itself. The exposition emphasises both the abstract framework and its concrete manifestations: every definition is illustrated by examples drawn from algebra, topology, and analysis.

Organisation. Chapters 1 and 2 lay the foundations: categories, functors, natural transformations, and the duality principle. Chapters 3 and 4 develop limits, colimits, and adjunctions. Later chapters treat representability, the Yoneda lemma, Kan extensions, abelian categories, and monoidal categories.

Exercises. Each chapter ends with a graded exercise set. Working through these is essential for mastering the material.

Notation and Conventions

Symbol	Meaning
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	Natural numbers (including 0), integers, rationals, reals, complex numbers
$\mathbb{K}$	An arbitrary field
Set, **Grp**, **Ab**	Categories of sets, groups, abelian groups
Ring, **Top**, $\mathbf{Vect}_{\mathbb{K}}$	Categories of rings, topological spaces, $\mathbb{K}$-vector spaces
$\mathbf{Mod}_R$	Category of left R-modules
Cat	Category of small categories
$\mathrm{Hom}_{\mathcal{C}}(A, B)$	Set of morphisms $A \to B$ in $\mathcal{C}$
$\mathcal{C}^{\mathrm{op}}$	Opposite (dual) category of $\mathcal{C}$
$F\colon \mathcal{C} \to \mathcal{D}$	Functor from $\mathcal{C}$ to $\mathcal{D}$
$\alpha\colon F \Rightarrow G$	Natural transformation from F to G
id_A	Identity morphism on A
$g \circ f$	Composition: first f, then g
$[\mathcal{C}, \mathcal{D}]$ or $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$	Functor category

Throughout, “category” means a locally small category unless otherwise stated. We use the terms “morphism”, “arrow”, and “map” interchangeably. All rings are assumed to have a multiplicative identity.

Chapter 1

Categories, Functors, and Natural Transformations

Categories, functors, and natural transformations constitute the three fundamental notions of category theory. A category axiomatises the idea of "objects with structure-preserving maps between them". A functor is a morphism of categories, and a natural transformation is a morphism of functors. Already at this first level of abstraction one finds a striking pattern: the passage from objects to morphisms between objects repeats at every stage.

In this chapter we introduce these three notions carefully, give copious examples, and establish basic terminology—monomorphisms, epimorphisms, isomorphisms, initial and terminal objects—that will be used throughout the course.

1.1 Categories

Definition 1.1.1 (Category)**.** A **category** $\mathcal{C}$ consists of the following data:

(i) A collection $\mathrm{Ob}(\mathcal{C})$ of *objects*.

(ii) For every ordered pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* (or *arrows*) from A to B. We write $f\colon A \to B$ to indicate $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

(iii) For every triple of objects A, B, C, a *composition law*

$$\circ\colon \mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C), \qquad (g,f) \longmapsto g\circ f.$$

(iv) For every object A, an *identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$.

These data are subject to two axioms:

(C1) Associativity. For all $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(C2) Identity. For all $f\colon A \to B$,

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f.$$

Remark 1.1.2. We require the hom-sets to be pairwise disjoint: every morphism f has a uniquely determined *domain* (or *source*) and *codomain* (or *target*). Some authors encode this by defining a morphism as a triple (f, A, B).

Notation 1.1.3. We write $\mathrm{Hom}_{\mathcal{C}}(A,B)$, or $\mathcal{C}(A,B)$, or $\mathrm{Mor}_{\mathcal{C}}(A,B)$ interchangeably. When the ambient category is clear we simply write $\mathrm{Hom}(A,B)$.

1.2 First examples of categories

Example 1.2.1 (Algebraic categories)**.** The following are categories:

(i) **Set**: objects are sets, morphisms are functions.

(ii) **Grp**: objects are groups, morphisms are group homomorphisms.

(iii) **Ab**: objects are abelian groups, morphisms are group homomorphisms.

(iv) **Ring**: objects are (unital) rings, morphisms are ring homomorphisms (preserving 1).

(v) $\mathbf{Mod}_R$: for a ring R, objects are left R-modules, morphisms are R-linear maps.

(vi) $\mathbf{Vect}_{\mathbb{K}}$: objects are $\mathbb{K}$-vector spaces, morphisms are $\mathbb{K}$-linear maps. This is the special case $\mathbf{Mod}_{\mathbb{K}}$.

In each case, composition is the usual composition of functions and the identity morphism on an object A is the identity function $\mathrm{id}_A\colon A \to A$.

Example 1.2.2 (Topological category)**.** **Top**: objects are topological spaces, morphisms are continuous maps.

Example 1.2.3 (Ordered sets as categories)**.** Let $(P, \leq)$ be a preorder (a set equipped with a reflexive, transitive relation). Define a category $\mathcal{C}_P$ by:

- $\mathrm{Ob}(\mathcal{C}_P) = P$.
- For $a, b \in P$:

$$\mathrm{Hom}(a, b) = \begin{cases} \{*\} & \text{if } a \leq b, \\ \varnothing & \text{otherwise.} \end{cases}$$

Transitivity gives composition; reflexivity gives identities. If $(P, \leq)$ is a partial order, distinct objects are never isomorphic.

Example 1.2.4 (Monoids as categories)**.** A monoid $(M, \cdot, e)$ may be viewed as a category with a single object $*$ and $\mathrm{Hom}(*, *) = M$. Composition is the monoid operation and the identity morphism is e.
Conversely, every category with exactly one object is a monoid. A group is a one-object category in which every morphism is invertible.

Example 1.2.5 (The category **Cat**)**.** The category **Cat** has small categories as objects and functors (defined in efsec:functors) as morphisms.

1.3 Small and locally small categories

Definition 1.3.1 (Size conditions)**.** A category $\mathcal{C}$ is called:

(i) **small** if $\mathrm{Ob}(\mathcal{C})$ is a set (not a proper class);

(ii) **locally small** if for every pair of objects A, B, the collection $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.

A category that is not small is called **large**.

Remark 1.3.2. Every small category is locally small. The categories **Set**, **Grp**, **Top**, etc. are locally small but not small (their collection of objects is a proper class). Every preorder viewed as a category is small provided the underlying set is a set. Throughout this text, "category" means "locally small category" unless otherwise stated.

1.4 Monomorphisms, epimorphisms, and isomorphisms

Definition 1.4.1 (Monomorphism)**.** A morphism $f\colon A \to B$ in a category $\mathcal{C}$ is a **monomorphism** (or is *monic*) if for every pair of morphisms $g_1, g_2\colon C \to A$,

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

We denote a monomorphism by $f\colon A \rightarrowtail B$.

Definition 1.4.2 (Epimorphism)**.** A morphism $f\colon A \to B$ is an **epimorphism** (or is *epic*) if for every pair $h_1, h_2\colon B \to D$,

$$h_1 \circ f = h_2 \circ f \implies h_1 = h_2.$$

We denote an epimorphism by $f\colon A \twoheadrightarrow B$.

Definition 1.4.3 (Isomorphism)**.** A morphism $f\colon A \to B$ is an **isomorphism** if there exists a morphism $g\colon B \to A$ such that

$$g \circ f = \mathrm{id}_A \qquad \text{and} \qquad f \circ g = \mathrm{id}_B.$$

The morphism g is unique and is called the *inverse* of f, written f^{-1}. Two objects are *isomorphic*, $A \cong B$, if there exists an isomorphism between them.

Proposition 1.4.4. *Every isomorphism is both a monomorphism and an epimorphism.*

Proof. Let $f\colon A \to B$ be an isomorphism with inverse g. If $f \circ g_1 = f \circ g_2$, apply g on the left: $g_1 = g \circ f \circ g_1 = g \circ f \circ g_2 = g_2$, so f is monic. Dually, f is epic. □

Remark 1.4.5. The converse fails in general. In **Ring**, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both monic and epic, but it is not an isomorphism.

Example 1.4.6 (Mono and epi in concrete categories)**.** (i) In **Set**: monomorphisms are injections, epimorphisms are surjections, isomorphisms are bijections.

(ii) In **Grp**: monomorphisms are injective homomorphisms, epimorphisms are surjective homomorphisms.

(iii) In **Top**: monomorphisms are injective continuous maps; epimorphisms are surjective continuous maps (but *not* necessarily quotient maps).

Lemma 1.4.7. *Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms.*

(i) If $g \circ f$ is monic, then f is monic.

(ii) If $g \circ f$ is epic, then g is epic.

(iii) If both f and g are monic (resp. epic), then so is $g \circ f$.

Proof. (i) Suppose $g \circ f$ is monic and $f \circ h_1 = f \circ h_2$. Then $g \circ f \circ h_1 = g \circ f \circ h_2$, hence $h_1 = h_2$. Parts (ii) and (iii) are similar. □

1.5 Initial and terminal objects

Definition 1.5.1 (Initial object)**.** An object $I \in \mathcal{C}$ is **initial** if for every object $A \in \mathcal{C}$ there exists a unique morphism $I \to A$.

Definition 1.5.2 (Terminal object)**.** An object $T \in \mathcal{C}$ is **terminal** if for every object $A \in \mathcal{C}$ there exists a unique morphism $A \to T$.

Definition 1.5.3 (Zero object)**.** An object that is both initial and terminal is called a **zero object**.

Proposition 1.5.4. *If an initial (resp. terminal, resp. zero) object exists, it is unique up to unique isomorphism.*

Proof. Let I and I' be initial. By the universal property there exist unique morphisms $f\colon I \to I'$ and $g\colon I' \to I$. Then $g \circ f\colon I \to I$ must equal id_I (uniqueness of the morphism $I \to I$), and similarly $f \circ g = \mathrm{id}_{I'}$. The argument for terminal objects is dual. ☐

Example 1.5.5 (Initial and terminal objects)**.**
- (i) In **Set**: the empty set $\varnothing$ is initial; any singleton $\{*\}$ is terminal.
- (ii) In **Grp** and **Ab**: the trivial group $\{e\}$ is a zero object.
- (iii) In **Ring**: $\mathbb{Z}$ is initial (unique ring homomorphism $\mathbb{Z} \to R$ sending $1 \mapsto 1_R$); the zero ring $\{0\}$ is terminal.
- (iv) In **Top**: the empty space is initial; any one-point space is terminal.
- (v) In $\mathbf{Vect}_{\mathbb{K}}$: the zero space $\{0\}$ is a zero object.

Remark 1.5.6. If $\mathcal{C}$ has a zero object 0, then for any objects A, B there is a unique morphism $A \to 0 \to B$, called the *zero morphism* and denoted 0_{AB} (or simply 0).

1.6 Functors

Definition 1.6.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **(covariant) functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

(i) A map on objects: $A \mapsto F(A)$ for each $A \in \mathrm{Ob}(\mathcal{C})$.

(ii) A map on morphisms: for each $f\colon A \to B$ in $\mathcal{C}$, a morphism $F(f)\colon F(A) \to F(B)$ in $\mathcal{D}$.

These must satisfy:

(F1) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for every object A.

(F2) $F(g \circ f) = F(g) \circ F(f)$ for every composable pair f, g.

Definition 1.6.2 (Contravariant functor)**.** A **contravariant functor** $F\colon \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Equivalently, F reverses the direction of morphisms: $F(f)\colon F(B) \to F(A)$ when $f\colon A \to B$, and $F(g \circ f) = F(f) \circ F(g)$.

Example 1.6.3 (Forgetful functors)**.** (i) $U\colon \mathbf{Grp} \to \mathbf{Set}$: sends a group to its underlying set and a homomorphism to the underlying function.

(ii) $U\colon \mathbf{Top} \to \mathbf{Set}$: forgets the topology.

(iii) $U\colon \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Ab}$: forgets scalar multiplication, retaining only the additive group.

(iv) $U\colon \mathbf{Ring} \to \mathbf{Ab}$: sends a ring to its underlying additive group.

Example 1.6.4 (Free functors)**.** (i) $F\colon \mathbf{Set} \to \mathbf{Grp}$: sends a set S to the free group on S.

(ii) $F\colon \mathbf{Set} \to \mathbf{Vect}_{\mathbb{K}}$: sends a set S to the vector space with basis S.

(iii) $F\colon \mathbf{Set} \to \mathbf{Ab}$: sends a set S to the free abelian group $\mathbb{Z}^{(S)} = \bigoplus_{s \in S} \mathbb{Z}$.

Example 1.6.5 (Power-set functor)**.** There are two natural functors $\mathbf{Set} \to \mathbf{Set}$ associated with the power set:

(i) **Covariant:** $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ sends a set S to $\mathcal{P}(S)$ and a function $f\colon S \to T$ to the direct image $f_*\colon \mathcal{P}(S) \to \mathcal{P}(T)$, $A \mapsto f(A)$.

(ii) **Contravariant:** $\mathcal{P}^{\mathrm{op}}\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ sends $f\colon S \to T$ to the preimage $f^*\colon \mathcal{P}(T) \to \mathcal{P}(S)$, $B \mapsto f^{-1}(B)$.

Example 1.6.6 (Hom-functors)**.** For any locally small category $\mathcal{C}$ and object $A \in \mathcal{C}$:

(i) The **covariant hom-functor** $\mathrm{Hom}_{\mathcal{C}}(A, -)\colon \mathcal{C} \to \mathbf{Set}$ sends $B \mapsto \mathrm{Hom}(A, B)$ and $f\colon B \to C$ to $f_*\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$, $g \mapsto f \circ g$.

(ii) The **contravariant hom-functor** $\mathrm{Hom}_{\mathcal{C}}(-, B)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ sends $A \mapsto \mathrm{Hom}(A, B)$ and $f\colon A' \to A$ to $f^*\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B)$, $g \mapsto g \circ f$.

Example 1.6.7 (Fundamental group functor)**.** Let $\mathbf{Top}_*$ denote the category of pointed topological spaces and base-point-preserving continuous maps. The fundamental group construction defines a functor

$$\pi_1\colon \mathbf{Top}_* \longrightarrow \mathbf{Grp}, \qquad (X, x_0) \longmapsto \pi_1(X, x_0).$$

A continuous map $f\colon (X, x_0) \to (Y, y_0)$ induces the group homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $[\gamma] \mapsto [f \circ \gamma]$.

1.7 Faithful, full, and essentially surjective functors

Definition 1.7.1 (Faithful, full, fully faithful)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called:

(i) **faithful** if for every pair $A, B \in \mathcal{C}$ the map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

is injective;

(ii) **full** if every $F_{A,B}$ is surjective;

(iii) **fully faithful** if every $F_{A,B}$ is bijective.

Definition 1.7.2 (Essentially surjective)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective** (or *essentially surjective on objects*) if for every $D \in \mathcal{D}$ there exists $C \in \mathcal{C}$ with $F(C) \cong D$.

Example 1.7.3. (i) Every forgetful functor (e.g. $U\colon \mathbf{Grp} \to \mathbf{Set}$) is faithful. It is not full in general: not every function between the underlying sets of two groups is a homomorphism.

(ii) The inclusion functor $\mathbf{Ab} \hookrightarrow \mathbf{Grp}$ is fully faithful: a homomorphism between abelian groups is the same thing whether viewed in **Ab** or **Grp**.

(iii) A functor is an *equivalence of categories* if and only if it is fully faithful and essentially surjective (assuming the axiom of choice).

Proposition 1.7.4. *A fully faithful functor reflects isomorphisms: if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.*

Proof. Let $F(f)\colon F(A) \to F(B)$ be an isomorphism with inverse h. Since F is full, $h = F(g)$ for some $g\colon B \to A$. Then $F(g \circ f) = F(g) \circ F(f) = \mathrm{id}_{F(A)} = F(\mathrm{id}_A)$. Since F is faithful, $g \circ f = \mathrm{id}_A$. Similarly $f \circ g = \mathrm{id}_B$. □

1.8 Natural transformations

Definition 1.8.1 (Natural transformation)**.** Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\alpha\colon F \Rightarrow G$ is a family of morphisms in $\mathcal{D}$,

$$\{\alpha_A\colon F(A) \to G(A)\}_{A \in \mathrm{Ob}(\mathcal{C})},$$

called the *components* of α, such that for every morphism $f\colon A \to B$ in $\mathcal{C}$

the following *naturality square* commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\alpha_B"'] & G(B)
\end{tikzcd}$$

That is, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ for all $f\colon A \to B$.

Definition 1.8.2 (Natural isomorphism)**.** A natural transformation $\alpha\colon F \Rightarrow G$ is a **natural isomorphism** if every component α_A is an isomorphism. We then write $F \cong G$ and say F and G are *naturally isomorphic.*

Example 1.8.3 (Double dual)**.** Let $\mathbf{Vect}_{\mathbb{K}}^{\mathrm{fd}}$ denote the category of finite-dimensional $\mathbb{K}$-vector spaces. Define $\eta\colon \mathrm{Id} \Rightarrow (-)^{**}$ by

$$\eta_V\colon V \longrightarrow V^{**}, \qquad v \longmapsto \big(\varphi \mapsto \varphi(v)\big).$$

This is a natural isomorphism (each η_V is an isomorphism by dimension counting). Crucially, the isomorphism $V \cong V^{**}$ is *natural* in V, whereas the isomorphism $V \cong V^*$ (which requires choosing a basis) is not.

Example 1.8.4 (Determinant)**.** Let **Ring** be the category of commutative rings. Consider the functors GL_n, $(-)^\times\colon \mathbf{Ring} \to \mathbf{Grp}$ sending a commutative ring R to the group of invertible $n \times n$ matrices over R and to the group of units $R^\times$, respectively. The determinant gives a natural transformation

$$\det\colon \mathrm{GL}_n \Longrightarrow (-)^\times.$$

Naturality says: for every ring homomorphism $\varphi\colon R \to S$ and every $M \in \mathrm{GL}_n(R)$, $\varphi(\det M) = \det(\varphi(M))$.

Definition 1.8.5 (Vertical composition)**.** Let $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ be natural transformations between functors $F, G, H\colon \mathcal{C} \to \mathcal{D}$. Their **vertical composition** $\beta \circ \alpha\colon F \Rightarrow H$ has components

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A.$$

This is depicted schematically as:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{D}
\arrow[Rightarrow, from=F, to=G, "\alpha"] \arrow[Rightarrow, from=G, to=H, "\beta"]
\end{tikzcd}$$

Lemma 1.8.6. *Vertical composition is associative, and the identity natural transformation* $\mathrm{id}_F\colon F \Rightarrow F$ *(with components* $(\mathrm{id}_F)_A = \mathrm{id}_{F(A)}$*) serves as identity.*

Proof. Both statements follow immediately from the corresponding properties of morphism composition in $\mathcal{D}$. ☐

Definition 1.8.7 (Horizontal composition)**.** Let $\alpha\colon F \Rightarrow G$ between functors $\mathcal{C} \to \mathcal{D}$ and $\beta\colon H \Rightarrow K$ between functors $\mathcal{D} \to \mathcal{E}$. The **horizontal composition** (or *Godement product*) $\beta * \alpha\colon H \circ F \Rightarrow K \circ G$ has components

$$(\beta * \alpha)_A = \beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}.$$

The two expressions are equal by naturality of β:

$$\begin{tikzcd}
HF(A) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & HG(A) \arrow[d, "\beta_{G(A)}"] \\
KF(A) \arrow[r, "K(\alpha_A)"'] & KG(A)
\end{tikzcd}$$

1.9 Functor categories

Definition 1.9.1 (Functor category)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories with $\mathcal{C}$ small. The **functor category** $[\mathcal{C}, \mathcal{D}]$ (also written $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$) is the category whose:

- objects are functors $F\colon \mathcal{C} \to \mathcal{D}$;
- morphisms are natural transformations;

- composition is vertical composition of natural transformations.

Remark 1.9.2. If $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then $[\mathcal{C},\mathcal{D}]$ is locally small.

Example 1.9.3 (Presheaf categories)**.** A particularly important special case: let $\mathcal{C}$ be a small category. The functor category

$$\widehat{\mathcal{C}} \;=\; [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$

is called the **category of presheaves** on $\mathcal{C}$. Its objects are contravariant functors $\mathcal{C} \to \mathbf{Set}$, called *presheaves*. This category has remarkable properties (it is complete, cocomplete, and cartesian closed) that we shall explore later.

Proposition 1.9.4. *A natural transformation $\alpha\colon F \Rightarrow G$ is an isomorphism in $[\mathcal{C},\mathcal{D}]$ if and only if each component α_A is an isomorphism in $\mathcal{D}$.*

Proof. If α is a natural isomorphism, its inverse α^{-1} has components $(\alpha^{-1})_A = (\alpha_A)^{-1}$. One checks naturality of α^{-1} by applying $(-)^{-1}$ to the naturality squares of α. Conversely, if each α_A is an isomorphism, define $\beta_A = (\alpha_A)^{-1}$. The naturality of β follows from that of α: for $f\colon A \to B$,

$$\beta_B \circ G(f) = \beta_B \circ G(f) \circ \alpha_A \circ \beta_A = \beta_B \circ \alpha_B \circ F(f) \circ \beta_A = F(f) \circ \beta_A. \qquad \square$$

Definition 1.9.5 (Equivalence of categories)**.** An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms

$$\eta\colon \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} G \circ F \qquad \text{and} \qquad \varepsilon\colon F \circ G \overset{\sim}{\Longrightarrow} \mathrm{Id}_{\mathcal{D}}\,.$$

We write $\mathcal{C} \simeq \mathcal{D}$ and say $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*.

Theorem 1.9.6. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Proof. ($\Rightarrow$) Suppose F is an equivalence with quasi-inverse G and natural isomorphisms $\eta\colon \mathrm{Id} \overset{\sim}{\Rightarrow} GF$ and $\varepsilon\colon FG \overset{\sim}{\Rightarrow} \mathrm{Id}$.

Essentially surjective: For $D \in \mathcal{D}$, $F(G(D)) \cong D$ via ε_D.

Faithful: Suppose $F(f) = F(g)$ for $f, g\colon A \to B$. Then $GF(f) = GF(g)$. Naturality of η gives $\eta_B \circ f = GF(f) \circ \eta_A = GF(g) \circ \eta_A = \eta_B \circ g$, and since η_B is an isomorphism, $f = g$.

Full: Let $h\colon F(A) \to F(B)$. Consider $f = \eta_B^{-1} \circ G(h) \circ \eta_A\colon A \to B$. Then $F(f) = \varepsilon_{F(B)} \circ FG(h) \circ (\varepsilon_{F(A)})^{-1}$. By naturality of ε, $\varepsilon_{F(B)} \circ FG(h) = h \circ \varepsilon_{F(A)}$, so $F(f) = h$.

($\Leftarrow$) Assume F is fully faithful and essentially surjective. For each $D \in \mathcal{D}$, choose $G(D) \in \mathcal{C}$ with an isomorphism $\varepsilon_D\colon FG(D) \xrightarrow{\sim} D$. For $h\colon D \to D'$, define $G(h)$ to be the unique morphism (existing by full faithfulness) satisfying $F(G(h)) = \varepsilon_{D'}^{-1} \circ h \circ \varepsilon_D$. One verifies G is a functor and that ε and the induced η are natural isomorphisms. ☐

1.10 Exercises for Chapter 1

Exercise 1.10.1. Define a category **Rel** whose objects are sets and whose morphisms $A \to B$ are relations $R \subseteq A \times B$. What is composition? What are the identity morphisms? Show that **Rel** is indeed a category.

Exercise 1.10.2. A *groupoid* is a category in which every morphism is an isomorphism. Show that the following are groupoids:

(i) A group, viewed as a one-object category.

(ii) The *fundamental groupoid* $\Pi_1(X)$ of a topological space X: objects are points of X, morphisms $x \to y$ are homotopy classes of paths from x to y.

(iii) The *core* $\mathrm{core}(\mathcal{C})$ of any category $\mathcal{C}$: same objects, but only isomorphisms.

Exercise 1.10.3. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$ an object. The *slice category* (or *over category*) $\mathcal{C}/A$ has:

- Objects: morphisms $f\colon X \to A$ in $\mathcal{C}$.
- Morphisms: from $(f\colon X \to A)$ to $(g\colon Y \to A)$, a morphism $h\colon X \to$

Y in $\mathcal{C}$ such that $g \circ h = f$:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

Verify that $\mathcal{C}/A$ is a category. What is the terminal object? Describe $\mathbf{Set}/\{0,1\}$ explicitly.

Exercise 1.10.4. Let $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be functors. Define the *comma category* $(F \downarrow G)$ and show that slice categories and coslice categories are special cases.

Exercise 1.10.5. (i) Show that in any category, the composition of two monomorphisms is a monomorphism.

(ii) Show that in **Set**, a morphism is an epimorphism if and only if it is surjective.

(iii) Find an example of a morphism in a concrete category that is both monic and epic but not an isomorphism (beyond the example of $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in **Ring**).

Exercise 1.10.6. For any object A in a category $\mathcal{C}$, show that $\mathrm{End}_{\mathcal{C}}(A) = \mathrm{Hom}_{\mathcal{C}}(A, A)$ forms a monoid under composition. When is this monoid a group?

Exercise 1.10.7. Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be functors. Show that the composition $G \circ F\colon \mathcal{A} \to \mathcal{C}$ is again a functor. Verify that functor composition is associative and that the identity functor $\mathrm{Id}_{\mathcal{C}}$ is a two-sided identity.

Exercise 1.10.8. Let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor. Show that the following are equivalent:

(i) $\alpha\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow F$ is a natural transformation;

(ii) For every morphism $f\colon A \to B$, one has $F(f) \circ \alpha_A = \alpha_B \circ f$.

Exercise 1.10.9. Prove the *interchange law* for horizontal and vertical composition of natural transformations. Specifically, given

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "F_1"{name=U1}] \arrow[r, bend right=50, "F_2"'{name=D1}] & \mathcal{B} \arrow[r, bend left=50, "G_1"{name=U2}] \arrow[r, bend right=50, "G_2"'{name=D2}] & \mathcal{C}
\arrow[Rightarrow, from=U1, to=D1, "\alpha"]
\arrow[Rightarrow, from=U2, to=D2, "\beta"]
\end{tikzcd}$$

and natural transformations $\alpha' \colon F_2 \Rightarrow F_3$, $\beta' \colon G_2 \Rightarrow G_3$, show that

$$(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha).$$

Exercise 1.10.10. A category $\mathcal{C}$ is *skeletal* if isomorphic objects are equal. A *skeleton* of $\mathcal{C}$ is a full subcategory that is skeletal and contains exactly one object from each isomorphism class. Show that every category has a skeleton and that inclusion of the skeleton is an equivalence of categories.

Exercise 1.10.11. Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be functors, $\alpha \colon F \Rightarrow G$ a natural transformation, and $H \colon \mathcal{B} \to \mathcal{C}$ a functor. Define the *whiskering* $\alpha H \colon FH \Rightarrow GH$ by $(\alpha H)_B = \alpha_{H(B)}$. Show that αH is a natural transformation. Similarly define whiskering on the other side.

Chapter 2

Duality and the Dual Principle

One of the most powerful features of category theory is the *duality principle*: every categorical statement has a dual, obtained by reversing all morphisms. This single observation doubles the theorems one can prove “for free”. In this chapter we make this precise by introducing opposite categories, formulating the duality principle, and examining several concrete dualities. We also study the two-variable hom-functor, which will be essential in later chapters.

2.1 The opposite category

Definition 2.1.1 (Opposite category)**.** Let $\mathcal{C}$ be a category. The **opposite category** (or *dual category*) $\mathcal{C}^{\mathrm{op}}$ is defined by:

(i) $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$.

(ii) For objects A, B: $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$.

(iii) Composition in $\mathcal{C}^{\mathrm{op}}$: given $f^{\mathrm{op}}\colon A \to B$ and $g^{\mathrm{op}}\colon B \to C$ in $\mathcal{C}^{\mathrm{op}}$ (corresponding to $f\colon B \to A$ and $g\colon C \to B$ in $\mathcal{C}$), we set $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$.

(iv) The identity on A in $\mathcal{C}^{\mathrm{op}}$ is $(\mathrm{id}_A)^{\mathrm{op}} = \mathrm{id}_A$.

Proposition 2.1.2. *For any category $\mathcal{C}$, $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.*

Proof. The objects coincide. For morphisms, $\mathrm{Hom}_{(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}}(A,B) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(B,A) = \mathrm{Hom}_{\mathcal{C}}(A,B)$. Composition: $(f^{\mathrm{op}})^{\mathrm{op}} \circ (g^{\mathrm{op}})^{\mathrm{op}} = ((g^{\mathrm{op}} \circ f^{\mathrm{op}})^{\mathrm{op}}) = ((f \circ g)^{\mathrm{op}})^{\mathrm{op}} = f \circ g$. Everything reduces to the original data of $\mathcal{C}$. □

Example 2.1.3. (i) If $(P, \leq)$ is a poset viewed as a category, then P^{op} is the poset $(P, \geq)$.

(ii) If G is a group viewed as a one-object category, then G^{op} is the *opposite group*: same elements, with multiplication $a \cdot^{\mathrm{op}} b = b \cdot a$. For abelian groups, $G^{\mathrm{op}} \cong G$.

(iii) **Set**$^{\mathrm{op}}$ is a perfectly valid category, but it is not equivalent to any "familiar" category of structured sets (it is, however, equivalent to the category of complete atomic Boolean algebras, by a non-trivial theorem).

2.2 The duality principle

Definition 2.2.1 (Dual statement)**.** Let Σ be a statement formulated purely in the language of category theory (objects, morphisms, composition, identities). The **dual statement** Σ^{op} is obtained from Σ by:

- reversing the direction of every morphism;
- replacing each composition $g \circ f$ by $f \circ g$;
- interchanging domain and codomain.

Theorem 2.2.2 (Duality principle)**.** *If a statement Σ holds in every category, then so does its dual Σ^{op}. More generally, if Σ holds in a category $\mathcal{C}$, then Σ^{op} holds in $\mathcal{C}^{\mathrm{op}}$.*

Proof. A statement about $\mathcal{C}^{\mathrm{op}}$ is, by definition, a statement about $\mathcal{C}$ with all arrows reversed. Since $\mathcal{C}^{\mathrm{op}}$ is a legitimate category, any theorem valid for all categories applies to it. Unwinding the definitions in $\mathcal{C}^{\mathrm{op}}$ recovers the dual statement in $\mathcal{C}$. □

Remark 2.2.3. The duality principle immediately gives us dual pairs of concepts:

Concept	**Dual concept**
monomorphism	epimorphism
initial object	terminal object
product	coproduct
limit	colimit
kernel	cokernel
left adjoint	right adjoint

Once we prove a theorem about monomorphisms, the dual theorem about epimorphisms follows automatically.

Example 2.2.4. We showed in eflem:mono-epi-composition that if $g \circ f$ is monic then f is monic. By duality: if $g \circ f$ is epic, then g is epic. No additional proof is needed—it follows from the proof of the original statement applied to $\mathcal{C}^{\mathrm{op}}$.

2.3 Functors and duality

Proposition 2.3.1. *A contravariant functor $F \colon \mathcal{C} \to \mathcal{D}$ is the same thing as a covariant functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, and also the same as a covariant functor $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.*

Proof. A covariant functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ assigns to each morphism $f^{\mathrm{op}} \colon B \to A$ in $\mathcal{C}^{\mathrm{op}}$ (i.e. $f \colon A \to B$ in $\mathcal{C}$) a morphism $F(f^{\mathrm{op}}) \colon F(B) \to F(A)$ in $\mathcal{D}$, which is precisely a contravariant assignment. The preservation of composition and identities translates directly. The argument for $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ is analogous. ☐

Remark 2.3.2. Every functor $F \colon \mathcal{C} \to \mathcal{D}$ induces a functor $F^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ defined by $F^{\mathrm{op}}(A) = F(A)$ on objects and $F^{\mathrm{op}}(f^{\mathrm{op}}) = (F(f))^{\mathrm{op}}$ on morphisms. In fact, $(-)^{\mathrm{op}}$ is a functor **Cat** → **Cat**.

2.4 Concrete dualities

The abstract duality principle tells us about dual *concepts*, but there also exist concrete *equivalences* $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{D}$ for specific categories. These are called *concrete dualities* or *dualities of categories.*

Example 2.4.1 (Stone duality)**.** The category **Stone** of Stone spaces (compact, Hausdorff, totally disconnected topological spaces) and continuous maps is equivalent to $\mathbf{Bool}^{\mathrm{op}}$, where **Bool** is the category of Boolean algebras and their homomorphisms:

$$\mathbf{Stone} \ \simeq \ \mathbf{Bool}^{\mathrm{op}}.$$

The equivalence is implemented by the functor sending a Stone space to its Boolean algebra of clopen subsets, and conversely sending a Boolean algebra to its space of ultrafilters.

Example 2.4.2 (Pontryagin duality)**.** Let **LCA** be the category of locally compact abelian groups and continuous homomorphisms. Pontryagin duality provides a contravariant equivalence

$$(-)^{\wedge} \colon \mathbf{LCA}^{\mathrm{op}} \xrightarrow{\sim} \mathbf{LCA},$$

where $G^{\wedge} = \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{R}/\mathbb{Z})$ is the *Pontryagin dual.* The natural map $G \to G^{\wedge\wedge}$, $g \mapsto (\chi \mapsto \chi(g))$, is an isomorphism.

Example 2.4.3 (Gelfand duality)**.** There is an equivalence

$$\mathbf{CHaus}^{\mathrm{op}} \ \simeq \ \mathbf{cC^{*}Alg}_1,$$

where **CHaus** is the category of compact Hausdorff spaces and $\mathbf{cC^{*}Alg}_1$ is the category of commutative unital C^{*}-algebras. The functor sends $X \mapsto C(X, \mathbb{C})$ (continuous functions) and the quasi-inverse sends a C^{*}-algebra to its maximal ideal space (Gelfand spectrum).

Example 2.4.4 (Finite-dimensional vector space duality)**.** The dual space functor $(-)^{*} \colon \mathbf{Vect}_{\mathbb{K}}^{\mathrm{fd}} \to (\mathbf{Vect}_{\mathbb{K}}^{\mathrm{fd}})^{\mathrm{op}}$ is an equivalence of categories. Combined with $(\mathbf{Vect}_{\mathbb{K}}^{\mathrm{fd}})^{\mathrm{op}} \simeq \mathbf{Vect}_{\mathbb{K}}^{\mathrm{fd}}$ (via the double dual), we see that finite-dimensional vector spaces are self-dual.

2.5 The hom-functor in both variables

Definition 2.5.1 (Bifunctor)**.** Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be categories. A **bifunctor** is a functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$.

Remark 2.5.2. A bifunctor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ determines, for each $C \in \mathcal{C}$, a functor $F(C, -)\colon \mathcal{D} \to \mathcal{E}$ and, for each $D \in \mathcal{D}$, a functor $F(-, D)\colon \mathcal{C} \to \mathcal{E}$. Conversely, a family of functors in each variable that is "jointly functorial" determines a bifunctor.

Proposition 2.5.3. *For any locally small category $\mathcal{C}$, the assignment*

$$\mathrm{Hom}_{\mathcal{C}}(-, -)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$$

defined on objects by $(A, B) \mapsto \mathrm{Hom}_{\mathcal{C}}(A, B)$ and on morphisms by

$$(f\colon A' \to A,\ g\colon B \to B') \longmapsto \big(h \mapsto g{\circ}h{\circ}f\big)\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B'),$$

is a bifunctor (i.e. a functor from the product category $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathbf{Set}$).

Proof. We verify the functor axioms. On identities:

$$\mathrm{Hom}(\mathrm{id}_A, \mathrm{id}_B)(h) = \mathrm{id}_B \circ h \circ \mathrm{id}_A = h,$$

so $\mathrm{Hom}(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{\mathrm{Hom}(A,B)}$.

On composition: let $f'\colon A'' \to A'$, $f\colon A' \to A$, $g\colon B \to B'$, $g'\colon B' \to B''$. We need $\mathrm{Hom}(f \circ f', g' \circ g) = \mathrm{Hom}(f', g') \circ \mathrm{Hom}(f, g)$. The left side maps $h \mapsto (g' \circ g) \circ h \circ (f \circ f')$. The right side maps $h \mapsto g \circ h \circ f \mapsto g' \circ (g \circ h \circ f) \circ f'$, which is the same by associativity. □

Remark 2.5.4. The bifunctor $\mathrm{Hom}(-, -)$ encodes the two "partial" hom-functors from efex:hom-functors:

- Fixing the first argument: $\mathrm{Hom}(A, -)$ is covariant.
- Fixing the second argument: $\mathrm{Hom}(-, B)$ is contravariant (i.e. covariant from $\mathcal{C}^{\mathrm{op}}$).

The two partial functors determine the bifunctor (they agree on pairs of identities and satisfy a compatibility condition that is automatic from the bifunctor structure).

Proposition 2.5.5. *For any object A in a locally small category $\mathcal{C}$, the covariant hom-functor* $\operatorname{Hom}(A,-)\colon \mathcal{C} \to \mathbf{Set}$ *preserves all limits that exist in $\mathcal{C}$.*

Proof. This is a consequence of the universal property of limits and will be proved rigorously in efch:limits-colimits. The key idea is that a natural bijection $\operatorname{Hom}(A, \lim F) \cong \lim \operatorname{Hom}(A, F-)$ follows directly from the defining universal property of the limit. ☐

Definition 2.5.6 (Representable functor (preview))**.** A functor $F\colon \mathcal{C} \to \mathbf{Set}$ is **representable** if it is naturally isomorphic to $\operatorname{Hom}_{\mathcal{C}}(A,-)$ for some object $A \in \mathcal{C}$. The object A is called a *representing object*.
A contravariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable if $F \cong \operatorname{Hom}_{\mathcal{C}}(-,B)$ for some B.
Representable functors are central to category theory; we shall study them in depth in connection with the Yoneda lemma.

2.6 Exercises for Chapter 2

Exercise 2.6.1. Let $\mathcal{C}$ be a category.

(i) Verify carefully that $\mathcal{C}^{\mathrm{op}}$ as defined in efdef:opposite-cat satisfies the axioms of a category.

(ii) Show that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ (equality, not merely isomorphism).

Exercise 2.6.2. Show that A is an initial object in $\mathcal{C}$ if and only if A is a terminal object in $\mathcal{C}^{\mathrm{op}}$. Deduce the uniqueness (up to unique isomorphism) of terminal objects from the corresponding result for initial objects via duality.

Exercise 2.6.3. Show that f is a monomorphism in $\mathcal{C}$ if and only if f^{op} is an epimorphism in $\mathcal{C}^{\mathrm{op}}$.

Exercise 2.6.4. Let $\mathcal{C}$ be small and $\mathcal{D}$ any category. Construct a canonical isomorphism of categories

$$[\mathcal{C}, \mathcal{D}]^{\mathrm{op}} \;\cong\; [\mathcal{C}, \mathcal{D}^{\mathrm{op}}].$$

(Hint: what happens to naturality squares when you reverse arrows in the target?)

Exercise 2.6.5. A category $\mathcal{C}$ is called *self-dual* if $\mathcal{C} \simeq \mathcal{C}^{\mathrm{op}}$.

(i) Show that $\mathbf{Vect}_{\mathbb{K}}^{\mathrm{fd}}$ is self-dual.

(ii) Show that any groupoid is self-dual.

(iii) Is **Set** self-dual? Justify.

Exercise 2.6.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Define the product category $\mathcal{C} \times \mathcal{D}$ and show that

$$(\mathcal{C} \times \mathcal{D})^{\mathrm{op}} \;\cong\; \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}}.$$

Exercise 2.6.7. Let $\mathcal{C}$ be a locally small category. Give a detailed verification that $\mathrm{Hom}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is a functor by checking the functor axioms (preservation of identities and composition) explicitly.

Exercise 2.6.8. Let $F, G\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be contravariant functors. Spell out explicitly what a natural transformation $\alpha\colon F \Rightarrow G$ looks like: what are the components, and what does the naturality condition say? Draw the naturality square and compare it with the covariant case.

Exercise 2.6.9. Show that if $\mathcal{C} \simeq \mathcal{D}$, then $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{D}^{\mathrm{op}}$.

Exercise 2.6.10. Let L be a bounded lattice, viewed as a category (poset). Show that L^{op} is again a bounded lattice, with joins and meets interchanged. Describe the dual of the lattice of open sets of a topological space X.

Exercise 2.6.11. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$ an object. The *coslice category* (or *under category*) $A/\mathcal{C}$ has objects $f\colon A \to X$ and morphisms