Dynamical Systems

Lecture Notes

Master M1 — 2025–2026

Yaë Ulrich Gaba

"If we knew exactly the laws of nature,
we should need only a single moment."
— Henri Poincaré

March 25, 2026

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Preface

Motivation and Objectives

The theory of dynamical systems stands as one of the fundamental pillars of modern applied mathematics. Born from the pioneering work of Henri Poincaré on the three-body problem at the end of the nineteenth century, this discipline has experienced spectacular growth throughout the twentieth century, thanks in particular to the contributions of Aleksandr Lyapunov, George Birkhoff, Andrey Kolmogorov, Stephen Smale, and Edward Lorenz.

This course is intended for Master-level (M1–M2) students in mathematics, theoretical physics, or engineering, with a solid background in real analysis, linear algebra, and ordinary differential equations. Our objective is threefold:

1. **Master the theoretical foundations**: existence and uniqueness of solutions, Lyapunov stability, spectral theory of linear systems, center manifolds, and dimensional reduction.

2. **Understand qualitative phenomena**: limit cycles, bifurcations, deterministic chaos, strange attractors, and ergodic behavior.

3. **Develop computational skills**: numerical simulation in Python, computation of Lyapunov exponents, bifurcation diagrams, and attractor visualization.

Course Organization

The course is structured in eleven chapters, organized in a logical progression from fundamental definitions to the most advanced concepts.

Part I — Foundations (Chapters 1–3). Chapter 1 lays down the basic definitions: continuous and discrete dynamical systems, flows, orbits, and limit sets. Chapter 2 introduces the notion of Lyapunov stability and both direct and indirect methods. Chapter 3 covers the complete classification of phase portraits for planar linear systems.

Part II — Nonlinear Systems (Chapters 4–6). Chapter 4 develops linearization techniques and center manifold theory. Chapter 5 studies limit cycles and the Poincaré-Bendixson theorem. Chapter 6 presents bifurcation theory: saddle-node, pitchfork, and Hopf bifurcations.

Part III — Chaos and Ergodicity (Chapters 7–11). Chapter 7 introduces formal definitions of chaos. Chapter 8 treats Lyapunov exponents. Chapter 9 studies strange attractors, notably those of Lorenz and Hénon. Chapter 10 addresses discrete dynamical systems. Finally, Chapter 11 provides an introduction to ergodic theory.

Prerequisites

The essential prerequisites for this course are:

- **Real analysis**: topology of $\mathbb{R}^n$, uniform convergence, fixed-point theorems, parameter-dependent integrals.
- **Linear algebra**: eigenvalues, Jordan decomposition, matrix exponential, bilinear forms.
- **Ordinary differential equations**: Picard-Lindelöf theorem, maximal solutions, Grönwall's lemma.
- **Measure theory** (for Chapter 11): Borel measures, Lebesgue integration, convergence theorems.
- **Python programming** (for practical sessions): `numpy`, `scipy`, `matplotlib` libraries.

Conventions and Notation

Throughout this course, we adopt the following conventions:

- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$ denote respectively the sets of real numbers, complex numbers, natural numbers, and integers.
- $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$ (generally the Euclidean norm $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$).
- $B(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$ is the open ball centered at x_0 with radius r.
- $\dot{x}$ denotes $\frac{dx}{dt}$, the time derivative.
- $Df(x)$ or $J_f(x)$ denotes the Jacobian matrix of f at the point x.
- $\sigma(A)$ denotes the spectrum (set of eigenvalues) of matrix A.
- $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote the real and imaginary parts of $\lambda \in \mathbb{C}$.
- Theorems, definitions, and propositions are numbered by chapter.

Historical Perspective

The theory of dynamical systems has a rich and fascinating history, which we briefly summarize.

The Origins (1880–1920). Henri Poincaré, in his monumental work *Les méthodes nouvelles de la mécanique céleste* (1892–1899), laid the foundations of the qualitative approach to differential equations. He introduced the notions of phase portrait and Poincaré section, and foresaw the possibility of chaotic behavior in deterministic systems.

Stability Theory (1892–1950). Aleksandr Lyapunov developed his stability theory in his 1892 thesis, *The General Problem of the Stability of Motion*. His methods, both direct and indirect, remain among the most powerful tools of modern theory.

The Topological School (1920–1960). George Birkhoff, followed by Soviet mathematicians (Andronov, Pontryagin, Kolmogorov), developed the topological and ergodic approach. The KAM theorem (Kolmogorov-Arnold-Moser, 1954–1963) partially resolved the stability problem for Hamiltonian systems.

The Chaos Revolution (1963–present). Edward Lorenz discovered in 1963 the sensitivity to initial conditions in a simplified meteorological model. The works of Smale (horseshoe), Ruelle and Takens (strange attractors), Feigenbaum (universality), and May (population dynamics) transformed our understanding of deterministic systems.

Applications

Dynamical systems find applications in numerous domains:

- **Physics**: celestial mechanics, fluid mechanics (turbulence), laser physics, nonlinear electronic circuits.
- **Biology**: population dynamics (predator-prey models), epidemiology (SIR model), neuroscience (Hodgkin-Huxley model), circadian rhythms.
- **Chemistry**: oscillating reactions (Belousov-Zhabotinsky), enzyme kinetics.
- **Economics**: growth models, business cycles, financial market dynamics.
- **Engineering**: systems control, robotics, aeronautics (flutter), electrical networks.
- **Climatology**: climate models, weather forecasting, El Niño.

How to Use This Course

Each chapter contains:

- Rigorously stated **definitions** and **theorems**, with detailed proofs.
- **Examples** illustrating abstract concepts.
- **Remarks** providing complements or cautionary notes.
- **Exercises** of progressive difficulty.
- **Python simulations** for visualizing phenomena.

We strongly encourage the reader to:

1. Work through the proofs in detail.
2. Solve the proposed exercises.
3. Run and modify the Python codes to develop numerical intuition.
4. Establish connections between different chapters.

Main References

1. M.W. Hirsch, S. Smale, R.L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed., Academic Press, 2012.
2. S.H. Strogatz, *Nonlinear Dynamics and Chaos*, 2nd ed., Westview Press, 2015.
3. L. Perko, *Differential Equations and Dynamical Systems*, 3rd ed., Springer, 2001.
4. A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.
5. J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, 1983.
6. K.T. Alligood, T.D. Sauer, J.A. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer, 1996.

Acknowledgments

This course owes much to the classic texts cited above, as well as to numerous discussions with colleagues and students. We also thank the broader mathematical community for its rich heritage in this field, from Poincaré to the present day.

Happy reading and productive work!

Chapter 1

Dynamical Systems — Definitions and Examples

In 1889, Henri Poincaré entered a competition organized by King Oscar II of Sweden: prove the stability of the solar system. Not only did Poincaré fail to prove stability, but he discovered something far more profound: the existence of behaviors so complicated that any long-term prediction becomes impossible. This is where dynamical systems theory was born: the study of how the states of a system evolve over time, with all the richness — order, chaos, bifurcations — that this entails.

1.1 Introduction

A dynamical system is a mathematical model describing the temporal evolution of a state in a given space. This evolution may be governed by a differential equation (continuous time) or by an iterated map (discrete time). In this chapter, we lay down the fundamental definitions and illustrate these concepts through classical examples from physics, biology, and engineering.

1.2 Continuous Dynamical Systems

Definition 1.1 (Autonomous continuous dynamical system)**.** An **autonomous continuous dynamical system** on an open set $U \subseteq \mathbb{R}^n$ is defined by the ordinary differential equation

$$\dot{x} = f(x), \qquad x \in U,$$

where $f : U \to \mathbb{R}^n$ is a vector field of at least class C^1.

Definition 1.2 (Non-autonomous system)**.** A **non-autonomous dynamical system** has the form

$$\dot{x} = f(t, x), \qquad (t, x) \in \mathbb{R} \times U,$$

where f depends explicitly on time t.

Remark 1.3. Any non-autonomous system $\dot{x} = f(t, x)$ on $\mathbb{R}^n$ can be transformed into an autonomous system on $\mathbb{R}^{n+1}$ by setting $\dot{t} = 1$, yielding the augmented system:

$$\begin{pmatrix} \dot{x} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} f(t, x) \\ 1 \end{pmatrix}.$$

1.2.1 Existence and Uniqueness of Solutions

Theorem 1.4 (Picard-Lindelöf / Cauchy-Lipschitz)**.** *Let $f : U \to \mathbb{R}^n$ be locally Lipschitz on the open set $U \subseteq \mathbb{R}^n$. For every $x_0 \in U$, there exists a maximal interval $I(x_0) = (\alpha, \omega)$ with $-\infty \leq \alpha < 0 < \omega \leq +\infty$ and a unique maximal solution $\varphi(\cdot, x_0) : I(x_0) \to U$ of the initial value problem*

$$\dot{x} = f(x), \qquad x(0) = x_0.$$

Proof. The classical proof uses the Banach fixed-point theorem applied to the Picard operator

$$(T\varphi)(t) = x_0 + \int_0^t f(\varphi(s))\, ds$$

in the space $C([0, h], \overline{B}(x_0, r))$ equipped with the uniform norm, for $h > 0$ sufficiently small. The Lipschitz condition $\|f(x) - f(y)\| \leq L\, \|x - y\|$ ensures that T is a contraction for $h < 1/L$. Uniqueness on the maximal interval follows from the extension theorem and Grönwall's lemma. □

Theorem 1.5 (Behavior at boundaries of the maximal interval)**.** *Let $\varphi(\cdot, x_0) : (\alpha, \omega) \to U$ be the maximal solution. If $\omega < +\infty$, then for every compact set $K \subseteq U$, there exists $t_K < \omega$ such that $\varphi(t, x_0) \notin K$ for all $t > t_K$. In other words, the trajectory "blows up" or leaves every compact set in finite time.*

1.3 Flows and Semi-flows

Definition 1.6 (Flow)**.** The **flow** associated with the system $\dot{x} = f(x)$ is the map

$$\varphi : \mathcal{D} \subseteq \mathbb{R} \times U \to U, \qquad (t, x_0) \mapsto \varphi(t, x_0) = \varphi_t(x_0),$$

where $\mathcal{D}$ is the maximal open domain of definition. The flow satisfies:

1. $\varphi_0 = \mathrm{Id}_U$ (initial condition);
2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all admissible s, t (group property).

Proposition 1.7 (Regularity of the flow)**.** If $f \in C^k(U, \mathbb{R}^n)$ with $k \geq 1$, then the flow $\varphi : \mathcal{D} \to U$ is of class C^k in (t, x_0).

Definition 1.8 (Semi-flow)**.** A **semi-flow** is defined for $t \geq 0$ only. It satisfies the same properties as the flow, but only for $t, s \geq 0$. Parabolic partial differential equations naturally generate semi-flows.

1.4 Orbits, Trajectories, and Limit Sets

Definition 1.9 (Orbit)**.** The **orbit** (or **trajectory**) through x_0 is the set

$$\gamma(x_0) = \{\varphi(t, x_0) : t \in I(x_0)\} \subseteq U.$$

The **positive orbit** is $\gamma^+(x_0) = \{\varphi(t, x_0) : t \geq 0\}$ and the **negative orbit** is $\gamma^-(x_0) = \{\varphi(t, x_0) : t \leq 0\}$.

Definition 1.10 (Equilibrium points)**.** A point $x^* \in U$ is an **equilibrium point** (or **fixed point**, **stationary point**) of the system $\dot{x} = f(x)$ if $f(x^*) = 0$. Equivalently, $\varphi(t, x^*) = x^*$ for all t.

Definition 1.11 (Periodic orbit)**.** An orbit $\gamma(x_0)$ is **periodic** if there exists $T > 0$ such that $\varphi(T, x_0) = x_0$ and $\varphi(t, x_0) \neq x_0$ for $0 < t < T$. The smallest such T is called the **minimal period**.

Definition 1.12 (ω-limit set)**.** The **ω-limit set** of x_0 is

$$\omega(x_0) = \bigcap_{T \geq 0} \overline{\{\varphi(t, x_0) : t \geq T\}}.$$

A point y belongs to $\omega(x_0)$ if and only if there exists a sequence $t_n \to +\infty$ such that $\varphi(t_n, x_0) \to y$.

Definition 1.13 (α-limit set)**.** Analogously, the **α-limit set** is

$$\alpha(x_0) = \bigcap_{T \leq 0} \overline{\{\varphi(t, x_0) : t \leq T\}}.$$

Proposition 1.14 (Properties of limit sets)**.** Let $\gamma^+(x_0)$ be bounded. Then:

1. $\omega(x_0)$ is nonempty, compact, and connected;
2. $\omega(x_0)$ is invariant under the flow: $\varphi_t(\omega(x_0)) = \omega(x_0)$ for all t;
3. $\operatorname{dist}(\varphi(t, x_0), \omega(x_0)) \to 0$ as $t \to +\infty$.

1.5 Phase Space and Phase Portrait

Definition 1.15 (Phase portrait)**.** The **phase portrait** of a dynamical system is the partition of the phase space U into orbits. It provides a global qualitative description of the dynamics.

Remark 1.16. Two distinct orbits cannot cross (a consequence of the uniqueness of solutions). The phase space is therefore partitioned into disjoint orbits.

Example 1.17 (Simple harmonic oscillator)**.** Consider the simple harmonic oscillator

$$\ddot{q} + \omega^2 q = 0,$$

written as a first-order system with $x_1 = q$, $x_2 = \dot{q}$:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\omega^2 x_1. \end{cases}$$

The system matrix is $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$, with eigenvalues $\lambda = \pm i\omega$. The orbits are ellipses centered at the origin. The phase portrait is a **center**.

Example 1.18 (Simple pendulum)**.** The undamped simple pendulum satisfies

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

Setting $x_1 = \theta$, $x_2 = \dot{\theta}$:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{g}{l}\sin x_1. \end{cases}$$

The equilibrium points are $(\theta^*, 0)$ with $\theta^* = k\pi$, $k \in \mathbb{Z}$. Points $\theta^* = 2k\pi$ are centers (downward positions) and points $\theta^* = (2k+1)\pi$ are saddles (upward positions).

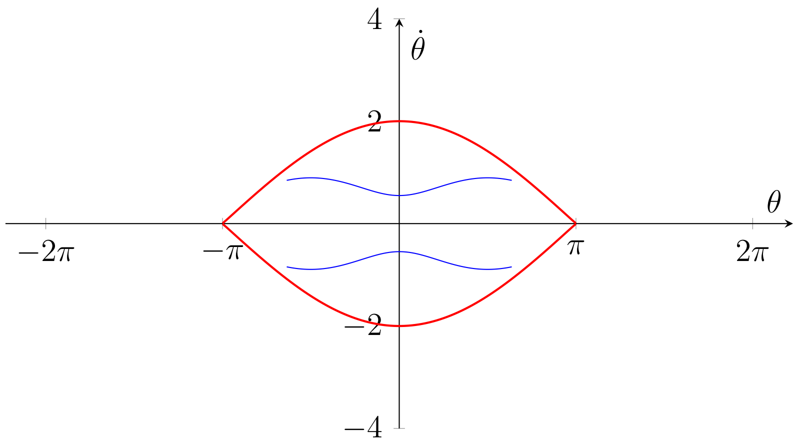
Intuition

The phase portrait of the pendulum reveals two types of motion:

- Closed trajectories near centers correspond to oscillations (librations);
- Open trajectories far from centers correspond to full rotations of the pendulum.

The trajectories separating these two regions are the **separatrices**, which connect the saddle points.

Phase portrait of the simple pendulum



1.6 Discrete Dynamical Systems

Definition 1.19 (Discrete dynamical system)**.** A **discrete dynamical system** is defined by the iteration of a map $f : U \to U$:

$$x_{n+1} = f(x_n), \qquad n \in \mathbb{N}.$$

The orbit of x_0 is the sequence $(x_0, f(x_0), f^2(x_0), \dots)$ where $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$.

Definition 1.20 (Fixed and periodic points)**.** A point x^* is a **fixed point** of f if $f(x^*) = x^*$. A point x^* is **periodic of period** p if $f^p(x^*) = x^*$ and $f^k(x^*) \neq x^*$ for $1 \leq k < p$.

Example 1.21 (Logistic map)**.** The **logistic map** is defined by

$$f_r(x) = rx(1-x), \qquad x \in [0,1], \quad r \in [0,4].$$

The fixed points satisfy $x = rx(1-x)$, giving $x^* = 0$ and $x^* = 1 - 1/r$ (for $r > 1$).

1.7 First Integrals and Conservative Systems

Definition 1.22 (First integral)**.** A function $H : U \to \mathbb{R}$ of class C^1 is a **first integral** (or **constant of motion**) of the system $\dot{x} = f(x)$ if H is constant along every orbit:

$$\frac{d}{dt} H(\varphi(t, x_0)) = \nabla H(\varphi(t, x_0)) \cdot f(\varphi(t, x_0)) = 0 \quad \forall t.$$

Theorem 1.23 (Dimensional reduction)**.** *If H is a first integral such that $\nabla H(x) \neq 0$ on a level set $\Sigma_c = \{x \in U : H(x) = c\}$, then Σ_c is an $(n-1)$-dimensional submanifold invariant under the flow. The dynamics thus reduces to the study of a system on Σ_c.*

Definition 1.24 (Hamiltonian system)**.** A **Hamiltonian system** on $\mathbb{R}^{2n}$ is defined by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \dots, n,$$

where $H : \mathbb{R}^{2n} \to \mathbb{R}$ is the **Hamiltonian function** (energy). The Hamiltonian H is automatically a first integral.

Example 1.25 (Pendulum as a Hamiltonian system)**.** The simple pendulum (Example 1.18) is Hamiltonian with

$$H(\theta, p) = \frac{p^2}{2} - \frac{g}{l} \cos\theta,$$

where $p = \dot{\theta}$. The level curves of H coincide with the orbits of the phase portrait.

1.8 Dissipative Systems and Liouville's Volume

Theorem 1.26 (Liouville's formula)**.** *Let $\Omega(t) = \varphi_t(\Omega_0)$ be the image of a measurable domain $\Omega_0 \subseteq U$ under the flow. Then*

$$\frac{d}{dt} \mathrm{Vol}(\Omega(t)) = \int_{\Omega(t)} \operatorname{div} f(x)\, dx.$$

Corollary 1.27. *If $\operatorname{div} f(x) = 0$ for all $x \in U$, the flow preserves volume (**incompressible** system). This holds for every Hamiltonian system.*

Definition 1.28 (Dissipative system)**.** A system is **dissipative** if $\operatorname{div} f(x) < 0$ on U. In this case, volumes in phase space contract over time.

Example 1.29 (Damped oscillator)**.** The damped harmonic oscillator

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\omega^2 x_1 - 2\gamma x_2, \quad \gamma > 0,$$

has divergence $\operatorname{div} f = -2\gamma < 0$. It is a dissipative system. Volumes contract at the rate $e^{-2\gamma t}$.

1.9 Famous Dynamical Systems

Example 1.30 (Lotka-Volterra system)**.** The Lotka-Volterra predator-prey model is

$$\dot{x} = \alpha x - \beta xy, \qquad \dot{y} = \delta xy - \gamma y,$$

with $\alpha, \beta, \gamma, \delta > 0$. The nontrivial equilibrium point is $(\gamma/\delta, \alpha/\beta)$. This system possesses the first integral $H(x, y) = \delta x - \gamma \ln x + \beta y - \alpha \ln y$.

Example 1.31 (Lorenz system)**.** The Lorenz system, introduced in 1963, is

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \rho x - y - xz, \qquad \dot{z} = xy - \beta z,$$

with classical parameter values $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. The divergence is $\operatorname{div} f = -\sigma - 1 - \beta < 0$: the system is dissipative.

Simulation of the Lorenz system

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

def lorenz(t, state, sigma=10, rho=28, beta=8/3):
    x, y, z = state
    return [sigma*(y - x), rho*x - y - x*z, x*y - beta*z]

sol = solve_ivp(lorenz, [0, 50], [1, 1, 1], max_step=0.01,
                dense_output=True)
t = np.linspace(0, 50, 10000)
xyz = sol.sol(t)

fig = plt.figure(figsize=(10, 7))
ax = fig.add_subplot(111, projection='3d')
ax.plot(xyz[0], xyz[1], xyz[2], lw=0.5, color='steelblue')
ax.set_xlabel('x'); ax.set_ylabel('y'); ax.set_zlabel('z')
ax.set_title("Lorenz Attractor")
plt.tight_layout()
plt.savefig("lorenz_attractor.pdf")
plt.show()
```

1.10 Topological Equivalence and Conjugacy

Definition 1.32 (Topological equivalence)**.** Two dynamical systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are **topologically equivalent** if there exists a homeomorphism $h : U \to V$ that maps orbits of f to orbits of g while preserving orientation (direction of traversal).

Definition 1.33 (Topological conjugacy)**.** Two systems are **topologically conjugate** if the change of variable h also respects the time parameterization: $h \circ \varphi_t^f = \varphi_t^g \circ h$ for all t.

Remark 1.34. Topological conjugacy is stronger than topological equivalence. Conjugacy preserves the periods of periodic orbits, unlike mere equivalence.

1.11 Exercises

Exercise 1.1 (Equilibrium points). Find all equilibrium points of the system

$$\dot{x} = y - x^2, \qquad \dot{y} = x - 2.$$

Determine the nature of each equilibrium point by computing the Jacobian matrix and its eigenvalues.

Exercise 1.2 (First integral). Show that $H(x, y) = x^2 + y^2 - 2\ln(1 + x^2 + y^2)$ is a first integral of the system

$$\dot{x} = -y\,\frac{x^2 + y^2}{1 + x^2 + y^2}, \qquad \dot{y} = x\,\frac{x^2 + y^2}{1 + x^2 + y^2}.$$

Exercise 1.3 (Explicit flow). Compute explicitly the flow of the linear system $\dot{x} = Ax$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Verify the group property $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

Exercise 1.4 (Dissipative system). Compute the divergence of the Rössler system

$$\dot{x} = -y - z, \qquad \dot{y} = x + ay, \qquad \dot{z} = b + z(x - c),$$

and determine for which parameter values a, b, c the system is dissipative.

Exercise 1.5 (Logistic map). For the logistic map $f_r(x) = rx(1 - x)$:

1. Find the fixed points and study their stability as a function of r.
2. Find the period-2 orbits and study their stability.
3. Show that for $r > 4$, the orbit of almost every initial point in $[0, 1]$ escapes to $-\infty$.

Exercise 1.6 (Volume and Liouville). Consider the system $\dot{x} = f(x)$ on $\mathbb{R}^3$ with $f(x, y, z) = (y, -x - \epsilon y, z - \epsilon z^3)$.

1. Compute $\operatorname{div} f$.
2. For which values of ϵ is the system dissipative? Conservative?
3. Qualitatively describe the behavior of volumes over time.

Chapter 2

Lyapunov Stability

2.1 Introduction

In 1892, a young Russian mathematician named Aleksandr Mikhailovich Lyapunov defended his doctoral thesis, *The General Problem of the Stability of Motion*, at the University of Kharkov. The question he posed was disarmingly simple: if a system sits at equilibrium and we nudge it slightly, does it return, drift away, or do something else entirely? The answer, Lyapunov showed, does not require solving the differential equations explicitly. Instead, one can construct an auxiliary function—an "energy-like" quantity—and study how it evolves along trajectories.

This idea, now called the *direct method* or *second method* of Lyapunov, was revolutionary. It bypassed the hopeless task of finding closed-form solutions and replaced it with a conceptual argument: if you can find a function that decreases along every trajectory, the system must be converging. The catch, of course, is finding such a function—an art as much as a science, and one that remains at the frontier of research to this day. Alongside the direct method, Lyapunov also developed the *indirect method*, which exploits linearisation near the equilibrium. Together, these two approaches form the backbone of stability theory.

2.2 Definitions of Stability

Consider the autonomous system $\dot{x} = f(x)$ where $f : U \to \mathbb{R}^n$ is of class C^1 and x^* is an equilibrium point: $f(x^*) = 0$. Without loss of generality, we assume $x^* = 0$ (by translation).

Definition 2.1 (Lyapunov stability)**.** The equilibrium point $x^* = 0$ is **stable in the sense of Lyapunov** if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x_0\| < \delta \implies \|\varphi(t, x_0)\| < \epsilon \quad \forall t \geq 0.$$

Definition 2.2 (Asymptotic stability)**.** The equilibrium 0 is **asymptotically stable** if it is Lyapunov stable and there exists $\delta_0 > 0$ such that

$$\|x_0\| < \delta_0 \implies \lim_{t \to +\infty} \varphi(t, x_0) = 0.$$

Definition 2.3 (Global asymptotic stability)**.** The point 0 is **globally asymptotically stable** (GAS) if it is stable and $\lim_{t \to +\infty} \varphi(t, x_0) = 0$ for all $x_0 \in U$.

Definition 2.4 (Instability)**.** The point 0 is **unstable** if it is not Lyapunov stable.

Definition 2.5 (Exponential stability)**.** The point 0 is **exponentially stable** if there exist constants $M > 0$, $\alpha > 0$, and $\delta > 0$ such that

$$\|x_0\| < \delta \implies \|\varphi(t, x_0)\| \leq M \|x_0\| e^{-\alpha t} \quad \forall t \geq 0.$$

Remark 2.6. Exponential stability implies asymptotic stability, which implies Lyapunov stability. The converses are false in general.

Intuition

- **Lyapunov stability**: a ball at the bottom of a bowl stays near the bottom if slightly perturbed.
- **Asymptotic stability**: the ball returns to the bottom due to friction.
- **Instability**: a ball atop a hill — any perturbation drives it away.

2.3 Basin of Attraction

Definition 2.7 (Basin of attraction)**.** The **basin of attraction** of an asymptotically stable equilibrium x^* is the set

$$\mathcal{B}(x^*) = \{x_0 \in U : \lim_{t \to +\infty} \varphi(t, x_0) = x^*\}.$$

Proposition 2.8. The basin of attraction $\mathcal{B}(x^*)$ is an open set invariant under the flow, and its boundary $\partial\mathcal{B}(x^*)$ is also invariant.

2.4 Indirect Method: Linearization

Theorem 2.9 (Stability by linearization — Lyapunov indirect)**.** *Let $A = Df(0)$ be the Jacobian matrix of f at the equilibrium $x^* = 0$.*

1. *If all eigenvalues of A have strictly negative real parts ($Re(\lambda_i) < 0$ for all i), then 0 is* ***asymptotically stable****.*
2. *If at least one eigenvalue of A has strictly positive real part ($Re(\lambda_j) > 0$ for some j), then 0 is* ***unstable****.*
3. *If all eigenvalues satisfy $Re(\lambda_i) \leq 0$ with at least one eigenvalue on the imaginary axis, linearization is* ***inconclusive****: the direct method must be used.*

Proof sketch. Write $f(x) = Ax + g(x)$ with $g(x) = o(\|x\|)$ near 0. If all eigenvalues of A have negative real parts, there exists a positive definite matrix P such that $A^\top P + PA = -Q$ with Q positive definite (matrix Lyapunov equation). The function $V(x) = x^\top Px$ is then a Lyapunov function for the linearized system, and the term $g(x) = o(\|x\|)$ does not alter the conclusion in a sufficiently small neighborhood of the origin. □

Example 2.10 (Damped pendulum). The linearized damped pendulum near $\theta = 0$ gives

$$A = \begin{pmatrix} 0 & 1 \\ -g/l & -c \end{pmatrix},$$

with $c > 0$ the damping coefficient. The characteristic polynomial is $\lambda^2 + c\lambda + g/l = 0$ with discriminant $\Delta = c^2 - 4g/l$. Both eigenvalues have negative real parts (since $c > 0$ and $g/l > 0$), so the equilibrium is asymptotically stable.

2.5 Lyapunov's Direct Method

Definition 2.11 (Lyapunov function). Let $V : U \to \mathbb{R}$ be a C^1 function defined on a neighborhood of $x^* = 0$ with $V(0) = 0$. The **orbital derivative** (or Lie derivative) of V along the field f is

$$\dot{V}(x) = \nabla V(x) \cdot f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x)\, f_i(x).$$

We say that V is:

- **positive definite** if $V(x) > 0$ for all $x \neq 0$ in a neighborhood of 0;
- **positive semi-definite** if $V(x) \geq 0$;
- **negative definite** if $V(x) < 0$ for all $x \neq 0$.

Theorem 2.12 (Lyapunov stability theorem). *If there exists a positive definite function V such that $\dot{V}(x) \leq 0$ in a neighborhood of 0, then 0 is **Lyapunov stable**.*

Proof. Let $\epsilon > 0$ be small enough that $\overline{B}(0, \epsilon) \subseteq U$. Set $c = \min_{\|x\|=\epsilon} V(x) > 0$ (since V is positive definite and continuous on the compact set $\{\|x\| = \epsilon\}$). By continuity of V and $V(0) = 0$, there exists $\delta > 0$ such that $V(x) < c$ for $\|x\| < \delta$. If $\|x_0\| < \delta$, then $V(\varphi(t, x_0)) \leq V(x_0) < c$ for all $t \geq 0$ (since $\dot{V} \leq 0$). Therefore $\varphi(t, x_0)$ cannot reach the sphere $\|x\| = \epsilon$, proving $\|\varphi(t, x_0)\| < \epsilon$. □

Theorem 2.13 (Lyapunov asymptotic stability theorem). *If there exists a positive definite function V such that $\dot{V}(x) < 0$ for all $x \neq 0$ in a neighborhood of 0 (i.e., $\dot{V}$ is negative definite), then 0 is **asymptotically stable**.*

Theorem 2.14 (Barbashin-Krasovskii theorem). *If $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite, radially unbounded ($V(x) \to +\infty$ as $\|x\| \to +\infty$), and $\dot{V}(x) < 0$ for all $x \neq 0$, then 0 is **globally asymptotically stable**.*

Radially unbounded Lyapunov functions

For *global* asymptotic stability, it is essential that V be radially unbounded. Without this condition, the trajectory could escape to infinity while V remains decreasing.

2.6 LaSalle's Invariance Principle

When $\dot{V}$ is only negative semi-definite, the asymptotic stability theorem does not apply directly. LaSalle's invariance principle often allows one to conclude nonetheless.

Theorem 2.15 (LaSalle's invariance principle)**.** *Let $\Omega \subseteq U$ be a compact positively invariant set. Let $V : \Omega \to \mathbb{R}$ be of class C^1 with $\dot{V}(x) \leq 0$ on Ω. Set $S = \{x \in \Omega : \dot{V}(x) = 0\}$ and let M be the largest invariant subset contained in S. Then*

$$\omega(x_0) \subseteq M \qquad \forall x_0 \in \Omega.$$

In particular, every trajectory starting in Ω converges to M.

Proof sketch. Since V is decreasing and bounded below on the compact set Ω, $V(\varphi(t, x_0))$ converges to a limit c. The ω-limit set $\omega(x_0)$ is nonempty (since Ω is compact), invariant, and $V \equiv c$ on $\omega(x_0)$. Therefore $\dot{V} = 0$ on $\omega(x_0)$, giving $\omega(x_0) \subseteq S$. By invariance of $\omega(x_0)$, we obtain $\omega(x_0) \subseteq M$. □

Example 2.16 (Damped pendulum — LaSalle)**.** Consider the damped pendulum:

$$\dot{\theta} = \omega, \qquad \dot{\omega} = -\frac{g}{l}\sin\theta - c\omega, \quad c > 0.$$

The energy function $V(\theta, \omega) = \frac{1}{2}\omega^2 + \frac{g}{l}(1 - \cos\theta)$ satisfies

$$\dot{V} = -c\omega^2 \leq 0.$$

The set $S = \{\dot{V} = 0\} = \{\omega = 0\}$. On S, we have $\dot{\omega} = -\frac{g}{l}\sin\theta$, which vanishes only if $\theta = k\pi$. The largest invariant set in S is $M = \{(0, 0), (\pi, 0)\}$. By LaSalle, every bounded trajectory converges to M. Near the equilibrium $(0, 0)$, we obtain asymptotic stability.

2.7 Instability Theorems

Theorem 2.17 (Lyapunov instability theorem)**.** *If there exists a C^1 function V with $V(0) = 0$ such that $\dot{V}(x) > 0$ in a neighborhood of 0 (except at 0), and if in every neighborhood of 0 there exists a point x with $V(x) > 0$, then 0 is* ***unstable****.*

Theorem 2.18 (Chetaev's theorem)**.** *Let $V : U \to \mathbb{R}$ be of class C^1 with $V(0) = 0$. Suppose there exists an open set G with $0 \in \partial G$ and:*

1. *$V(x) > 0$ for all $x \in G$;*
2. *$V(x) = 0$ on $\partial G \cap U$;*
3. *$\dot{V}(x) > 0$ for all $x \in G$.*

Then 0 is ***unstable****.*

2.8 Construction of Lyapunov Functions

The main difficulty of the direct method is *finding* an appropriate Lyapunov function. Here are several strategies.

2.8.1 Quadratic Functions

For the system $\dot{x} = Ax + g(x)$ with $g(x) = o(\|x\|)$, we seek $V(x) = x^\top P x$ with P a symmetric positive definite solution of the matrix Lyapunov equation:

$$A^\top P + PA = -Q,$$

where Q is symmetric positive definite. This equation has a unique positive definite solution P if and only if A is a Hurwitz matrix (all eigenvalues with negative real parts).

Matrix Lyapunov equation

$$A^\top P + PA = -Q \qquad \Longrightarrow \qquad P = \int_0^{+\infty} e^{A^\top t}\, Q\, e^{At}\, dt.$$

This integral converges if and only if A is a Hurwitz matrix.

2.8.2 Energy Functions

For mechanical systems, the total energy (kinetic + potential) is often a good candidate. If the system is dissipative, $\dot{V}$ will be negative or semi-negative.

2.8.3 Combinations and Estimates

For more complex systems, one can construct V by linear combination of simple terms, or by the backstepping method in control theory.

2.9 Stability of Periodic Orbits

Definition 2.19 (Orbital stability)**.** A periodic orbit Γ is **orbitally stable** if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\operatorname{dist}(x_0, \Gamma) < \delta \implies \operatorname{dist}(\varphi(t, x_0), \Gamma) < \epsilon \quad \forall t \geq 0.$$

Definition 2.20 (Asymptotic orbital stability)**.** The orbit Γ is **asymptotically orbitally stable** if it is orbitally stable and $\operatorname{dist}(\varphi(t, x_0), \Gamma) \to 0$ as $t \to +\infty$ for x_0 sufficiently close to Γ.

Remark 2.21. Orbital stability is more natural than Lyapunov stability for periodic orbits, since even in the stable case, two nearby points on the orbit separate linearly in time along the orbit direction.

2.10 Lyapunov Functions and Control

Remark 2.22. In control theory, Lyapunov functions play a central role. For a controlled system $\dot{x} = f(x, u)$, one seeks a state feedback $u = k(x)$ such that a Lyapunov function V satisfies $\dot{V} < 0$ in closed loop. This is the principle of **Control Lyapunov Functions** (CLF).

 CHAPTER 2. LYAPUNOV STABILITY

2.11 Exercises

Exercise 2.1 (Stability by linearization)**.** Study the stability of the equilibrium points of the system

$$\dot{x} = y + x(1 - x^2 - y^2), \qquad \dot{y} = -x + y(1 - x^2 - y^2).$$

Exercise 2.2 (Lyapunov function construction)**.** For the system $\dot{x} = -x + 2x^2 y$, $\dot{y} = -y$, show that $V(x, y) = x^2 + y^2$ is a Lyapunov function in a suitable neighborhood of the origin and conclude regarding stability.

Exercise 2.3 (LaSalle)**.** Apply LaSalle's invariance principle to the system

$$\dot{x} = y, \qquad \dot{y} = -x^3 - y^3,$$

with the function $V(x, y) = \frac{x^4}{4} + \frac{y^2}{2}$.

Exercise 2.4 (Instability via Chetaev)**.** Show that the origin is unstable for the system

$$\dot{x} = y + x^3, \qquad \dot{y} = x + y^3,$$

using the function $V(x, y) = xy$.

Exercise 2.5 (Matrix Lyapunov equation)**.** Solve the equation $A^\top P + PA = -I$ for $A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$. Verify that P is positive definite.

Exercise 2.6 (Global stability)**.** Show that the origin is globally asymptotically stable for $\dot{x} = -x^3$, $\dot{y} = -y$ using $V(x, y) = \frac{x^4}{4} + \frac{y^2}{2}$.

Chapter 3

Linear Systems and Phase Portrait Classification

3.1 Introduction

Before confronting the whirlwinds of chaos and the complexity of nonlinear systems, one must master the linear case—the one where everything can be computed explicitly. The phase portrait of a planar linear system is entirely determined by the eigenvalues of its matrix: stable or unstable nodes, spiralling foci, elliptic centres, saddle points. This classification, inherited from the work of Poincaré and his student Ivar Bendixson at the end of the nineteenth century, constitutes the visual dictionary that every dynamicist consults first.

But the linear system is more than a pedagogical exercise. Thanks to the Hartman-Grobman theorem, the linearised phase portrait near a hyperbolic equilibrium faithfully reproduces the local behaviour of the nonlinear system. This is why linear systems constitute the fundamental building block of dynamical systems theory. This chapter develops the complete classification of phase portraits for planar systems $\dot{x} = Ax$, $x \in \mathbb{R}^2$.

3.2 General Solution and Matrix Exponential

Theorem 3.1 (Solution of the linear system)**.** *The solution of $\dot{x} = Ax$ with initial condition $x(0) = x_0$ is*

$$x(t) = e^{At}\, x_0,$$

where the ***matrix exponential*** *is defined by*

$$e^{At} = \sum_{k=0}^{+\infty} \frac{(At)^k}{k!} = I + At + \frac{A^2t^2}{2!} + \cdots$$

Proposition 3.2 (Properties of e^{At})**.**

1. $e^{A\cdot 0} = I$;
2. $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$;
3. $e^{A(t+s)} = e^{At}\, e^{As}$;
4. $(e^{At})^{-1} = e^{-At}$;
5. If P is invertible, $e^{(P^{-1}AP)t} = P^{-1}e^{At}P$;

6. $\det(e^{At}) = e^{\operatorname{tr}(A)\,t}$.

Computing e^{At} for $A \in \mathbb{R}^{2\times 2}$

If A has eigenvalues λ_1, λ_2:

- **Distinct real eigenvalues**: $e^{At} = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} I + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} A$.
- **Repeated eigenvalue** λ: $e^{At} = e^{\lambda t}(I + (A - \lambda I)t)$ if $A \neq \lambda I$.
- **Complex eigenvalues** $\lambda = \alpha \pm i\beta$: $e^{At} = e^{\alpha t}\left(\cos(\beta t)\, I + \frac{\sin(\beta t)}{\beta}(A - \alpha I)\right)$.

3.3 Classification in $\mathbb{R}^2$

Consider the system $\dot{x} = Ax$ where $A \in \mathbb{R}^{2\times 2}$ and $\det(A) \neq 0$ (the origin is the unique equilibrium). The relevant invariants are:

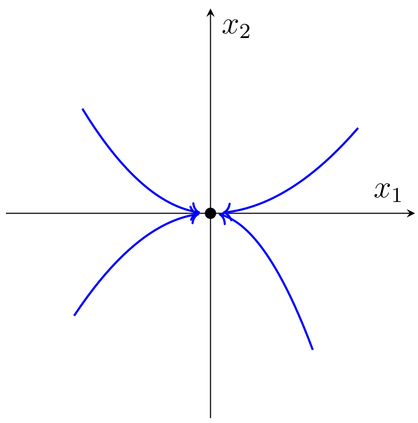
- $\tau = \operatorname{tr}(A) = \lambda_1 + \lambda_2$;
- $\delta = \det(A) = \lambda_1 \lambda_2$;
- $\Delta = \tau^2 - 4\delta$ (discriminant of the characteristic polynomial).

3.3.1 Stable Node ($\lambda_1 < \lambda_2 < 0$)

Definition 3.3 (Stable node)**.** The phase portrait is a **stable node** when both eigenvalues are real and strictly negative. All orbits converge to the origin, tangent to the eigenspace of the eigenvalue with largest modulus.

Conditions: $\delta > 0$, $\tau < 0$, $\Delta > 0$.

Stable node



3.3.2 Unstable Node ($0 < \lambda_1 < \lambda_2$)

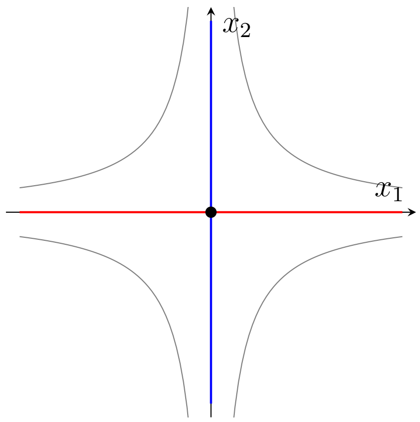
Definition 3.4 (Unstable node)**.** The phase portrait is symmetric to the stable node: orbits diverge from the origin. Conditions: $\delta > 0$, $\tau > 0$, $\Delta > 0$.

3.3.3 Saddle Point ($\lambda_1 < 0 < \lambda_2$)

Definition 3.5 (Saddle point)**.** The phase portrait is a **saddle point** when the eigenvalues have opposite signs. Orbits approach the origin along the stable eigenspace and recede along the unstable eigenspace.

Conditions: $\delta < 0$.

Saddle point



3.3.4 Stable Focus ($\lambda = \alpha \pm i\beta$, $\alpha < 0$)

Definition 3.6 (Stable focus)**.** The phase portrait is a **stable focus** (or stable spiral) when the eigenvalues are complex conjugates with negative real part. Orbits spiral toward the origin.

Conditions: $\Delta < 0$, $\tau < 0$.

3.3.5 Unstable Focus ($\alpha > 0$)

Symmetric portrait: diverging spirals. Conditions: $\Delta < 0$, $\tau > 0$.

3.3.6 Center ($\lambda = \pm i\beta$, $\alpha = 0$)

Definition 3.7 (Center)**.** The phase portrait is a **center** when the eigenvalues are purely imaginary. Orbits are closed ellipses around the origin. The equilibrium is stable but not asymptotically stable.

Conditions: $\Delta < 0$, $\tau = 0$.

Centers and nonlinear perturbations

A linear center is *structurally unstable*: an arbitrarily small nonlinear perturbation can turn it into a focus (stable or unstable). This is a critical case for linearization.

3.3.7 Degenerate Node and Star Node

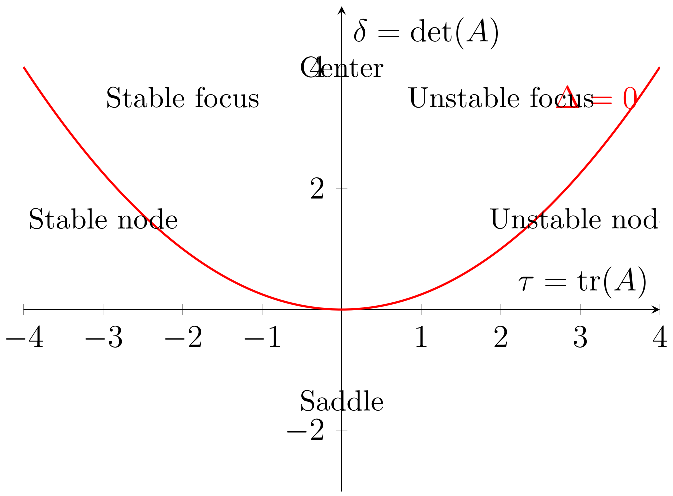
Definition 3.8 (Star node)**.** If $A = \lambda I$ ($\lambda < 0$), every line through the origin is an orbit: this is a **stable star node**.

Definition 3.9 (Degenerate node)**.** If A has a double eigenvalue $\lambda < 0$ with only one eigenvector (Jordan block), the phase portrait is a **stable degenerate node**. All orbits are tangent to the eigenspace.

3.4 The (τ, δ) Diagram

The complete classification is summarized in the (τ, δ) plane:

Phase portrait classification



Theorem 3.10 (Complete classification in dimension 2)**.** *Let* $A \in \mathbb{R}^{2\times 2}$ *with* $\det(A) \neq 0$. *The nature of the phase portrait of* $\dot{x} = Ax$ *is entirely determined by* $\tau = \operatorname{tr}(A)$ *and* $\delta = \det(A)$*:*

Type	***Conditions***	***Stability***
Stable node	$\delta > 0,\ \tau < 0,\ \Delta \geq 0$	*Asymp. stable*
Unstable node	$\delta > 0,\ \tau > 0,\ \Delta \geq 0$	*Unstable*
Saddle	$\delta < 0$	*Unstable*
Stable focus	$\Delta < 0,\ \tau < 0$	*Asymp. stable*
Unstable focus	$\Delta < 0,\ \tau > 0$	*Unstable*
Center	$\Delta < 0,\ \tau = 0$	*Stable (not asymp.)*

3.5 Linear Systems in Higher Dimensions

3.5.1 Decomposition into Invariant Subspaces

Theorem 3.11 (Spectral decomposition)**.** *Let* $A \in \mathbb{R}^{n\times n}$. *The space* $\mathbb{R}^n$ *decomposes as a direct sum of invariant subspaces:*

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c,$$

where:

- E^s *(**stable** subspace): generalized eigenvalues with real part* < 0*;*
- E^u *(**unstable** subspace): generalized eigenvalues with real part* > 0*;*

- E^c *(**center** subspace): generalized eigenvalues with real part* $= 0$.

Proposition 3.12 (Asymptotic behavior).
1. For $x_0 \in E^s$: $\left\|e^{At}x_0\right\| \to 0$ exponentially as $t \to +\infty$.
2. For $x_0 \in E^u$: $\left\|e^{At}x_0\right\| \to +\infty$ exponentially as $t \to +\infty$.
3. For $x_0 \in E^c$: $\left\|e^{At}x_0\right\|$ grows at most polynomially.

3.5.2 Jordan Form and Exponential

Proposition 3.13 (Exponential of a Jordan block)**.** If $J_k(\lambda) = \lambda I_k + N_k$ is a Jordan block of size k, then

$$e^{J_k(\lambda)t} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{k-2}}{(k-2)!} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & t \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

3.6 Non-autonomous Linear Systems

Definition 3.14 (Fundamental matrix)**.** For the system $\dot{x} = A(t)\,x$, a **fundamental matrix** is a matrix $\Phi(t) \in \mathbb{R}^{n\times n}$ whose columns form a fundamental set of solutions. It satisfies $\dot{\Phi}(t) = A(t)\,\Phi(t)$ and $\det(\Phi(t)) \neq 0$.

Theorem 3.15 (Liouville-Jacobi formula)**.** $\det(\Phi(t)) = \det(\Phi(t_0))\,\exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))\,ds\right)$.

3.7 Applications

Example 3.16 (RLC circuit)**.** A series RLC circuit is governed by

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0,$$

yielding with $x_1 = q$, $x_2 = \dot{q}$:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}.$$

Depending on the values of R, L, C:

- $R^2 < 4L/C$: stable focus (damped oscillations);
- $R^2 = 4L/C$: degenerate node (critical damping);
- $R^2 > 4L/C$: stable node (overdamping).

Phase portraits for different matrices

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
```

```
matrices = {
    "Stable node": np.array([[-2, 0], [0, -1]]),
    "Saddle": np.array([[-1, 0], [0, 1]]),
    "Stable focus": np.array([[-0.5, 2], [-2, -0.5]]),
    "Center": np.array([[0, 1], [-1, 0]])
}

fig, axes = plt.subplots(2, 2, figsize=(10, 10))
for ax, (name, A) in zip(axes.flat, matrices.items()):
    for angle in np.linspace(0, 2*np.pi, 20, endpoint=False):
        x0 = 2 * np.array([np.cos(angle), np.sin(angle)])
        sol = solve_ivp(lambda t, x: A @ x, [0, 10], x0,
                        max_step=0.05)
        ax.plot(sol.y[0], sol.y[1], 'b-', lw=0.5)
    ax.set_title(name)
    ax.set_xlim(-3, 3); ax.set_ylim(-3, 3)
    ax.set_aspect('equal'); ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.savefig("phase_portraits_linear.pdf")
plt.show()
```

3.8 Exercises

Exercise 3.1 (Matrix exponential)**.** Compute e^{At} for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (Jordan block). Sketch the phase portrait.

Exercise 3.2 (Classification)**.** Classify the phase portrait of $\dot{x} = Ax$ for

$$A = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix}$$

as a function of the parameter $a \in \mathbb{R}$.

Exercise 3.3 (Dimension 3)**.** Find the stable, unstable, and center subspaces for $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

Exercise 3.4 (Periodic system)**.** Consider the periodic system $\dot{x} = A(t)x$ with $A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}$ where $\omega(t) = \omega_0(1 + \epsilon \cos(t))$. Discuss stability using Floquet theory.

Exercise 3.5 (Topological equivalence)**.** Show that two stable nodes with distinct eigenvalues are topologically equivalent but not necessarily C^1-conjugate.

Exercise 3.6 (Stability diagram)**.** For the two-parameter system $A = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}$, draw in the (a, b) plane the regions corresponding to different types of phase portraits.

Chapter 4

Nonlinear Systems — Linearization and Center Manifold

We now leave the reassuring world of linear systems and enter the infinitely richer realm of nonlinear dynamics. Here, surprises abound: a pendulum can oscillate or spin, a population can explode or collapse, an electrical circuit can generate self-sustained oscillations. The Hartman-Grobman theorem (1960) assures us that near a *hyperbolic* equilibrium, the nonlinear phase portrait qualitatively resembles the linearised one. But what happens when eigenvalues fall on the imaginary axis? This is where *centre manifold theory*, developed by Victor Pliss (1964) and Jack Carr (1981), isolates the essential dynamics on a submanifold of minimal dimension.

4.1 Introduction

Nonlinear systems exhibit a phenomenological richness far beyond linear systems: multiple equilibria, limit cycles, chaos. To understand *local* behavior near an equilibrium, we use linearization (Hartman-Grobman theorem) and, in degenerate cases, center manifold theory.

4.2 The Hartman-Grobman Theorem

Theorem 4.1 (Hartman-Grobman)**.** *Let $\dot{x} = f(x)$ with $f \in C^1$, $f(0) = 0$, and $A = Df(0)$. If A is* ***hyperbolic*** *(no eigenvalues on the imaginary axis), then there exists a homeomorphism h defined in a neighborhood of 0 that conjugates the flow of $\dot{x} = f(x)$ to the linear flow $\dot{y} = Ay$:*

$$h(\varphi_t(x)) = e^{At}\, h(x)$$

for t sufficiently small and x in a neighborhood of 0.

Remark 4.2. The Hartman-Grobman theorem guarantees that the local phase portrait topology is entirely determined by the linearization when the equilibrium is hyperbolic. The homeomorphism h is generally not differentiable.

Definition 4.3 (Hyperbolic equilibrium)**.** An equilibrium x^* is **hyperbolic** if the Jacobian matrix $Df(x^*)$ has no eigenvalue with zero real part.

Definition 4.4 (Non-hyperbolic equilibrium)**.** An equilibrium is **non-hyperbolic** (or **degenerate**) if at least one eigenvalue of $Df(x^*)$ has zero real part. In this case, linearization does not determine stability: additional tools are needed.

4.3 Local Invariant Manifolds

Theorem 4.5 (Stable and unstable manifolds)**.** *Let $x^* = 0$ be a hyperbolic equilibrium of $\dot{x} = f(x)$ with $A = Df(0)$. Let E^s and E^u be the stable and unstable subspaces of A. Then there exist local manifolds:*

- $W^s_{\text{loc}}(0)$*: **local stable manifold**, tangent to E^s at 0, of dimension $\dim E^s$;*
- $W^u_{\text{loc}}(0)$*: **local unstable manifold**, tangent to E^u at 0, of dimension $\dim E^u$.*

These manifolds are of class C^k if $f \in C^k$ and are locally invariant under the flow.

Definition 4.6 (Global manifolds)**.** The **global stable and unstable manifolds** are defined by:

$$W^s(0) = \{x \in U : \varphi(t, x) \to 0 \text{ as } t \to +\infty\},$$
$$W^u(0) = \{x \in U : \varphi(t, x) \to 0 \text{ as } t \to -\infty\}.$$

Example 4.7 (Saddle in dimension 2)**.** For the system $\dot{x} = -x + x^2$, $\dot{y} = y$, the origin is a saddle with $E^s = \{y = 0\}$, $E^u = \{x = 0\}$. The local stable manifold is the curve $y = 0$, $|x| < 1$, and the unstable manifold is the $x = 0$ axis.

4.4 Center Manifold Theory

When the equilibrium is non-hyperbolic, the essential dynamics is captured by the **center manifold**.

Theorem 4.8 (Existence of the center manifold)**.** *Let $\dot{x} = f(x)$ with $f(0) = 0$ and $A = Df(0)$. Suppose A has n_s eigenvalues with negative real part, n_u with positive real part, and n_c with zero real part, where $n_s + n_u + n_c = n$. Then there exists a local manifold $W^c_{\text{loc}}(0)$:*

- *of dimension n_c;*
- *tangent to E^c at 0;*
- *locally invariant under the flow;*
- *of class C^k if $f \in C^k$ (but not necessarily analytic even if f is analytic).*

Non-uniqueness of the center manifold

Unlike stable and unstable manifolds, the center manifold is generally **not unique**. Two different center manifolds may agree to all orders but differ by exponentially small terms.

Theorem 4.9 (Reduction to the center manifold)**.** *If $n_u = 0$ (no unstable direction), the stability of the equilibrium $x^* = 0$ for the full system is determined by the stability of 0 in the dynamics restricted to the center manifold W^c.*

4.4.1 Practical Computation of the Center Manifold

Consider the system in canonical form:

$$\dot{x} = Cx + F(x, y), \qquad \dot{y} = Sy + G(x, y),$$

where $x \in \mathbb{R}^{n_c}$, $y \in \mathbb{R}^{n_s}$, C has all eigenvalues on the imaginary axis, S has all eigenvalues with negative real part, and F, G are at least second order.

The center manifold is the graph of a function $y = h(x)$ with $h(0) = 0$, $Dh(0) = 0$. The function h satisfies the partial differential equation:

$$Dh(x)\,[Cx + F(x, h(x))] = Sh(x) + G(x, h(x)).$$

In practice, one expands h in a Taylor series and identifies coefficients order by order.

Example 4.10 (Center manifold in dimension 2)**.** Consider the system

$$\dot{x} = xy, \qquad \dot{y} = -y + x^2.$$

The origin is a non-hyperbolic equilibrium: $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, $E^c = \{y = 0\}$, $E^s = \{x = 0\}$.

We seek $y = h(x) = ax^2 + bx^3 + \cdots$ Substituting into the center manifold equation:

$$h'(x) \cdot x\,h(x) = -h(x) + x^2.$$

At order 2: $0 = -ax^2 + x^2$, so $a = 1$. At order 3: the left side contributes $h'(x) \cdot x \cdot h(x) = 2x \cdot x \cdot x^2 + \cdots = 2x^4 + \cdots$, which is order 4. So $b = 0$ at order 3.

The dynamics on the center manifold is:

$$\dot{x} = x \cdot h(x) = x \cdot x^2 + \cdots = x^3 + \cdots$$

Since $\dot{x} = x^3 > 0$ for $x > 0$ and $\dot{x} < 0$ for $x < 0$, the equilibrium is **unstable**.

4.5 Normal Forms

Definition 4.11 (Poincaré normal form)**.** A **normal form** is a simplification of the nonlinear system obtained through successive polynomial changes of variables that eliminate non-resonant terms.

Theorem 4.12 (Normal form theorem)**.** *Let $\dot{x} = Ax + f_2(x) + f_3(x) + \cdots$ with f_k homogeneous of degree k. For each $k \geq 2$, there exists a polynomial change of variable $x = y + \phi_k(y)$ that eliminates the non-resonant terms of order k in f_k.*

Definition 4.13 (Resonance)**.** A monomial $x^m = x_1^{m_1} \cdots x_n^{m_n}$ in the j-th component is **resonant** if

$$\langle m, \lambda \rangle - \lambda_j = 0,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the eigenvalue vector and $\langle m, \lambda \rangle = m_1\lambda_1 + \cdots + m_n\lambda_n$.

4.6 Smooth Linearization

Theorem 4.14 (Sternberg)**.** *If $x^* = 0$ is hyperbolic and the eigenvalues of $Df(0)$ satisfy no resonance relation, then there exists a local C^∞ diffeomorphism conjugating the nonlinear system to the linear system $\dot{y} = Ay$.*

Remark 4.15. Sternberg's theorem improves upon Hartman-Grobman by providing a diffeomorphism (not merely a homeomorphism) under non-resonance conditions.

4.7 Application: Population Dynamics

Example 4.16 (Competition between two species)**.** The Lotka-Volterra competition model is

$$\dot{x}_1 = x_1(1 - x_1 - \alpha x_2), \qquad \dot{x}_2 = x_2(1 - x_2 - \beta x_1),$$

with $\alpha, \beta > 0$. The equilibrium points are $(0,0)$, $(1,0)$, $(0,1)$ and, if $\alpha \neq 1$ and $\beta \neq 1$, the coexistence point

$$x^* = \left(\frac{1-\alpha}{1-\alpha\beta}, \frac{1-\beta}{1-\alpha\beta} \right).$$

Linearization at each equilibrium determines the local dynamics.

Stable and unstable manifolds of a saddle

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

def system(t, state):
    x, y = state
    return [x - x**2, -y + x*y]

fig, ax = plt.subplots(figsize=(8, 6))

# Stable manifold (integrate backward from near saddle)
for eps in [0.01, -0.01]:
    sol = solve_ivp(lambda t, s: [-v for v in system(t, s)],
                    [0, 5], [0, eps], max_step=0.01)
    ax.plot(sol.y[0], sol.y[1], 'b-', lw=2,
            label='$W^s$' if eps > 0 else '')

# Unstable manifold (integrate forward)
for eps in [0.01, -0.01]:
    sol = solve_ivp(system, [0, 5], [eps, 0], max_step=0.01)
    ax.plot(sol.y[0], sol.y[1], 'r-', lw=2,
            label='$W^u$' if eps > 0 else '')

ax.set_xlim(-1, 2); ax.set_ylim(-1, 2)
ax.set_xlabel('$x$'); ax.set_ylabel('$y$')
ax.legend(); ax.grid(True, alpha=0.3)
ax.set_title("Stable and unstable manifolds")
plt.tight_layout()
plt.savefig("stable_unstable_manifolds.pdf")
plt.show()
```

4.8 Exercises

Exercise 4.1 (Hartman-Grobman)**.** Verify that the equilibrium $(0,0)$ of the system $\dot{x} = -x + y^2$, $\dot{y} = -2y + x^2$ is hyperbolic. What is the type of the local phase portrait?

Exercise 4.2 (Center manifold)**.** Compute the center manifold and determine the stability of the origin for

$$\dot{x} = -x^2 y, \qquad \dot{y} = -y + x^2.$$

Exercise 4.3 (Transcritical bifurcation)**.** Study the system $\dot{x} = \mu x - x^2$, $\dot{y} = -y$ near $(\mu, x, y) = (0, 0, 0)$ by center manifold reduction.

Exercise 4.4 (Normal form)**.** Put the system $\dot{x} = -y + ax^2 + bxy$, $\dot{y} = x + cx^2 + dxy$ into normal form up to order 3.

Exercise 4.5 (Global manifolds)**.** Numerically plot the stable and unstable manifolds of the equilibrium $(1, 0)$ for $\dot{x} = x(1 - x) - xy$, $\dot{y} = y(x - 1)$.

Exercise 4.6 (Non-uniqueness)**.** Show that for the system $\dot{x} = x^2$, $\dot{y} = -y$, the center manifold is not unique by exhibiting two functions $h_1(x)$ and $h_2(x)$ satisfying the center manifold equation.

CHAPTER 4. NONLINEAR SYSTEMS — LINEARIZATION AND CENTER MANIFOLD

Chapter 5

Limit Cycles and the Poincaré-Bendixson Theorem

The heart beats. An electronic circuit oscillates. A predator-prey population fluctuates periodically. These phenomena of self-sustained oscillation are ubiquitous in nature and technology. Mathematically, they correspond to *limit cycles*: isolated periodic orbits toward which neighbouring trajectories converge (or diverge). The Poincaré-Bendixson theorem (1901) is the fundamental result guaranteeing the existence of such cycles in planar systems: if a trajectory remains confined to a compact region with no equilibrium point, it must converge to a limit cycle. This result, specific to dimension 2, has no analogue in higher dimensions—and that is where chaos enters the picture.

5.1 Introduction

Limit cycles are isolated periodic orbits in phase space — closed orbits that do not belong to a continuous family of closed orbits. They constitute one of the most important phenomena unique to nonlinear systems, since linear systems can only produce centers (continuous families of closed orbits). The Poincaré-Bendixson theorem provides a powerful criterion for proving the existence of limit cycles in the plane.

5.2 Definitions

Definition 5.1 (Limit cycle)**.** A **limit cycle** is a periodic orbit Γ that is the ω-limit set or α-limit set of at least one orbit not belonging to Γ.

Definition 5.2 (Types of limit cycles)**.** A limit cycle Γ is:

- **stable** (attracting) if $\Gamma = \omega(x_0)$ for all x_0 in a neighborhood of Γ;
- **unstable** (repelling) if $\Gamma = \alpha(x_0)$ for all x_0 in a neighborhood of Γ;
- **semi-stable** if Γ is attracting on one side and repelling on the other.

Example 5.3 (Limit cycle in the Van der Pol oscillator)**.** The Van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \qquad \mu > 0,$$

possesses a unique stable limit cycle. For small μ, this cycle is close to the circle of radius 2 centered at the origin. For large μ, the cycle takes on a characteristic **relaxation oscillation** shape.

5.3 Non-existence Criterion: Bendixson-Dulac

Theorem 5.4 (Bendixson criterion)**.** *Let $D \subseteq \mathbb{R}^2$ be a simply connected domain. If* $\operatorname{div} f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ *has constant sign (and is not identically zero) on D, then the system $\dot{x} = f(x)$ has* ***no periodic orbit*** *entirely contained in D.*

Proof. By Green's theorem, if Γ were a periodic orbit enclosing a domain Ω, we would have

$$\iint_\Omega \operatorname{div} f \, dx_1 \, dx_2 = \oint_\Gamma (f_1 \, dx_2 - f_2 \, dx_1) = 0,$$

since on Γ, $dx_i = f_i \, dt$, and the line integral simplifies. But if $\operatorname{div} f$ has constant sign and is not identically zero, the double integral is nonzero: contradiction. □

Theorem 5.5 (Dulac criterion)**.** *The same result holds if* $\operatorname{div} f$ *is replaced by* $\operatorname{div}(\rho f)$ *where $\rho : D \to \mathbb{R}$ is a strictly positive C^1 function (Dulac function).*

Example 5.6 (Lotka-Volterra system)**.** For $\dot{x} = x(\alpha - \beta y)$, $\dot{y} = y(\delta x - \gamma)$ in the first quadrant, taking $\rho(x, y) = \frac{1}{xy}$ gives $\operatorname{div}(\rho f) = 0$, which is inconclusive. However, the existence of a first integral shows that orbits are closed (center) and not limit cycles.

5.4 The Poincaré-Bendixson Theorem

Theorem 5.7 (Poincaré-Bendixson)**.** *Let $\dot{x} = f(x)$ be a C^1 system on an open subset of $\mathbb{R}^2$. Let $\gamma^+(x_0)$ be a bounded positive semi-orbit contained in a compact set K with no equilibrium points. Then $\omega(x_0)$ is a periodic orbit (and hence a limit cycle if $x_0 \notin \omega(x_0)$).*

Remark 5.8. More generally, if K contains finitely many equilibria, the ω-limit set $\omega(x_0)$ is either:

1. an equilibrium point;
2. a periodic orbit;
3. a set consisting of equilibria connected by heteroclinic or homoclinic connections.

Dimension 2 only

The Poincaré-Bendixson theorem is specific to dimension 2. In dimension $n \geq 3$, bounded ω-limit sets can be far more complicated (strange attractors, for instance).

Corollary 5.9 (Existence of limit cycles)**.** *Let $K \subset \mathbb{R}^2$ be a compact, positively invariant, annular set (without holes) containing no equilibrium points. Then K contains at least one limit cycle.*

Intuition

Strategy for proving the existence of a limit cycle:

1. Construct an annular region K that is positively invariant (the vector field points inward on the boundary).
2. Verify that there are no equilibrium points in K.

3. Conclude by Poincaré-Bendixson.

5.5 Poincaré Map and Multipliers

Definition 5.10 (Poincaré section)**.** A **Poincaré section** is a curve Σ transverse to the vector field, i.e., $f(x) \cdot n(x) \neq 0$ for all $x \in \Sigma$, where n is a normal vector to Σ.

Definition 5.11 (Poincaré map)**.** The **Poincaré map** (or first return map) $P : \Sigma \to \Sigma$ maps a point $x \in \Sigma$ to the next intersection of its orbit with Σ.

Proposition 5.12. A periodic orbit Γ corresponds to a fixed point of P. The stability of Γ is determined by $|P'(x^*)|$:

- $|P'(x^*)| < 1$: stable cycle;
- $|P'(x^*)| > 1$: unstable cycle;
- $|P'(x^*)| = 1$: indeterminate case.

Definition 5.13 (Floquet multiplier)**.** For a periodic orbit of period T of the system $\dot{x} = f(x)$ in dimension n, the **Floquet multipliers** are the eigenvalues of the monodromy matrix

$$M = \Phi(T),$$

where $\Phi(t)$ solves $\dot{\Phi} = Df(\gamma(t))\,\Phi$, $\Phi(0) = I$. One multiplier always equals 1 (tangent direction). The orbit is stable if all other multipliers have modulus < 1.

5.6 Index Theory

Definition 5.14 (Index of a closed curve)**.** Let γ be a simple closed curve in $\mathbb{R}^2$ not passing through any equilibrium. The **index of** γ with respect to the field f is the number of turns made by $f(x)$ as x traverses γ counterclockwise:

$$I_\gamma = \frac{1}{2\pi} \oint_\gamma d\theta, \qquad \theta = \arctan \frac{f_2(x)}{f_1(x)}.$$

Theorem 5.15 (Properties of the index)**.** *1. If γ encloses no equilibria, $I_\gamma = 0$.*

2. The index of a node or focus is $+1$.

3. The index of a saddle is -1.

4. The index of a periodic orbit is $+1$.

5. The index is additive: if γ encloses several equilibria, $I_\gamma = \sum I_{x_i^*}$.

Corollary 5.16. *Inside a limit cycle, the sum of indices of equilibria equals $+1$. In particular, a limit cycle must enclose at least one equilibrium.*

5.7 Detailed Examples

Example 5.17 (Van der Pol oscillator — existence of the cycle)**.** Consider the system in Liénard coordinates:

$$\dot{x} = y - F(x), \qquad \dot{y} = -x,$$

with $F(x) = \mu(x^3/3 - x)$, $\mu > 0$. The unique equilibrium is the origin (unstable focus for $\mu > 0$). We construct an annular region:

- Inner boundary: small circle of radius r (field points outward since the equilibrium is unstable);
- Outer boundary: energy level curve chosen large enough (the system is globally dissipative).

By Poincaré-Bendixson, a limit cycle exists in this region.

Van der Pol limit cycle

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

def van_der_pol(t, state, mu=1.0):
    x, y = state
    return [y, mu*(1 - x**2)*y - x]

fig, axes = plt.subplots(1, 3, figsize=(15, 5))
for ax, mu in zip(axes, [0.5, 1.0, 5.0]):
    for r in [0.1, 0.5, 1.0, 3.0, 5.0]:
        for theta in np.linspace(0, 2*np.pi, 4, endpoint=False):
            x0 = [r*np.cos(theta), r*np.sin(theta)]
            sol = solve_ivp(van_der_pol, [0, 50], x0,
                            args=(mu,), max_step=0.05)
            ax.plot(sol.y[0], sol.y[1], 'b-', lw=0.3, alpha=0.5)
    ax.set_title(f"$\\mu = {mu}$")
    ax.set_xlabel("$x$"); ax.set_ylabel("$\\dot{x}$")
    ax.set_xlim(-6, 6); ax.set_ylim(-8, 8)
    ax.set_aspect('auto'); ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.savefig("van_der_pol_cycles.pdf")
plt.show()
```

Example 5.18 (FitzHugh-Nagumo model)**.** The simplified FitzHugh-Nagumo neuron model is

$$\dot{v} = v - \frac{v^3}{3} - w + I, \qquad \dot{w} = \epsilon(v + a - bw),$$

with $\epsilon \ll 1$. This system possesses a limit cycle for certain values of the injected current I, modeling the action potential.

5.8 Liénard's Theorem

Theorem 5.19 (Liénard)**.** *Consider the Liénard equation* $\ddot{x} + f(x)\dot{x} + g(x) = 0$ *with* $F(x) = \int_0^x f(s)\,ds$*. Under the conditions:*

1. f *and* g *continuous,* g *odd with* $xg(x) > 0$ *for* $x \neq 0$*;*
2. F *odd,* $F(x) \to +\infty$ *as* $x \to +\infty$*;*
3. F *is negative for* $0 < x < a$*, positive and increasing for* $x > a$ *(for some* $a > 0$*);*

then the system has a ***unique stable limit cycle****.*

5.9 Exercises

Exercise 5.1 (Bendixson)**.** Show that the system $\dot{x} = x + e^{-y}$, $\dot{y} = y + e^x$ has no periodic orbit in $\mathbb{R}^2$.

Exercise 5.2 (Poincaré-Bendixson)**.** Prove the existence of a limit cycle for the system $\dot{r} = r(1 - r^2) + \epsilon r \cos\theta$, $\dot{\theta} = 1$ for sufficiently small ϵ.

Exercise 5.3 (Index)**.** Compute the index of the origin for the system $\dot{x} = x^2 - y^2$, $\dot{y} = 2xy$.

Exercise 5.4 (Poincaré map)**.** For the system in polar coordinates $\dot{r} = r(1 - r)(2 - r)$, $\dot{\theta} = 1$, find all limit cycles and determine their stability.

Exercise 5.5 (Van der Pol)**.** Show by the averaging method that for $\mu \ll 1$, the limit cycle of the Van der Pol oscillator is close to the circle of radius 2.

Exercise 5.6 (Uniqueness)**.** Let $\dot{x} = y$, $\dot{y} = -x + y(1 - x^2 - 3y^2)$. Show that there exists a unique limit cycle using Liénard's theorem.

Chapter 6

Bifurcations

Imagine a physical system that depends on a parameter: the load on a bridge, the temperature of a fluid, the birth rate of a population. As the parameter varies, the system's behaviour changes gradually—until a critical threshold where everything shifts: a stable equilibrium disappears, a limit cycle appears, the number of solutions changes abruptly. This sudden qualitative change is a *bifurcation*, and its classification—saddle-node, transcritical, pitchfork, Hopf—constitutes one of the most beautiful chapters in dynamical systems theory, inherited from Poincaré's work and systematised by René Thom in his *catastrophe theory* (1972).

6.1 Introduction

A **bifurcation** occurs when the qualitative structure of a dynamical system's phase portrait changes as a parameter varies. This chapter studies the most fundamental local bifurcations: saddle-node, transcritical, pitchfork, and Hopf.

6.2 General Framework

Definition 6.1 (Bifurcation)**.** Consider the parametric system $\dot{x} = f(x, \mu)$, where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^p$ is a parameter vector. A value μ_0 is a **bifurcation value** if the phase portrait of $f(\cdot, \mu)$ is not topologically equivalent for μ near μ_0 on either side.

Definition 6.2 (Local bifurcation)**.** A bifurcation is **local** if it is associated with the change of stability of an equilibrium point, i.e., an eigenvalue of the Jacobian crossing the imaginary axis.

Definition 6.3 (Codimension)**.** The **codimension** of a bifurcation is the minimum number of independent parameters needed to observe it generically.

6.3 Saddle-Node Bifurcation

Definition 6.4 (Saddle-node bifurcation)**.** The **saddle-node bifurcation** (or **fold**) is the creation or annihilation of two equilibrium points as a parameter crosses a critical value.

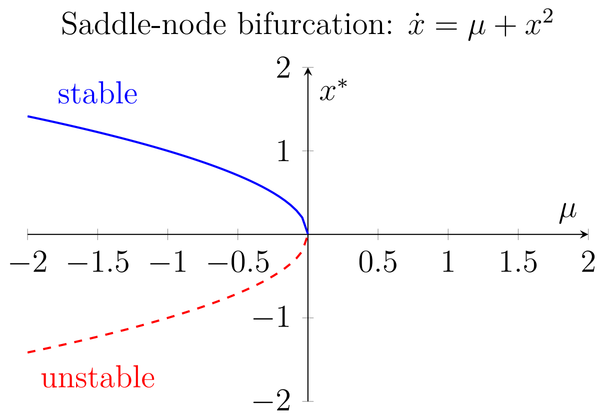
Theorem 6.5 (Saddle-node normal form). *The normal form of the saddle-node bifurcation in dimension 1 is*

$$\dot{x} = \mu + x^2.$$

- *For $\mu < 0$: two equilibria $x^* = \pm\sqrt{-\mu}$ (one stable, one unstable).*
- *For $\mu = 0$: one semi-stable equilibrium $x^* = 0$.*
- *For $\mu > 0$: no equilibria.*

Theorem 6.6 (Saddle-node conditions). *Let $\dot{x} = f(x, \mu)$ with $f(x_0, \mu_0) = 0$. The saddle-node bifurcation occurs at (x_0, μ_0) if:*

1. $\frac{\partial f}{\partial x}(x_0, \mu_0) = 0$ *(zero eigenvalue);*
2. $\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$ *(transversality);*
3. $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0$ *(non-degeneracy).*



6.4 Transcritical Bifurcation

Definition 6.7 (Transcritical bifurcation). In a **transcritical bifurcation**, two equilibria exchange their stability at the bifurcation point, without creation or disappearance.

Theorem 6.8 (Transcritical normal form). *The normal form is $\dot{x} = \mu x - x^2$.*

- *For $\mu < 0$: $x^* = 0$ stable, $x^* = \mu$ unstable.*
- *For $\mu > 0$: $x^* = 0$ unstable, $x^* = \mu$ stable.*

The two equilibrium branches cross at $\mu = 0$ and exchange stability.

Example 6.9 (Logistic model with constant harvesting). The model $\dot{N} = rN(1-N/K)-H$ exhibits a saddle-node bifurcation as H increases: beyond a critical threshold, the population collapses.

6.5 Pitchfork Bifurcation

Definition 6.10 (Pitchfork bifurcation)**.** The **pitchfork bifurcation** occurs in systems possessing a symmetry $x \mapsto -x$. An equilibrium loses stability and gives birth to two new equilibria.

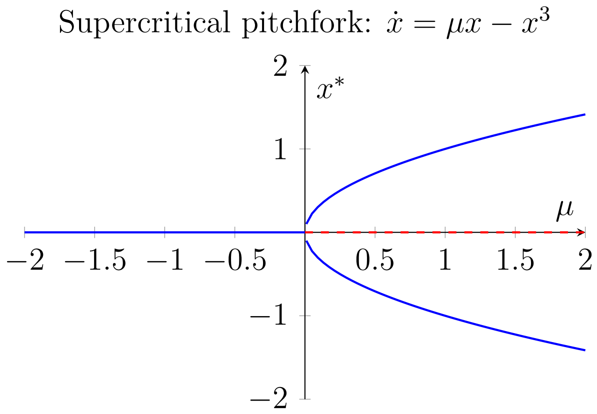
Theorem 6.11 (Supercritical pitchfork)**.** *The normal form is* $\dot{x} = \mu x - x^3$.

- *For* $\mu \leq 0$*: only* $x^* = 0$ *exists, stable.*
- *For* $\mu > 0$*:* $x^* = 0$ *unstable, two new stable equilibria* $x^* = \pm\sqrt{\mu}$.

Theorem 6.12 (Subcritical pitchfork)**.** *The normal form is* $\dot{x} = \mu x + x^3$.

- *For* $\mu < 0$*:* $x^* = 0$ *stable, two unstable equilibria* $x^* = \pm\sqrt{-\mu}$.
- *For* $\mu \geq 0$*: only* $x^* = 0$ *exists, unstable.*

The subcritical bifurcation is ***dangerous*** *because the equilibrium abruptly loses stability with no local stable alternative.*



Example 6.13 (Euler buckling)**.** The deflection of a beam under axial load P is governed by

$$EI\frac{d^2\theta}{ds^2} + P\sin\theta = 0.$$

Linearized, this yields a supercritical pitchfork bifurcation at $P_c = \pi^2 EI/L^2$ (Euler critical load).

6.6 Hopf Bifurcation

Definition 6.14 (Hopf bifurcation)**.** The **Hopf bifurcation** occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis. It gives birth to a limit cycle (or destroys one).

Theorem 6.15 (Hopf)**.** *Let* $\dot{x} = f(x, \mu)$ *with* $f(0, \mu) = 0$ *for all* μ *and* $A(\mu) = D_x f(0, \mu)$ *having eigenvalues* $\lambda(\mu) = \alpha(\mu) \pm i\beta(\mu)$*. Suppose:*

1. $\alpha(0) = 0$, $\beta(0) = \omega_0 \neq 0$ *(eigenvalues on the imaginary axis);*

2. *$\alpha'(0) \neq 0$ (transversality condition: eigenvalues cross the imaginary axis with nonzero speed).*

Then:

- ***Supercritical Hopf** (first Lyapunov coefficient $\ell_1 < 0$): a **stable** limit cycle is born for $\mu > 0$ (the equilibrium becomes unstable);*
- ***Subcritical Hopf** ($\ell_1 > 0$): an **unstable** limit cycle exists for $\mu < 0$ (the equilibrium is still stable), and disappears at $\mu = 0$.*

First Lyapunov coefficient

For the system $\dot{z} = (\alpha + i\omega)z + c_1 z|z|^2 + \cdots$ in complex coordinates, the first Lyapunov coefficient is $\ell_1 = \mathrm{Re}(c_1)$. If $\ell_1 < 0$, the bifurcation is supercritical.

Example 6.16 (Stuart-Landau oscillator)**.** The model $\dot{z} = (\mu + i\omega)z - z|z|^2$ in polar coordinates (r, θ) gives

$$\dot{r} = \mu r - r^3, \qquad \dot{\theta} = \omega.$$

For $\mu > 0$, a stable limit cycle of radius $r = \sqrt{\mu}$ exists. This is the prototypical supercritical Hopf bifurcation.

Bifurcation diagram and Hopf

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

def hopf_system(t, state, mu):
    x, y = state
    r2 = x**2 + y**2
    return [mu*x - y - x*r2, x + mu*y - y*r2]

fig, axes = plt.subplots(1, 3, figsize=(15, 5))
mus = [-0.5, 0.0, 0.5]
for ax, mu in zip(axes, mus):
    for r0 in [0.1, 0.3, 0.8, 1.5, 2.0]:
        for th in [0, np.pi/2, np.pi, 3*np.pi/2]:
            x0 = [r0*np.cos(th), r0*np.sin(th)]
            sol = solve_ivp(hopf_system, [0, 30], x0,
                            args=(mu,), max_step=0.05)
            ax.plot(sol.y[0], sol.y[1], 'b-', lw=0.3)
    ax.set_title(f"$\\mu = {mu}$")
    ax.set_xlim(-2, 2); ax.set_ylim(-2, 2)
    ax.set_aspect('equal'); ax.grid(True, alpha=0.3)
plt.suptitle("Supercritical Hopf bifurcation", fontsize=14)
plt.tight_layout()
plt.savefig("hopf_bifurcation.pdf")
plt.show()
```

6.7 Bifurcation Diagrams

Definition 6.17 (Bifurcation diagram)**.** A **bifurcation diagram** represents the equilibria (or cycle amplitudes) as a function of the parameter μ. Stable branches are drawn as solid lines, unstable branches as dashed lines.

6.8 Global Bifurcations

Definition 6.18 (Homoclinic bifurcation)**.** A **homoclinic bifurcation** occurs when a homoclinic orbit (connection from a saddle to itself) appears or disappears. It can generate a limit cycle of large period.

Definition 6.19 (SNIC bifurcation)**.** The **SNIC** (Saddle-Node on an Invariant Circle) bifurcation occurs when a saddle-node appears on a limit cycle, destroying it. The period of the cycle tends to infinity at the bifurcation.

Theorem 6.20 (Period scaling)**.**

- *SNIC bifurcation:* $T \sim \frac{1}{\sqrt{\mu-\mu_c}}$;
- *Homoclinic bifurcation:* $T \sim -\frac{1}{\lambda_s} \ln |\mu - \mu_c|$, *where* λ_s *is the stable eigenvalue of the saddle.*

6.9 Bifurcations in Higher Dimensions

Remark 6.21. In dimension $n > 2$, the same local bifurcations occur via center manifold reduction. The conditions are identical but apply to the reduced system. Specific bifurcations also appear:

- **Bogdanov-Takens** (codimension 2): double zero eigenvalue;
- **Hopf-Hopf** (codimension 2): two pairs of purely imaginary eigenvalues;
- **Cusp** (codimension 2): degeneracy of the saddle-node.

6.10 Physical Applications

Example 6.22 (Laser)**.** A semiconductor laser is modeled by rate equations:

$$\dot{n} = J - n - (1 + 2n)P, \qquad \dot{P} = \gamma \left[(1 + 2n)P - P + \beta n\right].$$

The laser transition (below to above threshold) is a transcritical bifurcation. In certain models with optical feedback, a Hopf bifurcation generates oscillations.

6.11 Exercises

Exercise 6.1 (Saddle-node in dimension 2)**.** Study the bifurcations of the system $\dot{x} = \mu - x^2$, $\dot{y} = -y$. Draw the bifurcation diagram.

Exercise 6.2 (Pitchfork with imperfection)**.** Consider $\dot{x} = h + \mu x - x^3$ with h small. Show that the imperfection h breaks the pitchfork symmetry and draw the diagram in the (μ, x^*) plane for $h > 0$.

 CHAPTER 6. BIFURCATIONS

Exercise 6.3 (Hopf in a chemical system)**.** For the Brusselator $\dot{x} = 1 - (b+1)x + ax^2y$, $\dot{y} = bx - ax^2y$, find the Hopf bifurcation value as a function of a and b.

Exercise 6.4 (Complete diagram)**.** For $\dot{x} = \mu x + x^3 - x^5$, draw the complete bifurcation diagram, identify saddle-node and pitchfork bifurcations, and determine the presence of hysteresis.

Exercise 6.5 (Hopf bifurcation — computation)**.** Compute the first Lyapunov coefficient for the system $\dot{x} = \mu x - y - x(x^2 + y^2)$, $\dot{y} = x + \mu y - y(x^2 + y^2)$ and determine whether the bifurcation is supercritical or subcritical.

Exercise 6.6 (Homoclinic bifurcation)**.** Consider the system $\dot{x} = y$, $\dot{y} = x - x^2 + \mu y$. Show that a homoclinic orbit exists for a particular value of μ and discuss the creation/destruction of a limit cycle.

Chapter 7

Chaos — Definitions and Examples

7.1 Introduction

In 1963, Edward Lorenz, a meteorologist at MIT, discovered by accident that his simplified model of atmospheric convection produced radically different results depending on whether an initial condition was rounded to three or six decimal places. This phenomenon — *sensitivity to initial conditions* — would become the symbol of deterministic chaos, popularized under the name "butterfly effect." But what exactly is chaos? The question is subtler than it appears. Robert Devaney proposed a rigorous definition in 1989 combining three ingredients: sensitivity to initial conditions, topological transitivity, and density of periodic orbits. Li and Yorke, in their celebrated 1975 paper "Period Three Implies Chaos," had given yet another. This chapter explores these definitions, their relationships, and the fundamental properties of chaotic systems — those deterministic systems that, paradoxically, produce the appearance of randomness.

7.2 Sensitivity to Initial Conditions

Definition 7.1 (Sensitivity to initial conditions)**.** A dynamical system $f : X \to X$ on a metric space (X, d) has **sensitivity to initial conditions** (SIC) if there exists $\delta > 0$ such that for every $x \in X$ and every $\epsilon > 0$, there exists y with $d(y, x) < \epsilon$ and $n > 0$ such that

$$d(f^n(x), f^n(y)) > \delta.$$

Intuition

Sensitivity to initial conditions is often called the *butterfly effect*: an infinitesimal perturbation can lead to radically different trajectories. This is the fundamental discovery of Lorenz (1963).

Remark 7.2. Sensitivity alone does not define chaos. The map $x \mapsto 2x$ on $\mathbb{R}$ is sensitive to initial conditions but is not chaotic — orbits simply escape to infinity without any recurrence.

7.3 Devaney Chaos

Definition 7.3 (Devaney chaos)**.** A continuous map $f : X \to X$ on a metric space (X, d) is **chaotic in the sense of Devaney** if:

1. **Topological transitivity**: for all non-empty open sets $U, V \subseteq X$, there exists $n \in \mathbb{N}$ with $f^n(U) \cap V \neq \emptyset$.

2. **Dense periodic orbits**: the set of periodic points is dense in X.

3. **Sensitivity to initial conditions**.

Theorem 7.4 (Banks et al., 1992)**.** *If $f : X \to X$ is topologically transitive with dense periodic orbits on an infinite metric space, then f automatically has sensitivity to initial conditions. Thus condition (3) is* ***redundant*** *in Devaney's definition.*

Proof sketch. Since periodic orbits are dense and X is infinite, there exist periodic points p, q with $d(p, q) > 0$. Setting $\delta = d(p, q)/4$, for any $x \in X$ and $\epsilon > 0$, transitivity yields iterates that visit neighbourhoods of both p and q, producing the required separation. □

7.4 Li-Yorke Chaos

Definition 7.5 (Li-Yorke pair)**.** Two points x, y form a **Li-Yorke pair** if

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \quad \text{and} \quad \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

Definition 7.6 (Li-Yorke chaos)**.** The system is **Li-Yorke chaotic** if there exists an uncountable **scrambled set** S such that every pair of distinct points in S is a Li-Yorke pair.

Theorem 7.7 (Li-Yorke, 1975)**.** *Let $f : [0, 1] \to [0, 1]$ be continuous. If f has a periodic orbit of period 3, then f has periodic orbits of every period $n \in \mathbb{N}^*$, and f is Li-Yorke chaotic.*

Remark 7.8. The famous statement "Period three implies chaos" summarises this theorem. More generally, Sharkovskii's ordering determines which periods force which others.

7.5 Sharkovskii's Theorem

Theorem 7.9 (Sharkovskii, 1964)**.** *Define the Sharkovskii ordering on $\mathbb{N}^*$:*

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd \cdots \rhd 4\cdot 3 \rhd 4\cdot 5 \rhd \cdots \rhd \cdots \rhd 8 \rhd 4 \rhd 2 \rhd 1.$$

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a periodic orbit of period m, then f has periodic orbits of every period n with $m \rhd n$. Moreover, for each m there exists a continuous map with a period-m orbit but no period-n orbit for any $n \rhd m$.

7.6 Fundamental Examples

7.6.1 The shift map

Definition 7.10 (Sequence space and shift)**.** The space of binary sequences is $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ with metric $d(s,t) = \sum_{i=0}^{\infty} |s_i - t_i| / 2^i$. The **shift** $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma(s_0, s_1, s_2, \ldots) = (s_1, s_2, s_3, \ldots)$.

Theorem 7.11. *The shift σ on Σ_2 is Devaney chaotic:*

1. *It is topologically transitive.*
2. *Periodic points are dense.*
3. *It has sensitivity to initial conditions with $\delta = 1$.*

7.6.2 The tent map

Definition 7.12 (Tent map)**.** The **tent map** is $T : [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2(1-x) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Proposition 7.13. The tent map is semi-conjugate to the shift on Σ_2 and is Devaney chaotic.

7.6.3 The logistic map at $r = 4$

Theorem 7.14. *The logistic map $f_4(x) = 4x(1-x)$ on $[0,1]$ is topologically conjugate to the tent map via $h(x) = \frac{2}{\pi} \arcsin(\sqrt{x})$. It is therefore Devaney chaotic.*

7.7 Smale's Horseshoe

Definition 7.15 (Smale horseshoe)**.** The **Smale horseshoe** is a diffeomorphism of the plane that stretches a square, folds it into a horseshoe shape, and maps it back into the original square. The maximal invariant set is a Cantor set on which the dynamics is conjugate to the shift on Σ_2.

Theorem 7.16 (Smale, 1967)**.** *On the invariant set of the horseshoe, the dynamics is:*

1. *Devaney chaotic;*
2. *uniformly hyperbolic;*
3. *structurally stable (persists under small perturbations).*

7.8 Routes to Chaos

Definition 7.17 (Period-doubling cascade)**.** The **Feigenbaum cascade** is an infinite sequence of period-doubling bifurcations $1 \to 2 \to 4 \to 8 \to \cdots$ leading to chaos.

Theorem 7.18 (Feigenbaum universality)**.** *The bifurcation values* μ_n *satisfy*

$$\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = \delta_F \approx 4.6692\ldots$$

The constant δ_F *is* ***universal****: it does not depend on the particular family of maps.*

Feigenbaum Constants

$$\delta_F \approx 4.669\,201\,609\ldots \quad \text{(parameter space scaling ratio)}$$
$$\alpha_F \approx -2.502\,907\,875\ldots \quad \text{(spatial scaling factor)}$$

Other routes to chaos:

- **Intermittency** (Pomeau-Manneville): alternation of laminar phases and chaotic bursts.
- **Quasi-periodicity** (Ruelle-Takens): destruction of an invariant torus.
- **Crisis**: collision of a chaotic attractor with a repeller.

7.9 Chaos in Continuous Systems

Minimum dimension for continuous chaos

An autonomous continuous system $\dot{x} = f(x)$ can only be chaotic in dimension $n \geq 3$. In dimension 2, the Poincaré-Bendixson theorem forbids chaos.

Example 7.19 (Mel'nikov's theorem)**.** Mel'nikov's theorem detects chaos in perturbed Hamiltonian systems. For the forced pendulum $\ddot{x} + \sin x = \epsilon(\gamma \cos \omega t - \delta \dot{x})$, the Mel'nikov function predicts the existence of transverse homoclinic orbits (hence a Smale horseshoe) for sufficiently small ϵ.

Demonstrating sensitivity to initial conditions

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

def lorenz(t, state, sigma=10, rho=28, beta=8/3):
    x, y, z = state
    return [sigma*(y - x), rho*x - y - x*z, x*y - beta*z]

x0_1 = [1.0, 1.0, 1.0]
x0_2 = [1.0 + 1e-10, 1.0, 1.0]
```

```
t_span, t_eval = [0, 50], np.linspace(0, 50, 5000)
sol1 = solve_ivp(lorenz, t_span, x0_1, t_eval=t_eval, max_step=0.01)
sol2 = solve_ivp(lorenz, t_span, x0_2, t_eval=t_eval, max_step=0.01)

fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(12, 8))
ax1.plot(sol1.t, sol1.y[0], 'b-', lw=0.5, label='IC 1')
ax1.plot(sol2.t, sol2.y[0], 'r-', lw=0.5, label='IC 2')
ax1.legend(); ax1.set_title('Sensitivity to initial conditions')

diff = np.sqrt(np.sum((sol1.y - sol2.y)**2, axis=0))
ax2.semilogy(sol1.t, diff, 'k-', lw=0.5)
ax2.set_ylabel(r'$\|\Delta x(t)\|$')
ax2.set_title('Exponential divergence')
plt.tight_layout(); plt.savefig("sensitivity.pdf")
```

7.10 Exercises

Exercise 7.1 (Shift and chaos)**.** Show that the shift σ on Σ_2 is topologically transitive by explicitly constructing a dense orbit.

Exercise 7.2 (Period three)**.** Show that $f(x) = x^2 - 7/4$ on $\mathbb{R}$ has a period-3 orbit and deduce that it is Li-Yorke chaotic.

Exercise 7.3 (Tent map periodic points)**.** Find all periodic points of period ≤ 3 of the tent map and verify that they are all unstable.

Exercise 7.4 (Logistic map)**.** For $f_r(x) = rx(1-x)$, numerically demonstrate the period-doubling cascade and estimate the Feigenbaum constant.

Exercise 7.5 (Horseshoe)**.** Describe geometrically the construction of the Smale horseshoe and show that the invariant set is a Cantor product.

Exercise 7.6 (Three-dimensional chaos)**.** Why can the Rössler system $\dot{x} = -y - z$, $\dot{y} = x + ay$, $\dot{z} = b + z(x - c)$ be chaotic despite having only three equations? Verify numerically for $a = 0.2$, $b = 0.2$, $c = 5.7$.

CHAPTER 7. CHAOS — DEFINITIONS AND EXAMPLES

Chapter 8

Lyapunov Exponents

8.1 Introduction

How does one measure chaos? Sensitivity to initial conditions is a qualitative notion: two nearby orbits diverge, but how fast? Alexander Lyapunov, in his 1892 thesis, had already introduced characteristic exponents to study the stability of differential equations. But it was in the 1960s–1970s, with the work of Valery Oseledets (the multiplicative ergodic theorem, 1968), that these exponents found their definitive formulation for general dynamical systems. The maximal Lyapunov exponent quantifies the rate of exponential divergence of nearby orbits: if it is positive, the system is chaotic. The full spectrum of exponents reveals the geometry of the attractor, distinguishing directions of expansion, contraction, and neutrality.

Intuition

Imagine two particles very close together in a flow. If they separate exponentially fast, the Lyapunov exponent is positive and the system is chaotic. If they converge, the exponent is negative and the system is stable. The exponent measures precisely this exponential rate.

8.2 Definition for Discrete Systems

Definition 8.1 (Lyapunov exponent for a one-dimensional map)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^1$ and x_0 an initial point. The **Lyapunov exponent** at x_0 is

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)| ,$$

where $x_k = f^k(x_0)$, whenever this limit exists.

Remark 8.2. By the chain rule, $|(f^n)'(x_0)| = \prod_{k=0}^{n-1} |f'(x_k)|$, so $\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln |(f^n)'(x_0)|$. The number e^λ is the average stretching factor per iteration.

Example 8.3 (Logistic map)**.** For $f_4(x) = 4x(1-x)$, the invariant measure is $\mu(\mathrm{d}x) = \frac{1}{\pi\sqrt{x(1-x)}}\mathrm{d}x$. The Lyapunov exponent is

$$\lambda = \int_0^1 \ln |4 - 8x| \, \frac{\mathrm{d}x}{\pi\sqrt{x(1-x)}} = \ln 2.$$

8.3 Higher-Dimensional Lyapunov Exponents

Definition 8.4 (Lyapunov matrix)**.** Consider $\dot{x} = f(x)$ in $\mathbb{R}^n$ with fundamental matrix $\Phi(t, x_0)$ of the variational equation $\dot{Y} = Df(\varphi(t, x_0))Y$. The **Lyapunov matrix** is

$$\Lambda(x_0) = \lim_{t\to\infty} \left[\Phi(t, x_0)^\top \Phi(t, x_0)\right]^{1/(2t)}.$$

Definition 8.5 (Lyapunov spectrum)**.** The **Lyapunov exponents** of the system at x_0 are the logarithms of the eigenvalues of $\Lambda(x_0)$, denoted

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

They measure the rates of expansion in the n principal directions.

Geometric Interpretation of the Exponents

- $\lambda_i > 0$: exponential stretching in the i-th direction.
- $\lambda_i = 0$: neither contraction nor expansion (flow direction).
- $\lambda_i < 0$: exponential contraction.
- $\lambda_1 > 0$: the system is **chaotic**.
- $\sum_{i=1}^n \lambda_i < 0$: the system is **dissipative**.

8.4 Oseledets Multiplicative Ergodic Theorem

Theorem 8.6 (Oseledets, 1968)**.** *Let $f : M \to M$ be a diffeomorphism preserving an ergodic probability measure μ, and let $A(x) = Df(x)$. Suppose $\int \ln^+ \|A(x)\| \, d\mu(x) < \infty$. Then, for μ-almost every x, there exist real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_s$ and a decomposition*

$$\mathbb{R}^n = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_s(x)$$

such that for every $v \in E_i(x) \setminus \{0\}$:

$$\lim_{n\to\pm\infty} \frac{1}{n} \ln \|Df^n(x) \cdot v\| = \lambda_i.$$

Moreover, $Df(x) \cdot E_i(x) = E_i(f(x))$ and $\dim E_i(x)$ is constant μ-almost everywhere.

Remark 8.7. Oseledets' theorem generalises Lyapunov exponents to any linear cocycle over an ergodic system. The integers $m_i = \dim E_i(x)$ are the **multiplicities** of the exponents.

8.5 Fundamental Properties

Proposition 8.8 (Exponent along the flow)**.** For an autonomous continuous system $\dot{x} = f(x)$, the flow direction $f(x)$ is always associated with a zero Lyapunov exponent. For a chaotic attractor in dimension n, the spectrum satisfies $\lambda_1 > 0 = \lambda_2 > \lambda_3 \geq \cdots \geq \lambda_n$.

Proposition 8.9 (Liouville formula)**.** The sum of Lyapunov exponents satisfies

$$\sum_{i=1}^{n} \lambda_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T \operatorname{tr}\big(Df(\varphi(t, x_0))\big)\, dt.$$

For the Lorenz system ($\sigma = 10$, $\rho = 28$, $\beta = 8/3$): $\lambda_1 + \lambda_2 + \lambda_3 = -(\sigma + 1 + \beta) \approx -13.67$.

Theorem 8.10 (Pesin's formula)**.** *Let f be a $\mathcal{C}^2$ diffeomorphism of a compact manifold preserving an ergodic measure μ absolutely continuous with respect to Lebesgue. Then*

$$h_\mu(f) = \sum_{\lambda_i > 0} m_i\, \lambda_i,$$

where the sum is over positive exponents counted with multiplicity.

Exponents and dimension

The Kaplan-Yorke formula relates Lyapunov exponents to the attractor dimension: $D_{KY} = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}$, where j is the largest integer with $\sum_{i=1}^{j} \lambda_i \geq 0$. This is conjectured to equal the information dimension.

8.6 Numerical Computation

Definition 8.11 (Benettin's algorithm)**.** The standard algorithm for computing the first p Lyapunov exponents:

1. Initialise an orthonormal basis $\{v_1, \dots, v_p\}$.
2. Simultaneously integrate the orbit and p tangent vectors over an interval τ.
3. Apply Gram-Schmidt to re-orthogonalise. Record the norms before orthogonalisation.
4. Repeat steps 2–3 for N iterations.
5. Estimate $\lambda_i \approx \frac{1}{N\tau} \sum_{k=1}^{N} \ln \left\| v_i^{(k)} \right\|$.

Lyapunov exponents of the Lorenz system

```
import numpy as np
from scipy.integrate import solve_ivp

def lorenz(t, state, sigma=10, rho=28, beta=8/3):
    x, y, z = state[:3]
    dxdt = [sigma*(y - x), rho*x - y - x*z, x*y - beta*z]
    J = np.array([[-sigma, sigma, 0],
                  [rho - z, -1, -x],
                  [y, x, -beta]])
    Y = state[3:].reshape(3, 3)
    dYdt = J @ Y
```

```
    return dxdt + dYdt.flatten().tolist()

n = 3
x0 = [1.0, 1.0, 1.0] + list(np.eye(n).flatten())
dt, N_steps = 0.01, 10000
lyap = np.zeros(n)

for i in range(N_steps):
    sol = solve_ivp(lorenz, [0, dt], x0, max_step=dt/2)
    x0[:3] = sol.y[:3, -1]
    Y = sol.y[3:, -1].reshape(n, n)
    Q, R = np.linalg.qr(Y)
    lyap += np.log(np.abs(np.diag(R)))
    x0[3:] = Q.flatten()

lyap /= (N_steps * dt)
print(f"Lyapunov exponents: {lyap}")
# Typical result: [0.91, 0.00, -14.57]
```

8.7 Applications to Chaos Detection

Proposition 8.12 (Chaos criterion)**.** A deterministic system is chaotic if and only if its largest Lyapunov exponent is strictly positive: $\lambda_1 > 0$.

Example 8.13 (Classification by Lyapunov exponents)**.** For a three-dimensional system:

- $(-, -, -)$: stable fixed point.
- $(0, -, -)$: stable limit cycle.
- $(0, 0, -)$: quasi-periodic torus T^2.
- $(+, 0, -)$: strange attractor (chaos).

Lemma 8.14 (Predictability time)**.** *The Lyapunov time $\tau_L = 1/\lambda_1$ defines the characteristic time scale beyond which prediction becomes impossible. After time t, an initial uncertainty δ_0 grows as $\delta(t) \sim \delta_0\, e^{\lambda_1 t}$.*

Example 8.15 (Weather prediction)**.** For the atmosphere, $\lambda_1 \approx 1/(2 \text{ days})$. An initial error of 10^{-6} reaches the synoptic scale (~ 1) after roughly $t = \frac{\ln(10^6)}{\lambda_1} \approx 28$ days, setting the theoretical limit of weather prediction.

8.8 Conditional Exponents and Synchronisation

Definition 8.16 (Conditional Lyapunov exponents)**.** For coupled systems $\dot{x} = f(x)$ and $\dot{y} = g(x, y)$, the **conditional Lyapunov exponents** of the slave system y are defined via $\dot{\delta y} = D_y g(x(t), y(t)) \cdot \delta y$. They determine whether synchronisation $y(t) \to x(t)$ is stable.

Theorem 8.17 (Chaotic synchronisation)**.** *Two identical coupled chaotic systems synchronise if and only if all transverse conditional Lyapunov exponents are strictly negative.*

8.9 Exercises

Exercise 8.1 (Tent map exponent)**.** Analytically compute the Lyapunov exponent of the tent map $T(x) = 1 - |2x - 1|$ on $[0, 1]$ with respect to Lebesgue measure.

Exercise 8.2 (Arnold's cat map spectrum)**.** For Arnold's cat map $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ (mod 1), compute both Lyapunov exponents and verify that their sum is zero (conservative system).

Exercise 8.3 (Kaplan-Yorke dimension)**.** For the Lorenz system with standard parameters, the exponents are approximately $\lambda_1 \approx 0.91$, $\lambda_2 = 0$, $\lambda_3 \approx -14.57$. Compute the Kaplan-Yorke dimension.

Exercise 8.4 (Numerical convergence)**.** Implement Benettin's algorithm for the Hénon map $H(x, y) = (1 - 1.4x^2 + y,\ 0.3x)$ and study the convergence of the exponents as a function of the number of iterations.

Exercise 8.5 (Pesin's formula)**.** Numerically verify Pesin's formula for Arnold's cat map by independently computing the metric entropy and the sum of positive exponents.

 CHAPTER 8. LYAPUNOV EXPONENTS

Chapter 9

Strange Attractors

9.1 Introduction

In 1971, David Ruelle and Floris Takens coined the term "strange attractor" to describe the geometric objects that govern the dynamics of chaotic systems. The Lorenz attractor, discovered in 1963, is the prototype: a butterfly-shaped set of fractal dimension ≈ 2.06 on which trajectories wind without ever retracing their path. The Hénon attractor, the Rössler attractor, and many others followed. What makes these objects "strange" is the coexistence of two contradictory behaviours: attraction (nearby orbits converge toward the attractor) and chaos (once on the attractor, orbits diverge exponentially). Geometrically, this coexistence produces a self-similar fractal structure, the result of repeated stretching and folding of phase space.

Intuition

A strange attractor works like a dough-kneading machine: it stretches nearby orbits (creating sensitivity to initial conditions) and then folds them back into a bounded volume (ensuring compactness). The result is an infinitely fine layered structure — a fractal.

9.2 Formal Definitions

Definition 9.1 (Attractor)**.** Let φ_t be a flow on $\mathbb{R}^n$. A compact invariant set A is an **attractor** if there exists an open neighbourhood $U \supset A$ (basin of attraction) such that:

1. $\varphi_t(U) \subset U$ for all $t > 0$;
2. $A = \bigcap_{t>0} \varphi_t(U)$;
3. A is topologically transitive (contains a dense orbit).

Definition 9.2 (Strange attractor)**.** An attractor is **strange** if it has a fractal structure (non-integer dimension) and exhibits sensitivity to initial conditions (at least one positive Lyapunov exponent).

CHAPTER 9. STRANGE ATTRACTORS

Strange $\neq$ chaotic

There exist strange non-chaotic attractors (fractal dimension but non-positive Lyapunov exponents) and chaotic non-strange attractors (chaos on a smooth torus). The two notions are logically independent.

9.3 The Lorenz Attractor

Definition 9.3 (Lorenz system)**.** The **Lorenz system** (1963) is

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = xy - \beta z, \end{cases}$$

with classical parameters $\sigma = 10$, $\rho = 28$, $\beta = 8/3$.

Theorem 9.4 (Tucker, 2002)**.** *For the classical parameters $(\sigma, \rho, \beta) = (10, 28, 8/3)$, the Lorenz system possesses a* ***robust strange attractor****. This result, proved using rigorous computer-assisted methods, confirmed a conjecture open since 1963.*

Proposition 9.5 (Properties of the Lorenz attractor)**.** The Lorenz attractor satisfies:

1. **Dissipativity**: $\nabla \cdot f = -(\sigma + 1 + \beta) < 0$, so volumes contract exponentially.
2. **Symmetry**: invariance under $(x, y, z) \mapsto (-x, -y, z)$.
3. **Lyapunov spectrum**: $\lambda_1 \approx 0.91$, $\lambda_2 = 0$, $\lambda_3 \approx -14.57$.
4. **Fractal dimension**: $D_{KY} \approx 2.06$.

Equilibria of the Lorenz System

$$C_0 = (0, 0, 0), \quad \text{unstable for } \rho > 1.$$

$$C_\pm = (\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1), \quad \text{unstable for } \rho > \rho_H = \sigma\frac{\sigma + \beta + 3}{\sigma - \beta - 1}.$$

9.4 The Rössler System

Definition 9.6 (Rössler system)**.** The **Rössler system** (1976) is

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c), \end{cases}$$

with typical parameters $a = 0.2$, $b = 0.2$, $c = 5.7$.

Remark 9.7. Unlike the two-lobed Lorenz attractor, the Rössler attractor has a single band structure with a simpler folding mechanism. It is often used as the minimal model for single-band chaos.

Proposition 9.8. For standard parameters, the Rössler system exhibits a period-doubling cascade as c increases: periodic orbit ($c \approx 2.5$), doubling ($c \approx 3.5$), chaos ($c \approx 4.2$), periodic window ($c \approx 5.2$), fully developed chaos ($c \approx 5.7$).

9.5 The Hénon Map

Definition 9.9 (Hénon map)**.** The **Hénon map** is the planar diffeomorphism

$$H(x, y) = (1 - ax^2 + y,\ bx),$$

with classical parameters $a = 1.4$, $b = 0.3$.

Proposition 9.10 (Properties of the Hénon map)**.**

1. The Jacobian is constant: $\det DH = -b$. The map contracts areas by a factor $|b| = 0.3$.
2. The attractor has fractal dimension $D \approx 1.26$.
3. Lyapunov exponents are $\lambda_1 \approx 0.42$ and $\lambda_2 \approx -1.62$.
4. The attractor has a Cantor structure in the transverse direction.

Theorem 9.11 (Benedicks-Carleson, 1991)**.** *For a set of parameters (a, b) of positive Lebesgue measure with b small, the Hénon map possesses a strange attractor carrying a unique ergodic SRB (Sinai-Ruelle-Bowen) measure.*

9.6 Fractal Dimension

Definition 9.12 (Box-counting dimension)**.** Let $A \subset \mathbb{R}^n$ be bounded. The **box-counting dimension** is

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{-\ln \epsilon},$$

where $N(\epsilon)$ is the minimum number of boxes of side ϵ needed to cover A.

Definition 9.13 (Hausdorff dimension)**.** The **Hausdorff dimension** of A is

$$D_H = \inf\left\{d \geq 0 : \mathcal{H}^d(A) = 0\right\},$$

where $\mathcal{H}^d(A) = \lim_{\epsilon \to 0} \inf\left\{\sum_i (\operatorname{diam} U_i)^d : A \subset \bigcup_i U_i,\ \operatorname{diam} U_i < \epsilon\right\}$.

Proposition 9.14 (Dimension inequality)**.** For any bounded set $A \subset \mathbb{R}^n$:

$$D_H(A) \leq \underline{D}_0(A) \leq \overline{D}_0(A).$$

Example 9.15 (Triadic Cantor set)**.** The triadic Cantor set has dimension $D_H = D_0 = \frac{\ln 2}{\ln 3} \approx 0.631$. At step n, there are $N(\epsilon) = 2^n$ intervals of length $\epsilon = 3^{-n}$, giving $D_0 = \lim \frac{n \ln 2}{n \ln 3}$.

Definition 9.16 (Correlation dimension)**.** The **correlation dimension** is

$$D_2 = \lim_{\epsilon \to 0} \frac{\ln C(\epsilon)}{\ln \epsilon},$$

where $C(\epsilon) = \lim_{N \to \infty} \frac{1}{N^2} \#\{(i, j) : \|x_i - x_j\| < \epsilon\}$ is the Grassberger-Procaccia correlation integral.

CHAPTER 9. STRANGE ATTRACTORS

9.7 Strange Attractors in Physical Systems

Example 9.17 (Rayleigh-Bénard convection)**.** The Lorenz system is a truncation of the Navier-Stokes equations for Rayleigh-Bénard convection between two heated horizontal plates. The parameter ρ corresponds to the normalised Rayleigh number.

Example 9.18 (Chua's circuit)**. Chua's circuit** is a simple electronic circuit (capacitors, inductor, nonlinear resistor) that produces a double-scroll strange attractor. It is the first physical system for which chaos was rigorously proved.

Visualising the Lorenz attractor

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

def lorenz(t, state, sigma=10, rho=28, beta=8/3):
    x, y, z = state
    return [sigma*(y - x), rho*x - y - x*z, x*y - beta*z]

sol = solve_ivp(lorenz, [0, 100], [1, 1, 1],
                max_step=0.01, t_eval=np.linspace(0, 100, 100000))
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')
ax.plot(sol.y[0], sol.y[1], sol.y[2], lw=0.3, color='darkblue')
ax.set_xlabel('$x$'); ax.set_ylabel('$y$'); ax.set_zlabel('$z$')
ax.set_title('Lorenz Attractor')
plt.savefig("lorenz_attractor.pdf")
```

9.8 Exercises

Exercise 9.1 (Dissipativity)**.** Show that the Lorenz system is dissipative and that any initial volume V_0 evolves as $V(t) = V_0 e^{-(\sigma+1+\beta)t}$. Deduce the existence of a zero-Lebesgue-measure attractor.

Exercise 9.2 (Box-counting dimension)**.** Compute the box-counting dimension of $\{0\} \cup \{1/n : n \in \mathbb{N}^*\}$ and of the Menger sponge.

Exercise 9.3 (Hénon attractor)**.** Write a program that plots the Hénon attractor for $a = 1.4$, $b = 0.3$ by iterating 10^6 times. Estimate its correlation dimension using the Grassberger-Procaccia method.

Exercise 9.4 (Poincaré section)**.** For the Rössler system, plot the Poincaré section in the plane $z = z_{\max}/2$ and show that it resembles a one-dimensional map.

Exercise 9.5 (Structural stability)**.** Is the Lorenz attractor structurally stable? Discuss by comparing with Smale's horseshoe and justify the notion of a robust attractor (in the sense of Morales-Pacífico-Pujals).

Chapter 10

Discrete Dynamical Systems

10.1 Introduction

Discrete dynamical systems, defined by the iteration of a map $x_{n+1} = f(x_n)$, provide the simplest framework for studying chaos. This chapter explores the fundamental phenomena: period-doubling cascades, universality, symbolic dynamics, and topological entropy.

Intuition

Even a map as simple as $f(x) = rx(1-x)$ generates, as r increases, the full complexity of chaos: bifurcations, self-similarity, and universality. Discrete systems are the ideal laboratory for understanding fundamental mechanisms.

10.2 The Logistic Map

Definition 10.1 (Logistic family)**.** The **logistic family** is $f_r : [0,1] \to [0,1]$ defined by $f_r(x) = rx(1-x)$, $r \in [0,4]$.

Proposition 10.2 (Fixed points and stability)**.**

1. The fixed point $x^* = 0$ exists for all r; it is stable if $r < 1$.
2. The fixed point $x^* = 1 - 1/r$ exists for $r > 1$; it is stable if $1 < r < 3$ (since $f_r'(x^*) = 2 - r$).
3. At $r = 3$, a period-doubling bifurcation creates a period-2 cycle.

Theorem 10.3 (Period-doubling cascade)**.** *There exists an increasing sequence $r_1 < r_2 < r_3 < \cdots$ with $r_1 = 3$ such that at $r = r_n$, the period-2^{n-1} cycle loses stability and a stable period-2^n cycle appears via a flip bifurcation. The sequence r_n converges to $r_\infty \approx 3.5699\ldots$*

Theorem 10.4 (Feigenbaum universality)**.** *The ratio $\delta_n = \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$ converges to the universal Feigenbaum constant*

$$\delta_F = \lim_{n\to\infty} \delta_n = 4.669\,201\,609\ldots$$

This constant is the same for every unimodal family with a quadratic maximum. The spatial scaling factor converges to $\alpha_F = -2.502\,907\,875\ldots$

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Regimes of the Logistic Map

$$\begin{aligned} r \in [0,1] &: \text{convergence to } 0. \\ r \in (1,3) &: \text{convergence to } x^* = 1 - 1/r. \\ r \in (3, 1+\sqrt{6}) &: \text{stable period-2 cycle.} \\ r = r_\infty \approx 3.5699 &: \text{accumulation of doublings.} \\ r \in (r_\infty, 4) &: \text{mixture of chaos and periodic windows.} \\ r = 4 &: \text{fully developed chaos on } [0,1]. \end{aligned}$$

10.3 Sharkovskii's Theorem

Definition 10.5 (Sharkovskii ordering)**.** The **Sharkovskii ordering** on $\mathbb{N}^*$ is:

$$3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \cdots \triangleright 2{\cdot}3 \triangleright 2{\cdot}5 \triangleright 2{\cdot}7 \triangleright \cdots \\ \triangleright 4{\cdot}3 \triangleright 4{\cdot}5 \triangleright \cdots \triangleright \cdots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.$$

Theorem 10.6 (Sharkovskii, 1964)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point of period m, then f has periodic points of every period n with $m \triangleright n$ in the Sharkovskii ordering. Moreover, for each m, there exists a continuous map having a period-m point but no period-n point for $n \triangleright m$.*

Corollary 10.7 (Period three implies all periods)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a period-3 point, then f has periodic points of every period $n \in \mathbb{N}^*$.*

Remark 10.8. Sharkovskii's ordering applies only in dimension one. In dimension two, a diffeomorphism can have a period-3 point without having a fixed point (e.g., rotation by $2\pi/3$).

Lemma 10.9 (Intermediate value lemma)**.** *Let I, J be closed intervals and f continuous. If $f(I) \supset J$, then there exists a sub-interval $K \subset I$ with $f(K) = J$. In particular, if $f(I) \supset I$, then f has a fixed point in I.*

10.4 Symbolic Dynamics

Definition 10.10 (Sequence space)**.** Let $\mathcal{A} = \{0, 1, \ldots, k-1\}$ be an alphabet of k symbols. The **sequence space** is $\Sigma_k = \mathcal{A}^{\mathbb{N}}$, with metric $d(s,t) = \sum_{i=0}^{\infty} |s_i - t_i| / k^i$.

Definition 10.11 (Shift map)**.** The **shift** $\sigma : \Sigma_k \to \Sigma_k$ is $\sigma(s_0, s_1, s_2, \ldots) = (s_1, s_2, s_3, \ldots)$.

Theorem 10.12 (Properties of the shift)**.** *The shift σ on Σ_k satisfies:*

1. *σ is continuous and surjective (but not injective).*
2. *Period-n points are exactly the n-periodic sequences; there are k^n of them.*
3. *Periodic points are dense in Σ_k.*
4. *σ is topologically transitive.*

5. σ *is Devaney chaotic.*

Definition 10.13 (Subshift of finite type)**.** Given a transition matrix $A \in \{0,1\}^{k\times k}$, the **subshift of finite type** is $\Sigma_A = \{s \in \Sigma_k : A_{s_i,s_{i+1}} = 1 \text{ for all } i\}$, with $\sigma_A = \sigma|_{\Sigma_A}$.

Proposition 10.14 (Itinerary coding)**.** Let $f : [0,1] \to [0,1]$ be unimodal with critical point c. The **itinerary** of x is $s(x) = (s_0, s_1, \ldots)$ where $s_n = 0$ if $f^n(x) < c$ and $s_n = 1$ if $f^n(x) > c$. The map $x \mapsto s(x)$ semi-conjugates f to the shift on a subspace of Σ_2.

10.5 Topological Entropy

Definition 10.15 (Topological entropy)**.** Let $f : X \to X$ be continuous on a compact metric space. The **topological entropy** is

$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \ln s(n, \epsilon, f),$$

where $s(n, \epsilon, f)$ is the maximum number of (n, ϵ)-separated points. The limit is independent of $\epsilon > 0$ sufficiently small.

Theorem 10.16 (Entropy of the shift)**.** *The topological entropy of* σ *on* Σ_k *is* $h_{\text{top}}(\sigma) = \ln k$*. For the subshift of finite type* σ_A*,* $h_{\text{top}}(\sigma_A) = \ln \rho(A)$*, where* $\rho(A)$ *is the spectral radius of* A*.*

Proposition 10.17 (Entropy of the logistic map)**.** For $f_4(x) = 4x(1-x)$, the topological entropy is $h_{\text{top}}(f_4) = \ln 2$. More generally, for the tent map of slope s: $h_{\text{top}} = \max(0, \ln s)$.

Theorem 10.18 (Variational principle)**.** *For any continuous map* f *on a compact metric space:*

$$h_{\text{top}}(f) = \sup_{\mu\in\mathcal{M}_f} h_\mu(f),$$

where the supremum is over all f*-invariant probability measures and* $h_\mu(f)$ *is the metric entropy.*

Entropy and complexity

$h_{\text{top}} > 0$ means the number of distinguishable orbits grows exponentially with time. This is a more robust indicator of chaos than the maximal Lyapunov exponent, as it is a topological invariant (independent of the metric and measure).

10.6 Advanced Examples

Example 10.19 (Bernoulli shift)**.** The sawtooth map $B_k(x) = kx \pmod 1$ on $[0, 1)$ is exactly conjugate to the shift on Σ_k via the base-k expansion. Its entropy is $\ln k$ and its Lyapunov exponent is $\ln k$.

Example 10.20 (Quadratic Julia sets)**.** For $f_c(z) = z^2 + c$ with $c \in \mathbb{C}$, the Julia set $J_c = \partial\{z : f_c^n(z) \not\to \infty\}$ is either connected (if c is in the Mandelbrot set) or totally disconnected (a Cantor set). The dynamics on J_c displays rich chaotic phenomena.

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10.7 Exercises

Exercise 10.1 (Logistic bifurcations)**.** For the logistic map $f_r(x) = rx(1-x)$, analytically find the value of r at which the period-2 cycle appears and the values of the cycle points.

Exercise 10.2 (Sharkovskii)**.** Construct a continuous map $f : [0,1] \to [0,1]$ having a period-6 point but no period-3 point. Show that it must have period-2 points.

Exercise 10.3 (Symbolic dynamics)**.** Show that the shift on Σ_2 is topologically mixing: for all open sets U, V, there exists N with $\sigma^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

Exercise 10.4 (Entropy)**.** Compute the topological entropy of the subshift of finite type defined by $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (Fibonacci shift). Show that $h_{\text{top}} = \ln \varphi$, where $\varphi = (1+\sqrt{5})/2$.

Exercise 10.5 (Bifurcation diagram)**.** Write a program plotting the bifurcation diagram of the logistic map for $r \in [2.5, 4]$. Visually identify the period-doubling cascade and periodic windows (period 3, 5, 6, etc.).

Exercise 10.6 (Kneading theory)**.** Determine the kneading itinerary of the critical point of f_r for $r = 3.8$ and $r = 4$. Deduce which periods are forced.

Chapter 11

Introduction to Ergodic Theory

11.1 Introduction

Ergodic theory studies the long-term statistical behaviour of dynamical systems. It addresses the fundamental question: do time averages coincide with space averages? This chapter presents measure-preserving transformations, the fundamental ergodic theorems, and their applications.

Intuition

Consider a gas in a box. Rather than tracking every molecule, we measure macroscopic quantities (temperature, pressure) that are averages. Boltzmann's ergodic hypothesis asserts that the time average of an observable along a trajectory equals its average over the entire phase space. This is the foundation of statistical mechanics.

11.2 Measure-Preserving Transformations

Definition 11.1 (Measure-preserving dynamical system)**.** A **measure-preserving dynamical system** is a tuple $(X, \mathcal{B}, \mu, T)$ where $(X, \mathcal{B}, \mu)$ is a probability space and $T : X \to X$ is a measurable transformation that **preserves the measure**: $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$.

Example 11.2 (Irrational circle rotation)**.** Let $T_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by $T_\alpha(x) = x + \alpha \pmod 1$ with $\alpha \notin \mathbb{Q}$. Lebesgue measure on $[0, 1)$ is invariant under T_α.

Example 11.3 (Doubling map)**.** The map $T(x) = 2x \pmod 1$ on $[0, 1)$ preserves Lebesgue measure. It is a non-invertible endomorphism with entropy $\ln 2$.

Example 11.4 (Baker's transformation)**.** The **baker's transformation** on $[0, 1)^2$ is

$$B(x, y) = \begin{cases} (2x, y/2) & \text{if } 0 \le x < 1/2, \\ (2x - 1, (y + 1)/2) & \text{if } 1/2 \le x < 1. \end{cases}$$

It preserves Lebesgue measure and models kneading: horizontal stretching, vertical compression, and folding.

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11.3 Ergodicity

Definition 11.5 (Invariant set)**.** A set $A \in \mathcal{B}$ is T**-invariant** if $T^{-1}(A) = A$ (modulo a null set).

Definition 11.6 (Ergodic transformation)**.** T is **ergodic** if every T-invariant set has measure 0 or 1. Equivalently, every measurable T-invariant function ($\varphi \circ T = \varphi$ a.e.) is constant a.e.

Theorem 11.7 (Ergodicity of irrational rotations)**.** *The rotation T_α is ergodic with respect to Lebesgue measure if and only if α is irrational.*

Proof. Let $\varphi \in L^2([0,1))$ be invariant: $\varphi(x+\alpha) = \varphi(x)$ a.e. Expanding in Fourier series $\varphi(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$, invariance gives $c_n e^{2\pi i n\alpha} = c_n$ for all n. If $\alpha \notin \mathbb{Q}$, then $e^{2\pi i n\alpha} \neq 1$ for $n \neq 0$, so $c_n = 0$ for all $n \neq 0$. Thus $\varphi = c_0$ is constant a.e. □

11.4 Birkhoff's Ergodic Theorem

Theorem 11.8 (Birkhoff, 1931)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system and $\varphi \in L^1(X,\mu)$. Then, for μ-almost every x:*

$$\overline{\varphi}(x) := \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \varphi(T^n(x))$$

exists and $\overline{\varphi}$ is T-invariant. Moreover, $\int_X \overline{\varphi}\, \mathrm{d}\mu = \int_X \varphi\, \mathrm{d}\mu$.

Corollary 11.9 (Ergodic case)**.** *If T is ergodic, then for every $\varphi \in L^1$ and μ-almost every x:*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \varphi(T^n(x)) = \int_X \varphi\, \mathrm{d}\mu.$$

Time averages equal space averages.

Fundamental Ergodic Theorems

- **Von Neumann (1932)**: L^2 convergence of $\frac{1}{N}\sum_{n=0}^{N-1} \varphi \circ T^n$ to the projection onto T-invariant functions.
- **Birkhoff (1931)**: almost sure convergence (stronger).
- **Kingman (1968)**: subadditive ergodic theorem for sequences (a_n) with $a_{m+n} \leq a_m + a_n \circ T^m$.

Example 11.10 (Weyl equidistribution)**.** Applying Birkhoff to T_α with $\varphi = \mathbb{1}_{[a,b]}$ yields Weyl's theorem: for $\alpha \notin \mathbb{Q}$,

$$\lim_{N\to\infty} \frac{\#\{0 \leq n < N : \{n\alpha\} \in [a,b]\}}{N} = b - a.$$

The sequence $(n\alpha \bmod 1)_{n\geq 0}$ is equidistributed modulo 1.

11.5 Mixing

Definition 11.11 (Weak and strong mixing)**.** T is **weakly mixing** if for all $A, B \in \mathcal{B}$:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left|\mu(T^{-n}(A) \cap B) - \mu(A)\mu(B)\right| = 0.$$

T is **(strongly) mixing** if:

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

Proposition 11.12 (Implications)**.**

$$\text{strong mixing} \implies \text{weak mixing} \implies \text{ergodicity}.$$

Neither implication reverses in general.

Example 11.13. The doubling map $T(x) = 2x \pmod 1$ and the baker's transformation are strongly mixing. The irrational rotation is ergodic but **not** mixing (not even weakly).

Theorem 11.14 (Spectral characterisation of mixing)**.** *T is strongly mixing if and only if for all $\varphi, \psi \in L^2(X, \mu)$:*

$$\lim_{n\to\infty} \int_X (\varphi \circ T^n) \cdot \psi \,\mathrm{d}\mu = \int_X \varphi \,\mathrm{d}\mu \cdot \int_X \psi \,\mathrm{d}\mu.$$

11.6 Ergodic Decomposition

Theorem 11.15 (Ergodic decomposition)**.** *Let $T : X \to X$ preserve μ on a standard probability space. Then there exists an (essentially unique) decomposition*

$$\mu = \int_Y \mu_y \,\mathrm{d}\nu(y)$$

*where ν is a probability measure on a measurable space Y and each μ_y is a T-invariant **ergodic** probability measure.*

Remark 11.16. Ergodic decomposition means that every measure-preserving system can be decomposed into irreducible ergodic components. It is the analogue of decomposing a Hilbert space into irreducible subspaces.

11.7 Metric Entropy

Definition 11.17 (Kolmogorov-Sinai entropy)**.** Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a measurable partition of X. The **entropy** of $\mathcal{P}$ is $H(\mathcal{P}) = -\sum_{i=1}^{k} \mu(P_i) \ln \mu(P_i)$. The **metric entropy** of T with respect to $\mathcal{P}$ is

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}\right),$$

and the **Kolmogorov-Sinai entropy** is $h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T, \mathcal{P})$.

Theorem 11.18 (Kolmogorov-Sinai)**.** *If $\mathcal{P}$ is a generating partition (i.e., $\bigvee_{n=0}^{\infty} T^{-n}\mathcal{P}$ generates $\mathcal{B}$ modulo null sets), then $h_\mu(T) = h_\mu(T, \mathcal{P})$.*

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11.8 Applications to Number Theory

Example 11.19 (Continued fractions and the Gauss map)**.** The **Gauss map** $G : (0, 1] \to [0, 1)$ defined by $G(x) = \{1/x\}$ (fractional part of $1/x$) encodes the continued fraction algorithm. The Gauss measure $\mathrm{d}\mu_G = \frac{1}{\ln 2} \frac{\mathrm{d}x}{1+x}$ is G-invariant and ergodic.

Corollary 11.20 (Khintchine's theorem)**.** *For almost every $x \in (0, 1)$ with continued fraction expansion $x = [a_1, a_2, \ldots]$:*

$$\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\ln k/\ln 2} \approx 2.685\ldots$$

(Khintchine's constant).

Example 11.21 (Lévy's theorem)**.** By the ergodic theorem applied to G:

$$\lim_{n\to\infty} \frac{1}{n} \ln q_n = \frac{\pi^2}{12 \ln 2} \approx 1.1866\ldots$$

where q_n is the denominator of the n-th convergent. This gives the typical rate of approximation by continued fractions.

11.9 Exercises

Exercise 11.1 (Ergodicity)**.** Show that the baker's transformation is ergodic using Fourier coefficients.

Exercise 11.2 (Poincaré recurrence theorem)**.** Prove that if T preserves μ and $\mu(A) > 0$, then for μ-almost every $x \in A$, there exists $n \geq 1$ with $T^n(x) \in A$.

Exercise 11.3 (Mixing of the doubling map)**.** Show that $T(x) = 2x \pmod 1$ is strongly mixing using the spectral characterisation with the functions $e^{2\pi i k x}$.

Exercise 11.4 (Entropy)**.** Compute the metric entropy of the Gauss map $G(x) = \{1/x\}$ with respect to the Gauss measure. (Answer: $h_{\mu_G}(G) = \pi^2/(6 \ln 2)$.)

Exercise 11.5 (Equidistribution)**.** Use Birkhoff's theorem to show that the sequence $(\sin(n) \bmod 1)_{n\geq 1}$ is not necessarily equidistributed, and discuss conditions on α for $(n^\alpha \bmod 1)$ to be equidistributed.

Exercise 11.6 (Gauss map verification)**.** Numerically verify Khintchine's constant by computing the geometric mean of the continued fraction coefficients of random numbers.

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