Fixed Point Theory

Lecture Notes

M1–PhD — 2025–2026

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"Every continuous function from a closed disk to itself has at least one fixed point." — L.E.J. Brouwer

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Contents

Preface

Fixed point theory stands as one of the fundamental pillars of modern mathematical analysis. Its influence extends well beyond the boundaries of functional analysis, permeating such diverse fields as topology, algebra, optimization, game theory, mathematical economics, and partial differential equations.

Objectives

This course has been designed for Master's and PhD students in mathematics. It aims to present in a rigorous and comprehensive manner the major fixed point theorems, their proofs, their generalizations, and their applications.

The main pedagogical objectives are:

- Master the complete proofs of the classical theorems (Banach, Brouwer, Schauder, Kakutani, Tarski–Knaster);
- Understand the deep connections between these results and the underlying mathematical structures (metric, topological, order-theoretic);
- Apply these theorems to the resolution of concrete problems: differential equations, integral equations, Nash equilibria;
- Explore modern extensions: generalized metric spaces, random fixed points, multivalued correspondences.

Organization

The course is structured in eleven chapters organized into three natural parts.

Part I: Metric Theorems (Chapters 1–4)

The first part is devoted to metric-type results. After an introductory chapter that sets the historical motivations and fundamental definitions, Chapter 2 presents the celebrated Banach Contraction Principle (1922) and its direct generalizations. Chapter 3 explores metric extensions due to Edelstein, Kannan, Ćirić, and Caristi, which relax the contraction hypothesis in various ways. Chapter 4 introduces generalized metric spaces (b-metrics, G-metrics) and examines how the classical theorems extend to these settings.

Part II: Topological Theorems (Chapters 5–7)

The second part addresses theorems of a topological nature. Chapter 5 is dedicated to Brouwer's theorem (1911), a true jewel of algebraic topology, with several proofs (combinatorial via Sperner's lemma, analytical, homological). Chapter 6 treats Schauder's theorem (1930), the infinite-dimensional generalization of Brouwer's theorem, and its variants (Schauder–Tychonoff, Darbo). Chapter 7 presents Kakutani's theorem (1941) for multivalued correspondences, an indispensable tool in game theory and mathematical economics.

Part III: Extensions and Applications (Chapters 8–11)

The third part gathers advanced topics. Chapter 8 studies fixed points in ordered spaces (theorems of Tarski, Knaster–Tarski, Bourbaki–Witt). Chapter 9 is a cross-cutting applications chapter: ordinary differential equations, partial differential equations, Nash equilibria, economic models. Chapter 10 presents the Markov–Kakutani theorem and its consequences for common fixed points of families of transformations. Finally, Chapter 11 introduces random fixed point theory, a rapidly developing area.

Prerequisites

The reader should possess solid knowledge of:

- **Functional analysis**: Banach spaces, Hilbert spaces, bounded linear operators, the Hahn–Banach theorem, weak compactness;
- **General topology**: compact spaces, connectedness, separation theorems, complete metric spaces;
- **Measure theory**: measurable spaces, measurable functions, Lebesgue integration (for Chapter 11);
- **Linear algebra**: finite-dimensional vector spaces, determinants, multilinear forms.

Notation

Throughout this course, we use the following notation:

- $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$: the sets of real, natural, integer, rational, and complex numbers;
- (X, d): a metric space, where d denotes the metric;
- $\|\cdot\|$: a norm on a normed vector space;
- $\langle \cdot, \cdot \rangle$: an inner product on a Hilbert space;
- $B(x, r)$: the open ball with center x and radius r;
- $\overline{B}(x, r)$: the closed ball with center x and radius r;
- $\overline{A}$: the closure of a set A;

- $\mathrm{co}(A)$, $\overline{\mathrm{co}}(A)$: the convex hull and closed convex hull of A;
- $\mathrm{Fix}(T)$: the set of fixed points of the mapping T;
- $\mathrm{Lip}(T)$: the Lipschitz constant of T.

Conventions

Unless otherwise stated:

- Vector spaces are over the field $\mathbb{R}$ of real numbers;
- Topological spaces are assumed to be Hausdorff;
- Maps take values in $\mathbb{R}$ or in a Banach space;
- The symbol $\square$ marks the end of a proof;
- The symbol $\lozenge$ marks the end of an example or remark.

Reading guide

This course may be approached in several ways depending on the reader's level and interests:

- **Foundational track** (M1): Chapters 1, 2, 5, and 9. This track covers the Banach contraction principle and Brouwer's theorem with their main applications.
- **Advanced track** (M2): Add Chapters 3, 6, 7, and 8. This completes the training with metric extensions, Schauder's theorem, Kakutani's theorem, and ordered spaces.
- **Research track** (PhD): The entire course, with particular attention to Chapters 4, 10, and 11, which touch on current research themes.

Acknowledgments

This course owes much to the classical references in the field, notably the works of Granas and Dugundji [2], Goebel and Kirk [10], Zeidler [8], Agarwal, Meehan, and O'Regan [11], and Smart [4]. We also thank the many colleagues and students whose remarks have helped improve this text.

How to use this course

Each chapter contains:

- Numbered, precise, and motivated **definitions**;
- **Theorems** with complete proofs;

- Illustrative **examples** and **counter-examples** showing the necessity of hypotheses;
- **Remarks** illuminating subtle points;
- **Exercises** of varied difficulty, some with hints.

Special boxes (key formulas, warnings, intuition, algorithms) allow quick identification of essential information.

The author
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Chapter 1

Introduction and Motivations

Consider a barista stirring a cocktail in a glass. After vigorously swirling the liquid, she can be certain that at least one point in the liquid ends up exactly at its original position. This result, known as Brouwer's theorem, illustrates a simple but profound idea: under certain conditions, every transformation has at least one *fixed point*, a point that does not move. This idea runs through mathematics: it appears in analysis, topology, algebra, economics (Nash equilibrium), computer science (program semantics), and game theory. This course systematically explores fixed-point theorems, their proofs, and their applications.

1.1 What is a fixed point?

The definition is disarmingly simple — but do not be fooled.

Definition 1.1 (Fixed point)**.** Let X be a set and $T : X \to X$ a mapping. An element $x^* \in X$ is a *fixed point* of T if $T(x^*) = x^*$. The set of fixed points of T is denoted

$$\mathrm{Fix}(T) = \{x \in X : T(x) = x\}.$$

This seemingly elementary definition leads to remarkably deep questions as soon as X is endowed with additional structure (metric, topology, order) and conditions are imposed on T.

Remark 1.2. The equation $T(x) = x$ can be reformulated as $F(x) = 0$ by setting $F = T - \mathrm{Id}$. Conversely, any equation $F(x) = 0$ can be reduced to a fixed point problem by setting $T(x) = x - F(x)$ (or $T(x) = x + \alpha F(x)$ for a suitably chosen parameter α). This duality underlies many numerical methods.

1.2 Historical context

Fixed point theory has a rich and fascinating history.

1.2.1 Origins: Picard and Banach

The method of successive approximations dates back to the work of Émile Picard (1890) on ordinary differential equations. Picard showed that the equation

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

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has a unique solution under a Lipschitz condition on f with respect to y, by constructing the sequence of iterates

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds.$$

It was Stefan Banach who, in his doctoral thesis (1920, published 1922), formalized this idea by stating the *Contraction Principle* in the general framework of complete metric spaces.

1.2.2 Brouwer's theorem

In 1911, L. E. J. Brouwer proved that in finite dimensions, every continuous map from a compact convex set into itself has a fixed point. This result, purely topological in nature (no metric hypothesis), relies on arguments from topological degree theory and algebraic topology.

1.2.3 Infinite-dimensional extensions

Juliusz Schauder (1930) generalized Brouwer's theorem to infinite-dimensional Banach spaces by replacing compactness of the set with compactness of the operator. Shizuo Kakutani (1941) extended the result to multivalued correspondences (set-valued mappings).

1.2.4 Order-theoretic approach

Alfred Tarski (1955) established a fixed point theorem in complete lattices, based solely on monotonicity of the mapping, with no metric or topological hypothesis. This result, often called the Knaster–Tarski theorem, finds applications in theoretical computer science and logic.

1.3 Elementary examples

Example 1.3 (Fixed point in dimension 1)**.** Let $f : [0,1] \to [0,1]$ be a continuous map. By the Intermediate Value Theorem, f has a fixed point.

Indeed, set $g(x) = f(x) - x$. Then $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. By the Intermediate Value Theorem, there exists $x^* \in [0,1]$ such that $g(x^*) = 0$, i.e., $f(x^*) = x^*$.

Example 1.4 (Non-existence of fixed points)**.** The map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$ has no fixed point. This shows the necessity of working in a "sufficiently small" set (bounded, compact, etc.).

Example 1.5 (Multiplicity of fixed points)**.** The identity map $T(x) = x$ has every point as a fixed point. The map $T(x) = x^2$ on $\mathbb{R}$ has exactly two fixed points: 0 and 1.

Example 1.6 (Rotation without fixed point)**.** The rotation $R_\theta : S^1 \to S^1$ by angle $\theta \neq 0 \pmod{2\pi}$ on the unit circle has no fixed point. This shows that convexity is essential in Brouwer's theorem: the circle S^1 is compact but not convex.

1.4 Formulation as a functional equation

Many classical mathematical problems can be reformulated as fixed point problems.

1.4.1 Integral equations

The Fredholm integral equation of the second kind

$$u(t) = f(t) + \lambda \int_a^b K(t,s)\, u(s)\, ds$$

is of the form $u = Tu$ where the operator T is defined by

$$(Tu)(t) = f(t) + \lambda \int_a^b K(t,s)\, u(s)\, ds.$$

1.4.2 Differential equations

The Cauchy problem $y' = f(t,y)$, $y(t_0) = y_0$ is equivalent, under regularity assumptions, to the fixed point problem $y = Ty$ where

$$(Ty)(t) = y_0 + \int_{t_0}^t f(s, y(s))\, ds.$$

1.4.3 Partial differential equations

Boundary value problems for elliptic PDEs are often reformulated as fixed point problems in Sobolev spaces. For example, the problem

$$-\Delta u = g(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

becomes $u = (-\Delta)^{-1} g(u) = T(u)$.

1.4.4 Nash equilibria

A Nash equilibrium in an n-player game can be viewed as a fixed point of a best-response correspondence. If S_i is the strategy set of player i and $BR_i(s_{-i})$ is their best response to the other players' strategies, then a Nash equilibrium is a profile $s^* = (s_1^*, \ldots, s_n^*)$ such that $s_i^* \in BR_i(s_{-i}^*)$ for all i.

1.5 Three families of theorems

Fixed point theorems fall into three broad families, according to the mathematical structure employed.

The three approaches

1. **Metric approach**: one assumes that T contracts distances. The fixed point is unique and obtained by iteration.
   Prototype: Banach's theorem.

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2. **Topological approach**: one assumes continuity of T and compactness of the domain. Existence is guaranteed, but neither uniqueness nor constructibility. Prototype: Brouwer's theorem.

3. **Order-theoretic approach**: one assumes monotonicity of T in a complete lattice. Existence is guaranteed, and the set of fixed points itself forms a complete lattice.
Prototype: Tarski's theorem.

1.6 Preliminaries and background

1.6.1 Metric spaces

Definition 1.7 (Metric space)**.** A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to [0, +\infty)$ satisfies for all $x, y, z \in X$:

1. $d(x, y) = 0 \Leftrightarrow x = y$ (separation),

2. $d(x, y) = d(y, x)$ (symmetry),

3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition 1.8 (Completeness)**.** A metric space (X, d) is *complete* if every Cauchy sequence in X converges. Recall that a sequence (x_n) is Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

Definition 1.9 (Lipschitz mapping)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $T : X \to Y$ is *Lipschitz* with constant $L \geq 0$ if

$$d_Y(T(x), T(y)) \leq L\, d_X(x, y) \quad \forall x, y \in X.$$

If $L < 1$, T is called a *contraction.* If $L = 1$, T is called *nonexpansive.*

1.6.2 Banach spaces

Definition 1.10 (Banach space)**.** A *Banach space* is a normed vector space $(E, \|\cdot\|)$ that is complete with respect to the induced metric $d(x, y) = \|x - y\|$.

Definition 1.11 (Compactness)**.** A subset K of a topological space is *compact* if every open cover of K has a finite subcover. In a metric space, this is equivalent to sequential compactness: every sequence in K has a convergent subsequence with limit in K.

Remark 1.12. In infinite dimensions, closed bounded balls are never compact (Riesz's theorem). This is why topological fixed point theorems in infinite dimensions (Schauder) require compactness hypotheses on the operator rather than on the domain.

1.6.3 Convexity

Definition 1.13 (Convex set)**.** A subset C of a vector space E is *convex* if for all $x, y \in C$ and all $t \in [0, 1]$, we have $(1-t)x + ty \in C$.

Definition 1.14 (Convex hull)**.** The *convex hull* of a set $A \subset E$ is the smallest convex set containing A:

$$\mathrm{co}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}^*,\ x_i \in A,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

1.7 Picard iteration

The most elementary method for finding a fixed point is Picard iteration (the method of successive approximations).

Picard iteration

Input: Mapping $T : X \to X$, initial point $x_0 \in X$.
Procedure: Define the sequence (x_n) by $x_{n+1} = T(x_n)$ for $n \geq 0$.
Output: If (x_n) converges to x^* and T is continuous, then x^* is a fixed point of T.

Remark 1.15. Convergence of Picard iteration is not guaranteed in general. The Banach theorem (Chapter 2) provides sufficient conditions.

1.8 The one-dimensional fixed point theorem

Theorem 1.16 (One-dimensional fixed point theorem)**.** *Let $f : [a, b] \to [a, b]$ be a continuous map. Then f has at least one fixed point in $[a, b]$.*

Proof. Consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - x$. Since $f([a, b]) \subset [a, b]$, we have $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$. By the Intermediate Value Theorem, there exists $x^* \in [a, b]$ such that $g(x^*) = 0$, i.e., $f(x^*) = x^*$. ☐

Necessary hypotheses

Both hypotheses are essential:

- **Continuity**: the map $f : [0, 1] \to [0, 1]$ defined by $f(x) = 0$ if $x > 1/2$ and $f(x) = 1$ if $x \leq 1/2$ has no fixed point.
- **Invariance** ($f([a, b]) \subset [a, b]$): the map $f(x) = x + 1$ on $[0, 1]$ has no fixed point since $f([0, 1]) = [1, 2] \not\subset [0, 1]$.

1.9 Topological degree: a preview

The topological degree is an algebraic invariant that "counts" the solutions of an equation $F(x) = p$ with multiplicity and sign. For a C^1 map $F : \overline{\Omega} \to \mathbb{R}^n$, with Ω a bounded open

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subset of $\mathbb{R}^n$ and $p \notin F(\partial\Omega)$, the Brouwer degree is defined by

$$\deg(F, \Omega, p) = \sum_{x \in F^{-1}(p)} \operatorname{sgn} \det(DF(x)).$$

Theorem 1.17 (Existence via nonzero degree)**.** *If* $\deg(F, \Omega, p) \neq 0$*, then the equation* $F(x) = p$ *has at least one solution in* Ω*.*

This connection between algebraic topology and existence of solutions will be developed in detail in Chapter 5.

1.10 Course outline

Course architecture

Ch.	Title	Main result
1	Introduction	Motivations, preliminaries
2	Banach	Contraction Principle
3	Metric extensions	Edelstein, Kannan, Ćirić, Caristi
4	Generalized spaces	b-metrics, G-metrics
5	Brouwer	Topological fixed point (finite dim.)
6	Schauder	Topological fixed point (infinite dim.)
7	Kakutani	Multivalued correspondences
8	Ordered spaces	Tarski–Knaster
9	Applications	ODE, PDE, games, economics
10	Markov–Kakutani	Common fixed points
11	Random fixed points	Measurable theory

1.11 Exercises

Exercise 1.1. Let $f : [0,1] \to [0,1]$ be an increasing (not necessarily continuous) map. Show that f has a fixed point. *Hint: consider* $x^* = \sup\{x \in [0,1] : f(x) \geq x\}$.

Exercise 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $|f(x) - f(y)| < |x - y|$ for all $x \neq y$. Show that f has at most one fixed point. Give an example where f has no fixed point.

Exercise 1.3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine isometry (i.e., $\|T(x) - T(y)\| = \|x - y\|$ for all x, y). Show that either T has a fixed point or T is a translation with no fixed point.

Exercise 1.4. Show that the rotation of $\mathbb{R}^2$ by angle $\pi/4$ about the origin has a unique fixed point. More generally, what can be said about the fixed points of a rotation by angle θ in dimension n?

Exercise 1.5. Let (X, d) be a compact metric space and $T : X \to X$ an isometry (i.e., $d(T(x), T(y)) = d(x, y)$ for all x, y). Show that T is surjective. Deduce that if X is moreover convex in a normed space, then T has a fixed point.

Exercise 1.6. Let A be an $n \times n$ matrix with nonnegative entries such that the sum of the entries in each column equals 1 (stochastic matrix). Show that A has an eigenvector associated to the eigenvalue 1 in the simplex $\Delta_n = \{x \in \mathbb{R}^n : x_i \geq 0, \sum x_i = 1\}$. Interpret this result as a fixed point theorem.

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Chapter 2

Banach Contraction Principle and Generalizations

In 1922, Stefan Banach, a largely self-taught Polish mathematician, published in his doctoral thesis a theorem of dazzling simplicity: if a mapping “brings points closer together” and the space is complete, then there exists a unique fixed point, computable by iteration. This result has become one of the most widely used tools in all of analysis: it proves existence and uniqueness for ODEs (Picard–Lindelöf), justifies Newton’s method, and underpins the solution of many integral and functional equations.

2.1 The fundamental theorem

The Banach Contraction Principle is arguably the most widely used fixed point theorem in mathematics. Its power lies in the conjunction of three properties: existence, uniqueness, and a constructive method of computation.

Theorem 2.1 (Banach Contraction Principle, 1922)**.** *Let (X, d) be a complete metric space and $T : X \to X$ a contraction, i.e., there exists $k \in [0, 1)$ such that*

$$d(T(x), T(y)) \leq k\, d(x, y) \quad \forall x, y \in X.$$

Then:

1. *T has a unique fixed point $x^* \in X$;*
2. *For every $x_0 \in X$, the sequence of iterates $(T^n(x_0))_{n \geq 0}$ converges to x^*;*
3. *The following error estimates hold:*

$$d(T^n(x_0), x^*) \leq \frac{k^n}{1-k}\, d(x_0, T(x_0)) \quad \textit{(a priori)}, \tag{2.1}$$

$$d(T^n(x_0), x^*) \leq \frac{k}{1-k}\, d(T^{n-1}(x_0), T^n(x_0)) \quad \textit{(a posteriori)}. \tag{2.2}$$

Proof. **Existence and convergence.** Fix $x_0 \in X$ and set $x_n = T^n(x_0)$ for $n \geq 0$. For $n \geq 1$,

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq k\, d(x_n, x_{n-1}) \leq \cdots \leq k^n\, d(x_1, x_0).$$

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For $m > n$, by the triangle inequality:

$$d(x_m, x_n) \le \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \le \sum_{i=n}^{m-1} k^i\, d(x_1, x_0) \le \frac{k^n}{1-k}\, d(x_1, x_0).$$

Since $k < 1$, the right-hand side tends to 0 as $n \to \infty$, so (x_n) is a Cauchy sequence. By completeness of (X, d), it converges to some $x^* \in X$.

Fixed point. By continuity of T (every contraction is Lipschitz, hence continuous):

$$T(x^*) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

Uniqueness. If y^* is another fixed point, then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \le k\, d(x^*, y^*).$$

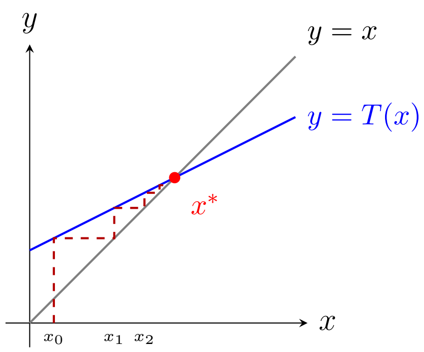
Since $k < 1$, this implies $d(x^*, y^*) = 0$, hence $x^* = y^*$.

A priori estimate. Letting $m \to \infty$ in $d(x_m, x_n) \le \frac{k^n}{1-k} d(x_1, x_0)$ yields (2.1).

A posteriori estimate. For $m > n$:

$$d(x_m, x_n) \le \sum_{i=0}^{m-n-1} k^i\, d(x_{n+1}, x_n) \le \frac{1}{1-k}\, d(x_{n+1}, x_n).$$

Passing to the limit $m \to \infty$ and replacing $d(x_{n+1}, x_n)$ by $d(T(x_n), T(x_{n-1})) \le k\, d(x_n, x_{n-1})$ yields (2.2). □



Banach Contraction Principle

Let (X, d) be complete, $T : X \to X$ with $d(Tx, Ty) \le k\, d(x, y)$, $k < 1$. Then:

$$\exists!\, x^* \in X : T(x^*) = x^*, \qquad d(T^n x_0, x^*) \le \frac{k^n}{1-k}\, d(x_0, Tx_0).$$

2.2 Necessity of hypotheses

Example 2.2 (Necessity of completeness)**.** Consider $X = (0, 1]$ with the usual metric and $T(x) = x/2$. Then T is a contraction with constant $1/2$, but T has no fixed point in X (the fixed point 0 does not belong to X). The space $(0, 1]$ is not complete.

Example 2.3 (Necessity of $k < 1$)**.** Let $X = [1, +\infty)$ with the usual metric and $T(x) = x + 1/x$. Then for $x \neq y$:

$$|T(x) - T(y)| = |x - y| \cdot \left|1 - \frac{1}{xy}\right| < |x - y|$$

so T is weakly contractive ($d(Tx, Ty) < d(x, y)$ for $x \neq y$), but there is no uniform $k < 1$. The map T has no fixed point in $[1, +\infty)$.

Strict contraction vs. weak contraction

The condition $d(Tx, Ty) < d(x, y)$ for $x \neq y$ (weak contraction) does **not** suffice to guarantee existence of a fixed point, even in a complete space. One needs the uniform condition $d(Tx, Ty) \leq k\, d(x, y)$ with $k < 1$ independent of x and y.

2.3 Rate of convergence

Proposition 2.4 (Geometric convergence)**.** Under the hypotheses of Theorem 2.1, the convergence of (x_n) to x^* is at least geometric with ratio k:

$$d(x_n, x^*) \leq k\, d(x_{n-1}, x^*) \quad \forall n \geq 1.$$

Proof. $d(x_n, x^*) = d(T(x_{n-1}), T(x^*)) \leq k\, d(x_{n-1}, x^*)$. □

Remark 2.5. In practice, the a posteriori estimate (2.2) is more useful because it does not require knowledge of $d(x_0, Tx_0)$ but only the last variation $d(x_{n-1}, x_n)$, which is computed during the iteration.

2.4 Classical applications

2.4.1 Picard–Lindelöf theorem

Theorem 2.6 (Picard–Lindelöf)**.** *Let $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^n$, and $f : [t_0 - a, t_0 + a] \times \overline{B}(y_0, b) \to \mathbb{R}^n$ be continuous and Lipschitz in y with constant L. Set $M = \sup \|f\|$ and $\delta = \min(a, b/M)$. If $L\delta < 1$, then the Cauchy problem*

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

has a unique solution on $[t_0 - \delta, t_0 + \delta]$.

Proof. Let $X = C([t_0 - \delta, t_0 + \delta], \overline{B}(y_0, b))$ with the supremum norm. Define the Picard operator:

$$(Ty)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

We verify that T maps X into X: for $y \in X$,

$$\|(Ty)(t) - y_0\| \leq \int_{t_0}^{t} \|f(s, y(s))\|\, ds \leq M\delta \leq b.$$

We verify that T is a contraction: for $y, z \in X$,

$$\|(Ty)(t) - (Tz)(t)\| \leq \int_{t_0}^{t} L\, \|y(s) - z(s)\|\, ds \leq L\delta\, \|y - z\|_\infty .$$

Since $L\delta < 1$, the Banach principle applies and provides existence and uniqueness of the solution. □

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2.4.2 Fredholm integral equations

Theorem 2.7 (Fredholm equation)**.** *Let $K \in C([a,b]^2)$ and $f \in C([a,b])$. If $|\lambda| \cdot \sup_t \int_a^b |K(t,s)|\, ds < 1$, then the equation*

$$u(t) = f(t) + \lambda \int_a^b K(t,s)\, u(s)\, ds$$

has a unique solution in $C([a,b])$.

Proof. The operator $(Tu)(t) = f(t) + \lambda \int_a^b K(t,s)u(s)\, ds$ is a contraction on $(C([a,b]), \|\cdot\|_\infty)$ with constant $k = |\lambda| \sup_t \int_a^b |K(t,s)|\, ds < 1$. ☐

2.4.3 Local inversion theorem

Theorem 2.8 (Local inversion via Banach)**.** *Let E be a Banach space and $F : E \to E$ be C^1 with $DF(x_0)$ invertible. Then F is a local diffeomorphism near x_0.*

Proof sketch. Without loss of generality, $x_0 = 0$ and $DF(0) = \mathrm{Id}$, so $F(x) = x - \varphi(x)$ with $D\varphi(0) = 0$. For y near 0, the equation $F(x) = y$ becomes $x = y + \varphi(x) = T_y(x)$. By continuity of $D\varphi$, T_y is a contraction on a sufficiently small ball around 0. ☐

2.5 Direct generalizations

2.5.1 Contractions on closed subsets

Proposition 2.9 (Contraction on a closed set)**.** Let (X,d) be a complete metric space and $F \subset X$ a nonempty closed set. If $T : F \to F$ is a contraction, then T has a unique fixed point in F.

Proof. A closed subset of a complete space is complete, and we apply Theorem 2.1. ☐

2.5.2 Contractive iterates

Theorem 2.10 (Contractive iterates)**.** *Let (X,d) be a complete metric space and $T : X \to X$ continuous. If T^N is a contraction for some $N \geq 1$, then T has a unique fixed point.*

Proof. By Banach's theorem applied to T^N, there exists a unique x^* with $T^N(x^*) = x^*$. Then $T^N(T(x^*)) = T(T^N(x^*)) = T(x^*)$, so $T(x^*)$ is also a fixed point of T^N. By uniqueness, $T(x^*) = x^*$.

Conversely, every fixed point of T is a fixed point of T^N, so T has a unique fixed point. ☐

Example 2.11. The map $T : C([0,1]) \to C([0,1])$ defined by $(Tf)(x) = \int_0^x f(t)\, dt$ is not a contraction, but T^N is a contraction for N large enough since $\left\|T^N f\right\|_\infty \leq \frac{\|f\|_\infty}{N!}$.

2.5.3 Banach principle with parameter

Theorem 2.12 (Continuous dependence on a parameter)**.** *Let (X, d) be a complete metric space, Λ a topological space, and $T : X \times \Lambda \to X$ such that:*

1. *For every $\lambda \in \Lambda$, $T(\cdot, \lambda)$ is a contraction with constant $k < 1$ (uniform in λ);*
2. *For every $x \in X$, $T(x, \cdot) : \Lambda \to X$ is continuous.*

Then the map $\lambda \mapsto x^*(\lambda)$, where $x^*(\lambda)$ is the unique fixed point of $T(\cdot, \lambda)$, is continuous.

Proof. Let $\lambda_0 \in \Lambda$ and $\varepsilon > 0$. We have:

$$\begin{aligned} d(x^*(\lambda), x^*(\lambda_0)) &= d(T(x^*(\lambda), \lambda), T(x^*(\lambda_0), \lambda_0)) \\ &\le d(T(x^*(\lambda), \lambda), T(x^*(\lambda_0), \lambda)) + d(T(x^*(\lambda_0), \lambda), T(x^*(\lambda_0), \lambda_0)) \\ &\le k\, d(x^*(\lambda), x^*(\lambda_0)) + d(T(x^*(\lambda_0), \lambda), T(x^*(\lambda_0), \lambda_0)). \end{aligned}$$

Hence $(1-k)\, d(x^*(\lambda), x^*(\lambda_0)) \le d(T(x^*(\lambda_0), \lambda), T(x^*(\lambda_0), \lambda_0))$. The right-hand side tends to 0 as $\lambda \to \lambda_0$ by hypothesis (2). □

2.6 Generalized Banach–Caccioppoli theorem

Theorem 2.13 (Contraction on a ball)**.** *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$, and $T : \overline{B}(x_0, r) \to X$ a contraction with constant $k < 1$. If*

$$d(x_0, T(x_0)) < (1 - k)r,$$

then T has a unique fixed point in $\overline{B}(x_0, r)$.

Proof. It suffices to show that $T(\overline{B}(x_0, r)) \subset \overline{B}(x_0, r)$. For $x \in \overline{B}(x_0, r)$:

$$d(T(x), x_0) \le d(T(x), T(x_0)) + d(T(x_0), x_0) \le k\, d(x, x_0) + d(T(x_0), x_0) \le kr + (1-k)r = r.$$

Then apply Banach's theorem to $T : \overline{B}(x_0, r) \to \overline{B}(x_0, r)$. □

2.7 Nonexpansive mappings

Definition 2.14 (Nonexpansive mapping)**.** A mapping $T : X \to X$ is *nonexpansive* if $d(T(x), T(y)) \le d(x, y)$ for all $x, y \in X$.

Remark 2.15. A nonexpansive mapping need not have a fixed point, even in a complete space. For example, the translation $T(x) = x + 1$ on $\mathbb{R}$ is nonexpansive with no fixed point.

Theorem 2.16 (Browder–Göhde–Kirk, 1965)**.** *Let C be a bounded, closed, convex subset of a Hilbert space H. If $T : C \to C$ is nonexpansive, then T has a fixed point.*

Proof sketch. For $\lambda \in (0, 1)$, fix $a \in C$ and define $T_\lambda(x) = \lambda a + (1 - \lambda)T(x)$. Then T_λ is a contraction with constant $1 - \lambda$ and maps C into C by convexity. Let x_λ be its unique fixed point:

$$x_\lambda = \lambda a + (1 - \lambda)T(x_\lambda).$$

As $\lambda \to 0^+$, one shows (using the Hilbert space structure) that (x_λ) converges weakly to a fixed point of T. □

2.8 Boyd–Wong generalized contractions

Theorem 2.17 (Boyd–Wong, 1969)**.** *Let (X, d) be a complete metric space and $T : X \to X$ satisfying*

$$d(T(x), T(y)) \leq \psi(d(x, y)) \quad \forall x, y \in X,$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and satisfies $\psi(t) < t$ for all $t > 0$. Then T has a unique fixed point.

Proof. Fix $x_0 \in X$ and set $x_n = T^n(x_0)$. The sequence (d_n) defined by $d_n = d(x_n, x_{n+1})$ is decreasing since $d_{n+1} \leq \psi(d_n) < d_n$ when $d_n > 0$. It therefore converges to a limit $\ell \geq 0$.

If $\ell > 0$, by upper semicontinuity from the right of ψ: $\ell \leq \limsup_n \psi(d_n) \leq \psi(\ell) < \ell$, a contradiction. Hence $\ell = 0$.

We show that (x_n) is Cauchy. Argue by contradiction: there would exist $\varepsilon > 0$ and subsequences $m(k) > n(k) \to \infty$ with $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$. Choose $m(k)$ minimal, so $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$. Then

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) < d_{m(k)-1} + \varepsilon.$$

So $d(x_{m(k)}, x_{n(k)}) \to \varepsilon$. By the contractive condition:

$$d(x_{m(k)+1}, x_{n(k)+1}) \leq \psi(d(x_{m(k)}, x_{n(k)})).$$

Passing to the limit: $\varepsilon \leq \psi(\varepsilon) < \varepsilon$, a contradiction. Hence (x_n) is Cauchy.

By completeness, $x_n \to x^*$, and by continuity of T, $T(x^*) = x^*$. Uniqueness follows as in Banach's theorem. □

2.9 Meir–Keeler contractions

Theorem 2.18 (Meir–Keeler, 1969)**.** *Let (X, d) be a complete metric space and $T : X \to X$ satisfying: for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(T(x), T(y)) < \varepsilon.$$

Then T has a unique fixed point.

Remark 2.19. The Meir–Keeler condition implies $d(Tx, Ty) < d(x, y)$ for $x \neq y$ (take $\varepsilon = d(x, y)$), but it is strictly stronger than a simple weak contraction. Every Banach contraction (constant $k < 1$) satisfies the Meir–Keeler condition with $\delta = \varepsilon(1 - k)/k$.

2.10 Matkowski's fixed point theorem

Theorem 2.20 (Matkowski, 1975)**.** *Let (X, d) be a complete metric space and $T : X \to X$ satisfying*

$$d(T(x), T(y)) \leq \psi(d(x, y)) \quad \forall x, y \in X,$$

where $\psi : [0, \infty) \to [0, \infty)$ is increasing and satisfies $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point.

Remark 2.21. Note that if ψ is increasing and $\psi^n(t) \to 0$, then $\psi(t) < t$ for all $t > 0$. Indeed, if $\psi(t_0) \geq t_0$ for some $t_0 > 0$, then by monotonicity $\psi^n(t_0) \geq t_0$ for all n, contradicting $\psi^n(t_0) \to 0$.

2.11 Exercises

Exercise 2.1. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = \frac{1}{2}(x + a/x)$ for $x > 0$, where $a > 0$. Show that for every $x_0 > 0$, the sequence $(T^n(x_0))$ converges to $\sqrt{a}$. Is this a Banach contraction?

Exercise 2.2. Show that the map $T : C([0,1]) \to C([0,1])$ defined by $(Tf)(x) = \int_0^x f(t)\,dt$ is not a contraction, but T^n is a contraction for $n \geq 2$. Deduce that T has a unique fixed point (the zero function).

Exercise 2.3. Let (X, d) be a complete metric space and $T : X \to X$ such that $d(T(x), T(y)) \leq \alpha\, d(x, T(x)) + \beta\, d(y, T(y))$ with $\alpha + \beta < 1$. Show that T has a unique fixed point. *This is Kannan's condition, studied in Chapter 3.*

Exercise 2.4. Let A be an $n \times n$ matrix with $\|A\| < 1$ (operator norm). Show that $(I - A)$ is invertible and $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ (Neumann series). Prove this using the Banach principle.

Exercise 2.5 (Nadler's theorem)**.** Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$ a set-valued mapping with nonempty compact values that is contractive for the Hausdorff metric:

$$H(T(x), T(y)) \leq k\, d(x, y), \quad k < 1.$$

Show that T has a fixed point, i.e., there exists x^* with $x^* \in T(x^*)$.

Exercise 2.6. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 with $\|Df(x)\| \leq k < 1$ for all x (operator norm). Show that f is a contraction and deduce that the equation $x = f(x)$ has a unique solution.

Exercise 2.7. Study the convergence of Newton's method as a special case of the contraction principle. Specifically, let $g : \mathbb{R} \to \mathbb{R}$ be C^2 with $g(x^*) = 0$ and $g'(x^*) \neq 0$. Show that the iteration $x_{n+1} = x_n - g(x_n)/g'(x_n)$ converges to x^* for x_0 sufficiently close to x^*, and that the convergence is quadratic.

Chapter 3

Metric Extensions

3.1 Introduction

Banach's theorem is elegant, but its hypothesis—a strict, uniform contraction—is sometimes too restrictive. Starting in the 1960s, a whole community of mathematicians embarked on a quest: weaken the assumptions while preserving the conclusion. Edelstein showed that a *local* contraction suffices if the space is compact. Kannan (1968) discovered that a fixed point can exist even when T is not continuous—a surprise that profoundly changed our understanding of this phenomenon. Chatterjea proposed his own variant, Ćirić unified these results in a general framework, and Caristi introduced an entirely different approach based on decreasing energy functions.

Each extension illuminates a different facet of the mechanism that forces a fixed point to exist, and together they paint a remarkably rich theory.

3.2 Edelstein's theorem

Theorem 3.1 (Edelstein, 1962)**.** *Let (X, d) be a compact metric space and $T : X \to X$ a weak contraction, i.e.,*

$$d(T(x), T(y)) < d(x, y) \quad \forall x, y \in X,\ x \neq y.$$

Then T has a unique fixed point, and for every $x_0 \in X$, the sequence $(T^n(x_0))$ converges to this fixed point.

Proof. Define $\varphi : X \to [0, \infty)$ by $\varphi(x) = d(x, Tx)$. This function is continuous on the compact space X, so it attains its minimum at some $x^* \in X$.

Suppose $\varphi(x^*) > 0$, i.e., $x^* \neq Tx^*$. Then

$$\varphi(Tx^*) = d(Tx^*, T^2x^*) < d(x^*, Tx^*) = \varphi(x^*),$$

contradicting the minimality of $\varphi(x^*)$. Hence $\varphi(x^*) = 0$ and $Tx^* = x^*$.

Uniqueness is immediate: if y^* is another fixed point, then $d(x^*, y^*) = d(Tx^*, Ty^*) < d(x^*, y^*)$, a contradiction.

For convergence, let $x_0 \in X$ and $x_n = T^n(x_0)$. The sequence $(d(x_n, x^*))$ is decreasing since $d(x_{n+1}, x^*) = d(Tx_n, Tx^*) < d(x_n, x^*)$ when $x_n \neq x^*$. By compactness, (x_n) has a convergent subsequence $x_{n_k} \to z$. Then $d(z, x^*) = \lim d(x_{n_k}, x^*) = \inf_n d(x_n, x^*)$. If $z \neq x^*$, $d(Tz, x^*) < d(z, x^*)$, but $d(x_{n_k+1}, x^*) \to d(Tz, x^*)$, contradicting the fact that $(d(x_n, x^*))$ is decreasing with limit $d(z, x^*)$. Hence $z = x^*$ and $x_n \to x^*$. □

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Compactness is essential

Without compactness, Edelstein's theorem fails. The example $T(x) = x + 1/x$ on $[1, +\infty)$ (a complete but noncompact space) is a weak contraction with no fixed point (see Chapter 2).

3.3 Kannan contractions

Definition 3.2 (Kannan mapping)**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is a *Kannan contraction* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \le \alpha\big[d(x, Tx) + d(y, Ty)\big] \quad \forall x, y \in X.$$

Remark 3.3. Unlike a Banach contraction, a Kannan contraction need not be continuous. Indeed, Kannan's condition controls the distance $d(Tx, Ty)$ only through the displacements $d(x, Tx)$ and $d(y, Ty)$.

Theorem 3.4 (Kannan, 1968)**.** *Let (X, d) be a complete metric space and $T : X \to X$ a Kannan contraction with constant $\alpha \in [0, 1/2)$. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ and $x_n = T^n(x_0)$. We have:

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le \alpha\big[d(x_n, Tx_n)+d(x_{n-1}, Tx_{n-1})\big] = \alpha\big[d(x_n, x_{n+1})+d(x_{n-1}, x_n)\big].$$

Hence $(1-\alpha)\, d(x_{n+1}, x_n) \le \alpha\, d(x_n, x_{n-1})$, so

$$d(x_{n+1}, x_n) \le \frac{\alpha}{1-\alpha}\, d(x_n, x_{n-1}) = \beta\, d(x_n, x_{n-1})$$

with $\beta = \alpha/(1-\alpha) < 1$ since $\alpha < 1/2$.

Thus $d(x_{n+1}, x_n) \le \beta^n d(x_1, x_0)$. The sequence (x_n) is Cauchy (same argument as for Banach) and converges to $x^* \in X$.

We verify that x^* is a fixed point:

$$d(x^*, Tx^*) \le d(x^*, x_{n+1})+d(x_{n+1}, Tx^*) = d(x^*, x_{n+1})+d(Tx_n, Tx^*) \le d(x^*, x_{n+1})+\alpha[d(x_n, x_{n+1})+d(x^*, T$$

Hence $(1-\alpha)\, d(x^*, Tx^*) \le d(x^*, x_{n+1}) + \alpha\, d(x_n, x_{n+1}) \to 0$, and $Tx^* = x^*$.

Uniqueness: if x^*, y^* are two fixed points, then $d(x^*, y^*) = d(Tx^*, Ty^*) \le \alpha[d(x^*, Tx^*)+ d(y^*, Ty^*)] = 0$. ☐

Kannan vs. Banach

Banach and Kannan contractions are independent classes:

- There exist Banach contractions that are not Kannan contractions;
- There exist Kannan contractions that are not Banach contractions (and that are not even continuous).

Nevertheless, both conditions guarantee existence and uniqueness of a fixed point in a complete metric space.

Example 3.5 (Discontinuous Kannan map)**.** Let $X = [0, 1]$ and T defined by $T(x) = x/4$ for $x \in [0, 1)$ and $T(1) = 1/6$. Then T is a Kannan contraction (with suitable α) but T is not continuous at $x = 1$. Its unique fixed point is $x^* = 0$.

3.4 Chatterjea contractions

Definition 3.6 (Chatterjea mapping)**.** A mapping $T : X \to X$ is a *Chatterjea contraction* if there exists $\beta \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \beta\big[d(x, Ty) + d(y, Tx)\big] \quad \forall x, y \in X.$$

Theorem 3.7 (Chatterjea, 1972)**.** *Let (X, d) be a complete metric space and $T : X \to X$ a Chatterjea contraction. Then T has a unique fixed point.*

Proof. Let $x_n = T^n(x_0)$. We have:

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \leq \beta[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] \\ &= \beta[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] = \beta\, d(x_{n-1}, x_{n+1}) \\ &\leq \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

Hence $(1 - \beta)d(x_{n+1}, x_n) \leq \beta\, d(x_n, x_{n-1})$, and we conclude as for Kannan with the ratio $\beta/(1 - \beta) < 1$.

The convergence of (x_n) to a fixed point x^* and uniqueness are proved analogously. □

3.5 Ćirić's contraction

Definition 3.8 (Ćirić quasi-contraction)**.** A mapping $T : X \to X$ is a *Ćirić quasi-contraction* if there exists $q \in [0, 1)$ such that

$$d(Tx, Ty) \leq q \cdot \max\{d(x, y),\, d(x, Tx),\, d(y, Ty),\, d(x, Ty),\, d(y, Tx)\}$$

for all $x, y \in X$.

Remark 3.9. Ćirić's condition simultaneously generalizes those of Banach ($q \cdot d(x, y)$), Kannan ($q \cdot [d(x, Tx) + d(y, Ty)]/2$), and Chatterjea ($q \cdot [d(x, Ty) + d(y, Tx)]/2$).

Theorem 3.10 (Ćirić, 1974)**.** *Let (X, d) be a complete metric space and $T : X \to X$ a Ćirić quasi-contraction with constant $q \in [0, 1)$. Then T has a unique fixed point x^*, and for every $x_0 \in X$, $T^n(x_0) \to x^*$.*

Proof. Set $x_n = T^n(x_0)$ and $d_n = d(x_n, x_{n+1})$. We have:

$$d_n = d(Tx_{n-1}, Tx_n) \leq q \cdot \max\{d_{n-1}, d_{n-1}, d_n, d(x_{n-1}, x_{n+1}), 0\}.$$

By the triangle inequality, $d(x_{n-1}, x_{n+1}) \leq d_{n-1} + d_n$, so

$$d_n \leq q \cdot \max\{d_{n-1}, d_n, d_{n-1} + d_n\}.$$

If $d_n > d_{n-1}$, then $\max = d_{n-1} + d_n \leq 2d_n$, giving $d_n \leq 2qd_n$, impossible for $q < 1/2$. A more delicate argument handles the general case and shows $d_n \leq q \cdot d_{n-1}$ when $d_n \leq d_{n-1}$.

The sequence is Cauchy and converges to x^*. To verify x^* is a fixed point:

$$d(Tx^*, x^*) \leq d(Tx^*, Tx_n) + d(x_{n+1}, x^*) \leq q \cdot \max\{d(x^*, x_n), d(x^*, Tx^*), d_n, d(x^*, x_{n+1}), d(x_n, Tx^*)\} + d(x_{n+$$

Passing to the limit, $d(Tx^*, x^*) \leq q \cdot d(x^*, Tx^*)$, hence $d(Tx^*, x^*) = 0$.

Uniqueness: if $y^* = Ty^*$, then $d(x^*, y^*) = d(Tx^*, Ty^*) \leq q \cdot d(x^*, y^*)$, hence $d(x^*, y^*) = 0$. □

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3.6 Caristi's theorem

Caristi's theorem is remarkable because it assumes no continuity condition on T.

Theorem 3.11 (Caristi, 1976)**.** *Let (X, d) be a complete metric space and $\varphi : X \to [0, \infty)$ a lower semicontinuous function. If $T : X \to X$ satisfies*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \quad \forall x \in X,$$

then T has a fixed point.

Proof. Define a partial order on X by

$$x \preceq y \iff d(x, y) \leq \varphi(x) - \varphi(y).$$

We verify this is indeed a partial order:

- Reflexivity: $d(x, x) = 0 \leq \varphi(x) - \varphi(x) = 0$.
- Antisymmetry: if $x \preceq y$ and $y \preceq x$, then $d(x, y) = 0$.
- Transitivity: if $x \preceq y$ and $y \preceq z$, then $d(x, z) \leq d(x, y) + d(y, z) \leq [\varphi(x) - \varphi(y)] + [\varphi(y) - \varphi(z)] = \varphi(x) - \varphi(z)$.

The hypothesis says $x \preceq Tx$ for every x.

By Zorn's lemma, it suffices to show every chain $(x_\alpha)_{\alpha \in A}$ in $(X, \preceq)$ has an upper bound. Let (x_α) be a totally ordered chain. The function $\alpha \mapsto \varphi(x_\alpha)$ is decreasing and bounded below by 0, so the net $(\varphi(x_\alpha))$ has an infimum ℓ. One can extract a generalized Cauchy subnet: for $\alpha \leq \beta$, $d(x_\alpha, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta) \to 0$. By completeness, it converges to $\bar{x}$. By lower semicontinuity, $\varphi(\bar{x}) \leq \ell$. One verifies that $\bar{x}$ is an upper bound of the chain.

By Zorn's lemma, there exists a maximal element x^*. Since $x^* \preceq Tx^*$ and x^* is maximal, we have $Tx^* = x^*$. □

Caristi's condition

$d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ with φ l.s.c. and $\varphi \geq 0$ in (X, d) complete $\implies$ T has a fixed point.

3.7 Equivalence of Caristi and Ekeland's variational principle

Theorem 3.12 (Ekeland's variational principle, 1974)**.** *Let (X, d) be a complete metric space and $\varphi : X \to (-\infty, +\infty]$ lower semicontinuous, bounded below, and not identically $+\infty$. For every $\varepsilon > 0$ and every $x_0 \in X$ with $\varphi(x_0) \leq \inf_X \varphi + \varepsilon$, there exists $x^* \in X$ such that:*

1. *$\varphi(x^*) \leq \varphi(x_0)$;*
2. *$d(x_0, x^*) \leq 1$;*
3. *$\varphi(x^*) < \varphi(x) + \varepsilon\, d(x, x^*)$ for all $x \neq x^*$.*

Proposition 3.13. Caristi's theorem and Ekeland's variational principle are equivalent.

Sketch. **Ekeland** ⇒ **Caristi**: If T satisfies Caristi's condition, let x^* be given by Ekeland with suitable ε. Condition (3) with $x = Tx^*$ and Caristi's condition lead to $Tx^* = x^*$.

Caristi ⇒ **Ekeland**: More technical, using the construction of a decreasing sequence for φ in an appropriate ball. □

3.8 Zamfirescu contractions

Definition 3.14 (Zamfirescu contraction)**.** $T : X \to X$ is a *Zamfirescu contraction* if there exist $a \in [0, 1)$, $b \in [0, 1/2)$, $c \in [0, 1/2)$ such that for all x, y, at least one of the following holds:

1. $d(Tx, Ty) \leq a\, d(x, y)$;
2. $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;
3. $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Theorem 3.15 (Zamfirescu, 1972)**.** *Every Zamfirescu contraction on a complete metric space has a unique fixed point.*

Proof. Set $\delta = \max\{a, b/(1-b), c/(1-c)\}$. Then $\delta < 1$ and $d(Tx, Ty) \leq \delta\, d(x, y)$ for all x, y, reducing to Banach's theorem.

Indeed, if condition (1) holds, $d(Tx, Ty) \leq a\, d(x, y)$. If condition (2) holds, $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \leq b[d(x, y) + 2d(y, Ty)] + b\, d(Tx, Ty)$, giving $(1-b)d(Tx, Ty) \leq b[d(x, y) + 2d(y, Ty)]$, and careful reasoning shows $d(Tx, Ty) \leq \frac{2b}{1-b} d(x, y)$. A similar argument applies for (3). □

3.9 Subrahmanyam's characterization

Theorem 3.16 (Subrahmanyam, 1975)**.** *A metric space (X, d) is complete if and only if every Kannan contraction $T : X \to X$ has a fixed point.*

Remark 3.17. This result is remarkable because it shows that Kannan's condition *characterizes* metric completeness, unlike Banach's condition. There exist incomplete metric spaces in which every Banach contraction has a fixed point.

3.10 Exercises

Exercise 3.1. Show that $T(x) = \sin(x)$ is a weak contraction on $[0, \pi]$ but not a Banach contraction. Apply Edelstein's theorem.

Exercise 3.2. Construct a Kannan contraction that is not continuous and verify directly that it has a unique fixed point.

Exercise 3.3. Show that every Banach contraction is a Ćirić quasi-contraction, but the converse is false.

Exercise 3.4. Apply Caristi's theorem to the operator $T : C([0, 1]) \to C([0, 1])$ defined by $(Tf)(t) = \frac{1}{2} f(t)$, with $\varphi(f) = \|f\|_\infty$.

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Exercise 3.5. Prove in detail the equivalence between Caristi's theorem and Ekeland's variational principle.

Exercise 3.6. Let (X, d) be complete and $T : X \to X$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ with $\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon$ (Meir–Keeler condition). Show that this condition is stronger than Edelstein's weak contraction but weaker than Banach contraction.

Chapter 4

Generalized Metric Spaces

4.1 Introduction

The notion of metric space, as Fréchet defined it in 1906, rests on three simple axioms. But what happens if we relax one of them? If the triangle inequality only holds up to a multiplicative constant, we obtain a *b-metric space*. If the distance from a point to itself is not necessarily zero, we enter the world of *partial metric spaces*, invented by Steve Matthews in 1994 to model computational domains. Each generalisation arises from a concrete need—fractal analysis, program semantics, measure theory—and each raises the same question: does the fixed point theorem survive?

This chapter shows that the answer is yes, with subtle adaptations, and presents b-metric spaces, partial metric spaces, and G-metric spaces, with their associated fixed point theorems.

4.2 b-Metric spaces

Definition 4.1 (b-metric space)**.** Let X be a set and $s \geq 1$ a real number. A function $d : X \times X \to [0, +\infty)$ is a *b-metric* (with constant s) if for all $x, y, z \in X$:

1. $d(x, y) = 0 \Leftrightarrow x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s\big[d(x, y) + d(y, z)\big]$ (b-triangle inequality).

The pair (X, d) is then called a *b-metric space*.

Remark 4.2. For $s = 1$, we recover the classical definition of a metric space. The constant s measures the "defect" of the triangle inequality.

Example 4.3 (Natural b-metric)**.** Let $X = \mathbb{R}$ and $d(x, y) = (x - y)^2$. Then (X, d) is a b-metric space with $s = 2$. Indeed, $(x-z)^2 = ((x-y)+(y-z))^2 \leq 2(x-y)^2+2(y-z)^2 = 2[d(x, y) + d(y, z)]$ by the inequality $(a + b)^2 \leq 2(a^2 + b^2)$.

However, (X, d) is not a classical metric space since $d(0, 2) = 4 > d(0, 1) + d(1, 2) = 1 + 1 = 2$.

Example 4.4 (ℓ^p space with $0 < p < 1$)**.** The space ℓ^p for $0 < p < 1$ equipped with $d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ is a b-metric space with $s = 2^{1/p-1}$, but it is not a metric space.

Definition 4.5 (Completeness in a b-metric space)**.** The notions of Cauchy sequence and completeness are defined as in the classical case: (x_n) is Cauchy if $d(x_m, x_n) \to 0$ as $m, n \to \infty$, and (X, d) is complete if every Cauchy sequence converges.

Difficulties specific to b-metrics

In a b-metric space:

- The induced topology may not be metrizable in the classical sense;
- A b-metric is not necessarily continuous (the function $(x, y) \mapsto d(x, y)$ may not be continuous for the induced topology);
- Open balls may not be open sets in the induced topology.

4.2.1 Contraction principle in b-metric spaces

Theorem 4.6 (Banach for b-metrics)**.** *Let (X, d) be a complete b-metric space with constant $s \geq 1$, and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq k\, d(x, y) \quad \forall x, y \in X$$

with $0 \leq k < 1/s$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and $x_n = T^n(x_0)$. We have $d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$. For $m > n$, applying the b-triangle inequality recursively:

$$\begin{aligned} d(x_n, x_m) &\leq s\, d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} d(x_{m-1}, x_m) \\ &\leq \sum_{i=0}^{m-n-1} s^{i+1} k^{n+i} d(x_1, x_0) = sk^n \sum_{i=0}^{m-n-1} (sk)^i\, d(x_1, x_0). \end{aligned}$$

Since $sk < 1$, the geometric series converges and $d(x_n, x_m) \leq \frac{sk^n}{1-sk} d(x_1, x_0) \to 0$. So (x_n) is Cauchy and converges to $x^* \in X$.

Using: $d(x^*, Tx^*) \leq s[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)] \leq s[d(x^*, x_{n+1}) + k\, d(x_n, x^*)] \to 0$.

Uniqueness follows the same argument as in the classical case. □

Remark 4.7. The condition $k < 1/s$ is more restrictive than $k < 1$. For $s = 1$ (classical metric space), we recover the usual condition $k < 1$.

Theorem 4.8 (Improved version, Czerwik 1998)**.** *If (X, d) is a complete b-metric space and $T : X \to X$ satisfies $d(Tx, Ty) \leq k\, d(x, y)$ with $k < 1$ (without the restriction $k < 1/s$), then T has a unique fixed point, provided d is continuous.*

4.3 Partial metric spaces

Definition 4.9 (Partial metric space)**.** A *partial metric space* is a pair (X, p) where $p : X \times X \to [0, +\infty)$ satisfies for all $x, y, z \in X$:

1. $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
2. $p(x, x) \leq p(x, y)$ (small self-distance);

3. $p(x, y) = p(y, x)$ (symmetry);
4. $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$ (modified triangle inequality).

Remark 4.10. The essential difference from a classical metric is that $p(x, x)$ can be strictly positive. This is motivated by theoretical computer science: $p(x, x)$ represents the "size" or "cost" of the information x.

Example 4.11. On $X = [0, +\infty)$, $p(x, y) = \max(x, y)$ is a partial metric. We have $p(x, x) = x \ge 0$, with equality only at $x = 0$.

Theorem 4.12 (Matthews, 1994)**.** *Let (X, p) be a complete partial metric space and $T : X \to X$ a contraction, i.e., $p(Tx, Ty) \le k\,p(x, y)$ with $k \in [0, 1)$. Then T has a unique fixed point x^*, and $p(x^*, x^*) = 0$.*

Proof. The partial metric p induces a classical metric $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, and T is also a contraction for d_p. By Banach's theorem applied to (X, d_p) (which is complete if (X, p) is), T has a unique fixed point x^*. Moreover, $p(x^*, x^*) = p(Tx^*, Tx^*) \le k\,p(x^*, x^*) \le k^2 p(x^*, x^*) \le \ldots$, hence $p(x^*, x^*) = 0$. □

4.4 G-Metric spaces

Definition 4.13 (G-metric space (Mustafa–Sims))**.** A *G-metric space* is a pair (X, G) where $G : X \times X \times X \to [0, +\infty)$ satisfies for all $x, y, z, a \in X$:

1. $G(x, y, z) = 0 \Leftrightarrow x = y = z$;
2. $0 < G(x, x, y)$ for $x \ne y$;
3. $G(x, x, y) \le G(x, y, z)$ for $z \ne y$;
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all variables);
5. $G(x, y, z) \le G(x, a, a) + G(a, y, z)$ (rectangle inequality).

Example 4.14. For any metric space (X, d), one can define $G(x, y, z) = d(x, y) + d(y, z) + d(x, z)$ or $G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$. Both define G-metrics.

Theorem 4.15 (Mustafa–Sims, 2006)**.** *Let (X, G) be a complete G-metric space and $T : X \to X$ satisfying*

$$G(Tx, Ty, Tz) \le k\,G(x, y, z) \quad \forall x, y, z \in X$$

with $k \in [0, 1)$. Then T has a unique fixed point.

Proof. Fix x_0 and set $x_n = T^n(x_0)$. We have:

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \le k\,G(x_{n-1}, x_n, x_n) \le k^n\,G(x_0, x_1, x_1).$$

For $m > n$:

$$G(x_n, x_m, x_m) \le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m) \le \sum_{i=n}^{m-1} k^i G(x_0, x_1, x_1) \le \frac{k^n}{1-k} G(x_0, x_1, x_1).$$

So (x_n) is G-Cauchy and converges to $x^* \in X$.

$G(x^*, Tx^*, Tx^*) \le G(x^*, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tx^*, Tx^*) \le G(x^*, x_{n+1}, x_{n+1}) + k\,G(x_n, x^*, x^*) \to 0$. Hence $Tx^* = x^*$. □

Remark 4.16. Jleli and Samet (2012) showed that many fixed point results in G-metric spaces can be deduced from classical results in ordinary metric spaces via the associated metric $d_G(x, y) = G(x, y, y) + G(x, x, y)$. This observation has sparked debate about the actual utility of the G-metric concept.

4.5 Rectangular metric spaces

Definition 4.17 (Rectangular metric space (Branciari))**.** A rectangular metric space is a pair (X, d) where $d : X \times X \to [0, +\infty)$ satisfies axioms (1) and (2) of a metric space, and where the triangle inequality is replaced by the *rectangular inequality*: for all distinct x, y and all $u, v \in X \setminus \{x, y\}$ distinct,

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, y).$$

Theorem 4.18 (Branciari, 2000)**.** *Let (X, d) be a complete rectangular metric space and $T : X \to X$ a Banach contraction ($d(Tx, Ty) \leq k\, d(x, y)$, $k < 1$). Then T has a unique fixed point.*

Subtleties

Branciari's original proof contained gaps, notably because in a rectangular space, limits are not necessarily unique and the topology can be pathological. Corrections were provided by Sarma et al. (2009).

4.6 Modular metric spaces

Definition 4.19 (Modular metric space)**.** Let X be a set. A function $\omega : (0, \infty) \times X \times X \to [0, +\infty)$ is a *metric modular* if for all $x, y, z \in X$ and all $\lambda, \mu > 0$:

1. $\omega(\lambda, x, y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$;
2. $\omega(\lambda, x, y) = \omega(\lambda, y, x)$;
3. $\omega(\lambda + \mu, x, z) \leq \omega(\lambda, x, y) + \omega(\mu, y, z)$.

Theorem 4.20. *Let $X_\omega = \{x \in X : \omega(\lambda, x, x_0) < \infty$ for some $\lambda > 0\}$ be the associated modular metric space. If X_ω is ω-complete and $T : X_\omega \to X_\omega$ satisfies $\omega(\lambda, Tx, Ty) \leq k\, \omega(\lambda, x, y)$ for all $\lambda > 0$ with $k < 1$, then T has a unique fixed point.*

4.7 Comparison of generalizations

Summary of generalized spaces

Space	Modification	FP condition
Metric	–	$k < 1$
b-metric	$d(x,z) \leq s[d(x,y) + d(y,z)]$	$k < 1/s$ or $k < 1$
Partial	$p(x,x) \geq 0$	$k < 1$
G-metric	3 variables	$k < 1$
Rectangular	Rectangular ineq.	$k < 1$

4.8 Exercises

Exercise 4.1. Show that $d(x,y) = |x-y|^2$ defines a b-metric space on $\mathbb{R}$ with $s = 2$, but not a metric space.

Exercise 4.2. Verify that ℓ^p for $0 < p < 1$ with $d(x,y) = \sum |x_n - y_n|^p$ is a complete b-metric space and compute the constant s.

Exercise 4.3. Let (X,p) be a partial metric space. Show that $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a classical metric on X.

Exercise 4.4. Let (X,d) be a metric space. Show that $G(x,y,z) = d(x,y) + d(y,z) + d(x,z)$ defines a G-metric space. What is the relationship between completeness of (X,d) and that of (X,G)?

Exercise 4.5. Construct a rectangular metric space that is not a metric space. *Hint: consider a four-point set with a carefully chosen metric.*

Exercise 4.6. Let (X,d) be a complete b-metric space with constant $s = 2$ and $T : X \to X$ with $d(Tx,Ty) \leq \frac{1}{3}d(x,y)$. Show that T has a unique fixed point and compute the a priori error estimate.

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Chapter 5

Brouwer Fixed Point Theorem

5.1 Introduction and statement

Stir your coffee in a cup, as vigorously as you like. When the liquid settles, at least one point ends up exactly where it started. This astonishing fact is a consequence of Brouwer's fixed point theorem (1911), one of the jewels of topology. L. E. J. Brouwer proved it using combinatorial methods (Sperner's lemma), and since then the theorem has found applications in game theory (Nash equilibrium), mathematical economics (Arrow-Debreu theorem), and nonlinear analysis.

Unlike the Banach principle, it is purely topological in nature: no metric hypothesis on T is required, only continuity suffices.

Theorem 5.1 (Brouwer, 1911)**.** *Let $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. Every continuous map $f : B^n \to B^n$ has at least one fixed point.*

More generally:

Theorem 5.2 (Brouwer, general version)**.** *Let K be a nonempty compact convex subset of $\mathbb{R}^n$. Every continuous map $f : K \to K$ has a fixed point.*

Remark 5.3. The two versions are equivalent. Indeed, every nonempty compact convex subset of $\mathbb{R}^n$ is homeomorphic to B^m for some $m \leq n$ (if K has dimension m).

5.2 The no-retraction lemma

Brouwer's theorem is equivalent to the following topological result.

Definition 5.4 (Retraction)**.** Let $A \subset X$ be a subspace. A *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$.

Theorem 5.5 (No retraction)**.** *There is no continuous retraction of B^n onto $S^{n-1} = \partial B^n$.*

Proposition 5.6. Theorem 5.1 and Theorem 5.5 are equivalent.

Proof. **No retraction $\Rightarrow$ Brouwer.** Suppose by contradiction that $f : B^n \to B^n$ continuous has no fixed point. Then $f(x) \neq x$ for all $x \in B^n$. Define $r : B^n \to S^{n-1}$ by sending x to the intersection of the ray from $f(x)$ through x with the sphere S^{n-1}. This map is continuous (explicit formulas show this) and $r(x) = x$ for $x \in S^{n-1}$, contradicting the no-retraction theorem.

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Brouwer $\Rightarrow$ No retraction. Suppose there exists a retraction $r : B^n \to S^{n-1}$. Define $f : B^n \to B^n$ by $f(x) = -r(x)$. Then f is continuous and $f(B^n) \subset S^{n-1} \subset B^n$. If $f(x^*) = x^*$, then $x^* \in S^{n-1}$ and $-r(x^*) = x^*$, so $-x^* = x^*$, hence $x^* = 0$, contradicting $x^* \in S^{n-1}$. □

5.3 Proof via Sperner's lemma

We first present the combinatorial proof, due to Knaster, Kuratowski, and Mazurkiewicz (1929), which uses Sperner's lemma.

5.3.1 Sperner's lemma

Definition 5.7 (Triangulation and Sperner labeling)**.** Let $\sigma = [v_0, \dots, v_n]$ be an n-simplex in $\mathbb{R}^n$. A *triangulation* of σ is a simplicial complex $\mathcal{T}$ whose union is σ. A *Sperner labeling* is a map $\ell : V(\mathcal{T}) \to \{0, 1, \dots, n\}$ (where $V(\mathcal{T})$ is the vertex set of $\mathcal{T}$) such that:

- $\ell(v_i) = i$ for each vertex v_i of σ;
- If v lies on the face $[v_{i_1}, \dots, v_{i_k}]$, then $\ell(v) \in \{i_1, \dots, i_k\}$.

Lemma 5.8 (Sperner, 1928)**.** *Let σ be an n-simplex with a triangulation $\mathcal{T}$ and a Sperner labeling. Then the number of subsimplices of $\mathcal{T}$ whose vertices carry all labels $\{0, 1, \dots, n\}$ is odd (and hence nonzero).*

Proof in dimension $n = 1$. Consider the segment $[v_0, v_1]$ subdivided into subsegments. The vertices are labeled 0 or 1, with v_0 labeled 0 and v_1 labeled 1. Traversing the segment from left to right, the number of label changes is odd (one goes from an initial 0 to a final 1). □

Proof in general dimension (sketch). One uses a parity argument. Consider the dual graph: each n-simplex of $\mathcal{T}$ is a vertex, and two vertices are connected if they share an $(n-1)$-dimensional face carrying labels $\{0, \dots, n-1\}$. One shows that the vertices of odd degree are exactly the completely labeled simplices and those with a boundary face carrying labels $\{0, \dots, n-1\}$. By the handshaking lemma and by induction (Sperner in dimension $n-1$), the number of completely labeled simplices is odd. □

5.3.2 Proof of Brouwer's theorem via Sperner

Proof of Theorem 5.1. Let $\sigma = [e_0, e_1, \dots, e_n]$ be the standard simplex in $\mathbb{R}^{n+1}$ (where e_i are the basis vectors). Let $f : \sigma \to \sigma$ be continuous. For each $m \geq 1$, consider the barycentric triangulation of σ with mesh $\leq 1/m$.

Define the labeling: for a vertex v of the triangulation, write $v = \sum \lambda_i e_i$ in barycentric coordinates. Set $\ell(v) = \min\{i : f(v)_i \leq v_i\}$, where $f(v)_i$ and v_i are the i-th barycentric coordinates. Such an i exists since $\sum f(v)_i = \sum v_i = 1$ implies $f(v)_i \leq v_i$ for at least one i.

This labeling is a Sperner labeling (if v lies on the face $\{i : v_i = 0\}^c$, then $\ell(v) \in \{i : v_i > 0\}$).

By Sperner's lemma, there exists a completely labeled subsimplex σ_m. Its vertices $v_m^0, \dots, v_m^n$ satisfy $f(v_m^i)_i \leq (v_m^i)_i$ and $\mathrm{diam}(\sigma_m) \leq 1/m$.

By compactness, extract a subsequence such that $v_m^i \to x^*$ for all i. By continuity of f and passing to the limit: $f(x^*)_i \le x_i^*$ for all i. Since $\sum f(x^*)_i = \sum x_i^* = 1$, we have $f(x^*)_i = x_i^*$ for all i, hence $f(x^*) = x^*$. □

5.4 Analytical proof

We present an analytical proof of the no-retraction theorem based on differential calculus.

Analytical proof of Theorem 5.5. **Step 1: reduction to the C^∞ case.** By the Weierstrass approximation theorem, any continuous map $r : B^n \to S^{n-1}$ with $r|_{S^{n-1}} = \mathrm{Id}$ can be uniformly approximated by C^∞ maps. By a truncation argument, we may assume r is C^∞.

Step 2: use of the change-of-variables formula. If $r : B^n \to S^{n-1}$ is C^∞ with $r|_{S^{n-1}} = \mathrm{Id}$, consider the differential form $\omega = \sum_{i=1}^n (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ on S^{n-1}. Then $\int_{S^{n-1}} \omega = \mathrm{Vol}(S^{n-1}) \neq 0$.

But $r^*\omega = \omega$ on S^{n-1}, and by Stokes' theorem:

$$\int_{S^{n-1}} \omega = \int_{S^{n-1}} r^*\omega = \int_{B^n} d(r^*\omega).$$

Now r takes values in S^{n-1}, which has dimension $n-1$, so $r^*\omega$ is exact on B^n and $d(r^*\omega) = r^*(d\omega) = 0$ since $d\omega = n \, dx_1 \wedge \cdots \wedge dx_n$ and r maps into an $(n-1)$-dimensional manifold. Contradiction. □

5.5 Consequences of Brouwer's theorem

5.5.1 Fixed point theorem for simplices

Corollary 5.9. *Every continuous map from a simplex to itself has a fixed point.*

5.5.2 Borsuk–Ulam theorem (dimension 1)

Theorem 5.10 (Borsuk–Ulam in dimension 1)**.** *For every continuous map $f : S^1 \to \mathbb{R}$, there exists $x \in S^1$ such that $f(x) = f(-x)$.*

Proof. Set $g(x) = f(x) - f(-x)$. Then $g(-x) = -g(x)$. If g never vanishes, g has constant sign, say $g(x_0) > 0$, but $g(-x_0) = -g(x_0) < 0$, a contradiction by the Intermediate Value Theorem (on an arc connecting x_0 to $-x_0$). □

5.5.3 Ham Sandwich theorem

Theorem 5.11 (Ham Sandwich)**.** *Let $A_1, \ldots, A_n$ be bounded measurable subsets of $\mathbb{R}^n$, each of positive measure. There exists a hyperplane that bisects each A_i into two parts of equal measure.*

Remark 5.12. This theorem is a consequence of the Borsuk–Ulam theorem.

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5.6 Counter-examples: necessity of hypotheses

Example 5.13 (Non-convexity)**.** The rotation by angle $\pi/3$ on the circle S^1 is a continuous map from S^1 to S^1 with no fixed point. The circle is compact but not convex.

Example 5.14 (Non-compactness)**.** The translation $T(x) = x + e_1$ on $\mathbb{R}^n$ is continuous with no fixed point. The space $\mathbb{R}^n$ is convex but not compact (nor bounded).

Example 5.15 (Infinite dimension)**.** Consider the unit ball B of ℓ^2 and the shift

$$T(x_1, x_2, \ldots) = (\sqrt{1 - \|x\|^2}, x_1, x_2, \ldots).$$

One verifies that $T : B \to B$ is continuous but has no fixed point (if $Tx = x$, then $x_1 = \sqrt{1 - \|x\|^2}$ and $x_{n+1} = x_n$ for all n, so $x = (c, c, c, \ldots)$ which is not in ℓ^2 unless $c = 0$, but $c = \sqrt{1 - 0} = 1$, contradiction). This shows that Brouwer's theorem does not extend to infinite dimensions (see Schauder's theorem in Chapter 6).

5.7 Brouwer's topological degree

Definition 5.16 (Brouwer degree)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \overline{\Omega} \to \mathbb{R}^n$ continuous, and $p \notin f(\partial\Omega)$. The *Brouwer degree* $\deg(f, \Omega, p) \in \mathbb{Z}$ is the unique integer satisfying:

1. **Normalization**: $\deg(\mathrm{Id}, \Omega, p) = 1$ if $p \in \Omega$;
2. **Additivity**: if $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and $p \notin f(\partial\Omega_i)$, then $\deg(f, \Omega, p) = \deg(f, \Omega_1, p) + \deg(f, \Omega_2, p)$;
3. **Homotopy invariance**: if $H : [0, 1] \times \overline{\Omega} \to \mathbb{R}^n$ is continuous with $p \notin H(t, \partial\Omega)$ for all t, then $\deg(H(t, \cdot), \Omega, p)$ is constant in t.

Theorem 5.17 (Brouwer via degree)**.** *Let $f : B^n \to B^n$ be continuous. Then* $\deg(\mathrm{Id} - f, B^n, 0) = 1$*, so f has a fixed point.*

Proof. The homotopy $H(t, x) = x - tf(x)$ satisfies $H(0, x) = x$ and $H(1, x) = x - f(x)$. For $x \in S^{n-1}$ and $t \in [0, 1]$: $H(t, x) = 0$ would imply $x = tf(x)$, hence $1 = \|x\| = t\,\|f(x)\| \leq t$, so $t = 1$ and $f(x) = x$, meaning x is a fixed point on the boundary. Even in this case, the degree is well defined and equals 1 by normalization.

Hence $\deg(\mathrm{Id} - f, B^n, 0) = \deg(\mathrm{Id}, B^n, 0) = 1 \neq 0$, and the equation $x - f(x) = 0$ has a solution. □

5.8 Lefschetz fixed point theorem

Theorem 5.18 (Lefschetz)**.** *Let X be a compact absolute neighborhood retract (ANR) and $f : X \to X$ continuous. If the Lefschetz number*

$$\Lambda(f) = \sum_{k=0}^{\infty} (-1)^k \mathrm{tr}(f_{*k} : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q}))$$

is nonzero, then f has a fixed point.

Remark 5.19. Brouwer's theorem is a special case: for $X = B^n$, all homology groups are trivial except $H_0 \cong \mathbb{Q}$, and $f_{*0} = \mathrm{Id}$, so $\Lambda(f) = 1 \neq 0$.

5.9 Exercises

Exercise 5.1. Prove Brouwer's theorem directly (without Sperner's lemma) in dimension $n = 1$, i.e., that every continuous $f : [0, 1] \to [0, 1]$ has a fixed point.

Exercise 5.2. Prove Sperner's lemma in dimension $n = 2$ using the parity argument (dual graph).

Exercise 5.3. Use Brouwer's theorem to show that every $n \times n$ matrix with nonnegative real entries whose column sums are 1 has an eigenvector associated to the eigenvalue 1.

Exercise 5.4. Show that Brouwer's theorem implies the fundamental theorem of algebra: every polynomial with complex coefficients of degree $n \geq 1$ has a root. *Hint: use the topological degree.*

Exercise 5.5. Let A be a real $n \times n$ matrix. Show that there exist $\lambda \in \mathbb{R}$ and $x \in S^{n-1}$ such that $Ax = \lambda x$ (i.e., A has a "generalized eigenvector" on the sphere).

Exercise 5.6. Explicitly construct a continuous map $f : B^\infty \to B^\infty$ with no fixed point, where B^∞ is the closed unit ball of an infinite-dimensional Hilbert space.

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Chapter 6

Schauder Fixed Point Theorem and Variants

6.1 Motivation

Brouwer's theorem is a gem of finite-dimensional topology, but it fails spectacularly in infinite dimensions: the closed unit ball of an infinite-dimensional Banach space is not compact, and continuous maps without fixed points exist. Juliusz Schauder, a Polish mathematician of the Lwów school, overcame this obstacle in 1930 by replacing compactness of the set with *compactness of the operator*: if T maps a closed bounded convex set into a relatively compact subset of itself, then T has a fixed point. This result is the pillar of existence methods for nonlinear differential and integral equations.

Brouwer's theorem (Chapter 5) applies only in finite dimensions. In infinite dimensions, the closed unit ball is not compact, and continuous maps without fixed points exist. Schauder resolved this problem by replacing compactness of the set with compactness of the operator.

6.2 Compact operators

Definition 6.1 (Compact operator). Let E be a Banach space and $C \subset E$. A continuous operator $T : C \to E$ is *compact* if $T(B)$ is relatively compact (i.e., $\overline{T(B)}$ is compact) for every bounded subset $B \subset C$.

Definition 6.2 (Completely continuous operator). An operator $T : C \to E$ is *completely continuous* if it is continuous and maps every bounded set to a relatively compact set.

Remark 6.3. In a Banach space, every completely continuous operator is compact. The converse holds when the domain is bounded.

Example 6.4 (Fredholm integral operator). The operator $(Tu)(t) = \int_0^1 K(t,s)u(s)\,ds$ on $C([0,1])$ is compact whenever $K \in C([0,1]^2)$. This is a consequence of the Arzelà–Ascoli theorem.

6.3 Schauder's theorem

Theorem 6.5 (Schauder, 1930). *Let E be a Banach space, $C \subset E$ a closed, bounded, convex subset, and $T : C \to C$ a compact operator. Then T has at least one fixed point.*

The proof relies on the following approximation lemma.

Lemma 6.6 (Schauder approximation)**.** *Let K be a compact subset of a Banach space E and $\varepsilon > 0$. There exists a finite-dimensional subspace $F \subset E$ and a continuous map $P_\varepsilon : K \to \mathrm{co}(K) \cap F$ such that $\|P_\varepsilon(x) - x\| < \varepsilon$ for all $x \in K$.*

Proof. By compactness, there exist $x_1, \dots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$. Define $\mu_i(x) = \max(0, \varepsilon - \|x - x_i\|)$ and

$$P_\varepsilon(x) = \frac{\sum_{i=1}^n \mu_i(x)\, x_i}{\sum_{i=1}^n \mu_i(x)}.$$

This map is well defined (the denominator is positive since x is within distance $< \varepsilon$ of at least one x_i), continuous, and $P_\varepsilon(x) \in \mathrm{co}(\{x_1, \dots, x_n\})$. Moreover,

$$\|P_\varepsilon(x) - x\| = \left\| \frac{\sum \mu_i(x)(x_i - x)}{\sum \mu_i(x)} \right\| \le \frac{\sum \mu_i(x)\, \|x_i - x\|}{\sum \mu_i(x)} < \varepsilon$$

since $\mu_i(x) > 0$ implies $\|x_i - x\| < \varepsilon$. □

Proof of Theorem 6.5. Set $K = \overline{T(C)}$, which is compact. For each $n \ge 1$, let $P_n = P_{1/n}$ be the Schauder approximation from Lemma 6.6. The map $P_n \circ T : C \to \mathrm{co}(K) \cap F_n$ is continuous with values in a finite-dimensional space.

Set $C_n = C \cap F_n$, which is convex, closed, and bounded in F_n. The map $P_n \circ T|_{C_n} : C_n \to C_n$ is continuous (adjusting the domain if necessary to ensure invariance). By Brouwer's theorem, there exists $x_n \in C_n$ with $P_n(T(x_n)) = x_n$.

We have $\|x_n - T(x_n)\| = \|P_n(T(x_n)) - T(x_n)\| < 1/n$. By compactness of T, the sequence $(T(x_n))$ has a convergent subsequence: $T(x_{n_k}) \to y$. Then $x_{n_k} \to y$ as well, and by continuity of T: $T(y) = \lim T(x_{n_k}) = y$. Since C is closed, $y \in C$. □

Schauder's theorem

C closed bounded convex in a Banach space, $T : C \to C$ compact $\implies \exists\, x^* \in C : T(x^*) = x^*$.

6.4 Schauder–Tychonoff version

Theorem 6.7 (Schauder–Tychonoff)**.** *Let E be a separated locally convex space, $C \subset E$ a nonempty compact convex subset, and $T : C \to C$ continuous. Then T has a fixed point.*

Remark 6.8. This theorem generalizes both Brouwer (finite dimension) and Schauder (Banach space). The local convexity hypothesis on the ambient space is essential.

6.5 Leray–Schauder theorem

Definition 6.9 (Leray–Schauder degree)**.** Let E be a Banach space, $\Omega \subset E$ a bounded open set, and $T : \overline{\Omega} \to E$ a compact operator with $x \neq T(x)$ for all $x \in \partial\Omega$. The *Leray–Schauder degree* $\deg_{LS}(\mathrm{Id} - T, \Omega, 0) \in \mathbb{Z}$ is defined by finite-dimensional approximation.

Theorem 6.10 (Leray–Schauder, 1934). *Let E be a Banach space, $\Omega \subset E$ a bounded open set containing 0, and $T : \overline{\Omega} \to E$ a compact operator. Suppose that*

$$x \neq \lambda T(x) \quad \forall x \in \partial\Omega,\ \forall \lambda \in [0,1].$$

Then T has a fixed point in Ω.

Proof sketch. The homotopy $H(\lambda, x) = x - \lambda T(x)$ satisfies $H(\lambda, x) \neq 0$ for $x \in \partial\Omega$ and $\lambda \in [0,1]$. By homotopy invariance of the degree:

$$\deg_{LS}(\mathrm{Id} - T, \Omega, 0) = \deg_{LS}(\mathrm{Id}, \Omega, 0) = 1.$$

Since the degree is nonzero, the equation $x - T(x) = 0$ has a solution. □

Corollary 6.11 (Leray–Schauder alternative). *Let E be a Banach space and $T : E \to E$ a completely continuous operator. Then at least one of the following holds:*

1. *The equation $x = T(x)$ has a solution;*
2. *The set $\{x \in E : x = \lambda T(x) \text{ for some } \lambda \in (0,1)\}$ is unbounded.*

6.6 Measure of noncompactness and Darbo's theorem

Definition 6.12 (Kuratowski measure of noncompactness). Let A be a bounded subset of a metric space. The *Kuratowski measure of noncompactness* is

$$\alpha(A) = \inf\left\{\varepsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i \text{ with } \mathrm{diam}(A_i) \leq \varepsilon\right\}.$$

Proposition 6.13 (Properties of α). For bounded subsets A, B of a Banach space:

1. $\alpha(A) = 0 \Leftrightarrow \overline{A}$ is compact;
2. $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$;
3. $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$;
4. $\alpha(\overline{\mathrm{co}}(A)) = \alpha(A)$;
5. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;
6. $\alpha(\lambda A) = |\lambda|\, \alpha(A)$.

Definition 6.14 (Condensing map). A continuous operator $T : C \to C$ (where C is closed bounded convex) is *α-condensing* if there exists $k \in [0,1)$ such that $\alpha(T(A)) \leq k\,\alpha(A)$ for every bounded $A \subset C$.

Theorem 6.15 (Darbo, 1955). *Let E be a Banach space, $C \subset E$ a nonempty closed bounded convex set, and $T : C \to C$ an α-condensing operator. Then T has a fixed point.*

Proof. Define recursively $C_0 = C$ and $C_{n+1} = \overline{\text{co}}(T(C_n))$. The sequence (C_n) is decreasing $(C_{n+1} \subset C_n)$ since $T(C_n) \subset C_n$ implies $C_{n+1} = \overline{\text{co}}(T(C_n)) \subset \overline{\text{co}}(C_n) = C_n$.

Set $\alpha_n = \alpha(C_n)$. Then $\alpha_{n+1} = \alpha(\overline{\text{co}}(T(C_n))) = \alpha(T(C_n)) \le k\alpha_n$. Hence $\alpha_n \le k^n \alpha_0 \to 0$.

Set $C_\infty = \bigcap_{n\ge 0} C_n$. As a decreasing intersection of closed sets in a complete space with $\alpha(C_n) \to 0$, the set C_∞ is nonempty and compact (by generalized Cantor). Moreover, C_∞ is convex. We have $T(C_\infty) \subset T(C_n) \subset C_{n+1}$ for all n, so $T(C_\infty) \subset C_\infty$.

By Schauder's theorem applied to $T : C_\infty \to C_\infty$ (continuous on a compact convex set), T has a fixed point. □

Remark 6.16. Schauder's theorem is a special case of Darbo's: if T is compact, then $\alpha(T(A)) = 0$ for every bounded A, so T is α-condensing with $k = 0$.

6.7 Sadovskii's theorem

Theorem 6.17 (Sadovskii, 1967). *Let C be a closed, bounded, convex subset of a Banach space, and $T : C \to C$ continuous with $\alpha(T(A)) < \alpha(A)$ for every $A \subset C$ with $\alpha(A) > 0$. Then T has a fixed point.*

Remark 6.18. Sadovskii generalizes Darbo by replacing the condition $\alpha(T(A)) \le k\alpha(A)$ $(k < 1)$ with $\alpha(T(A)) < \alpha(A)$. The proof uses a Zorn's lemma argument.

6.8 Krasnoselskii's theorem

Theorem 6.19 (Krasnoselskii, 1958). *Let C be a closed, bounded, convex subset of a Banach space E. Let $A : C \to E$ be a compact operator and $B : C \to E$ a contraction with constant $k < 1$. If $A(x)+B(y) \in C$ for all $x, y \in C$, then the equation $x = A(x)+B(x)$ has a solution in C.*

Proof. For each $x \in C$, the map $y \mapsto A(x) + B(y)$ is a contraction from C to C (by hypothesis). By Banach's theorem, it has a unique fixed point $y(x)$: $y(x) = A(x) + B(y(x))$. The map $x \mapsto y(x)$ is continuous (by continuous dependence of the fixed point on the parameter) and compact (since $y(x) = A(x) + B(y(x))$ and A is compact). By Schauder's theorem, $y(\cdot)$ has a fixed point $x^* = y(x^*)$, whence $x^* = A(x^*) + B(x^*)$. □

Remark 6.20. Krasnoselskii's theorem is particularly useful for integral equations of the form $x = Ax + Bx$ where A is an integral operator (compact) and B is a contraction.

6.9 Applications

Example 6.21 (Nonlinear integral equation). Consider the equation

$$u(t) = g(t) + \int_0^1 K(t,s)\, h(s, u(s))\, ds, \quad t \in [0,1]$$

where $g \in C([0,1])$, $K \in C([0,1]^2)$, and $h : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and bounded. The operator $(Tu)(t) = g(t)+\int_0^1 K(t,s)h(s,u(s))\, ds$ is compact on the ball $\overline{B}(0,R) \subset C([0,1])$ for R large enough, and $T(\overline{B}(0,R)) \subset \overline{B}(0,R)$ provided $R \ge \|g\|_\infty + \|K\|_\infty \|h\|_\infty$. By Schauder's theorem, the equation has a solution.

6.10 Exercises

Exercise 6.1. Show that the Volterra operator $(Tu)(t) = \int_0^t K(t,s)u(s)\,ds$ is compact on $C([0,1])$ when $K \in C([0,1]^2)$.

Exercise 6.2. Verify that $\alpha(\overline{B}(0,r)) = 2r$ in an infinite-dimensional Banach space, and $\alpha(\overline{B}(0,r)) = 0$ in finite dimensions.

Exercise 6.3. Use Schauder's theorem to prove existence of a solution to the Cauchy–Peano problem: $y' = f(t,y)$, $y(0) = y_0$, where f is continuous (but not necessarily Lipschitz).

Exercise 6.4. Show that Darbo's theorem implies Schauder's theorem.

Exercise 6.5. Apply Krasnoselskii's theorem to show existence of a solution to the equation $u(t) = \frac{1}{3}u(t) + \int_0^1 K(t,s)g(u(s))\,ds$ in $C([0,1])$, under suitable hypotheses on K and g.

Exercise 6.6. Show that in a reflexive Banach space, the closed unit ball is weakly compact. Deduce a version of Schauder's theorem for the weak topology.

Chapter 7

Kakutani's Theorem and Correspondences

In 1941, Shizuo Kakutani, a young Japanese mathematician trained at Osaka University, published a result that appeared to be a mere technical extension of Brouwer's theorem: instead of requiring a function to have a fixed point, he showed that a *correspondence* — a map that assigns to each point not a single value but an *entire set* of values — also possesses a fixed point, provided the images are convex and the correspondence is upper semicontinuous. The result could have remained confined to topological circles. But nine years later, John Nash used it as the centrepiece of his proof that non-cooperative games possess equilibria, work that would earn him the Nobel Prize in Economics in 1994. Kakutani's theorem thus became one of the pillars of game theory and mathematical economics.

Intuition

Kakutani's theorem generalizes Brouwer's theorem to set-valued maps (correspondences). Instead of a function $f\colon C \to C$, one considers a correspondence $F\colon C \rightrightarrows C$ that associates to each point a *set* of values. Kakutani shows that such a correspondence has a fixed point (a point x^* with $x^* \in F(x^*)$) under convexity and semicontinuity hypotheses. This result is fundamental in game theory, where it is used to prove the existence of Nash equilibria.

7.1 Correspondences and Semicontinuity

Definition 7.1 (Correspondence (set-valued map))**.** Let X and Y be topological spaces. A *correspondence* (or set-valued map) $F\colon X \rightrightarrows Y$ is a map $F\colon X \to \mathcal{P}(Y)$, i.e., for each $x \in X$, $F(x)$ is a subset of Y. The *graph* of F is $\mathrm{Gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$.

Definition 7.2 (Upper hemicontinuity (u.h.c.))**.** A correspondence $F\colon X \rightrightarrows Y$ is *upper hemicontinuous* (or *upper semicontinuous*) if for every open set $V \subseteq Y$, the set

$$F^+(V) = \{x \in X : F(x) \subseteq V\}$$

is open in X.

Definition 7.3 (Lower hemicontinuity (l.h.c.))**.** A correspondence $F\colon X \rightrightarrows Y$ is *lower hemicontinuous* if for every open set $V \subseteq Y$, the set

$$F^{-}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$$

is open in X.

Proposition 7.4 (Closed graph characterization)**.** Let $F\colon X \rightrightarrows Y$ be a compact-valued correspondence and Y a compact Hausdorff space. Then F is u.h.c. if and only if its graph $\mathrm{Gr}(F)$ is closed in $X \times Y$.

Proof. ($\Rightarrow$): Let (x_α, y_α) be a net in $\mathrm{Gr}(F)$ converging to (x, y). Suppose $y \notin F(x)$. Since $F(x)$ is compact and Y is Hausdorff, there exists an open set V containing $F(x)$ with $y \notin \overline{V}$. By u.h.c., $F^{+}(V)$ is a neighborhood of x, so $F(x_\alpha) \subseteq V$ for large α, hence $y_\alpha \in V$. In the limit, $y \in \overline{V}$, contradiction.

($\Leftarrow$): Let V be open with $F(x) \subseteq V$. If $F^{+}(V)$ were not a neighborhood of x, there would exist a net $x_\alpha \to x$ with $F(x_\alpha) \not\subseteq V$. One could find $y_\alpha \in F(x_\alpha) \setminus V$. By compactness of Y, one can extract a subnet converging to $y \in Y \setminus V$. Then $(x, y) \in \mathrm{Gr}(F)$ since the graph is closed, so $y \in F(x) \subseteq V$, contradiction. $\square$

Remark 7.5. For a single-valued function $f\colon X \to Y$ (viewed as a correspondence $F(x) = \{f(x)\}$), upper hemicontinuity coincides with ordinary continuity.

7.2 Kakutani's Theorem

Theorem 7.6 (Kakutani, 1941)**.** *Let C be a nonempty compact convex subset of $\mathbb{R}^n$ and $F\colon C \rightrightarrows C$ a correspondence such that:*

1. *for every $x \in C$, $F(x)$ is nonempty and convex;*
2. *F is upper hemicontinuous (closed graph).*

Then F has a fixed point: there exists $x^* \in C$ with $x^* \in F(x^*)$.

Proof. For each $n \in \mathbb{N}^*$, consider a simplicial triangulation of C with mesh less than $1/n$, with vertices $v_1^n, \ldots, v_{k_n}^n$. For each vertex v_i^n, choose $w_i^n \in F(v_i^n)$. Define the piecewise affine map $f_n\colon C \to C$ by $f_n(v_i^n) = w_i^n$ and affine extension on each simplex.

Since C is convex compact and f_n is continuous, by Brouwer's theorem there exists $x_n \in C$ with $f_n(x_n) = x_n$.

By compactness, (x_n) has a convergent subsequence $x_{n_k} \to x^*$. We show that $x^* \in F(x^*)$. For every $\varepsilon > 0$, for large k, the mesh of the triangulation is less than ε, and x_{n_k} belongs to a simplex whose vertices $v_i^{n_k}$ satisfy $\|v_i^{n_k} - x_{n_k}\| < \varepsilon$. Since $f_{n_k}(x_{n_k}) = x_{n_k}$ is a convex combination of the $w_i^{n_k} \in F(v_i^{n_k})$, and since $\mathrm{Gr}(F)$ is closed (Proposition 7.4), we conclude $x^* \in F(x^*)$. $\square$

Attention

The convexity hypothesis on $F(x)$ is essential. Consider $C = [0, 1]$ and $F(x) = \{0, 1\} \setminus \{x\}$: this closed-graph correspondence has no fixed point because $F(x)$ is not convex.

7.3 The Glicksberg–Fan Extension

Theorem 7.7 (Glicksberg–Fan)**.** *Let E be a locally convex topological vector space and $C \subseteq E$ a nonempty compact convex set. Let $F\colon C \rightrightarrows C$ be a correspondence with nonempty, convex, closed values that is upper hemicontinuous. Then F has a fixed point.*

Remark 7.8. This theorem extends Kakutani to infinite dimensions, just as Schauder's theorem extends Brouwer's.

7.4 Application: Nash Equilibrium

Definition 7.9 (Normal form game)**.** A *normal form game* with n players is a triple $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ where:

- $N = \{1, \dots, n\}$ is the set of players;
- S_i is the strategy set of player i;
- $u_i\colon S_1 \times \cdots \times S_n \to \mathbb{R}$ is the utility function of player i.

The set of mixed strategies of player i is $\Delta(S_i) = \{\sigma_i \in \mathbb{R}^{S_i} : \sigma_i \geq 0, \sum_{s\in S_i} \sigma_i(s) = 1\}$.

Definition 7.10 (Nash equilibrium)**.** A mixed strategy profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ is a *Nash equilibrium* if for every player i and every mixed strategy $\sigma_i \in \Delta(S_i)$:

$$u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

where σ_{-i}^* denotes the strategies of the other players.

Definition 7.11 (Best response correspondence)**.** The *best response correspondence* of player i is

$$BR_i(\sigma_{-i}) = \arg\max_{\sigma_i\in\Delta(S_i)} u_i(\sigma_i, \sigma_{-i}).$$

Theorem 7.12 (Nash, 1950)**.** *Every finite normal form game (i.e., with S_i finite for all i) has at least one Nash equilibrium in mixed strategies.*

Proof. Let $C = \prod_{i=1}^n \Delta(S_i)$. This is a product of simplices, hence a nonempty compact convex subset of $\mathbb{R}^{\sum |S_i|}$.

Define the global best response correspondence:

$$BR(\sigma) = BR_1(\sigma_{-1}) \times \cdots \times BR_n(\sigma_{-n}).$$

We verify Kakutani's hypotheses:

Nonemptiness and convexity: $BR_i(\sigma_{-i})$ is nonempty (maximum of a continuous function on a compact set) and convex (since u_i is linear in σ_i: if σ_i, σ_i' both maximize $u_i(\cdot, \sigma_{-i})$, so does any convex combination).

Upper hemicontinuity: Berge's maximum theorem ensures that BR is u.h.c., since u_i is continuous and $\Delta(S_i)$ does not depend on σ_{-i}.

By Kakutani's theorem, BR has a fixed point σ^*, i.e., $\sigma_i^* \in BR_i(\sigma_{-i}^*)$ for all i. This is a Nash equilibrium. □

7.5 Minimax Theorems

Theorem 7.13 (Von Neumann's minimax theorem)**.** *Let A be a real $m \times n$ matrix. Then:*

$$\max_{\sigma \in \Delta_m} \min_{\tau \in \Delta_n} \sigma^T A \tau = \min_{\tau \in \Delta_n} \max_{\sigma \in \Delta_m} \sigma^T A \tau.$$

Proof. The inequality $\max\min \leq \min\max$ always holds. For equality, consider the two-player zero-sum game defined by matrix A. By Nash's theorem (or directly by Kakutani), this game has an equilibrium (σ^*, τ^*). The value of the game is $v = \sigma^{*T} A \tau^* = \max_\sigma \min_\tau \sigma^T A \tau = \min_\tau \max_\sigma \sigma^T A \tau$. □

Corollary 7.14. *Every zero-sum matrix game has a value, and both players have optimal mixed strategies.*

Summary

Theorem	Key hypotheses
Kakutani	C compact convex $\subseteq \mathbb{R}^n$, $F(x)$ convex, u.h.c.
Glicksberg–Fan	C compact convex in lctvs, $F(x)$ convex closed, u.h.c.
Nash	Finite n-player game
von Neumann	Zero-sum matrix game

7.6 Exercises

Exercise 7.1. Show that the correspondence $F\colon [0,1] \rightrightarrows [0,1]$ defined by $F(x) = [1-x, 1]$ if $x \leq 1/2$ and $F(x) = [0, 1-x]$ if $x > 1/2$ satisfies Kakutani's hypotheses. Find the fixed points explicitly.

Exercise 7.2. Let $C = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the closed unit ball. Show that the correspondence $F(x) = \{y \in C : \langle x, y \rangle \geq 0\}$ satisfies Kakutani's hypotheses and find the set of fixed points.

Exercise 7.3 (Berge's maximum theorem)**.** Let X, Y be topological spaces, $f\colon X \times Y \to \mathbb{R}$ continuous, and $\Gamma\colon X \rightrightarrows Y$ a u.h.c. correspondence with nonempty compact values. Show that the maximizer correspondence $M(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$ is u.h.c.

Exercise 7.4. Find the Nash equilibria in mixed strategies of Rock-Paper-Scissors, defined by the payoff matrix:

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Exercise 7.5. Show that the convexity hypothesis on $F(x)$ cannot be replaced by connectedness. *Hint: consider a circle in* $\mathbb{R}^2$.

Exercise 7.6. Let Γ be a two-player game where the strategy sets are compact convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and the utility functions are continuous and quasi-concave in each player's own strategy. Prove the existence of a Nash equilibrium in pure strategies.

Chapter 8

Fixed Points in Ordered Spaces

In 1928, Bronislaw Knaster and Alfred Tarski proved a result of extraordinary simplicity and reach: every monotone map on a complete lattice has a fixed point. Better still, the set of *all* fixed points itself forms a complete lattice. This theorem uses no metric, no topology, no continuity: only the order structure intervenes. It would find spectacular applications in theoretical computer science, where it underpins the denotational semantics of recursive programs (Dana Scott's work in the 1970s), and in logic, where it supports the theory of inductive definitions. Together with the Bourbaki–Witt transfinite iterates and Kleene's theorem for ω-cpo's, this chapter explores the third great family of fixed point theorems — the one founded on order.

Intuition

Fixed point theorems in ordered sets constitute the third great family of results, after the metric theorems (Banach) and the topological ones (Brouwer, Schauder). Here, no metric or topological structure is required: only the order structure intervenes. The fundamental result is the Knaster–Tarski theorem, which asserts that every monotone map on a complete lattice has a fixed point, and that the set of fixed points itself forms a complete lattice.

8.1 Background on Ordered Sets

Definition 8.1 (Partial order and lattice)**.** A *partially ordered set* (poset) is a pair $(P, \leq)$ where $\leq$ is a reflexive, antisymmetric, and transitive relation. A *lattice* is a poset in which every pair $\{a, b\}$ has a supremum $a \vee b$ and an infimum $a \wedge b$. A lattice is *complete* if every subset has a supremum and an infimum.

Example 8.2 (Fundamental complete lattices)**.**

- $(\mathcal{P}(X), \subseteq)$ for any set X.
- $([0, 1], \leq)$ with the usual order.
- The lattice of vector subspaces of a vector space.
- The lattice of equivalence relations on a set.

Definition 8.3 (Monotone and inflationary maps)**.** Let $(P, \leq)$ be a poset. A map $f: P \to P$ is *monotone* (or order-preserving) if $x \leq y \Rightarrow f(x) \leq f(y)$. It is *inflationary* if $x \leq f(x)$ for all x, and *deflationary* if $f(x) \leq x$ for all x.

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8.2 The Knaster–Tarski Theorem

Theorem 8.4 (Knaster–Tarski)**.** *Let $(L, \leq)$ be a complete lattice and $f\colon L \to L$ a monotone map. Then:*

1. *f has a least fixed point:* $\mathrm{lfp}(f) = \bigwedge\{x \in L : f(x) \leq x\}$.

2. *f has a greatest fixed point:* $\mathrm{gfp}(f) = \bigvee\{x \in L : x \leq f(x)\}$.

3. *The set* $\mathrm{Fix}(f) = \{x \in L : f(x) = x\}$ *is a complete lattice (under the induced order).*

Proof. Let $\mathrm{Pre}(f) = \{x \in L : f(x) \leq x\}$ (the pre-fixed points) and $a = \bigwedge \mathrm{Pre}(f)$.

Step 1: $f(a) \leq a$. Indeed, for every $x \in \mathrm{Pre}(f)$, we have $a \leq x$, so $f(a) \leq f(x) \leq x$. Thus $f(a)$ is a lower bound of $\mathrm{Pre}(f)$, hence $f(a) \leq a$.

Step 2: $a \leq f(a)$. From Step 1, $f(a) \leq a$, so by monotonicity $f(f(a)) \leq f(a)$. Thus $f(a) \in \mathrm{Pre}(f)$, hence $a \leq f(a)$.

Step 3: $f(a) = a$ by Steps 1 and 2.

Least fixed point: if b is a fixed point, then $b \in \mathrm{Pre}(f)$, so $a \leq b$.

Complete lattice of fixed points: Let $S \subseteq \mathrm{Fix}(f)$. Define $L_S = \{x \in L : x \geq s\ \forall s \in S \text{ and } f(x) \leq x\}$. Then L_S is nonempty ($\top \in L_S$) and $\bigwedge L_S$ is a fixed point of f that is the supremum of S in $\mathrm{Fix}(f)$. Dually, $\mathrm{Fix}(f)$ admits infima. □

Example 8.5 (Application in logic)**.** Let X be a set and $\Phi\colon \mathcal{P}(X) \to \mathcal{P}(X)$ a monotone operator. The least fixed point of Φ is the smallest set A with $\Phi(A) = A$. If Φ is a logical consequence operator, $\mathrm{lfp}(\Phi)$ is the set of deducible formulas.

8.3 The Bourbaki–Witt Theorem

Theorem 8.6 (Bourbaki–Witt)**.** *Let $(P, \leq)$ be a partially ordered set in which every chain has an upper bound, and let $f\colon P \to P$ be an inflationary map ($x \leq f(x)$ for all x). If P has a least element $\bot$, then f has a fixed point.*

Proof. Consider the collection $\mathcal{A}$ of all chains C in P such that:

- $\bot \in C$;
- C is closed under f;
- if $C' \subseteq C$ is a chain, any upper bound of C' in P that belongs to C is indeed an upper bound.

Define $C_0 = \bigcap_{C \in \mathcal{A}} C$. One shows that C_0 is a well-ordered chain. Let m be an upper bound of C_0 in P (which exists by hypothesis). Then $f(m) \geq m$. If $f(m) = m$, it is a fixed point. Otherwise, $C_0 \cup \{m\}$ would be a strictly larger chain, contradicting the minimality of C_0. □

Remark 8.7. The Bourbaki–Witt theorem is equivalent to Zorn's lemma in ZF + axiom of choice. Without the axiom of choice, the statement remains true for explicitly defined inflationary functions.

8.4 Application to Formal Concept Analysis

Definition 8.8 (Formal context)**.** A *formal context* is a triple (G, M, I) where G is a set of objects, M a set of attributes, and $I \subseteq G \times M$ an incidence relation (gIm means that object g has attribute m).

Definition 8.9 (Derivation operators)**.** For $A \subseteq G$ and $B \subseteq M$, define:

$$A' = \{m \in M : \forall g \in A,\ gIm\}, \qquad B' = \{g \in G : \forall m \in B,\ gIm\}.$$

Proposition 8.10 (Galois connection)**.** The operators $A \mapsto A'$ and $B \mapsto B'$ form a *Galois connection* between $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \supseteq)$:

1. $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$;
2. $A \subseteq A''$;
3. $A' = A'''$.

Definition 8.11 (Formal concept)**.** A *formal concept* is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. The set A is the *extent* and B the *intent* of the concept.

Theorem 8.12 (Concept lattice)**.** *The set of formal concepts of a context* (G, M, I)*, ordered by* $(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2$ *(equivalently* $B_2 \subseteq B_1$*), is a complete lattice.*

Proof. The closure operator $\varphi \colon \mathcal{P}(G) \to \mathcal{P}(G)$ defined by $\varphi(A) = A''$ is monotone, extensive ($A \subseteq A''$), and idempotent ($A'' = A''''$). The closed sets (fixed points of φ) form a complete lattice and correspond bijectively to formal concepts. □

8.5 Fixed Points and Closure Operators

Definition 8.13 (Closure operator)**.** A *closure operator* on a poset $(P, \leq)$ is a monotone, extensive, and idempotent map $\varphi \colon P \to P$.

Theorem 8.14. *Let* $(L, \leq)$ *be a complete lattice and* $\varphi \colon L \to L$ *a closure operator. Then* $\mathrm{Fix}(\varphi)$ *is a complete lattice (with suprema of* L *closed by* φ*, and infima unchanged).*

Proof. The fixed points of φ are the closed elements, i.e., those x with $\varphi(x) = x$. For $S \subseteq \mathrm{Fix}(\varphi)$, the infimum in $\mathrm{Fix}(\varphi)$ is $\bigwedge_L S$ (since $\varphi(\bigwedge_L S) \leq \varphi(s) = s$ for all $s \in S$, hence $\varphi(\bigwedge_L S) \leq \bigwedge_L S$, and by extensiveness we get equality). The supremum in $\mathrm{Fix}(\varphi)$ is $\varphi(\bigvee_L S)$. □

Fixed points in ordered sets

Theorem	Hypotheses	Conclusion
Knaster–Tarski	Complete lattice, f monotone	$\mathrm{Fix}(f)$ complete lattice
Bourbaki–Witt	Chains bounded, f inflat.	Fixed point exists
Closure op.	Complete lattice, φ closure	$\mathrm{Fix}(\varphi)$ complete lattice

CHAPTER 8. FIXED POINTS IN ORDERED SPACES

8.6 Exercises

Exercise 8.1. Show that the Knaster–Tarski theorem implies the Cantor–Bernstein theorem: if $f\colon A \hookrightarrow B$ and $g\colon B \hookrightarrow A$ are injections, then there exists a bijection $h\colon A \to B$. *Hint: apply the theorem to* $\varphi(S) = A \setminus g(B \setminus f(S))$ *on* $(\mathcal{P}(A), \subseteq)$.

Exercise 8.2. Let $(L, \leq)$ be a complete lattice and $f\colon L \to L$ monotone. Show that if $a \leq f(a)$ and $f(b) \leq b$ with $a \leq b$, then f has a fixed point in the interval $[a, b]$.

Exercise 8.3. Give an example of an ordered set (not complete) and a monotone map with no fixed point.

Exercise 8.4. Let $G = (\{1,2,3,4\}, E)$ be a graph and $f\colon \mathcal{P}(\{1,2,3,4\}) \to \mathcal{P}(\{1,2,3,4\})$ defined by $f(S) = S \cup \{v : \exists u \in S,\ (u,v) \in E\}$. Compute $\mathrm{lfp}(f)$ and $\mathrm{gfp}(f)$ for the graph $E = \{(1,2),(2,3),(3,4)\}$ starting from $\{1\}$.

Exercise 8.5 (Galois connections and fixed points)**.** Let (f, g) be a Galois connection between two complete lattices L_1 and L_2 ($f \dashv g$). Show that $g \circ f$ is a closure operator on L_1 and $f \circ g$ is a kernel operator on L_2.

Exercise 8.6. Construct the concept lattice of the formal context with $G = \{\text{dog}, \text{cat}, \text{fish}\}$, $M = \{\text{fur}, \text{swims}, 4\ \text{legs}\}$ and the natural incidence relation.

Chapter 9

Applications

Fixed point theorems are not abstract curiosities: they are the tools that, behind the scenes, make a large part of classical analysis work. The Picard–Lindelöf existence and uniqueness theorem for ordinary differential equations? It is Banach's contraction principle applied to an integral operator. Fredholm integral equations? Banach again. The implicit function theorem, pillar of differential calculus? The contraction principle, once more. This chapter gathers these classical applications, showing how a single abstract result — the existence of a fixed point for a contraction — irrigates entire branches of applied mathematics.

Intuition

This chapter illustrates the power of fixed point theorems by applying them to three classical areas of analysis: ordinary differential equations (via the Picard–Lindelöf theorem as a consequence of Banach's contraction principle), integral equations (Fredholm and Volterra), and the implicit function theorem. In each case, the problem is reformulated as a fixed point problem in an appropriate function space.

9.1 Ordinary Differential Equations: Picard–Lindelöf

Theorem 9.1 (Picard–Lindelöf (Cauchy–Lipschitz))**.** *Let* $f\colon [t_0-a, t_0+a]\times\overline{B}(y_0, b) \to \mathbb{R}^n$ *be a continuous map, Lipschitz in* y *with constant* L*:*

$$\|f(t, y_1) - f(t, y_2)\| \le L\,\|y_1 - y_2\|$$

for all t, y_1, y_2*. Let* $M = \sup\|f\|$ *and* $\delta = \min(a, b/M)$*. Then the Cauchy problem*

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0$$

has a unique solution $y \in C^1([t_0-\delta, t_0+\delta], \mathbb{R}^n)$*.*

Proof. The Cauchy problem is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds =: (Ty)(t).$$

Consider the space $E = \{y \in C([t_0-\delta, t_0+\delta], \mathbb{R}^n) : \|y(t) - y_0\| \le b\ \forall t\}$ equipped with the norm

$$\|y\|_\lambda = \sup_t e^{-\lambda|t-t_0|}\,\|y(t) - y_0\|$$

for $\lambda > L$ to be chosen.

T maps E into E: For $y \in E$, $\|(Ty)(t) - y_0\| \leq M\,|t - t_0| \leq M\delta \leq b$.

T is a contraction: For $y_1, y_2 \in E$,

$$\begin{aligned} e^{-\lambda|t-t_0|}\,\|(Ty_1)(t) - (Ty_2)(t)\| &\leq e^{-\lambda|t-t_0|} \int_{t_0}^{t} L\,\|y_1(s) - y_2(s)\|\,ds \\ &\leq L\,\|y_1 - y_2\|_\lambda \int_{t_0}^{t} e^{\lambda(|s-t_0|-|t-t_0|)} ds \\ &\leq \frac{L}{\lambda}\,\|y_1 - y_2\|_\lambda\,. \end{aligned}$$

Choosing $\lambda > L$, we get $\|Ty_1 - Ty_2\|_\lambda \leq \frac{L}{\lambda}\,\|y_1 - y_2\|_\lambda$ with $L/\lambda < 1$.

By Banach's contraction principle, T has a unique fixed point in E. □

Remark 9.2. The norm $\|\cdot\|_\lambda$ (Bielecki norm) is a classical trick that avoids having to restrict the time interval to obtain contraction.

Corollary 9.3 (Continuous dependence)**.** *The solution $y(t; t_0, y_0)$ depends continuously on the initial data (t_0, y_0).*

9.2 Fredholm Integral Equations

Definition 9.4 (Fredholm equation of the second kind)**.** Let $K \in C([a,b]^2, \mathbb{R})$ be a *kernel*, $f \in C([a,b], \mathbb{R})$, and $\lambda \in \mathbb{R}$. The *Fredholm equation of the second kind* is:

$$u(t) = f(t) + \lambda \int_a^b K(t,s)\,u(s)\,ds.$$

Theorem 9.5 (Existence and uniqueness for Fredholm)**.** *If $|\lambda| < \frac{1}{(b-a)\|K\|_\infty}$, then the Fredholm equation has a unique solution $u \in C([a,b], \mathbb{R})$.*

Proof. Define the operator $T\colon C([a,b]) \to C([a,b])$ by

$$(Tu)(t) = f(t) + \lambda \int_a^b K(t,s)\,u(s)\,ds.$$

Then:

$$\|Tu_1 - Tu_2\|_\infty \leq |\lambda|\,(b-a)\,\|K\|_\infty\,\|u_1 - u_2\|_\infty\,.$$

Under the hypothesis $|\lambda|\,(b-a)\,\|K\|_\infty < 1$, the operator T is a contraction on the Banach space $C([a,b])$, and Banach's theorem applies. □

Example 9.6. The equation $u(t) = \sin(t) + \frac{1}{4}\int_0^1 ts\,u(s)\,ds$ has kernel $K(t,s) = ts$ with $\|K\|_\infty = 1$. Since $|\lambda|\,(b-a)\,\|K\|_\infty = \frac{1}{4} < 1$, the solution exists and is unique.

9.3 Volterra Integral Equations

Definition 9.7 (Volterra equation of the second kind)**.** The *Volterra equation of the second kind* is:

$$u(t) = f(t) + \lambda \int_a^t K(t,s)\,u(s)\,ds.$$

The difference from Fredholm is that the upper limit of integration is t (not b).

Theorem 9.8 (Existence and uniqueness for Volterra)**.** *For any $\lambda \in \mathbb{R}$ and any continuous kernel K, the Volterra equation of the second kind has a unique solution $u \in C([a,b],\mathbb{R})$.*

Proof. The operator T defined by $(Tu)(t) = f(t) + \lambda \int_a^t K(t,s)u(s)ds$ is not necessarily a contraction for the uniform norm. However, we use the Bielecki norm:

$$\|u\|_\mu = \sup_{t\in[a,b]} e^{-\mu(t-a)} |u(t)| .$$

Then:

$$\begin{aligned} e^{-\mu(t-a)} |(Tu_1)(t) - (Tu_2)(t)| &\leq |\lambda| \|K\|_\infty \int_a^t e^{-\mu(t-s)} e^{-\mu(s-a)} |u_1(s) - u_2(s)| \, ds \\ &\leq |\lambda| \|K\|_\infty \|u_1 - u_2\|_\mu \int_a^t e^{-\mu(t-s)} ds \\ &\leq \frac{|\lambda| \|K\|_\infty}{\mu} \|u_1 - u_2\|_\mu . \end{aligned}$$

Choosing $\mu > |\lambda| \|K\|_\infty$, T is a contraction. □

Attention

Unlike Fredholm, the Volterra equation always has a unique solution, with no restriction on λ. This is a consequence of the "triangular" character of the Volterra operator.

9.4 The Implicit Function Theorem

Theorem 9.9 (Implicit function theorem via fixed point)**.** *Let $F\colon U \to \mathbb{R}^m$ be a C^1 map where U is an open subset of $\mathbb{R}^n \times \mathbb{R}^m$. Let $(a,b) \in U$ with $F(a,b) = 0$ and $D_yF(a,b)$ invertible. Then there exist neighborhoods V of a and W of b and a unique C^1 map $\varphi\colon V \to W$ such that*

$$F(x,\varphi(x)) = 0 \quad \forall x \in V.$$

Proof. We reduce to a fixed point problem. Set $A = D_yF(a,b)$ and $G(x,y) = y - A^{-1}F(x,y)$. Then $F(x,y) = 0$ is equivalent to $y = G(x,y)$.

We show that for x near a, $y \mapsto G(x,y)$ is a contraction near b. We have $D_yG(a,b) = I - A^{-1}D_yF(a,b) = 0$. By continuity of D_yG, there exists $r > 0$ such that $\|D_yG(x,y)\| \leq 1/2$ in a neighborhood of (a,b).

Thus, for fixed x near a, $y \mapsto G(x,y)$ is a contraction with constant $1/2$ on $\overline{B}(b,r)$ (adjusting r if necessary). By Banach's theorem, there exists a unique $y = \varphi(x)$ with $G(x,\varphi(x)) = \varphi(x)$, i.e., $F(x,\varphi(x)) = 0$.

The C^1 regularity of φ is obtained by differentiating $F(x,\varphi(x)) = 0$, giving $D\varphi(x) = -[D_yF(x,\varphi(x))]^{-1}D_xF(x,\varphi(x))$. □

Remark 9.10. This fixed point proof is more constructive than the classical proof (via the inverse function theorem) because it provides an approximation algorithm: the sequence $y_{n+1} = G(x,y_n)$ converges to $\varphi(x)$.

9.5 Complements: Newton's Method and Fixed Points

Proposition 9.11 (Newton's method as fixed point iteration)**.** Newton's method for solving $F(x) = 0$ in $\mathbb{R}^n$ reads:

$$x_{n+1} = x_n - [DF(x_n)]^{-1}F(x_n) =: T(x_n).$$

Under regularity conditions (Lipschitz DF, invertibility of $DF(x^*)$), T is not a global contraction, but it is locally contractive near the solution x^*, with a quadratic convergence rate: $\|x_{n+1} - x^*\| \le C\,\|x_n - x^*\|^2$.

Summary of applications

Problem	**Theorem used**	**Function space**
ODE (Picard–Lindelöf)	Banach	$C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$
Fredholm	Banach	$C([a, b], \mathbb{R})$
Volterra	Banach (Bielecki norm)	$C([a, b], \mathbb{R})$
Implicit functions	Banach	$\mathbb{R}^m$

9.6 Exercises

Exercise 9.1. Consider the ODE $y' = y^2$, $y(0) = 1$. Compute the first three Picard iterates $y_0(t) = 1$, $y_{n+1}(t) = 1 + \int_0^t y_n(s)^2 ds$. Compare with the exact solution $y(t) = \frac{1}{1-t}$.

Exercise 9.2. Solve the Fredholm equation $u(t) = 1 + \lambda \int_0^1 u(s)ds$ explicitly for all $\lambda \neq 1$. Verify that the solution diverges as $\lambda \to 1$.

Exercise 9.3. Solve the Volterra equation $u(t) = 1 + \int_0^t u(s)ds$ by recognizing an ODE. Verify the result by the method of successive iterations.

Exercise 9.4. Show that the equation $F(x, y) = x^2 + y^2 - 1 = 0$ defines y as a function of x near $(0, 1)$. Compute $\varphi'(0)$ and $\varphi''(0)$.

Exercise 9.5. Let $K(t, s) = e^{ts}$ and $f(t) = 1$. Study the convergence of the iterates of the operator T associated with the Fredholm equation $u(t) = 1 + \lambda \int_0^1 e^{ts}u(s)ds$ as a function of λ.

Exercise 9.6 (Systems of ODEs)**.** Apply the Picard–Lindelöf theorem to the system $y_1' = y_2$, $y_2' = -y_1$, $y_1(0) = 1$, $y_2(0) = 0$. Compute the Picard iterates explicitly and show they converge to $(\cos t, -\sin t)$.

Chapter 10

Markov-Kakutani Theorem

Intuition

The Markov-Kakutani theorem guarantees the existence of a common fixed point for a family of commuting continuous affine maps acting on a compact convex set in a locally convex topological vector space. This result, simple in its statement, has deep applications: it underlies the theory of invariant means and plays a central role in the study of amenable groups.

10.1 Statement and Proof of the Theorem

Definition 10.1 (Affine map)**.** Let E be a vector space and $C \subseteq E$ a convex set. A map $T\colon C \to C$ is *affine* if for all $x, y \in C$ and all $\lambda \in [0,1]$:

$$T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y).$$

Theorem 10.2 (Markov-Kakutani)**.** *Let E be a locally convex topological vector space and $C \subseteq E$ a nonempty compact convex set. Let $\mathcal{F}$ be a family of continuous affine maps $T\colon C \to C$ that pairwise commute ($T_1 \circ T_2 = T_2 \circ T_1$ for all $T_1, T_2 \in \mathcal{F}$). Then there exists $x^* \in C$ such that $T(x^*) = x^*$ for all $T \in \mathcal{F}$.*

Proof. Step 1: single map case. Let $T\colon C \to C$ be continuous and affine. For each $x \in C$ and $n \geq 1$, set

$$\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k(x).$$

Since C is convex and $T(C) \subseteq C$, we have $\sigma_n(x) \in C$. Define $C_n = \sigma_n(C) = \{\sigma_n(x) : x \in C\}$. Then C_n is compact (continuous image of a compact set) and nonempty.

We show that $(C_n)_{n\geq 1}$ has the finite intersection property, which implies $\bigcap_n \overline{C_n} \neq \emptyset$ by compactness of C.

Let $y \in \bigcap_n \overline{C_n}$. For every ε-neighborhood V of 0, there exists $x_n \in C$ with $\sigma_n(x_n) - y \in V$. One verifies that $T(\sigma_n(x_n)) - \sigma_n(x_n) = \frac{1}{n}(T^n(x_n) - x_n) \to 0$ since C is bounded. Thus $T(y) = y$.

Step 2: commuting family. For each $T \in \mathcal{F}$, define $\mathrm{Fix}(T) = \{x \in C : T(x) = x\}$. By Step 1, $\mathrm{Fix}(T)$ is nonempty. Moreover:

- $\mathrm{Fix}(T)$ is closed (preimage of the closed diagonal under $(x, T(x))$) and contained in the compact C, hence compact.

- $\mathrm{Fix}(T)$ is convex since T is affine: if $T(x) = x$ and $T(y) = y$, then $T(\lambda x+(1-\lambda)y) = \lambda x + (1-\lambda)y$.

If T_1, T_2 commute and $x \in \mathrm{Fix}(T_1)$, then $T_1(T_2(x)) = T_2(T_1(x)) = T_2(x)$, so $T_2(\mathrm{Fix}(T_1)) \subseteq \mathrm{Fix}(T_1)$. Thus T_2 restricted to $\mathrm{Fix}(T_1)$ is a continuous affine map of a compact convex set into itself, and by Step 1 it has a fixed point in $\mathrm{Fix}(T_1)$.

By transfinite induction (or the finite intersection property), $\bigcap_{T\in\mathcal{F}} \mathrm{Fix}(T) \neq \emptyset$. □

10.2 Amenable Groups

Definition 10.3 (Invariant mean)**.** Let G be a discrete group. A *left-invariant mean* on G is a linear functional $m\colon \ell^\infty(G) \to \mathbb{R}$ such that:

1. $m(\nVdash_G) = 1$ (normalization);
2. $m(f) \geq 0$ if $f \geq 0$ (positivity);
3. $m(L_g f) = m(f)$ for all $g \in G$ (invariance), where $(L_g f)(x) = f(g^{-1}x)$.

Definition 10.4 (Amenable group)**.** A discrete group G is *amenable* if it admits a left-invariant mean.

Theorem 10.5 (Abelian groups are amenable)**.** *Every discrete abelian group is amenable.*

Proof. Let G be an abelian group. The set of means

$$\mathcal{M} = \{m \in (\ell^\infty(G))^* : m \geq 0,\ m(\nVdash_G) = 1\}$$

is a nonempty compact convex set for the weak-$*$ topology (by the Banach–Alaoglu theorem).

For each $g \in G$, the map $T_g\colon \mathcal{M} \to \mathcal{M}$ defined by $(T_g m)(f) = m(L_{g^{-1}} f)$ is affine and continuous (for the weak-$*$ topology). It maps $\mathcal{M}$ into $\mathcal{M}$ since $L_{g^{-1}}$ preserves positivity and $L_{g^{-1}}\nVdash_G = \nVdash_G$.

Since G is abelian, $L_g \circ L_h = L_{gh} = L_{hg} = L_h \circ L_g$, so the T_g commute. By the Markov-Kakutani theorem, there exists $m^* \in \bigcap_{g\in G} \mathrm{Fix}(T_g)$, i.e., an invariant mean. □

Example 10.6 (Invariant mean on $\mathbb{Z}$)**.** The Banach mean on $\mathbb{Z}$ constructed above is not explicit. However, one can show that for any bounded sequence $(a_n)_{n\in\mathbb{Z}}$, if the Cesàro limit $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} a_n$ exists, then it coincides with $m^*((a_n))$.

10.3 Examples of Non-Amenable Groups

Theorem 10.7 (von Neumann)**.** *The free group on two generators F_2 is not amenable.*

Proof sketch. One uses the Banach–Tarski paradox: if F_2 were amenable, one could construct a finitely additive invariant measure on F_2, contradicting the fact that F_2 admits a paradoxical decomposition (a partition into four pieces that, by translations, reconstitute two copies of F_2). □

Proposition 10.8 (Properties of amenability)**.** 1. Subgroups of amenable groups are amenable.

2. Quotients of amenable groups are amenable.
3. Extensions of amenable groups by amenable groups are amenable.
4. Every finite group is amenable (take the uniform mean).
5. A group containing F_2 as a subgroup is not amenable.

10.4 Variants and Generalizations

Theorem 10.9 (Ryll-Nardzewski)**.** *Let E be a Banach space and $C \subseteq E$ a nonempty weakly compact convex set. Let $\mathcal{F}$ be a semigroup of affine isometries of C. Then $\bigcap_{T\in\mathcal{F}} \mathrm{Fix}(T) \neq \emptyset$.*

Remark 10.10. The Ryll-Nardzewski theorem generalizes Markov-Kakutani by replacing commutativity with a nonexpansiveness condition and weakening compactness to weak compactness.

Common fixed points

Theorem	Family	Space
Markov-Kakutani	Commuting affine maps	Compact convex, lctvs
Ryll-Nardzewski	Affine isometries	Weakly compact convex, Banach

10.5 Exercises

Exercise 10.1. Show that the Markov-Kakutani theorem is false without the commutativity hypothesis. *Hint: consider two rotations of the disk.*

Exercise 10.2. Apply the Markov-Kakutani theorem to show that every compact abelian group G admits a Haar measure (by taking C to be the set of probability measures on G).

Exercise 10.3. Show that the isometry group of the Euclidean plane is not amenable. *Hint: show that it contains F_2 as a subgroup.*

Exercise 10.4. Let $T\colon C \to C$ be a continuous affine map on a compact convex set C in a Banach space. Show directly (without Markov-Kakutani) that the Cesàro means $\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k(x)$ converge (at least along a subsequence) to a fixed point of T.

Exercise 10.5. Let G be a discrete amenable group acting on a compact space X by homeomorphisms. Show that there exists a Borel probability measure on X invariant under the action of G.

Exercise 10.6 (Amenability and Følner sequences)**.** Show that a countable group G is amenable if and only if it admits a *Følner sequence*: a sequence (F_n) of nonempty finite subsets of G such that for every $g \in G$,

$$\frac{|gF_n \triangle F_n|}{|F_n|} \to 0 \quad \text{as } n \to \infty.$$

Verify that $F_n = \{-n, \dots, n\}$ is a Følner sequence for $\mathbb{Z}$.

CHAPTER 10. MARKOV-KAKUTANI THEOREM

Chapter 11

Random Fixed Points

Intuition

Random fixed point theory extends classical results to the stochastic setting: one considers operators $T(\omega, \cdot)$ that depend on a random parameter $\omega \in \Omega$, and seeks a fixed point $x^*(\omega)$ that is itself measurable in ω. The main difficulties are of a measurability nature: one must ensure that constructions (iterations, limits) preserve measurability. These results find applications in stochastic analysis, mathematical economics, and stochastic game theory.

11.1 Random Operators

Definition 11.1 (Separable measurable space)**.** A topological space (X, τ) is *separable* if it contains a countable dense subset. We equip X with its Borel σ-algebra $\mathcal{B}(X) = \sigma(\tau)$.

Definition 11.2 (Random operator)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and (X, d) a separable metric space. A *random operator* is a map $T\colon \Omega \times X \to X$ such that:

1. for every $x \in X$, the map $\omega \mapsto T(\omega, x)$ is $(\mathcal{A}, \mathcal{B}(X))$-measurable;
2. for every $\omega \in \Omega$ (or $\mathbb{P}$-almost every ω), the map $x \mapsto T(\omega, x)$ satisfies certain properties (continuity, contraction, etc.).

Definition 11.3 (Random fixed point)**.** A *random fixed point* of T is a random variable $\xi\colon \Omega \to X$ (measurable) such that

$$T(\omega, \xi(\omega)) = \xi(\omega) \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

Remark 11.4. The existence of a fixed point for each ω (by a deterministic theorem) does not guarantee the existence of a random fixed point: the measurability of $\omega \mapsto x^*(\omega)$ is a nontrivial problem.

11.2 Measurable Selection Theorems

Definition 11.5 (Measurable correspondence)**.** A correspondence $F\colon \Omega \rightrightarrows X$ is *measurable* if for every open set $U \subseteq X$, the set $\{\omega : F(\omega) \cap U \neq \emptyset\}$ belongs to $\mathcal{A}$.

CHAPTER 11. RANDOM FIXED POINTS

Theorem 11.6 (Kuratowski–Ryll-Nardzewski measurable selection)**.** *Let $(\Omega, \mathcal{A})$ be a measurable space and (X, d) a complete separable metric space. If $F: \Omega \rightrightarrows X$ is a measurable correspondence with nonempty closed values, then F admits a* measurable selection*: there exists a measurable $f: \Omega \to X$ with $f(\omega) \in F(\omega)$ for every ω.*

Proof sketch. One constructs a sequence (f_n) of ε_n-approximate selections (with $\varepsilon_n \to 0$) by induction. Let $(x_k)_{k\geq 1}$ be a dense sequence in X. For each n, partition Ω into measurable sets $A_k^n = \{\omega : F(\omega) \cap B(x_k, \varepsilon_n) \neq \emptyset\} \setminus \bigcup_{j<k} A_j^n$ and set $f_n(\omega) = x_k$ for $\omega \in A_k^n$. The sequence (f_n) converges uniformly to a measurable selection f. □

11.3 Random Banach Fixed Point Theorem

Theorem 11.7 (Random contraction fixed point)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, (X, d) a complete separable metric space, and $T: \Omega \times X \to X$ a random operator such that:*

1. *for every $x \in X$, $\omega \mapsto T(\omega, x)$ is measurable;*
2. *for $\mathbb{P}$-almost every ω, $x \mapsto T(\omega, x)$ is a contraction with constant $k(\omega) < 1$.*

Then T has a unique random fixed point.

Proof. For each ω, the deterministic Banach theorem provides a unique fixed point $\xi(\omega) \in X$. It remains to show that ξ is measurable.

Fix $x_0 \in X$ and define the sequence of iterates:

$$\xi_0(\omega) = x_0, \qquad \xi_{n+1}(\omega) = T(\omega, \xi_n(\omega)).$$

By hypothesis, ξ_0 is measurable (constant). If ξ_n is measurable, then $\omega \mapsto T(\omega, \xi_n(\omega))$ is measurable as a composition of measurable functions (Carathéodory's theorem). So ξ_{n+1} is measurable.

For almost every ω, $\xi_n(\omega) \to \xi(\omega)$ (convergence from Banach's theorem). As a pointwise limit of measurable functions, ξ is measurable. □

Attention

The separability hypothesis on X is essential to guarantee joint measurability and measurability of the limit. Without this hypothesis, the result is false in general.

11.4 Random Schauder Fixed Point Theorem

Theorem 11.8 (Random Schauder fixed point)**.** *Let $(\Omega, \mathcal{A})$ be a measurable space, C a closed bounded convex subset of a separable Banach space E, and $T: \Omega \times C \to C$ a random operator such that:*

1. *for every $x \in C$, $\omega \mapsto T(\omega, x)$ is measurable;*
2. *for every ω, $x \mapsto T(\omega, x)$ is continuous;*
3. *for every ω, $T(\omega, \cdot)$ is compact (maps bounded sets to relatively compact sets).*

Then T has a random fixed point.

Proof sketch. By the deterministic Schauder theorem, for each ω, the set of fixed points $F(\omega) = \{x \in C : T(\omega, x) = x\}$ is nonempty and closed. One shows that the correspondence $\omega \mapsto F(\omega)$ is measurable. The Kuratowski–Ryll-Nardzewski measurable selection theorem then provides a measurable selection, which is the desired random fixed point. □

11.5 Applications in Stochastic Analysis

Proposition 11.9 (Random differential equations)**.** Consider the random differential equation

$$y'(t, \omega) = f(\omega, t, y(t, \omega)), \qquad y(t_0, \omega) = Y_0(\omega)$$

where $Y_0 : \Omega \to \mathbb{R}^n$ is a random variable and f satisfies:

- $f(\omega, \cdot, \cdot)$ is continuous in (t, y) for every ω;
- $f(\cdot, t, y)$ is measurable in ω for all t, y;
- f is Lipschitz in y uniformly in ω.

Then there exists a unique solution $y(\cdot, \omega)$ that is a measurable process.

Proof. The Picard operator $T(\omega, u)(t) = Y_0(\omega) + \int_{t_0}^{t} f(\omega, s, u(s))ds$ is a contractive random operator on $C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$ equipped with the Bielecki norm. Theorem 11.7 applies. □

Proposition 11.10 (Stochastic integral equations)**.** The random Fredholm integral equation

$$u(\omega, t) = f(\omega, t) + \lambda \int_a^b K(\omega, t, s)u(\omega, s)\, ds$$

has a unique random fixed point whenever $|\lambda| < 1/((b - a) \sup_\omega \|K(\omega, \cdot, \cdot)\|_\infty)$.

11.6 Itoh's Theorem

Theorem 11.11 (Itoh)**.** *Let $(\Omega, \mathcal{A})$ be a complete measurable space, C a closed separable subset of a Banach space, and $T : \Omega \times C \to C$ a nonexpansive random operator ($d(T(\omega, x), T(\omega, y)) \leq d(x, y)$) such that for every ω, $T(\omega, C)$ is contained in a compact subset of C. Then T has a random fixed point.*

Summary of random fixed point theorems

Theorem	**Hypothesis on $T(\omega, \cdot)$**	**Uniqueness**
Random Banach	Contraction	Yes
Random Schauder	Continuous, compact	No
Itoh	Nonexpansive, compact image	No

CHAPTER 11. RANDOM FIXED POINTS

11.7 Exercises

Exercise 11.1. Let $\Omega = [0, 1]$ with the Lebesgue measure and $X = \mathbb{R}$. Define $T(\omega, x) = \omega x/2$. Verify that T is a contractive random operator and determine the random fixed point explicitly.

Exercise 11.2. Let $T(\omega, x) = a(\omega)x + b(\omega)$ where $a, b\colon \Omega \to \mathbb{R}$ are measurable with $|a(\omega)| < 1$ a.s. Show that the random fixed point is $\xi(\omega) = b(\omega)/(1 - a(\omega))$. Verify its measurability.

Exercise 11.3. Let (X, d) be a compact metric space and $T\colon \Omega \times X \to X$ a continuous random operator. Show that the correspondence $F(\omega) = \{x : T(\omega, x) = x\}$ is measurable. *Hint: show that* $\{\omega : F(\omega) \cap U \neq \emptyset\}$ *is measurable for every open set* U.

Exercise 11.4. Consider the random ODE $y' = A(\omega)y$, $y(0) = 1$, where $A(\omega)$ is a real random variable. Reformulate as a fixed point problem and apply the random contraction fixed point theorem.

Exercise 11.5. Give an example of a random operator $T\colon \Omega \times [0, 1] \to [0, 1]$ continuous in x and measurable in ω, which has a fixed point for each ω, but for which no measurable selection of fixed points exists. *Hint: use a non-complete measurable space* $(\Omega, \mathcal{A})$.

Exercise 11.6 (Random iterations)**.** Let $(T_1, \ldots, T_k)$ be a family of contractions on $\mathbb{R}^n$ with probabilities $(p_1, \ldots, p_k)$. Define $x_{n+1} = T_{\sigma_n}(x_n)$ where (σ_n) is i.i.d. with law (p_i). Show that the sequence (x_n) converges a.s. to an invariant measure (random attractor), independently of the initial point x_0.

Bibliography

[1] Granas, A. and Dugundji, J., *Fixed Point Theory*, Springer, 2003.

[2] Granas, A. and Dugundji, J., *Fixed Point Theory*, Springer, 2003.

[3] Smart, D.R., *Fixed Point Theorems*, Cambridge, 1974.

[4] Smart, D.R., *Fixed Point Theorems*, Cambridge, 1974.

[5] Kirk, W.A. and Shahzad, N., *Fixed Point Theory in Distance Spaces*, Springer, 2014.

[6] Berinde, V., *Iterative Approximation of Fixed Points*, Springer, 2007.

[7] Zeidler, E., *Nonlinear Functional Analysis and Its Applications I*, Springer, 1986.

[8] Zeidler, E., *Nonlinear Functional Analysis and Its Applications I*, Springer, 1986.

[9] Border, K.C., *Fixed Point Theorems with Applications to Economics*, Cambridge, 1985.

[10] Goebel, K. and Kirk, W.A., *Topics in Metric Fixed Point Theory*, Cambridge, 1990.

[11] Agarwal, R.P., Meehan, M. and O'Regan, D., *Fixed Point Theory and Applications*, Cambridge, 2001.