Riemannian Geometry

Lecture Notes

Master M1 — 2025–2026

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"Geometry is not true, it is advantageous."
— Henri Poincaré

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Preface

Riemannian geometry stands as one of the fundamental pillars of modern mathematics, lying at the crossroads of analysis, algebra, and topology. Born from the visionary work of Bernhard Riemann in his celebrated 1854 *Habilitationsschrift*, entitled *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, this discipline has undergone spectacular development, culminating in results as profound as Perelman's proof of the Poincaré conjecture via Ricci flow.

Goals of this course

This course is aimed at Master's and PhD students in pure mathematics. It provides a rigorous and comprehensive presentation of the foundations of Riemannian geometry, from the definition of Riemannian manifolds through comparison theorems and an introduction to Ricci flow.

The main objectives are:

- Master the formalism of connections, covariant differentiation, and parallel transport.
- Understand in depth the Levi-Civita connection and its characterizing properties.
- Study geodesics, the exponential map, and minimality properties.
- Develop the tensor calculus of curvature: the Riemann tensor, sectional curvature, Ricci curvature, and scalar curvature.
- Establish the major global theorems: Bonnet–Myers, Hadamard–Cartan, and the comparison theorems of Rauch and Toponogov.
- Explore the geometry of symmetric spaces and submanifolds.
- Provide an introduction to Ricci flow, motivated by the Poincaré conjecture.

Prerequisites

The reader is assumed to be familiar with:

- The theory of smooth manifolds: atlases, charts, diffeomorphisms, tangent bundles, differential forms.
- Multilinear algebra and basic tensor calculus: tensor products, exterior algebra, contraction.

- Elements of general topology: compactness, connectedness, covering spaces.
- Ordinary differential equations: the Cauchy–Lipschitz (Picard–Lindelöf) theorem, smooth dependence on initial conditions.
- Basics of Lie group theory and Lie algebras.

Prior familiarity with the differential geometry of curves and surfaces in $\mathbb{R}^3$ is helpful but not essential; the relevant concepts will be revisited in the intrinsic framework.

Organization of the course

The course is organized into eleven chapters, structured as follows:

Chapter 1 — Riemannian Manifolds. We introduce Riemannian metrics, the associated metric space structure, and the fundamental examples: spheres S^n, hyperbolic spaces $\mathbb{H}^n$, flat tori, and Lie groups with bi-invariant metrics. Isometries and completeness are discussed.

Chapter 2 — Connections and Covariant Derivatives. We develop the general theory of connections on vector bundles: axiomatic definition, parallel transport, torsion, and holonomy.

Chapter 3 — The Levi-Civita Connection. We prove the fundamental theorem of Riemannian geometry: existence and uniqueness of the torsion-free, metric-compatible connection. Christoffel symbols are computed explicitly on examples.

Chapter 4 — Geodesics and the Exponential Map. We study geodesics as locally length-minimizing curves, the exponential map, the Gauss lemma, normal coordinates, and the injectivity radius.

Chapter 5 — Curvature: The Riemann Tensor. We introduce the Riemann curvature tensor via the non-commutativity of covariant differentiation, its fundamental symmetries, and the Bianchi identities.

Chapter 6 — Sectional, Ricci, and Scalar Curvature. We define the various contractions of the Riemann tensor and compute them on the model spaces.

Chapter 7 — Comparison Theorems. We present the theorems of Rauch, Toponogov, and Bishop–Gromov, relating curvature bounds to global geometry.

Chapter 8 — Symmetric Spaces. We study Riemannian symmetric spaces in the sense of Cartan, their connection to Lie groups, and the classification into compact and noncompact types.

Chapter 9 — Bonnet–Myers and Hadamard–Cartan. We prove these two fundamental theorems illustrating the influence of curvature sign on global topology.

Chapter 10 — Submanifold Geometry. We develop the theory of submanifolds: the second fundamental form, the Gauss–Codazzi–Ricci equations, hypersurfaces, and mean curvature.

Chapter 11 — Introduction to Ricci Flow. We present Hamilton's Ricci flow, local existence, self-similar solutions (solitons), and an overview of Perelman's strategy.

Conventions and notation

Throughout this course, we adopt the following conventions:

- Manifolds are smooth (C^∞), Hausdorff, second-countable, and connected unless otherwise stated.
- (M, g) denotes a Riemannian manifold, ∇ the Levi-Civita connection, R the curvature tensor.
- Sign convention for the Riemann tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

- The Einstein summation convention is used: a repeated index in contravariant and covariant positions is implicitly summed.
- $\mathfrak{X}(M)$ denotes the space of smooth vector fields on M.
- $\Gamma(E)$ denotes the space of smooth sections of a vector bundle E.
- T_pM denotes the tangent space at p, T_p^*M the cotangent space.
- $\langle \cdot, \cdot \rangle$ or $g(\cdot, \cdot)$ denotes the Riemannian inner product.
- $\|v\| = \sqrt{\langle v, v \rangle}$ denotes the Riemannian norm.
- $d(p, q)$ denotes the Riemannian distance between p and q.
- $B(p, r) = \{q \in M : d(p, q) < r\}$ denotes the geodesic ball.
- Ric denotes the Ricci tensor, S or Scal the scalar curvature.
- $K(\sigma)$ denotes the sectional curvature of the 2-plane σ.

Reference model spaces

The three families of model spaces of constant curvature:

1. **Sphere** $S^n(r)$: sectional curvature $K = 1/r^2$, $\pi_1 = 0$ for $n \geq 2$.
2. **Euclidean space** $\mathbb{R}^n$: curvature $K = 0$, flat.
3. **Hyperbolic space** $\mathbb{H}^n(r)$: curvature $K = -1/r^2$, simply connected, non-compact.

Guiding theme: curvature and topology

The central theme of this course is the relationship between *curvature* (a local, analytic datum) and *topology* (a global, qualitative datum). This interplay is manifested in the following results:

Curvature condition	**Topological consequence**	**Theorem**
$\mathrm{Ric} \geq (n-1)\kappa > 0$	M compact, π_1 finite	Bonnet–Myers
$K \leq 0$ everywhere	$\widetilde{M} \cong \mathbb{R}^n$	Hadamard–Cartan
$K \geq \delta > 0$	$\mathrm{diam}(M) \leq \pi/\sqrt{\delta}$	Bonnet–Myers
$K > 1/4$-pinched	M homeomorphic to S^n	Sphere theorem

Study recommendations

Riemannian geometry is a discipline where geometric intuition and analytical rigor must complement each other. We recommend:

- Always checking general formulas on concrete examples (S^2, $\mathbb{H}^2$, flat torus).
- Drawing geometric situations, even schematically.
- Solving the proposed exercises, which are an integral part of the course.
- Systematically comparing intrinsic and coordinate-based approaches.

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Chapter 1

Riemannian Manifolds

In 1854, Bernhard Riemann, only 27 years old, delivered his habilitation lecture at Göttingen before a jury that included Gauss himself. The title: *Über die Hypothesen, welche der Geometrie zu Grunde liegen* — "On the hypotheses which lie at the foundations of geometry." In one hour, Riemann overturned the foundations of geometry: he proposed replacing rigid Euclidean space with *curved spaces*, where the notion of distance varies from point to point. Gauss, it is said, was profoundly moved.

Riemann's central idea is elegant: instead of defining a global distance, one equips each point with an *inner product* on its tangent space. From this infinitesimal data, one can reconstruct distances, angles, areas, and even the curvature of space. It is this structure — a manifold endowed with such a field of inner products — that we call a *Riemannian manifold*.

1.1 Riemannian metrics

Let us formalize Riemann's intuition. On a smooth manifold M, each point p possesses a tangent space T_pM, which is a vector space of dimension n. A Riemannian metric consists of choosing an inner product on each of these tangent spaces, in a smooth way.

Definition 1.1 (Riemannian metric)**.** Let M be a smooth manifold of dimension n. A *Riemannian metric* on M is the assignment, for each point $p \in M$, of an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ varying smoothly: for all smooth vector fields $X, Y \in \mathfrak{X}(M)$, the function $p \mapsto g_p(X_p, Y_p)$ is C^∞.

The pair (M, g) is called a *Riemannian manifold*.

In local coordinates $(x^1, \dots, x^n)$, the metric is written:

$$g = g_{ij}\, dx^i \otimes dx^j, \qquad g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

The matrix $(g_{ij}(p))$ is symmetric positive definite at every point p.

Theorem 1.2 (Existence of Riemannian metrics)**.** *Every paracompact smooth manifold admits a Riemannian metric.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas of M and $\{\rho_\alpha\}$ a partition of unity subordinate to $\{U_\alpha\}$. On each chart domain U_α, define the pullback Euclidean metric $g_\alpha = \varphi_\alpha^*(\delta_{ij}\, dx^i \otimes dx^j)$. Then $g = \sum_\alpha \rho_\alpha\, g_\alpha$ is a Riemannian metric on M, since a convex combination of inner products is again an inner product. □

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Remark 1.3. Uniqueness fails: the space of Riemannian metrics on M is an infinite-dimensional open convex cone in $\Gamma(S^2T^*M)$.

1.2 Fundamental examples

Example 1.4 (Euclidean space)**.** The space $\mathbb{R}^n$ with the standard metric $g = \delta_{ij}\, dx^i \otimes dx^j$ is the simplest Riemannian manifold. We have $g_{ij} = \delta_{ij}$.

Example 1.5 (The sphere S^n)**.** The unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ carries the induced metric from the inclusion $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$: $g_{S^n} = \iota^* g_{\mathbb{R}^{n+1}}$.

In spherical coordinates on S^2, with $(\theta, \phi) \in (0, \pi) \times (0, 2\pi)$:

$$g_{S^2} = d\theta^2 + \sin^2\theta\, d\phi^2.$$

The metric matrix is $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$, with determinant $\det(g_{ij}) = \sin^2\theta$.

Example 1.6 (Hyperbolic space $\mathbb{H}^n$)**.** The *upper half-space model*: $\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n > 0\}$ with metric:

$$g_{\mathbb{H}^n} = \frac{(dx^1)^2 + \cdots + (dx^n)^2}{(x^n)^2}.$$

For $n = 2$: $g = \frac{dx^2+dy^2}{y^2}$ with $(x, y) \in \mathbb{R} \times \mathbb{R}_{>0}$.

The *Poincaré disk model*: $\mathbb{D}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$ with metric:

$$g_{\mathbb{D}^n} = \frac{4\left((dx^1)^2 + \cdots + (dx^n)^2\right)}{(1 - \|x\|^2)^2}.$$

These two models are isometric.

Example 1.7 (Flat torus)**.** The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ inherits the standard Euclidean metric by passing to the quotient. It is a compact flat manifold (zero curvature). More generally, for a lattice $\Lambda \subset \mathbb{R}^n$, the torus $\mathbb{R}^n/\Lambda$ is a flat Riemannian manifold.

Example 1.8 (Lie groups and bi-invariant metrics)**.** Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra. An inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ defines a left-invariant metric by:

$$g_a(u, v) = \langle (dL_{a^{-1}})_a\, u, (dL_{a^{-1}})_a\, v\rangle, \quad u, v \in T_aG,$$

where $L_a : G \to G$ is left translation.

The metric is *bi-invariant* if it is also right-invariant, which is equivalent to:

$$\langle \mathrm{ad}_X\, Y, Z\rangle + \langle Y, \mathrm{ad}_X\, Z\rangle = 0, \quad \forall\, X, Y, Z \in \mathfrak{g}.$$

Every compact Lie group admits such a metric (for compact semisimple groups, the negative of the Killing form works).

1.3 Length, distance, and volume

Definition 1.9 (Length of a curve)**.** Let $\gamma : [a, b] \to M$ be a piecewise smooth curve. Its *length* is:

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt.$$

Definition 1.10 (Riemannian distance)**.** The *Riemannian distance* between two points $p, q \in M$ is:

$$d(p, q) = \inf\{L(\gamma) : \gamma \text{ piecewise smooth curve from } p \text{ to } q\}.$$

Theorem 1.11 (Metric space structure)**.** *The distance d defines a metric space structure on M whose topology coincides with the manifold topology.*

Proof. The separation property $d(p, q) > 0$ for $p \neq q$ follows from working in local coordinates, where the Riemannian metric is equivalent to the Euclidean metric, so any curve joining distinct points has strictly positive length. The triangle inequality is immediate from concatenation of curves. Topological compatibility is proved by comparing metric balls with coordinate balls. □

Definition 1.12 (Volume element)**.** The *Riemannian volume form* is the n-form defined in oriented local coordinates by:

$$dV_g = \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^n.$$

The volume of a domain $\Omega \subset M$ is $\mathrm{Vol}(\Omega) = \int_\Omega dV_g$.

Volume formulas for model spaces

$$\mathrm{Vol}(S^n(r)) = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} r^n,$$

$$\mathrm{Vol}(B_{\mathbb{R}^n}(0, r)) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} r^n,$$

$$\mathrm{Vol}(B_{\mathbb{H}^n}(p, r)) = \omega_{n-1} \int_0^r \sinh^{n-1}(t) \, dt.$$

1.4 Isometries and conformal maps

Definition 1.13 (Isometry)**.** Let (M, g) and (N, h) be Riemannian manifolds. A diffeomorphism $\varphi : M \to N$ is an *isometry* if $\varphi^* h = g$, i.e.:

$$h_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)) = g_p(u, v), \quad \forall\, p \in M, \; u, v \in T_pM.$$

Theorem 1.14 (Isometry group – Myers–Steenrod)**.** *The isometry group* $\mathrm{Isom}(M, g)$ *of a Riemannian manifold is a Lie group of dimension at most* $\frac{n(n+1)}{2}$.

Remark 1.15. Equality $\dim \mathrm{Isom}(M, g) = \frac{n(n+1)}{2}$ holds if and only if (M, g) has constant sectional curvature.

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Example 1.16 (Isometries of S^n)**.** $\mathrm{Isom}(S^n) = O(n+1)$, of dimension $\frac{n(n+1)}{2}$.

Example 1.17 (Isometries of $\mathbb{H}^n$)**.** In the upper half-space model, for $n = 2$, the orientation-preserving isometries are the Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, giving $\mathrm{Isom}^+(\mathbb{H}^2) \cong PSL(2, \mathbb{R})$.

Definition 1.18 (Conformal map)**.** A diffeomorphism $\varphi : (M, g) \to (N, h)$ is *conformal* if there exists $\lambda \in C^\infty(M)$, $\lambda > 0$, such that $\varphi^* h = \lambda^2 g$.

1.5 Products and Riemannian coverings

Definition 1.19 (Product metric)**.** Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. The *Riemannian product* is $(M_1 \times M_2, g_1 \oplus g_2)$, where:

$$(g_1 \oplus g_2)_{(p_1,p_2)}((u_1, u_2), (v_1, v_2)) = (g_1)_{p_1}(u_1, v_1) + (g_2)_{p_2}(u_2, v_2).$$

Definition 1.20 (Warped product)**.** Let (B, g_B) be a base and (F, g_F) a fiber. For $f \in C^\infty(B)$, $f > 0$, the *warped product* is $B \times_f F$ with metric:

$$g = g_B + f^2 \, g_F.$$

Example 1.21. The space $\mathbb{R}^n \setminus \{0\}$ in polar coordinates is a warped product $\mathbb{R}_{>0} \times_r S^{n-1}$ with $g = dr^2 + r^2 g_{S^{n-1}}$.

Proposition 1.22 (Riemannian covering)**.** If $\pi : \widetilde{M} \to M$ is a covering map and (M, g) is a Riemannian manifold, then there exists a unique metric $\tilde{g}$ on $\widetilde{M}$ such that π is a local isometry: $\pi^* g = \tilde{g}$.

1.6 Completeness

Definition 1.23 (Metric completeness)**.** (M, g) is *complete* if the metric space (M, d) is Cauchy-complete, i.e., every Cauchy sequence converges.

Theorem 1.24 (Hopf–Rinow)**.** *Let (M, g) be a connected Riemannian manifold. The following are equivalent:*

1. *(M, d) is complete as a metric space.*
2. *Closed bounded subsets are compact.*
3. *There exists $p \in M$ such that $\exp_p$ is defined on all of T_pM.*
4. *For every $p \in M$, $\exp_p$ is defined on all of T_pM.*

Moreover, if any of these conditions holds, then any two points of M are joined by a minimizing geodesic.

Completeness vs. compactness

Completeness does *not* imply compactness: $\mathbb{R}^n$ and $\mathbb{H}^n$ are complete but noncompact. Conversely, every compact Riemannian manifold is complete.

1.7 Coordinate computations

Example 1.25 (Polar coordinates on $\mathbb{R}^2$)**.** The coordinate change $x = r\cos\theta$, $y = r\sin\theta$ gives:

$$g = dx^2 + dy^2 = dr^2 + r^2\, d\theta^2.$$

Hence $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ and $\sqrt{\det g} = r$.

Example 1.26 (Fubini–Study metric on $\mathbb{C}P^1$)**.** Identifying $\mathbb{C}P^1 \cong S^2$, in inhomogeneous coordinates $z = x + iy$:

$$g_{FS} = \frac{4\,(dx^2 + dy^2)}{(1 + x^2 + y^2)^2} = \frac{4\,|dz|^2}{(1 + |z|^2)^2}.$$

This is the constant curvature $K = 1$ metric on S^2.

1.8 Functorial operations on metrics

Proposition 1.27 (Pullback metric)**.** If $f : M \to N$ is an immersion and (N, h) is Riemannian, then $g = f^*h$ is a Riemannian metric on M (the *induced metric*).

Proposition 1.28 (Musical isomorphisms)**.** The metric induces canonical isomorphisms:

$$\begin{aligned} \flat &: TM \to T^*M, && v \mapsto g(v, \cdot), \\ \sharp &: T^*M \to TM, && \omega \mapsto \text{the unique } v \text{ with } g(v, \cdot) = \omega. \end{aligned}$$

In coordinates: $(v^\flat)_i = g_{ij} v^j$ and $(\omega^\sharp)^i = g^{ij}\omega_j$, where (g^{ij}) is the inverse matrix of (g_{ij}).

1.9 Exercises

Exercise 1.1. Show that the hyperbolic metric $g = \frac{dx^2 + dy^2}{y^2}$ on $\mathbb{H}^2$ gives the distance formula:

$$d((x_1, y_1), (x_2, y_2)) = \operatorname{arcosh}\left(1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2 y_1 y_2}\right).$$

Exercise 1.2. Compute the volume of the sphere $S^2(r)$ of radius r by integrating the volume form in spherical coordinates. Verify that $\mathrm{Vol}(S^2(r)) = 4\pi r^2$.

Exercise 1.3. Let $G = SO(3)$ with the bi-invariant metric $g(X, Y) = -\frac{1}{2}\mathrm{tr}(XY)$ for $X, Y \in \mathfrak{so}(3)$. Show that this metric makes $SO(3)$ a space of constant sectional curvature $K = 1/4$.

Exercise 1.4. Show that the warped product $M = \mathbb{R} \times_{\cosh t} \mathbb{R}$ with $g = dt^2 + \cosh^2(t)\, ds^2$ is isometric to an open subset of $\mathbb{H}^2$.

Exercise 1.5. Let (M, g) be a Riemannian manifold and $\lambda \in C^\infty(M)$, $\lambda > 0$. Set $\tilde{g} = \lambda^2 g$ (conformal change). Compute the relation between the volume forms $dV_{\tilde{g}}$ and dV_g.

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Chapter 2

Connections and Covariant Derivatives

How do you differentiate a vector field on a sphere? On $\mathbb{R}^n$, you differentiate component by component. But on a curved manifold, tangent spaces change from point to point, and comparing two vectors at different points has no intrinsic meaning. An additional ingredient is needed: a *connection*, which prescribes how to "transport" a vector along a curve. It was Tullio Levi-Civita who, in 1917, formalized this idea in the Riemannian setting, showing that there exists a unique connection compatible with the metric and torsion-free.

2.1 Motivation: directional derivatives are not intrinsic

On $\mathbb{R}^n$, the derivative of a vector field Y in the direction X is $(D_X Y)^i = X^j \frac{\partial Y^i}{\partial x^j}$. On a general manifold, this operation is not intrinsic: it depends on the choice of coordinates. The notion of *connection* provides the correct framework for differentiating sections of vector bundles.

2.2 Connections on vector bundles

Definition 2.1 (Linear connection)**.** Let $E \to M$ be a smooth vector bundle. A *connection* (or *covariant derivative*) on E is an $\mathbb{R}$-bilinear map:

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s,$$

satisfying the following axioms:

1. **$C^\infty(M)$-linearity in X:** $\nabla_{fX} s = f\, \nabla_X s$ for all $f \in C^\infty(M)$.
2. **Leibniz rule:** $\nabla_X(fs) = (Xf)\, s + f\, \nabla_X s$ for all $f \in C^\infty(M)$.

Remark 2.2. Axiom (1) shows that $\nabla_X s$ at a point p depends only on X_p (not on the entire field X). Axiom (2) shows that ∇ is *not* $C^\infty(M)$-linear in s: it is a first-order differential operator.

Proposition 2.3 (Locality)**.** If $s_1 = s_2$ on an open set $U \subset M$, then $\nabla_X s_1 = \nabla_X s_2$ on U. The connection is therefore a local operator.

2.3 Coordinate expression: Christoffel symbols

Let $(U, x^1, \dots, x^n)$ be a local chart and $(e_1, \dots, e_r)$ a local frame of E over U. The *connection coefficients* ω_{ij}^k are defined by:

$$\nabla_{\partial_i} e_j = \omega_{ij}^k \, e_k.$$

For the tangent bundle $E = TM$, these coefficients are denoted Γ_{ij}^k and called *Christoffel symbols*:

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \, \partial_k.$$

Proposition 2.4 (Coordinate formula)**.** If $Y = Y^j \partial_j$ and $X = X^i \partial_i$, then:

$$\nabla_X Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + \Gamma_{ij}^k Y^j \right) \partial_k.$$

In components: $(\nabla_X Y)^k = X^i(\partial_i Y^k + \Gamma_{ij}^k Y^j) = X^i \nabla_i Y^k$.

Proof. By the Leibniz rule:

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i \partial_i}(Y^j \partial_j) = X^i \nabla_{\partial_i}(Y^j \partial_j) \\ &= X^i \left[(\partial_i Y^j) \, \partial_j + Y^j \nabla_{\partial_i} \partial_j \right] = X^i \left[(\partial_i Y^j) \, \partial_j + Y^j \Gamma_{ij}^k \, \partial_k \right]. \end{aligned}$$

Relabeling the dummy index: $(\nabla_X Y)^k = X^i(\partial_i Y^k + \Gamma_{ij}^k Y^j)$. □

2.4 Change of coordinates

Proposition 2.5 (Transformation of Christoffel symbols)**.** Under a coordinate change $x^i \mapsto \bar{x}^i$, the Christoffel symbols transform as:

$$\bar{\Gamma}_{ij}^k = \frac{\partial \bar{x}^k}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} \Gamma_{mn}^l + \frac{\partial \bar{x}^k}{\partial x^l} \frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j}.$$

The inhomogeneous term shows that Γ_{ij}^k do *not* form a tensor.

Christoffel symbols are not tensors

The presence of the term $\frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j}$ reflects the fact that the connection encodes additional information beyond the manifold structure alone. This "correction" is precisely what enables covariant differentiation.

2.5 Parallel transport

Definition 2.6 (Parallel field along a curve)**.** Let $\gamma : [a, b] \to M$ be a smooth curve and $V(t)$ a vector field along γ (i.e. $V(t) \in T_{\gamma(t)}M$). We say V is *parallel* along γ if:

$$\frac{DV}{dt} := \nabla_{\dot{\gamma}(t)} V = 0.$$

In coordinates, the parallelism condition reads:

$$\frac{dV^k}{dt} + \Gamma^k_{ij}(\gamma(t))\,\dot\gamma^i(t)\,V^j(t) = 0, \quad k = 1, \dots, n.$$

This is a system of first-order linear ODEs.

Theorem 2.7 (Existence and uniqueness of parallel transport)**.** *Let $\gamma : [a, b] \to M$ be a smooth curve and $v_0 \in T_{\gamma(a)}M$. There exists a unique parallel field $V(t)$ along γ with $V(a) = v_0$. The map:*

$$P^{a\to b}_\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M, \quad v_0 \mapsto V(b),$$

is a linear isomorphism called parallel transport *along γ.*

Proof. Existence and uniqueness follow from the Cauchy–Lipschitz theorem applied to the linear system above. Linearity of $P^{a\to b}_\gamma$ follows from linearity of the system. Invertibility is obtained by considering parallel transport along γ traversed in reverse. □

Example 2.8 (Parallel transport on S^2)**.** Consider the spherical triangle formed by great circle arcs connecting the North Pole $N = (0, 0, 1)$ to $A = (1, 0, 0)$, then A to $B = (0, 1, 0)$, then B back to N. Parallel transport of a vector around this triangle rotates it by $\pi/2$. This angle equals the area of the spherical triangle, illustrating the Gauss–Bonnet theorem.

2.6 Torsion of a connection

Definition 2.9 (Torsion tensor)**.** The *torsion* of a connection ∇ on TM is the $(1, 2)$-tensor:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Proposition 2.10. *T is indeed a tensor: it is $C^\infty(M)$-bilinear and antisymmetric. In coordinates: $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$.*

Proof. We check $C^\infty(M)$-linearity:

$$\begin{aligned} T(fX, Y) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\ &= f\nabla_X Y - f\nabla_Y X - (Yf)X - f[X, Y] + (Yf)X \\ &= f\,T(X, Y). \end{aligned}$$

Antisymmetry is immediate. In coordinates: $T(\partial_i, \partial_j) = \Gamma^k_{ij}\partial_k - \Gamma^k_{ji}\partial_k$ since $[\partial_i, \partial_j] = 0$. □

Definition 2.11 (Torsion-free connection)**.** A connection is *torsion-free* (or *symmetric*) if $T \equiv 0$, i.e.: $\nabla_X Y - \nabla_Y X = [X, Y]$ for all X, Y. In coordinates: $\Gamma^k_{ij} = \Gamma^k_{ji}$.

2.7 Holonomy

Definition 2.12 (Holonomy group)**.** Let (M, ∇) be a manifold with connection and $p \in M$. The *holonomy group* of ∇ at p is:

$$\mathrm{Hol}_p(\nabla) = \{P_\gamma : T_pM \to T_pM : \gamma \text{ smooth loop at } p\} \subset GL(T_pM).$$

The *restricted holonomy group* $\mathrm{Hol}^0_p(\nabla)$ is the subgroup corresponding to contractible loops.

Theorem 2.13 (Ambrose–Singer)**.** *The restricted holonomy group* $\mathrm{Hol}_p^0(\nabla)$ *is a connected Lie subgroup of* $GL(T_pM)$ *whose Lie algebra is generated by elements of the form* $P_\gamma^{-1} \circ R(X,Y) \circ P_\gamma$*, where* γ *ranges over paths from* p *to a point* q *and* $X, Y \in T_qM$*.*

Example 2.14 (Holonomy of S^n)**.** For the sphere S^n ($n \geq 2$), $\mathrm{Hol}(S^n) = SO(n)$, since the sphere is an irreducible simply connected symmetric space.

2.8 Covariant derivative of tensors

Definition 2.15 (Extension to tensors)**.** Let ∇ be a connection on TM. We extend ∇ to all tensors by:

1. $\nabla_X f = Xf$ for $f \in C^\infty(M)$.
2. $\nabla_X(S \otimes T) = (\nabla_X S) \otimes T + S \otimes (\nabla_X T)$ (Leibniz rule).
3. ∇ commutes with contractions.
4. On 1-forms: $(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$.

Proposition 2.16 (Covariant derivative of the metric)**.** For a symmetric $(0,2)$-tensor g:

$$(\nabla_X g)(Y, Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

We say ∇ is *compatible with* g if $\nabla g = 0$, i.e. $(\nabla_X g)(Y,Z) = 0$ for all X, Y, Z.

Covariant derivative in coordinates

For a tensor of type (r,s) with components $T^{i_1 \cdots i_r}_{j_1 \cdots j_s}$:

$$\nabla_k T^{i_1 \cdots i_r}_{j_1 \cdots j_s} = \partial_k T^{i_1 \cdots i_r}_{j_1 \cdots j_s} + \sum_{m=1}^{r} \Gamma^{i_m}_{k\ell}\, T^{i_1 \cdots \ell \cdots i_r}_{j_1 \cdots j_s} - \sum_{m=1}^{s} \Gamma^{\ell}_{k j_m}\, T^{i_1 \cdots i_r}_{j_1 \cdots \ell \cdots j_s}.$$

2.9 Metric compatibility and consequences

Proposition 2.17 (Characterization of compatibility)**.** The following conditions are equivalent:

1. $\nabla g = 0$.
2. $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ for all $X, Y, Z \in \mathfrak{X}(M)$.
3. Parallel transport preserves the inner product: for any curve γ and all $u, v \in T_{\gamma(a)}M$: $\langle P_\gamma u, P_\gamma v\rangle_{\gamma(b)} = \langle u, v\rangle_{\gamma(a)}$.
4. In coordinates: $\partial_k g_{ij} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il}$.

Proof. (1) $\Leftrightarrow$ (2): This is the definition of $\nabla g = 0$.

(2) $\Rightarrow$ (3): If $V(t)$ and $W(t)$ are parallel along γ, then $\frac{d}{dt}\langle V, W\rangle = \langle \nabla_{\dot\gamma} V, W\rangle + \langle V, \nabla_{\dot\gamma} W\rangle = 0$, so $\langle V, W\rangle$ is constant.

(3) $\Rightarrow$ (2): Any vector field along γ can be written as a linear combination of parallel fields, and the condition follows. □

2.10 Affine space of connections

Proposition 2.18. The space of connections on TM is an affine space modeled on $\Gamma(T^*M \otimes T^*M \otimes TM)$: if ∇ and $\tilde{\nabla}$ are two connections, then $A(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$ is a $(1,2)$-tensor.

Proof. We verify $C^\infty(M)$-bilinearity: $A(fX,Y) = \tilde{\nabla}_{fX} Y - \nabla_{fX} Y = f(\tilde{\nabla}_X Y - \nabla_X Y) = fA(X,Y)$, and $A(X,fY) = fA(X,Y) + (Xf)Y - (Xf)Y = fA(X,Y)$. □

2.11 Exercises

Exercise 2.1. Show that parallel transport along a meridian of S^2 (from θ_0 to θ_1, with ϕ fixed) sends ∂_ϕ to ∂_ϕ (appropriately normalized).

Exercise 2.2. Compute the Christoffel symbols of the metric $g = dr^2 + r^2 d\theta^2$ in polar coordinates on $\mathbb{R}^2$. Verify that the torsion vanishes.

Exercise 2.3. Let ∇ be a metric-compatible connection and $V(t)$ a parallel field along γ. Show that $\|V(t)\|$ is constant.

Exercise 2.4. Show that if two connections ∇ and $\tilde{\nabla}$ are both torsion-free and compatible with the same metric g, then $\nabla = \tilde{\nabla}$. (This is the uniqueness of the Levi-Civita connection, proved in the next chapter.)

Exercise 2.5. Let G be a Lie group with left-invariant metric. Show that the connection defined by $\nabla_X Y = \frac{1}{2}[X,Y]$ for left-invariant X, Y is torsion-free. When is it metric-compatible?

Exercise 2.6. Compute the holonomy group of the flat torus $\mathbb{T}^2$. Compare with that of S^2.

Chapter 3

Levi-Civita Connection

On a curved surface, the Euclidean notion of "straight line" loses its meaning. But the idea of "the straightest possible line" survives: it is the geodesic. To define it, one needs to know how to differentiate vector fields along curves on the manifold—and this is where the Levi-Civita connection enters, named after the Italian mathematician Tullio Levi-Civita who, together with Gregorio Ricci-Curbastro, developed tensor calculus at the turn of the twentieth century. The fundamental result of this chapter is as surprising as it is elegant: there exists a *unique* way to differentiate that is compatible with the metric and torsion-free. This uniqueness is the fundamental theorem of Riemannian geometry.

3.1 The fundamental theorem of Riemannian geometry

Theorem 3.1 (Fundamental theorem). *Let (M, g) be a Riemannian manifold. There exists a unique connection ∇ on TM satisfying:*

1. ***Metric compatibility:*** $\nabla g = 0$, *i.e.* $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.
2. ***Torsion-freeness:*** $\nabla_X Y - \nabla_Y X = [X, Y]$.

This connection is called the Levi-Civita connection *of* (M, g).

Proof. **Uniqueness.** Assume ∇ satisfies (1) and (2). Writing compatibility for three cyclic permutations:

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Taking the combination $(1)+(2)-(3)$ and using torsion-freeness ($\nabla_X Y - \nabla_Y X = [X, Y]$), we obtain the *Koszul formula*:

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = {} & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & + \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle. \end{aligned} \tag{3.1}$$

The right-hand side is entirely determined by g and the Lie brackets, so $\nabla_X Y$ is uniquely determined by the nondegeneracy of g.

CHAPTER 3. LEVI-CIVITA CONNECTION

Existence. Define $\nabla_X Y$ by the Koszul formula (3.1). One verifies that this defines a connection (linearity in X, Leibniz rule in Y), that it is metric-compatible, and torsion-free. The verifications are direct but computational. □

Koszul formula

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle.$$

This is *the* fundamental formula of Riemannian geometry.

3.2 Christoffel symbols of the Levi-Civita connection

Proposition 3.2 (Christoffel symbol formula)**.** In local coordinates $(x^1, \ldots, x^n)$, the Christoffel symbols of the Levi-Civita connection are:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

Proof. Applying the Koszul formula to coordinate vector fields ∂_i, ∂_j, ∂_l (whose Lie brackets vanish):

$$2\langle \nabla_{\partial_i} \partial_j, \partial_l\rangle = \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}.$$

Hence $2\, g_{kl}\, \Gamma^k_{ij} = \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}$, and multiplying by $g^{lm}/2$ yields the formula. □

Remark 3.3. The symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ is immediate, reflecting torsion-freeness: $[\partial_i, \partial_j] = 0$ implies $\nabla_{\partial_i} \partial_j = \nabla_{\partial_j} \partial_i$.

3.3 Computations on fundamental examples

Example 3.4 (Sphere S^2 in spherical coordinates)**.** With $g = d\theta^2 + \sin^2\theta\, d\phi^2$:

$$g_{11} = 1, \quad g_{22} = \sin^2\theta, \quad g_{12} = 0.$$

The nonzero Christoffel symbols are:

$$\Gamma^1_{22} = -\sin\theta\cos\theta,$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \cot\theta.$$

All others vanish. Verification: $\Gamma^1_{22} = \frac{1}{2} g^{11}(-\partial_1 g_{22}) = -\frac{1}{2} \cdot 2\sin\theta\cos\theta = -\sin\theta\cos\theta$.

Example 3.5 (Hyperbolic space $\mathbb{H}^2$ (half-plane))**.** With $g = \frac{dx^2+dy^2}{y^2}$:

$$g_{11} = g_{22} = \frac{1}{y^2}, \quad g_{12} = 0, \quad g^{11} = g^{22} = y^2, \quad g^{12} = 0.$$

The nonzero symbols are:

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y},$$
$$\Gamma^2_{11} = \frac{1}{y},$$
$$\Gamma^2_{22} = -\frac{1}{y}.$$

Example 3.6 (Surface of revolution)**.** Let S be the surface of revolution parametrized by $(u, v) \mapsto (f(u)\cos v, f(u)\sin v, h(u))$ with $f > 0$. The induced metric is $g = (f'^2 + h'^2)\, du^2 + f^2\, dv^2$.

If the generating curve is arc-length parametrized ($f'^2 + h'^2 = 1$), then $g = du^2 + f(u)^2\, dv^2$ and the nonzero symbols are:

$$\Gamma^1_{22} = -ff', \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{f'}{f}.$$

Example 3.7 (Lie group with bi-invariant metric)**.** On a Lie group G with bi-invariant metric, for left-invariant fields $X, Y \in \mathfrak{g}$, the Koszul formula gives:

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Indeed, bi-invariance gives $\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0$ (ad-antisymmetry), and the terms $X\langle Y, Z\rangle$ etc. vanish since the fields and the metric are left-invariant. The Koszul formula reduces to $2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle = \langle [X,Y], Z\rangle$.

3.4 Gradient, divergence, and Laplacian

Definition 3.8 (Gradient)**.** The *gradient* of $f \in C^\infty(M)$ is the unique vector field $\operatorname{grad} f = (\nabla f)^\sharp$ such that:

$$g(\operatorname{grad} f, X) = Xf = df(X), \quad \forall\, X \in \mathfrak{X}(M).$$

In coordinates: $(\operatorname{grad} f)^i = g^{ij} \frac{\partial f}{\partial x^j}$.

Definition 3.9 (Divergence)**.** The *divergence* of a vector field $X \in \mathfrak{X}(M)$ is:

$$\operatorname{div} X = \operatorname{tr}(\nabla X) = \nabla_i X^i = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, X^i\right).$$

Definition 3.10 (Laplace–Beltrami operator)**.** The *Laplacian* of a function f is:

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x^j}\right).$$

Example 3.11 (Laplacian on S^2)**.** In spherical coordinates, $\sqrt{\det g} = \sin\theta$, so:

$$\Delta_{S^2} f = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial\phi^2}.$$

Example 3.12 (Laplacian on $\mathbb{H}^2$)**.** With $g = y^{-2}(dx^2 + dy^2)$:

$$\Delta_{\mathbb{H}^2} f = y^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

3.5 Covariant differentiation along curves

Definition 3.13. Let $\gamma : [a, b] \to M$ be a smooth curve and V a vector field along γ. The *covariant derivative* of V along γ is:

$$\frac{DV}{dt} = \nabla_{\dot\gamma} V.$$

In coordinates: $\left(\frac{DV}{dt}\right)^k = \frac{dV^k}{dt} + \Gamma^k_{ij}(\gamma(t))\,\dot\gamma^i(t)\,V^j(t).$

Proposition 3.14 (Leibniz rule along curves)**.** If V and W are vector fields along γ and ∇ is metric-compatible:

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle.$$

3.6 Covariant acceleration and geodesics

Definition 3.15 (Covariant acceleration)**.** The *covariant acceleration* of a curve γ is:

$$\frac{D\dot\gamma}{dt} = \nabla_{\dot\gamma}\dot\gamma.$$

In coordinates: $\left(\frac{D\dot\gamma}{dt}\right)^k = \ddot\gamma^k + \Gamma^k_{ij}\,\dot\gamma^i\,\dot\gamma^j.$

Definition 3.16 (Geodesic)**.** A curve γ is a *geodesic* if its covariant acceleration vanishes:

$$\frac{D\dot\gamma}{dt} = \nabla_{\dot\gamma}\dot\gamma = 0.$$

The geodesic equation in coordinates is the second-order ODE system:

$$\ddot\gamma^k + \Gamma^k_{ij}(\gamma)\,\dot\gamma^i\,\dot\gamma^j = 0, \quad k = 1, \ldots, n.$$

Geodesic equation

$$\ddot\gamma^k + \Gamma^k_{ij}\,\dot\gamma^i\,\dot\gamma^j = 0.$$

A system of n second-order ODEs, equivalent to $2n$ first-order ODEs on TM.

3.7 Levi-Civita connection of products and warped products

Proposition 3.17 (Riemannian product)**.** On $(M_1 \times M_2, g_1 \oplus g_2)$, if $X_1, Y_1 \in \mathfrak{X}(M_1)$ and $X_2, Y_2 \in \mathfrak{X}(M_2)$, then:

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = (\nabla^1_{X_1} Y_1, \nabla^2_{X_2} Y_2),$$

where ∇^1 and ∇^2 are the Levi-Civita connections of (M_1, g_1) and (M_2, g_2) respectively.

Proposition 3.18 (Warped product)**.** On the warped product $B \times_f F$ with $g = g_B + f^2 g_F$, the Levi-Civita connection satisfies, for $X, Y \in \mathfrak{X}(B)$ and $V, W \in \mathfrak{X}(F)$:

$$\begin{aligned} \nabla_X Y &= \nabla^B_X Y, \\ \nabla_X V = \nabla_V X &= \frac{Xf}{f} V, \\ \nabla_V W &= \nabla^F_V W - \frac{\langle V, W \rangle_F}{f} \operatorname{grad}_B f. \end{aligned}$$

3.8 Exercises

Exercise 3.1. Compute all Christoffel symbols for the metric $g = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$ of $\mathbb{R}^3$ in spherical coordinates.

Exercise 3.2. Verify that great circles on S^2 are geodesics using the Christoffel symbols computed above.

Exercise 3.3. Show that on a Lie group with bi-invariant metric, the geodesics through the identity e are exactly the one-parameter subgroups $t \mapsto \exp(tX)$.

Exercise 3.4. Compute the Laplacian of $f(x, y) = \ln y$ on $\mathbb{H}^2$ with metric $g = \frac{dx^2 + dy^2}{y^2}$.

Exercise 3.5. Let (M, g) be Riemannian and $\tilde{g} = e^{2\varphi} g$ a conformal change. Show that the Christoffel symbols are related by:

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \partial_j \varphi + \delta^k_j \partial_i \varphi - g_{ij} g^{kl} \partial_l \varphi.$$

Exercise 3.6. On the warped product $\mathbb{R} \times_{\cosh t} \mathbb{R}^{n-1}$ with $g = dt^2 + \cosh^2(t) \, g_{\mathbb{R}^{n-1}}$, compute the Christoffel symbols and verify that the curves $t \mapsto (t, x_0)$ are geodesics.

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Chapter 4

Geodesics and Exponential Map

What is the shortest path between two points on a sphere? A great circle, of course. But how does one formulate this question on an arbitrary Riemannian manifold? The answer involves *geodesics*: curves that do not turn, whose covariant acceleration vanishes. On a plane, these are straight lines; on a sphere, great circles; on a surface of revolution, curves computable via the Euler-Lagrange equations of the calculus of variations. The *exponential map*, which sends a tangent vector to the point reached at time 1 along the corresponding geodesic, provides a local coordinate system centred at each point—normal coordinates, where the metric is Euclidean to first order.

4.1 The geodesic equation

Definition 4.1 (Geodesic)**.** Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. A smooth curve $\gamma : I \to M$ is a *geodesic* if:

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

In local coordinates $(x^1, \dots, x^n)$:

$$\ddot\gamma^k(t) + \Gamma^k_{ij}(\gamma(t))\,\dot\gamma^i(t)\,\dot\gamma^j(t) = 0, \quad k = 1, \dots, n.$$

Proposition 4.2 (Constant speed parametrization)**.** If γ is a geodesic, then $\|\dot\gamma(t)\|$ is constant. In particular, geodesics are parametrized proportionally to arc length.

Proof. $\frac{d}{dt}\langle\dot\gamma, \dot\gamma\rangle = 2\langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 0$. □

Theorem 4.3 (Local existence and uniqueness)**.** *For every $p \in M$ and $v \in T_pM$, there exists $\varepsilon > 0$ and a unique geodesic $\gamma_v : (-\varepsilon, \varepsilon) \to M$ with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$.*

Proof. The geodesic equation is a second-order ODE system with smooth coefficients. The Cauchy–Lipschitz theorem guarantees local existence and uniqueness. □

Proposition 4.4 (Homogeneity)**.** For any $\lambda \in \mathbb{R}$ and $v \in T_pM$: $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$ wherever both sides are defined.

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4.2 The exponential map

Definition 4.5 (Exponential map)**.** The *exponential map* at $p \in M$ is:

$$\exp_p : \mathcal{D}_p \subset T_pM \to M, \quad v \mapsto \gamma_v(1),$$

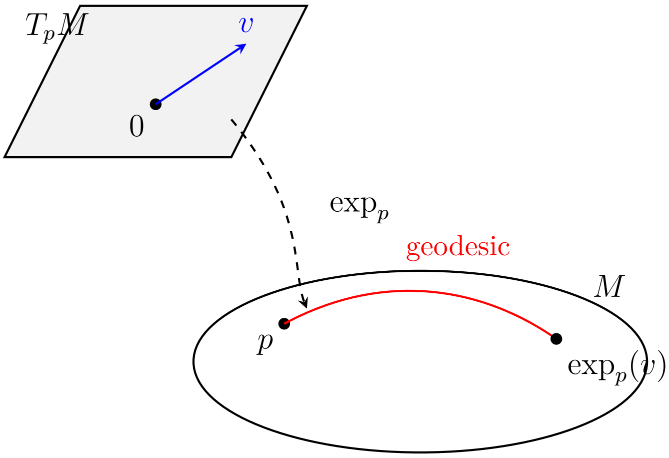
where $\mathcal{D}_p$ is the maximal domain of definition (a star-shaped open set containing 0) and γ_v is the geodesic with $\gamma_v(0) = p$, $\dot{\gamma}_v(0) = v$.

Proposition 4.6. The map $\exp_p$ is smooth and satisfies:

1. $\exp_p(0) = p$.
2. $(d\exp_p)_0 = \mathrm{Id}_{T_pM}$ (via the canonical identification $T_0(T_pM) \cong T_pM$).
3. $\exp_p(tv) = \gamma_v(t)$ for all t in the domain.

Proof. For (2): $(d\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} \gamma_v(t) = \dot{\gamma}_v(0) = v$. □

Corollary 4.7. *By the inverse function theorem,* $\exp_p$ *is a diffeomorphism from a neighborhood of* 0 *in* T_pM *onto a neighborhood of* p *in* M*.*



4.3 Normal coordinates

Definition 4.8 (Normal coordinates)**.** Let $(e_1, \ldots, e_n)$ be an orthonormal basis of T_pM. The *normal coordinates* centered at p are the coordinates x^i defined on a neighborhood U of p by:

$$q = \exp_p(x^1 e_1 + \cdots + x^n e_n).$$

Proposition 4.9 (Properties of normal coordinates)**.** In normal coordinates centered at p:

1. $g_{ij}(p) = \delta_{ij}$.
2. $\Gamma^k_{ij}(p) = 0$.
3. $\partial_l g_{ij}(p) = 0$.
4. Radial geodesics through p are the lines $t \mapsto (tv^1, \ldots, tv^n)$.

5. The Taylor expansion of the metric is:

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{ikjl}(p)\, x^k x^l + O(\|x\|^3).$$

Proof. (1) follows from the orthonormal basis choice. (2): radial geodesics satisfy $\gamma(t) = tv$, so $\ddot{\gamma}^k = 0$ and the geodesic equation gives $\Gamma^k_{ij}(p)v^i v^j = 0$ for all v, hence $\Gamma^k_{ij}(p) = 0$ by polarization. (3) follows from (2) and the Christoffel formula. □

Normal coordinates = linearization of geometry

In normal coordinates, the metric is Euclidean to first order: curvature effects appear only at second order. This is the Riemannian analogue of the fact that the Earth appears flat locally.

4.4 The Gauss lemma

Lemma 4.10 (Gauss lemma). *Let $p \in M$ and $v \in T_pM$ with $\exp_p(v)$ defined. Then:*

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w)\rangle = \langle v, w\rangle$$

for all $w \in T_pM$, identifying $T_v(T_pM) \cong T_pM$.

Proof. Set $q = \exp_p(v)$. The vector $(d\exp_p)_v(v) = \dot{\gamma}_v(1)$ where $\gamma_v(t) = \exp_p(tv)$. Consider the variation $F(t, s) = \exp_p(t(v + sw))$.

For fixed t, $F(t, \cdot)$ traces radial geodesics. The associated Jacobi field is $J(t) = \frac{\partial F}{\partial s}\big|_{s=0} = t\,(d\exp_p)_{tv}(w)$.

We compute:

$$\frac{d}{dt}\langle \dot{\gamma}_v(t), J(t)\rangle = \underbrace{\langle \nabla_{\dot{\gamma}}\dot{\gamma}, J\rangle}_{=0} + \langle \dot{\gamma}, \frac{DJ}{dt}\rangle.$$

One shows that $\frac{d}{dt}\langle \dot{\gamma}, J\rangle = \langle v, w\rangle$ by noting $J(0) = 0$ and $\frac{DJ}{dt}(0) = w$. Evaluating at $t = 1$ gives $\langle \dot{\gamma}_v(1), J(1)\rangle = \langle v, w\rangle$, and since $J(1) = (d\exp_p)_v(w)$ and $\dot{\gamma}_v(1) = (d\exp_p)_v(v)$, the result follows. □

Corollary 4.11 (Orthogonality of geodesic spheres). *Radial geodesics from p are orthogonal to geodesic spheres $S(p, r) = \{q : d(p, q) = r\}$ (for r sufficiently small).*

4.5 Minimizing geodesics

Theorem 4.12 (Short geodesics are minimizing). *For every $p \in M$, there exists $\varepsilon > 0$ such that for all $q \in B(p, \varepsilon)$, there is a unique minimizing geodesic from p to q, and this geodesic is contained in $B(p, \varepsilon)$.*

Proof. We use the Gauss lemma and comparison with normal coordinates. Let γ be the radial geodesic from p to $q = \exp_p(v)$ and $\sigma : [0, 1] \to M$ any curve from p to q. In normal coordinates, decompose $\dot{\sigma} = \dot{\sigma}^r + \dot{\sigma}^\perp$ into radial and tangential components to geodesic spheres. By the Gauss lemma:

$$L(\sigma) = \int_0^1 \sqrt{|\dot{\sigma}^r|^2 + |\dot{\sigma}^\perp|^2}\, dt \geq \int_0^1 |\dot{\sigma}^r|\, dt \geq |r(1) - r(0)| = \|v\| = L(\gamma).$$

□

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Definition 4.13 (Normal neighborhood)**.** An open set $U \ni p$ is a *normal neighborhood* of p if there is a star-shaped open set $V \subset T_pM$ containing 0 such that $\exp_p : V \to U$ is a diffeomorphism and every point of U is joined to p by a unique minimizing geodesic contained in U.

4.6 Injectivity radius

Definition 4.14 (Injectivity radius)**.** The *injectivity radius* at p is:

$$\mathrm{inj}(p) = \sup\{r > 0 : \exp_p|_{B(0,r)} \text{ is a diffeomorphism}\}.$$

The injectivity radius of M is $\mathrm{inj}(M) = \inf_{p\in M} \mathrm{inj}(p)$.

Example 4.15 (Injectivity radius of S^n)**.** $\mathrm{inj}(S^n) = \pi$: the exponential map ceases to be injective at the antipodal point.

Example 4.16 (Injectivity radius of the flat torus)**.** For $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\mathrm{inj}(\mathbb{T}^2) = 1/2$.

Example 4.17 (Injectivity radius of $\mathbb{H}^n$)**.** $\mathrm{inj}(\mathbb{H}^n) = +\infty$: the exponential map is a global diffeomorphism.

4.7 Conjugate points and cut locus

Definition 4.18 (Conjugate point)**.** A point $\gamma(t_0)$ is *conjugate* to $\gamma(0) = p$ along the geodesic γ if $(d\exp_p)_{t_0\dot\gamma(0)}$ is singular, i.e. there exists a nonzero Jacobi field J along γ with $J(0) = J(t_0) = 0$.

Definition 4.19 (Cut point)**.** The *cut point* of p along γ is the first point $\gamma(t_c)$ beyond which γ ceases to be minimizing:

$$t_c = \sup\{t > 0 : \gamma|_{[0,t]} \text{ is minimizing}\}.$$

The *cut locus* $\mathrm{Cut}(p)$ is the set of cut points for all geodesics from p.

Theorem 4.20 (Characterization of cut points)**.** *The cut point $\gamma(t_c)$ satisfies one of two conditions:*

1. *$\gamma(t_c)$ is the first conjugate point along γ.*
2. *There exists another minimizing geodesic from p to $\gamma(t_c)$.*

Example 4.21 (Cut locus of S^n)**.** For p the North Pole of S^n, $\mathrm{Cut}(p) = \{-p\}$ (the antipodal point).

4.8 Geodesics of model spaces

Example 4.22 (Geodesics of $\mathbb{R}^n$)**.** The geodesics are straight lines: $\gamma(t) = p + tv$.

Example 4.23 (Geodesics of S^n)**.** The geodesics are great circles, parametrized by: $\gamma(t) = \cos(t)\,p + \sin(t)\,v$, where $p \in S^n$ and $v \in T_pS^n$ with $\|v\| = 1$.

Example 4.24 (Geodesics of $\mathbb{H}^2$ (half-plane))**.** The geodesics are:

- Vertical half-lines $\{x = x_0, y > 0\}$.
- Semicircles centered on the axis $\{y = 0\}$.

For a vertical half-line parametrized by $\gamma(t) = (x_0, e^t)$: $\dot{\gamma} = (0, e^t)$, $\|\dot{\gamma}\|_g = \frac{e^t}{e^t} = 1$, and one verifies the geodesic equation is satisfied.

4.9 Energy functional and first variation

Definition 4.25 (Energy functional)**.** The *energy* of a curve $\gamma : [a, b] \to M$ is:

$$E(\gamma) = \frac{1}{2} \int_a^b \|\dot{\gamma}(t)\|^2 \, dt.$$

Proposition 4.26 (Cauchy–Schwarz inequality)**.** $L(\gamma)^2 \leq 2(b - a)\, E(\gamma)$, with equality if and only if $\|\dot{\gamma}\|$ is constant.

Theorem 4.27 (First variation formula)**.** *Let $\gamma_s : [a, b] \to M$ be a smooth variation of $\gamma_0 = \gamma$ with variation field $V = \frac{\partial \gamma_s}{\partial s}\big|_{s=0}$. Then:*

$$\left.\frac{d}{ds}\right|_{s=0} E(\gamma_s) = -\int_a^b \langle \nabla_{\dot{\gamma}} \dot{\gamma}, V \rangle \, dt + \langle \dot{\gamma}(b), V(b) \rangle - \langle \dot{\gamma}(a), V(a) \rangle.$$

Corollary 4.28. *The critical points of E among curves with fixed endpoints are exactly the geodesics. Geodesics are thus the* critical curves *of the energy (and length) functional.*

4.10 Geodesic completeness and Hopf–Rinow

Definition 4.29. (M, g) is *geodesically complete* if every geodesic can be extended for all time $t \in \mathbb{R}$.

We recall that the Hopf–Rinow theorem (Theorem 1.24) establishes the equivalence between metric completeness and geodesic completeness.

4.11 Exercises

Exercise 4.1. Compute the geodesics of the cylinder $S^1 \times \mathbb{R}$ with the product metric. Show they are helices (including circles and lines as limiting cases).

Exercise 4.2. Show that the geodesics of $\mathbb{H}^2$ in the Poincaré disk model are diameters and circular arcs orthogonal to the boundary $\partial\mathbb{D}$.

Exercise 4.3. Let $G = SU(2)$ with a bi-invariant metric. Identify $SU(2) \cong S^3$ and show that the geodesics are great circles.

Exercise 4.4. Compute the injectivity radius of the real projective space $\mathbb{R}P^n$ with the quotient metric from S^n.

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Exercise 4.5. Show that on a surface of revolution $g = du^2 + f(u)^2 dv^2$, geodesics satisfy Clairaut's relation: $f^2 \dot{v} = c$ (constant).

Exercise 4.6. Prove the first variation formula for length: if γ is arc-length parametrized:

$$\frac{d}{ds}\bigg|_{s=0} L(\gamma_s) = -\int_a^b \langle \nabla_{\dot{\gamma}} \dot{\gamma}, V \rangle \, dt + \langle \dot{\gamma}, V \rangle \bigg|_a^b.$$

Chapter 5

Curvature — Riemann Tensor

Curvature is the quantity that distinguishes one geometry from another. On a plane, parallel lines remain parallel; on a sphere, they eventually meet; on a saddle surface, they diverge. Riemann understood, as early as 1854, that this deviation of geodesics is measured by a tensorial object—the *curvature tensor*—which encodes, at each point and in each tangent plane, how space curves. This tensor, formalised by Christoffel and Ricci-Curbastro, is the central ingredient of Einstein's general relativity: the curvature of spacetime *is* gravitation.

5.1 The Riemann curvature tensor

Definition 5.1 (Curvature tensor)**.** The *Riemann curvature tensor* of the Levi-Civita connection ∇ is the map $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Curvature measures non-commutativity

The Riemann tensor measures the failure of covariant differentiation to commute: in flat space, $\nabla_X \nabla_Y = \nabla_Y \nabla_X$ (modulo the Lie bracket). Curvature is the obstruction to parallel transport being path-independent.

Proposition 5.2. *R is a $(1,3)$-tensor: it is $C^\infty(M)$-multilinear in X, Y, and Z.*

Proof. We check $C^\infty(M)$-linearity in Z (the others are similar):

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ) \\ &= \nabla_X\big((Yf)Z + f\nabla_Y Z\big) - \nabla_Y\big((Xf)Z + f\nabla_X Z\big) - ([X,Y]f)Z - f\nabla_{[X,Y]}Z. \end{aligned}$$

Expanding and simplifying: the terms involving Xf, Yf, XYf, YXf cancel (using $[X,Y]f = XYf - YXf$), leaving $fR(X,Y)Z$. □

5.2 Coordinate expression

In local coordinates, $R(\partial_i, \partial_j)\partial_k = R^l_{kij}\,\partial_l$, with:

$$R^l_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}.$$

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One also defines the *lowered curvature tensor*:

$$R_{ijkl} = g_{im} R^m_{jkl},$$

so that $R_{ijkl} = \langle R(\partial_k, \partial_l)\partial_j, \partial_i \rangle$.

Components of the Riemann tensor

$$R^l_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}.$$

In normal coordinates at p: $R^l_{kij}(p) = \partial_i \Gamma^l_{jk}(p) - \partial_j \Gamma^l_{ik}(p)$.

5.3 Symmetries of the Riemann tensor

Theorem 5.3 (Fundamental symmetries)**.** *The lowered curvature tensor R_{ijkl} possesses the following symmetries:*

1. ***Antisymmetry in** (k, l)**:** $R_{ijkl} = -R_{ijlk}$.*
2. ***Antisymmetry in** (i, j)**:** $R_{ijkl} = -R_{jikl}$.*
3. ***Pair symmetry:** $R_{ijkl} = R_{klij}$.*
4. ***First Bianchi identity (algebraic):** $R_{ijkl} + R_{iklj} + R_{iljk} = 0$.*

In intrinsic notation:

1. $R(X, Y) = -R(Y, X)$.
2. $\langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle$.
3. $\langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle$.
4. $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.

Proof. (1) is immediate from the definition ($R(X, Y) = -R(Y, X)$).

(2) Follows from metric compatibility. Indeed:

$$\begin{aligned} \langle R(X, Y)Z, Z \rangle &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, Z \rangle \\ &= X \langle \nabla_Y Z, Z \rangle - \langle \nabla_Y Z, \nabla_X Z \rangle - Y \langle \nabla_X Z, Z \rangle + \langle \nabla_X Z, \nabla_Y Z \rangle \\ &= \frac{1}{2} XY \langle Z, Z \rangle - \frac{1}{2} YX \langle Z, Z \rangle = \frac{1}{2}[X, Y]\langle Z, Z \rangle = \langle \nabla_{[X,Y]} Z, Z \rangle. \end{aligned}$$

Hence $\langle R(X, Y)Z, Z \rangle = 0$, which by polarization gives (2).

(3) follows from (1), (2), and (4).

(4) First Bianchi identity: using torsion-freeness and the definition, expand the three cyclic terms and verify the sum vanishes. □

Proposition 5.4 (Number of independent components)**.** In dimension n, the number of independent components of the Riemann tensor is:

$$\frac{n^2(n^2 - 1)}{12}.$$

For $n = 2$: 1; for $n = 3$: 6; for $n = 4$: 20.

5.4 Bianchi identities

Theorem 5.5 (Second Bianchi identity (differential))**.**

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

In coordinates: $\nabla_m R^l_{kij} + \nabla_i R^l_{kmj} + \nabla_j R^l_{kim} = 0.$

Proof. Work in normal coordinates at a point p where $\Gamma^k_{ij}(p) = 0$. The covariant derivative of R reduces to the partial derivative, and the result follows from a direct computation using commutativity of partial derivatives. □

Remark 5.6. The second Bianchi identity plays a fundamental role: when contracted, it yields the divergence-freeness of the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2} S g_{ij}$ in general relativity.

5.5 Curvature and parallel transport

Theorem 5.7 (Interpretation via parallel transport)**.** *Let X, Y be two commuting vector fields ($[X,Y] = 0$). Consider the infinitesimal parallelogram formed by the flows of sX and tY. Parallel transport of a vector Z around this parallelogram yields, to second order:*

$$Z_{\textit{final}} - Z_{\textit{initial}} = st\, R(X,Y)Z + O(s^2 + t^2).$$

Corollary 5.8 (Flatness)**.** *(M,g) is flat ($R \equiv 0$) if and only if parallel transport is path-independent, if and only if (M,g) is locally isometric to $\mathbb{R}^n$.*

5.6 Jacobi fields

Definition 5.9 (Jacobi field)**.** Let γ be a geodesic. A vector field J along γ is a *Jacobi field* if it satisfies the *Jacobi equation*:

$$\frac{D^2 J}{dt^2} + R(\dot\gamma, J)\dot\gamma = 0.$$

Proposition 5.10. Jacobi fields along a geodesic form a $2n$-dimensional vector space, determined by the initial conditions $J(0)$ and $\frac{DJ}{dt}(0)$.

Theorem 5.11 (Origin of Jacobi fields)**.** *Jacobi fields are the variation fields of families of geodesics: if γ_s is a smooth family of geodesics, then $J(t) = \frac{\partial \gamma_s}{\partial s}\big|_{s=0}$ is a Jacobi field along γ_0.*

Proof. Set $T = \dot\gamma_s$ and $J = \frac{\partial \gamma_s}{\partial s}$. Since $[T,J] = 0$ (symmetry of the variation) and $\nabla_T T = 0$ (geodesic equation):

$$\frac{D^2 J}{dt^2} = \nabla_T \nabla_T J = \nabla_T \nabla_J T = \nabla_J \nabla_T T + R(T,J)T = R(T,J)T = -R(\dot\gamma, J)\dot\gamma.$$

Hence $\frac{D^2 J}{dt^2} + R(\dot\gamma, J)\dot\gamma = 0$. □

Example 5.12 (Jacobi fields on S^n)**.** On S^n (curvature $K = 1$), along a unit-speed great circle γ, the Jacobi equation for an orthogonal field $J \perp \dot\gamma$ becomes:

$$J''(t) + J(t) = 0,$$

with solution $J(t) = A\cos t + B\sin t$. Conjugate points occur at $t = k\pi$, $k \in \mathbb{Z}^*$.

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Example 5.13 (Jacobi fields on $\mathbb{H}^n$)**.** On $\mathbb{H}^n$ (curvature $K = -1$), for $J \perp \dot{\gamma}$:

$$J''(t) - J(t) = 0,$$

with solution $J(t) = A\cosh t + B\sinh t$. There are *no* conjugate points.

5.7 Curvature computations on examples

Example 5.14 (Curvature of S^2)**.** In spherical coordinates:

$$R^1_{212} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1_{1m}\Gamma^m_{22} - \Gamma^1_{2m}\Gamma^m_{12}.$$

One obtains $R^1_{212} = \sin^2\theta$, hence $R_{1212} = g_{11}R^1_{212} = \sin^2\theta$. The Gaussian curvature is $K = \frac{R_{1212}}{g_{11}g_{22}-g_{12}^2} = \frac{\sin^2\theta}{\sin^2\theta} = 1$.

Example 5.15 (Curvature of $\mathbb{H}^2$)**.** With the metric $g = y^{-2}(dx^2 + dy^2)$, one computes $R_{1212} = -1/y^4$, $\det(g) = 1/y^4$, giving $K = -1$.

Example 5.16 (Curvature of a bi-invariant Lie group)**.** For left-invariant $X, Y, Z \in \mathfrak{g}$, with $\nabla_X Y = \frac{1}{2}[X, Y]$:

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z].$$

5.8 Second variation formula

Theorem 5.17 (Second variation of energy)**.** *Let $\gamma : [0, L] \to M$ be a unit-speed geodesic and γ_s a fixed-endpoint variation with variation field V. Then:*

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(\gamma_s) = \int_0^L \left(\left\| \frac{DV}{dt} \right\|^2 - \langle R(\dot{\gamma}, V)\dot{\gamma}, V\rangle \right) dt.$$

Definition 5.18 (Index form)**.** The *index form* of the geodesic γ is the bilinear form:

$$I(V, W) = \int_0^L \left(\langle \frac{DV}{dt}, \frac{DW}{dt} \rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, W\rangle \right) dt.$$

5.9 Exercises

Exercise 5.1. Verify the first Bianchi identity $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ for S^2 in spherical coordinates.

Exercise 5.2. Show that in dimension 2, the Riemann tensor is entirely determined by the Gaussian curvature K: $R_{1212} = K\,(g_{11}g_{22} - g_{12}^2) = K\det(g)$.

Exercise 5.3. Compute the Jacobi fields along a radial geodesic of the hyperbolic plane, and verify that there are no conjugate points.

Exercise 5.4. For the Lie group $SO(3)$ with bi-invariant metric, compute $R(X, Y)Z$ for an orthonormal basis (X, Y, Z) of $\mathfrak{so}(3)$.

Exercise 5.5. Show that if (M, g) is flat, then every Jacobi field has the form $J(t) = A + tB$ for parallel vectors A and B.

Exercise 5.6. Compute the Riemann tensor of the warped product $g = dt^2 + f(t)^2 g_{S^{n-1}}$ in terms of f and its derivatives.

Chapter 6

Sectional, Ricci, and Scalar Curvature

The Riemann tensor is a rich but complex object—in dimension n, it has $\frac{n^2(n^2-1)}{12}$ independent components. To extract more readable geometric information, one contracts this tensor in different ways. *Sectional curvature* measures curvature in each tangent plane—it is the direct generalisation of Gaussian curvature for surfaces. *Ricci curvature*, obtained by averaging over planes containing a given direction, controls the convergence or divergence of geodesics and appears in Einstein's equations. *Scalar curvature*, the final contraction, gives a single number at each point. These three levels of reading form the fundamental hierarchy of Riemannian geometry.

6.1 Sectional curvature

Definition 6.1 (Sectional curvature)**.** Let (M, g) be a Riemannian manifold and $\sigma \subset T_pM$ a 2-plane spanned by linearly independent vectors u, v. The *sectional curvature* of σ is:

$$K(\sigma) = K(u, v) = \frac{\langle R(u, v)v, u\rangle}{\|u\|^2 \|v\|^2 - \langle u, v\rangle^2}.$$

Proposition 6.2. The sectional curvature $K(u, v)$ depends only on the 2-plane $\sigma = \mathrm{Span}(u, v)$, not on the choice of basis.

Proof. If $(u', v') = (au + bv, cu + dv)$ is another basis of σ with $ad - bc \neq 0$, then by the symmetries of R: $\langle R(u', v')v', u'\rangle = (ad - bc)^2\langle R(u, v)v, u\rangle$ and $\|u'\|^2 \|v'\|^2 - \langle u', v'\rangle^2 = (ad - bc)^2(\|u\|^2 \|v\|^2 - \langle u, v\rangle^2)$. The ratio is therefore invariant. □

Theorem 6.3 (Determination of the Riemann tensor)**.** *The sectional curvature completely determines the Riemann tensor. More precisely, if $K(u, v)$ is known for all 2-planes, then R is uniquely determined.*

Proof. The function $(u, v) \mapsto \langle R(u, v)v, u\rangle$ is a quadratic form in each variable u and v. By the symmetries of R, knowing this form for all u, v determines R_{ijkl} by polarization. □

6.2 Spaces of constant curvature

Definition 6.4. (M, g) is a *space of constant (sectional) curvature* κ if $K(\sigma) = \kappa$ for every 2-plane σ at every point.

Theorem 6.5 (Form of the Riemann tensor)**.** *If $K \equiv \kappa$, then:*

$$R(X,Y)Z = \kappa\big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\big).$$

In coordinates: $R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk})$.

Spaces of constant curvature

Space	κ	Model	Topology
Sphere S^n	$+1$	$\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$	compact, $\pi_1 = 0$
Euclidean $\mathbb{R}^n$	0	$\mathbb{R}^n$	noncompact, $\pi_1 = 0$
Hyperbolic $\mathbb{H}^n$	-1	half-space	noncompact, $\pi_1 = 0$

Theorem 6.6 (Classification – Killing–Hopf)**.** *Let (M,g) be a complete simply connected Riemannian manifold of constant sectional curvature κ. Then (M,g) is isometric to:*

- $S^n(1/\sqrt{\kappa})$ *if* $\kappa > 0$,
- $\mathbb{R}^n$ *if* $\kappa = 0$,
- $\mathbb{H}^n(1/\sqrt{-\kappa})$ *if* $\kappa < 0$.

Every complete manifold of constant curvature is a quotient of these spaces by a discrete group of isometries acting freely and properly.

6.3 Ricci curvature

Definition 6.7 (Ricci tensor)**.** The *Ricci tensor* is the contraction of the Riemann tensor:

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\big(Z \mapsto R(Z,X)Y\big).$$

In coordinates: $R_{ij} = R^k_{ikj} = g^{kl}R_{kilj}$.

Proposition 6.8. The Ricci tensor is symmetric: $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$.

Proof. In an orthonormal basis: $R_{ij} = \sum_k R_{kikj}$ and $R_{ji} = \sum_k R_{kjki} = \sum_k R_{kikj} = R_{ij}$, using the pair symmetry $R_{kjki} = R_{kikj}$. □

Proposition 6.9 (Geometric interpretation)**.** If $(e_1, \ldots, e_n)$ is an orthonormal basis of T_pM with $e_1 = v/\|v\|$:

$$\mathrm{Ric}(v,v) = \|v\|^2 \sum_{i=2}^{n} K(v, e_i).$$

The Ricci curvature in direction v is the *sum of sectional curvatures* of 2-planes containing v.

Example 6.10 (Ricci of S^n)**.** All sectional curvatures equal 1, so $\mathrm{Ric}(v,v) = (n-1)\|v\|^2$, i.e. $\mathrm{Ric} = (n-1)g$.

Example 6.11 (Ricci of $\mathbb{H}^n$)**.** $\mathrm{Ric} = -(n-1)g$.

Example 6.12 (Ricci of a bi-invariant Lie group)**.** With $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$:

$$\mathrm{Ric}(X,Y) = -\frac{1}{4}\mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = -\frac{1}{4}B(X,Y),$$

where B is the Killing form of $\mathfrak{g}$.

6.4 Scalar curvature

Definition 6.13 (Scalar curvature)**.** The *scalar curvature* is the trace of the Ricci tensor:

$$S = \mathrm{Scal} = \mathrm{tr}_g \mathrm{Ric} = g^{ij} R_{ij}.$$

If $(e_1, \ldots, e_n)$ is orthonormal: $S = \sum_i \mathrm{Ric}(e_i, e_i) = 2 \sum_{i<j} K(e_i, e_j)$.

Example 6.14. For S^n: $S = n(n-1)$. For $\mathbb{H}^n$: $S = -n(n-1)$. For $\mathbb{R}^n$: $S = 0$.

6.5 Einstein manifolds

Definition 6.15 (Einstein manifold)**.** (M, g) is an *Einstein manifold* if the Ricci tensor is proportional to the metric:

$$\mathrm{Ric} = \lambda\, g$$

for a constant $\lambda \in \mathbb{R}$. Taking the trace: $S = n\lambda$, so $\lambda = S/n$.

Example 6.16. Spaces of constant curvature are Einstein manifolds with $\lambda = (n-1)\kappa$.

Theorem 6.17 (Schur)**.** *If $n \geq 3$ and the sectional curvature is constant at each point (i.e. $K_p(\sigma) = f(p)$ for all 2-planes σ at p), then f is constant on M.*

More generally, if $n \geq 3$ and $\mathrm{Ric} = f\, g$ with $f \in C^\infty(M)$, then f is constant.

Proof. From $\mathrm{Ric} = f\, g$ we get $S = nf$, and the contracted second Bianchi identity gives:

$$\nabla^j R_{ij} = \frac{1}{2} \nabla_i S.$$

But $\nabla^j R_{ij} = \nabla^j (f g_{ij}) = \nabla_i f$. Hence $\nabla_i f = \frac{n}{2} \nabla_i f$, giving $(1 - n/2)\nabla_i f = 0$. For $n \geq 3$, this implies $\nabla f = 0$. □

6.6 The Weyl tensor

Definition 6.18 (Weyl tensor)**.** For $n \geq 3$, the *Weyl tensor W* is the part of the Riemann tensor not determined by Ricci:

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}\left(R_{ik} g_{jl} - R_{il} g_{jk} + R_{jl} g_{ik} - R_{jk} g_{il}\right) - \frac{S}{(n-1)(n-2)}\left(g_{ik} g_{jl} - g_{il} g_{jk}\right).$$

Proposition 6.19. The Weyl tensor is conformally invariant: if $\tilde{g} = e^{2\varphi} g$, then $\tilde{W}^i_{jkl} = W^i_{jkl}$.

Proposition 6.20. $W \equiv 0$ if and only if $n \leq 3$, or if $n \geq 4$ and (M, g) is *conformally flat* (locally conformal to $\mathbb{R}^n$).

6.7 Curvature of warped products

Proposition 6.21 (Warped product curvature)**.** On $B \times_f F$ with $g = g_B + f^2 g_F$, when $\dim B = 1$ (i.e. $B = I \subset \mathbb{R}$, $g_B = dt^2$):

1. For V, W tangent to F: $K(V, W) = \frac{K_F(V,W)-(f')^2}{f^2}$.
2. For V tangent to F and ∂_t tangent to B: $K(\partial_t, V) = -\frac{f''}{f}$.

Example 6.22 (Curvature of $\mathbb{R}^n$ in polar coordinates)**.** $g = dr^2 + r^2 g_{S^{n-1}}$, so $f(r) = r$, $f' = 1$, $f'' = 0$. $K(\partial_r, V) = 0$ and $K(V, W) = \frac{1-1}{r^2} = 0$. Everything is flat.

Example 6.23 (Curvature of S^n in geodesic coordinates)**.** $g = dr^2 + \sin^2 r\, g_{S^{n-1}}$, $f(r) = \sin r$, $f'' = -\sin r$. $K(\partial_r, V) = 1$ and $K(V, W) = \frac{1-\cos^2 r}{\sin^2 r} = 1$.

Example 6.24 (Curvature of $\mathbb{H}^n$ in geodesic coordinates)**.** $g = dr^2 + \sinh^2 r\, g_{S^{n-1}}$, $f(r) = \sinh r$, $f'' = \sinh r$. $K(\partial_r, V) = -1$ and $K(V, W) = \frac{1-\cosh^2 r}{\sinh^2 r} = -1$.

6.8 Contracted Bianchi identity

Theorem 6.25 (Contracted Bianchi identity)**.**

$$\nabla^j R_{ij} = \frac{1}{2} \nabla_i S,$$

or equivalently, the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2} S g_{ij}$ *is divergence-free:* $\nabla^j G_{ij} = 0$.

Remark 6.26. This identity is fundamental in general relativity, where Einstein's equations read $G_{ij} = 8\pi T_{ij}$, and energy-momentum conservation $\nabla^j T_{ij} = 0$ is automatic.

6.9 Exercises

Exercise 6.1. Compute the sectional curvature, Ricci tensor, and scalar curvature of the torus $S^1(r_1) \times S^1(r_2)$ with the product metric.

Exercise 6.2. Show that if (M, g) has dimension 2, then $\mathrm{Ric} = K\, g$ and $S = 2K$.

Exercise 6.3. Show that $\mathbb{C}P^n$ with the Fubini–Study metric is an Einstein manifold with $\lambda = 2(n + 1)$.

Exercise 6.4. Compute the Weyl tensor of $S^2 \times S^2$ and show it is not conformally flat for $n = 4$.

Exercise 6.5. For the warped product $g = dt^2 + e^{2t} g_{\mathbb{R}^{n-1}}$, compute all sectional curvatures, the Ricci tensor, and the scalar curvature.

Exercise 6.6. Prove Schur's theorem in dimension 3 using the second Bianchi identity. Give a counterexample showing the theorem fails in dimension 2.

Chapter 7

Comparison Theorems

7.1 Introduction and motivation

One of the most fruitful ideas in Riemannian geometry rests on a remarkably simple principle: if one knows that the curvature of a manifold is bounded, then its geometry cannot deviate too far from that of a "model" space of constant curvature. This comparison programme, initiated by Harry Rauch in the 1950s and then developed by Marcel Berger, Wilhelm Klingenberg, Jeff Cheeger, and Detlef Gromoll over the following decades, has produced some of the deepest results in global geometry. Rauch's theorem compares the behaviour of Jacobi fields; Bishop–Gromov compares volumes; Toponogov compares triangles. Each extracts global geometric conclusions from a local hypothesis on curvature, illustrating the power of curvature as a controlling invariant.

7.2 Jacobi equation and model solutions

Recall that along a unit-speed geodesic γ, an orthogonal Jacobi field $J \perp \dot\gamma$ satisfies:

$$J'' + K(t)\, J = 0,$$

where $K(t)$ denotes the sectional curvature of the plane $(\dot\gamma, J)$.

Definition 7.1 (Model solutions)**.** For $\kappa \in \mathbb{R}$, the solution of $f'' + \kappa f = 0$ with $f(0) = 0$, $f'(0) = 1$ is:

$$\operatorname{sn}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, t) & \text{if } \kappa > 0, \\ t & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, t) & \text{if } \kappa < 0. \end{cases}$$

Comparison functions

$$\operatorname{sn}_\kappa(t) = \text{solution of } f'' + \kappa f = 0,\ f(0) = 0,\ f'(0) = 1,$$
$$\operatorname{cn}_\kappa(t) = \operatorname{sn}'_\kappa(t) = \text{solution of } f'' + \kappa f = 0,\ f(0) = 1,\ f'(0) = 0.$$

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7.3 Sturm comparison lemma

Lemma 7.2 (Sturm). *Let f and g be solutions of:*

$$f'' + K_1(t)\, f = 0, \qquad g'' + K_2(t)\, g = 0,$$

with $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$, and $K_1(t) \geq K_2(t)$.

Then $f(t) \leq g(t)$ on the interval $[0, t_1]$ where t_1 is the first zero of f.

Proof. Consider $h = f'g - fg'$. We have $h(0) = 0$ and:

$$h' = f''g - fg'' = -K_1 fg + K_2 fg = (K_2 - K_1) fg \leq 0,$$

as long as $f \geq 0$ and $g \geq 0$. Thus $h \leq 0$, i.e. $f'/f \leq g'/g$ (when $f, g > 0$). Integrating: $\ln f - \ln g$ is decreasing, and since $\lim_{t\to 0^+} f(t)/g(t) = 1$, we get $f \leq g$. □

7.4 Rauch comparison theorem

Theorem 7.3 (Rauch). *Let (M, g) and $(\bar{M}, \bar{g})$ be Riemannian manifolds, and $\gamma : [0, L] \to M$, $\bar{\gamma} : [0, L] \to \bar{M}$ unit-speed geodesics without conjugate points. Suppose:*

$$K_M(\dot{\gamma}, \cdot) \leq K_{\bar{M}}(\dot{\bar{\gamma}}, \cdot).$$

If J and $\bar{J}$ are orthogonal Jacobi fields along γ and $\bar{\gamma}$ with $J(0) = \bar{J}(0) = 0$ and $\|J'(0)\| = \|\bar{J}'(0)\|$, then:

$$\|J(t)\| \geq \|\bar{J}(t)\|, \quad \forall\, t \in [0, L].$$

Proof sketch. Consider the ratio $u(t) = \|J(t)\| / \operatorname{sn}_\kappa(t)$ where κ is the lower curvature bound. By the Sturm lemma, compare $\|J\|$ with the model solution. The key is the inequality:

$$\frac{d}{dt}\frac{\|J\|'}{\|J\|} \leq -K(t),$$

which follows from the Jacobi equation and the Cauchy–Schwarz inequality. The conclusion follows from Sturm comparison. □

Corollary 7.4 (Comparison with model space). *If $K_M \leq \kappa$ (upper bound):*

$$\|J(t)\| \geq \operatorname{sn}_\kappa(t)\ \|J'(0)\|.$$

If $K_M \geq \kappa$ (lower bound):

$$\|J(t)\| \leq \operatorname{sn}_\kappa(t)\ \|J'(0)\|.$$

7.5 Toponogov's theorem

Theorem 7.5 (Toponogov – global version). *Let (M, g) be a complete Riemannian manifold with $K \geq \kappa$. Let $\triangle(p, q, r)$ be a geodesic triangle in M (formed by three minimizing geodesics). Let $\bar{\triangle}(\bar{p}, \bar{q}, \bar{r})$ be a comparison triangle in M_κ (same side lengths). Then:*

Angle version: *The angles of the triangle in M are* greater than or equal to *the corresponding angles of the comparison triangle:*

$$\angle_p(q, r) \geq \bar{\angle}_{\bar{p}}(\bar{q}, \bar{r}).$$

Distance version: *For any point x on the side $[q, r]$ and the corresponding point $\bar{x}$ on $[\bar{q}, \bar{r}]$ (same proportions):*

$$d(p, x) \leq d(\bar{p}, \bar{x}).$$

Hypotheses for Toponogov

Toponogov's theorem requires:

1. Completeness of (M, g).
2. A *lower* bound on sectional curvature.
3. The sides of the triangle are *minimizing* geodesics.
4. If $\kappa > 0$, the perimeter does not exceed $2\pi/\sqrt{\kappa}$.

Corollary 7.6. *If $K \geq 0$ (nonnegative curvature), geodesic triangles are "fatter" than the corresponding Euclidean triangles: angles are larger and "medians" are shorter.*

7.6 Bishop–Gromov comparison theorem

Theorem 7.7 (Bishop–Gromov)**.** *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)\kappa$. Let $V_\kappa(r)$ be the volume of the ball of radius r in the model space M^n_κ. Then the function:*

$$r \mapsto \frac{\mathrm{Vol}(B(p,r))}{V_\kappa(r)}$$

is nonincreasing for $r > 0$, and tends to 1 as $r \to 0$.

In particular:

$$\mathrm{Vol}(B(p,r)) \leq V_\kappa(r), \quad \forall r > 0.$$

Proof sketch. In geodesic polar coordinates: $\mathrm{Vol}(B(p,r)) = \int_{S^{n-1}} \int_0^{\min(r,c(v))} \mathcal{A}(t,v)\, dt\, d\sigma(v)$, where $\mathcal{A}(t,v)$ is the Jacobian of $\exp_p$ and $c(v)$ the cut distance. By the Jacobi equation and the bound $\mathrm{Ric} \geq (n-1)\kappa$, one shows that $\mathcal{A}(t,v)/\mathcal{A}_\kappa(t)$ is nonincreasing in t, where $\mathcal{A}_\kappa(t) = \mathrm{sn}_\kappa(t)^{n-1}$ is the model Jacobian. □

Corollary 7.8 (Volume comparison)**.** *Under the hypotheses of the theorem, for $0 < r \leq R$:*

$$\frac{\mathrm{Vol}(B(p,R))}{\mathrm{Vol}(B(p,r))} \leq \frac{V_\kappa(R)}{V_\kappa(r)}.$$

Corollary 7.9 (Volume growth)**.** *If $\mathrm{Ric} \geq 0$, then $\mathrm{Vol}(B(p,r)) \leq \omega_n r^n$ (Euclidean volume).*

7.7 Application: Cheng's theorem

Theorem 7.10 (Cheng)**.** *Let (M^n, g) be complete with $\mathrm{Ric} \geq (n-1)\kappa > 0$. If $\mathrm{diam}(M) = \pi/\sqrt{\kappa}$ (equality in Bonnet–Myers), then M is isometric to $S^n(1/\sqrt{\kappa})$.*

7.8 Laplacian comparison theorem

Theorem 7.11 (Laplacian comparison)**.** *Let (M^n, g) satisfy $\mathrm{Ric} \geq (n-1)\kappa$. The distance function $r(x) = d(p,x)$ satisfies, in the sense of distributions:*

$$\Delta r \leq (n-1)\frac{\mathrm{cn}_\kappa(r)}{\mathrm{sn}_\kappa(r)}.$$

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For $\kappa = 0$: $\Delta r \leq \frac{n-1}{r}$. For $\kappa = 1$: $\Delta r \leq (n-1)\cot r$. For $\kappa = -1$: $\Delta r \leq (n-1)\coth r$.

7.9 Gromov's theorem on generators

Theorem 7.12 (Gromov)**.** *If (M^n, g) is complete with $\operatorname{Ric} \geq 0$, then the fundamental group $\pi_1(M)$ is generated by at most $C(n)$ elements, where $C(n)$ depends only on the dimension.*

7.10 Gromov–Hausdorff space

Definition 7.13 (Gromov–Hausdorff distance)**.** The *Gromov–Hausdorff distance* between two compact metric spaces (X, d_X) and (Y, d_Y) is:

$$d_{GH}(X, Y) = \inf\{d_H(\varphi(X), \psi(Y))\},$$

where the infimum is over all isometric embeddings $\varphi : X \to Z$, $\psi : Y \to Z$ into a metric space (Z, d_Z), and d_H is the Hausdorff distance.

Theorem 7.14 (Gromov's precompactness)**.** *The set of Riemannian manifolds (M^n, g) satisfying $\operatorname{Ric} \geq (n-1)\kappa$ and $\operatorname{diam}(M) \leq D$ is precompact in the Gromov–Hausdorff topology.*

7.11 Exercises

Exercise 7.1. Verify the Bishop–Gromov theorem for balls in S^n of curvature 1 by explicitly computing $\operatorname{Vol}(B(p, r))$ and comparing with $V_1(r)$.

Exercise 7.2. Show that if (M, g) is complete with $K \geq 1$, then $\operatorname{diam}(M) \leq \pi$.

Exercise 7.3. Apply Toponogov's theorem to show that if $K \geq 0$ and M is noncompact, then M contains a geodesic line (splitting theorem).

Exercise 7.4. Use volume comparison to show that if $\operatorname{Ric} \geq 0$ and $\operatorname{Vol}(B(p, r)) \geq c\, r^n$ for some $c > 0$ and all r, then M has at most finitely many ends.

Exercise 7.5. Compute the Laplacian comparison for the distance function on $\mathbb{H}^3$ and verify that $\Delta r = 2\coth r$.

Exercise 7.6. Show that if $K \leq 0$ and M is simply connected, then there are no conjugate points (using the Rauch comparison theorem).

Chapter 8

Symmetric Spaces

Élie Cartan, in the 1920s–1930s, undertook one of the most ambitious classifications in geometry: that of Riemannian spaces whose curvature is "the same at every point and in every direction" in the sense that the geodesic symmetry at each point is a global isometry. These *symmetric spaces* encompass the spaces of constant curvature (spheres, hyperbolic spaces), projective spaces, Grassmannians, and many others. Cartan showed that their classification reduces to that of semi-simple Lie algebras, establishing a deep bridge between geometry and algebra. Today, symmetric spaces appear in theoretical physics, number theory, and machine learning (optimization on Stiefel and Grassmann manifolds).

8.1 Definition and first examples

Definition 8.1 (Riemannian symmetric space)**.** A complete Riemannian manifold (M, g) is a *Riemannian symmetric space* if for every point $p \in M$, there exists an involutive isometry $s_p : M \to M$ such that:

1. $s_p(p) = p$ (fixed point).

2. $(ds_p)_p = -\operatorname{Id}_{T_pM}$ (the differential is $-\operatorname{Id}$).

The isometry s_p is called the *geodesic symmetry* at p.

Remark 8.2. The condition $(ds_p)_p = -\operatorname{Id}$ implies that s_p reverses all geodesics through p: $s_p(\gamma(t)) = \gamma(-t)$.

Example 8.3 (Model spaces)**.**

- $\mathbb{R}^n$: $s_p(x) = 2p - x$ (central symmetry).

- S^n: s_p is the reflection through the subspace passing through p and the center.

- $\mathbb{H}^n$: in the disk model, s_0 is the inversion $x \mapsto -x$.

Example 8.4 (Projective spaces)**.** $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ (quaternionic projective space) and the Cayley plane $\mathbb{O}P^2$ are compact symmetric spaces of rank 1.

Example 8.5 (Compact Lie groups)**.** Every compact Lie group G with a bi-invariant metric is a symmetric space. The symmetry at the identity is $s_e(g) = g^{-1}$.

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8.2 Symmetric pair structure

Definition 8.6 (Symmetric pair)**.** A *Riemannian symmetric pair* is a couple (G, K) where:

- G is a connected Lie group.
- K is a compact subgroup of G.
- There exists an involutive automorphism $\sigma : G \to G$ such that $(G^\sigma)_0 \subset K \subset G^\sigma$, where G^σ is the fixed-point subgroup of σ.

Theorem 8.7 (Fundamental correspondence)**.** *There is a bijective correspondence between:*

1. *Simply connected symmetric spaces* (M, g)*.*
2. *Effective symmetric pairs* (G, K, σ) *with* G *simply connected.*

The symmetric space is $M = G/K$ *and the metric is induced by an* $\mathrm{Ad}(K)$*-invariant inner product on* $\mathfrak{p}$*.*

8.3 Cartan decomposition

Definition 8.8 (Cartan decomposition)**.** Let $\sigma : G \to G$ be the involution and $\sigma_* : \mathfrak{g} \to \mathfrak{g}$ its differential. The Lie algebra decomposes into eigenspaces:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \{X \in \mathfrak{g} : \sigma_*(X) = X\}$ (+1 eigenspace) and $\mathfrak{p} = \{X \in \mathfrak{g} : \sigma_*(X) = -X\}$ (-1 eigenspace).

Proposition 8.9 (Bracket relations)**.** The Lie brackets satisfy:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Proof. If $X, Y \in \mathfrak{k}$, then $\sigma_*[X, Y] = [\sigma_* X, \sigma_* Y] = [X, Y]$, so $[X, Y] \in \mathfrak{k}$. If $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$, then $\sigma_*[X, Y] = [X, -Y] = -[X, Y]$, so $[X, Y] \in \mathfrak{p}$. If $X, Y \in \mathfrak{p}$, then $\sigma_*[X, Y] = [-X, -Y] = [X, Y]$, so $[X, Y] \in \mathfrak{k}$. □

The tangent space to $M = G/K$ at the base point $o = eK$ is identified with $\mathfrak{p}$, and the Riemannian metric is an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{p}$.

8.4 Curvature of symmetric spaces

Theorem 8.10 (Curvature formula)**.** *On the symmetric space* $M = G/K$*, the Levi-Civita connection satisfies, for* $X, Y \in \mathfrak{p} \cong T_o M$*:*

$$\nabla_X Y = \frac{1}{2}[X, Y]_{\mathfrak{p}} = 0,$$

since $[X, Y] \in \mathfrak{k}$ *and the projection onto* $\mathfrak{p}$ *vanishes. The curvature tensor is:*

$$R(X, Y)Z = -[[X, Y], Z], \quad X, Y, Z \in \mathfrak{p}.$$

Corollary 8.11. *The curvature of a symmetric space is entirely determined by the Lie brackets. In particular, $\nabla R = 0$ (the curvature is parallel).*

Theorem 8.12 (Characterization by $\nabla R = 0$)**.** *A complete simply connected Riemannian manifold (M, g) is a symmetric space if and only if $\nabla R = 0$.*

Proposition 8.13 (Sectional curvature)**.** For orthonormal $X, Y \in \mathfrak{p}$:

$$K(X,Y) = \langle [[X,Y],X],Y\rangle = \|[X,Y]\|^2 \geq 0$$

if the metric comes from the negative Killing form (compact type). For noncompact type, $K(X,Y) = -\|[X,Y]\|^2 \leq 0$.

8.5 Classification of symmetric spaces

Theorem 8.14 (de Rham decomposition)**.** *Every simply connected symmetric space decomposes uniquely as a product:*

$$M = M_0 \times M_1 \times \cdots \times M_k,$$

where $M_0 = \mathbb{R}^p$ is the Euclidean factor and the M_i $(i \geq 1)$ are irreducible symmetric spaces.

Definition 8.15 (Types of symmetric spaces)**.** Irreducible symmetric spaces are classified into:

- **Compact type:** $K \geq 0$, G compact. Example: S^n, $\mathbb{C}P^n$.
- **Noncompact type:** $K \leq 0$, G noncompact. Example: $\mathbb{H}^n$, $SL(n,\mathbb{R})/SO(n)$.
- **Duality:** to every compact space corresponds a "dual" noncompact space.

Examples of irreducible symmetric spaces

Space	G/K	**Type**	dim
Sphere S^n	$SO(n+1)/SO(n)$	compact	n
$\mathbb{C}P^n$	$SU(n+1)/S(U(1)\times U(n))$	compact	$2n$
Grassmannian	$SO(p+q)/S(O(p)\times O(q))$	compact	pq
$\mathbb{H}^n$	$SO_0(n,1)/SO(n)$	noncompact	n
$SL(n,\mathbb{R})/SO(n)$		noncompact	$\frac{n(n+1)}{2} - 1$

8.6 Rank and flat subspaces

Definition 8.16 (Rank)**.** The *rank* of a symmetric space is the maximal dimension of an abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ (i.e. $[\mathfrak{a},\mathfrak{a}] = 0$). Geometrically, it is the dimension of the largest totally geodesic flat subspace.

Example 8.17. S^n and $\mathbb{H}^n$ have rank 1. $SL(n,\mathbb{R})/SO(n)$ has rank $n-1$. The Grassmannian $G_{p,q}$ has rank $\min(p,q)$.

8.7 Geodesics and group structure

Proposition 8.18 (Geodesics)**.** The geodesics of $M = G/K$ through $o = eK$ are the curves $t \mapsto \exp(tX) \cdot o$ for $X \in \mathfrak{p}$.

Proposition 8.19 (Parallel transport)**.** Parallel transport along a geodesic $\gamma(t) = \exp(tX) \cdot o$ is given by the action of $\exp(tX)$ on $T_oM \cong \mathfrak{p}$.

Theorem 8.20 (Holonomy)**.** *The restricted holonomy group of an irreducible symmetric space is* $\mathrm{Hol}^0(M) = K$ *(the connected component). Berger's classification of holonomy groups of irreducible non-symmetric manifolds complements the symmetric space classification.*

8.8 Rank-one symmetric spaces

The compact simply connected rank-one symmetric spaces are:

1. $S^n = SO(n+1)/SO(n)$, curvature $K = 1$.
2. $\mathbb{C}P^n = SU(n+1)/S(U(1) \times U(n))$, curvature $\frac{1}{4} \leq K \leq 1$.
3. $\mathbb{H}P^n = Sp(n+1)/(Sp(1) \times Sp(n))$, curvature $\frac{1}{4} \leq K \leq 1$.
4. $\mathbb{O}P^2 = F_4/\operatorname{Spin}(9)$, curvature $\frac{1}{4} \leq K \leq 1$.

Remark 8.21. For $\mathbb{C}P^n$, the sectional curvature varies between $1/4$ and 1. The minimum $1/4$ is achieved on totally real planes, and the maximum 1 on holomorphic planes.

8.9 Exercises

Exercise 8.1. Show that S^n is a symmetric space by explicitly exhibiting the symmetry s_p at every point p.

Exercise 8.2. Compute the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for $S^2 = SO(3)/SO(2)$ and verify the bracket relations.

Exercise 8.3. Show that for the Lie group G with bi-invariant metric viewed as the symmetric space $(G \times G)/\operatorname{diag}(G)$, the sectional curvature satisfies $K(X,Y) = \frac{1}{4}\,\|[X,Y]\|^2$.

Exercise 8.4. Show that $\mathbb{C}P^n$ is a symmetric space and compute its sectional curvature. Verify that $1/4 \leq K \leq 1$.

Exercise 8.5. Show that $SL(2,\mathbb{R})/SO(2) \cong \mathbb{H}^2$ by exhibiting an explicit isometry.

Exercise 8.6. Determine the rank of the Grassmannian $G_{2,3}(\mathbb{R}) = SO(5)/(SO(2) \times SO(3))$.

Chapter 9

Bonnet-Myers and Synge Theorems

9.1 Introduction and motivation

Can one deduce the global shape of a universe from local measurements of its curvature? This is precisely what the Bonnet–Myers and Synge theorems accomplish, two jewels of global Riemannian geometry. Bonnet's theorem (1855), rediscovered and generalized by Myers in 1941, asserts that a complete manifold whose Ricci curvature is strictly positive is necessarily compact and of finite diameter. In other words, positive curvature "closes" the space upon itself. Synge's theorem (1936) goes further: a compact Riemannian manifold of even dimension with strictly positive sectional curvature is simply connected. These results illustrate spectacularly how curvature controls topology.

Intuition

The guiding principle is that positive curvature "brings geodesics closer together," preventing the manifold from extending to infinity. More precisely:

- Ricci curvature bounded below by a positive constant forces a **finite diameter**, hence compactness.
- Strictly positive sectional curvature in even dimensions constrains **orientability** and **simple connectivity**.

9.2 Jacobi fields and Ricci curvature

Definition 9.1 (Ricci curvature)**.** Let (M, g) be a Riemannian manifold of dimension n. The *Ricci curvature* is the trace of the curvature tensor:

$$\mathrm{Ric}(u, v) = \sum_{i=1}^{n} g(R(e_i, u)v, e_i),$$

where $(e_1, \ldots, e_n)$ is an orthonormal basis of T_pM.

Remark 9.2. The Ricci curvature $\mathrm{Ric}(v, v)$ for $\|v\| = 1$ is the sum of sectional curvatures $K(\sigma_i)$ of the planes spanned by v and the remaining basis vectors. It is thus an "average" of sectional curvature.

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Definition 9.3 (Jacobi field)**.** Let $\gamma : [0, L] \to M$ be a geodesic. A *Jacobi field* along γ is a vector field J along γ satisfying the Jacobi equation:

$$\frac{D^2 J}{dt^2} + R(J, \dot{\gamma})\dot{\gamma} = 0.$$

Definition 9.4 (Index form)**.** The *index form* of a geodesic $\gamma : [0, L] \to M$ is:

$$I(V, W) = \int_0^L \left[g\left(\frac{DV}{dt}, \frac{DW}{dt} \right) - g(R(V, \dot{\gamma})\dot{\gamma}, W) \right] dt,$$

defined on vector fields along γ vanishing at the endpoints.

9.3 The Bonnet-Myers theorem

Theorem 9.5 (Bonnet-Myers)**.** *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$. If the Ricci curvature satisfies:*

$$\mathrm{Ric}(v, v) \geq (n-1)\kappa > 0 \quad \forall\, v \in TM,\ \|v\| = 1,$$

then:

1. *The diameter satisfies* $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{\kappa}}$.
2. *M is compact.*
3. *The fundamental group $\pi_1(M)$ is finite.*

Proof. **Step 1: Diameter bound.** Let $p, q \in M$ and $\gamma : [0, L] \to M$ be a minimizing geodesic from p to q, parametrized by arc length. Suppose for contradiction that $L > \pi/\sqrt{\kappa}$.

Choose an orthonormal basis $(e_1, \ldots, e_{n-1}, \dot{\gamma}(0))$ of T_pM and let $E_i(t) = P_{0,t}(e_i)$ be the parallel translates along γ. Define the variation fields:

$$V_i(t) = \sin\left(\frac{\pi t}{L} \right) E_i(t).$$

Each V_i vanishes at $t = 0$ and $t = L$. The index form gives:

$$I(V_i, V_i) = \int_0^L \left[\frac{\pi^2}{L^2} \cos^2\left(\frac{\pi t}{L} \right) - \sin^2\left(\frac{\pi t}{L} \right) K(E_i, \dot{\gamma}) \right] dt.$$

Summing over $i = 1, \ldots, n-1$:

$$\sum_{i=1}^{n-1} I(V_i, V_i) = \int_0^L \sin^2\left(\frac{\pi t}{L} \right) \left[\frac{(n-1)\pi^2}{L^2} - \mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) \right] dt.$$

If $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) \geq (n-1)\kappa$ and $L > \pi/\sqrt{\kappa}$, then $(n-1)\pi^2/L^2 < (n-1)\kappa$, making the sum strictly negative. Hence at least one $I(V_i, V_i) < 0$, meaning γ is not minimizing — contradiction.

Step 2: Compactness. Since M is complete with finite diameter, it is bounded. By the Hopf-Rinow theorem, closed bounded subsets of a complete manifold are compact, so M is compact.

Step 3: Finiteness of $\pi_1(M)$. The universal cover $\widetilde{M}$ inherits the pullback metric, which satisfies the same Ricci curvature bound. By Steps 1–2, $\widetilde{M}$ is compact. The covering $\pi : \widetilde{M} \to M$ is therefore finite, so $|\pi_1(M)| < \infty$. □

Completeness is essential

Completeness is crucial: an open disk with a positively curved metric is not compact. Likewise, the *strict* positivity $\kappa > 0$ is essential: a paraboloid has $\mathrm{Ric} \geq 0$ but is non-compact.

Corollary 9.6 (Comparison with the sphere). *If (M^n, g) is complete with* $\mathrm{Ric} \geq (n-1)$*, then* $\mathrm{diam}(M) \leq \pi = \mathrm{diam}(S^n)$.

Example 9.7 (Sphere and projective spaces). The sphere S^n of radius r has $\mathrm{Ric} = \frac{n-1}{r^2} g$, so $\kappa = 1/r^2$ and Bonnet-Myers gives $\mathrm{diam}(S^n(r)) \leq \pi r$. Equality holds: the diameter of $S^n(r)$ is exactly πr.

Example 9.8 (Compact Lie groups). Every compact semisimple Lie group with the negative of the Killing form has $\mathrm{Ric} > 0$. Thus $SU(n)$, $SO(n)$, $Sp(n)$ are compact with finite fundamental group.

9.4 Cheng's maximal diameter theorem

Theorem 9.9 (Cheng). *Let (M^n, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)\kappa > 0$*. If* $\mathrm{diam}(M) = \pi/\sqrt{\kappa}$*, then M is isometric to the sphere* $S^n(1/\sqrt{\kappa})$.

Remark 9.10. This rigidity theorem shows that the Bonnet-Myers bound is sharp and only the sphere achieves it.

9.5 Synge's theorem

Theorem 9.11 (Synge). *Let (M, g) be a compact Riemannian manifold with strictly positive sectional curvature ($K > 0$). Then:*

1. *If* $\dim M$ *is even and M is orientable, then M is simply connected.*
2. *If* $\dim M$ *is odd, then M is orientable.*

Proof (even-dimensional case). Suppose $\dim M = 2m$ and M is orientable. Assume for contradiction that $\pi_1(M) \neq \{e\}$. Then there exists a closed geodesic $\gamma : [0, L] \to M$ of minimal length in its nontrivial free homotopy class.

Parallel transport along γ defines an isometry $P : T_{\gamma(0)}M \to T_{\gamma(0)}M$ preserving $\dot{\gamma}(0)$. The restriction $P|_{\dot{\gamma}(0)^\perp}$ is an isometry of the $(2m-1)$-dimensional (odd-dimensional) subspace. Since M is orientable, $\det(P) = 1$, and the restriction also has determinant 1. An isometry of an odd-dimensional space with determinant 1 has a fixed unit vector v.

The parallel vector field $V(t)$ along γ with $V(0) = v$ satisfies $V(L) = P(v) = v = V(0)$, so V is a periodic parallel field perpendicular to $\dot{\gamma}$. Computing the index form:

$$I(V, V) = -\int_0^L K(V, \dot{\gamma})\, dt < 0,$$

since $K > 0$. This means γ can be shortened by a variation in the direction of V, contradicting the minimality of γ in its homotopy class. ∎

Corollary 9.12. *The only even-dimensional manifolds with strictly positive sectional curvature and nontrivial fundamental group are the real projective spaces* $\mathbb{R}P^{2m}$ *(which are non-orientable).*

Example 9.13 (S^{2m} and $\mathbb{C}P^m$)**.** The sphere S^{2m} ($K = 1$) is simply connected, in agreement with Synge. The complex projective space $\mathbb{C}P^m$ ($K \in [1, 4]$) is orientable of even dimension $2m$, and indeed $\pi_1(\mathbb{C}P^m) = 0$.

Example 9.14 (S^3 and $\mathbb{R}P^3$)**.** In odd dimensions, Synge does not force simple connectivity: $\mathbb{R}P^3$ has $K > 0$ and $\pi_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z} \neq 0$, but it is orientable, consistent with the theorem.

9.6 Applications and topological consequences

Proposition 9.15 (Constraints on the fundamental group)**.** Let (M^n, g) be complete with $\mathrm{Ric} \geq (n-1)\kappa > 0$. Then:

1. $\pi_1(M)$ is finite (Bonnet-Myers).
2. If n is even and M is orientable, $\pi_1(M) = 0$ (Synge).
3. If n is odd and $K > 0$, M is orientable (Synge).

Theorem 9.16 (Bishop-Gromov volume comparison)**.** *If* (M^n, g) *is complete with* $\mathrm{Ric} \geq (n-1)\kappa$*, then for all* $p \in M$ *and* $0 < r \leq R$*:*

$$\frac{\mathrm{Vol}(B(p,R))}{\mathrm{Vol}(B(p,r))} \leq \frac{V_\kappa(R)}{V_\kappa(r)},$$

where $V_\kappa(r)$ *is the volume of the ball of radius* r *in the space form of constant curvature* κ*.*

Summary of curvature and topology

Hypothesis	Conclusion	Theorem
$\mathrm{Ric} \geq (n-1)\kappa > 0$, complete	$\mathrm{diam} \leq \pi/\sqrt{\kappa}$, compact	Bonnet-Myers
$\mathrm{Ric} \geq (n-1)\kappa > 0$, complete	$\pi_1(M)$ finite	Bonnet-Myers
$K > 0$, compact, even dim, orientable	$\pi_1(M) = 0$	Synge
$K > 0$, compact, odd dim	M orientable	Synge
$\mathrm{Ric} \geq (n-1)\kappa$, complete	Volume comparison	Bishop-Gromov

9.7 Exercises

Exercise 9.1. Show that $\mathbb{R}P^n$ with the quotient metric from S^n has $K = 1$. For even n, verify that $\mathbb{R}P^n$ is non-orientable with $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$, consistent with Synge's theorem.

Exercise 9.2. Let (M^n, g) be complete with $\mathrm{Ric} \geq (n-1)$. Show that $\mathrm{Vol}(M) \leq \mathrm{Vol}(S^n)$.

Exercise 9.3. Let $M = S^2 \times S^2$ with the product metric. Compute the Ricci curvature and verify that $\mathrm{Ric} > 0$. Deduce that $S^2 \times S^2$ is compact and simply connected (by Synge, $\dim = 4$ is even).

Exercise 9.4 (Optimality of Bonnet-Myers)**.** Construct a family of metrics on S^n with $\mathrm{Ric} \geq (n-1)\kappa$ for a given κ, whose diameter approaches $\pi/\sqrt{\kappa}$.

Exercise 9.5. Show that the torus $\mathbb{T}^n$ admits no metric with strictly positive Ricci curvature. *Hint:* use $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$.

Exercise 9.6. Let (M, g) be a compact 3-manifold with $K > 0$. Show that M is orientable. Can one conclude that M is simply connected? Justify using the example of $\mathbb{R}P^3$.

CHAPTER 9. BONNET-MYERS AND SYNGE THEOREMS

Chapter 10

Submanifolds and the Second Fundamental Form

10.1 Riemannian submanifolds

Definition 10.1 (Riemannian submanifold)**.** Let (M, g) be a Riemannian manifold of dimension n and $\Sigma \subset M$ a submanifold of dimension $k < n$. The *induced metric* on Σ is $\bar{g} = \iota^* g$, where $\iota : \Sigma \hookrightarrow M$ is the inclusion. The pair $(\Sigma, \bar{g})$ is called a *Riemannian submanifold.*

Definition 10.2 (Tangent bundle decomposition)**.** At each point $p \in \Sigma$, the tangent bundle of M decomposes orthogonally:

$$T_pM = T_p\Sigma \oplus N_p\Sigma,$$

where $N_p\Sigma = (T_p\Sigma)^\perp$ is the *normal bundle* of Σ in M. We write $(\cdot)^\top$ and $(\cdot)^\perp$ for the corresponding projections.

Intuition

Think of a surface Σ^2 in $\mathbb{R}^3$. At each point, the tangent space of $\mathbb{R}^3$ decomposes into the tangent plane of Σ and the normal direction ν. The second fundamental form measures how Σ "bends" inside $\mathbb{R}^3$, i.e., how the normal ν varies along Σ.

10.2 Induced connection and second fundamental form

Definition 10.3 (Gauss formula)**.** Let ∇ be the Levi-Civita connection of (M, g) and $\bar{\nabla}$ that of $(\Sigma, \bar{g})$. For all vector fields X, Y tangent to Σ:

$$\nabla_X Y = \bar{\nabla}_X Y + \mathrm{I\!I}(X, Y),$$

where $\bar{\nabla}_X Y = (\nabla_X Y)^\top$ and $\mathrm{I\!I}(X, Y) = (\nabla_X Y)^\perp$.

Definition 10.4 (Second fundamental form)**.** The *second fundamental form* is the symmetric bilinear map:

$$\mathrm{I\!I} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \Gamma(N\Sigma), \quad \mathrm{I\!I}(X, Y) = (\nabla_X Y)^\perp.$$

Proposition 10.5 (Symmetry of $\mathrm{I\!I}$)**.** The second fundamental form is symmetric: $\mathrm{I\!I}(X,Y) = \mathrm{I\!I}(Y,X)$.

Proof. We have $\mathrm{I\!I}(X,Y) - \mathrm{I\!I}(Y,X) = (\nabla_X Y - \nabla_Y X)^\perp = [X,Y]^\perp$. Since X and Y are tangent to Σ, the Lie bracket $[X,Y]$ is also tangent, so $[X,Y]^\perp = 0$. □

Definition 10.6 (Shape operator (Weingarten map))**.** For a normal field $\xi \in \Gamma(N\Sigma)$, the *shape operator* is the endomorphism $A_\xi : T\Sigma \to T\Sigma$ defined by:

$$A_\xi(X) = -(\nabla_X \xi)^\top.$$

It is related to $\mathrm{I\!I}$ by: $g(\mathrm{I\!I}(X,Y),\xi) = g(A_\xi(X),Y)$.

Proposition 10.7 (Self-adjointness of A_ξ)**.** The shape operator is self-adjoint: $\bar{g}(A_\xi(X),Y) = \bar{g}(X, A_\xi(Y))$.

Proof. $\bar{g}(A_\xi(X),Y) = g(\mathrm{I\!I}(X,Y),\xi) = g(\mathrm{I\!I}(Y,X),\xi) = \bar{g}(A_\xi(Y),X)$, by symmetry of $\mathrm{I\!I}$. □

10.3 Mean curvature and minimal submanifolds

Definition 10.8 (Mean curvature)**.** Let $(e_1,\dots,e_k)$ be an orthonormal basis of $T_p\Sigma$. The *mean curvature vector* is:

$$\vec{H} = \frac{1}{k}\sum_{i=1}^{k} \mathrm{I\!I}(e_i,e_i) = \frac{1}{k}\mathrm{tr}(\mathrm{I\!I}).$$

Definition 10.9 (Minimal submanifold)**.** A submanifold Σ is *minimal* if $\vec{H} = 0$, i.e., $\mathrm{tr}(\mathrm{I\!I}) = 0$.

Remark 10.10*.* The term “minimal” is historical. Minimal submanifolds are critical points of the volume functional: $\delta\,\mathrm{Vol}(\Sigma) = 0$. They do not necessarily minimize volume.

Example 10.11 (Catenoid)**.** The surface of revolution in $\mathbb{R}^3$ parametrized by:

$$\Phi(u,v) = (\cosh u \cos v, \cosh u \sin v, u), \quad (u,v) \in \mathbb{R} \times [0,2\pi),$$

is a minimal surface with $H = 0$.

Example 10.12 (Sphere $S^{n-1} \subset \mathbb{R}^n$)**.** The sphere $S^{n-1}(r)$ of radius r has second fundamental form $\mathrm{I\!I}(X,Y) = -\frac{1}{r}g(X,Y)\,\nu$, where ν is the outward unit normal. The mean curvature is $\vec{H} = -\frac{1}{r}\nu$: the sphere is not minimal.

10.4 Gauss and Codazzi equations

Theorem 10.13 (Gauss equation)**.** *Let X,Y,Z,W be vector fields tangent to Σ. The curvature of Σ is related to that of M by:*

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) \\ + g(\mathrm{I\!I}(X,Z),\mathrm{I\!I}(Y,W)) - g(\mathrm{I\!I}(Y,Z),\mathrm{I\!I}(X,W)).\end{aligned}$$

Proof. By definition, $\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$. Using the Gauss formula $\nabla_X Y = \bar{\nabla}_X Y + \mathrm{II}(X,Y)$ and expanding:

$$\begin{aligned}\nabla_X(\bar{\nabla}_Y Z + \mathrm{II}(Y,Z)) &= \bar{\nabla}_X\bar{\nabla}_Y Z + \mathrm{II}(X,\bar{\nabla}_Y Z)\\ &\quad - A_{\mathrm{II}(Y,Z)}X + \nabla^{\perp}_X \mathrm{II}(Y,Z).\end{aligned}$$

Antisymmetrizing in X, Y and projecting onto the tangential component yields:

$$(\bar{R}(X,Y)Z)^{\top} = (R(X,Y)Z)^{\top} + A_{\mathrm{II}(X,Z)}Y - A_{\mathrm{II}(Y,Z)}X.$$

Taking the inner product with $W \in T\Sigma$ and using $g(A_\xi Y, W) = g(\mathrm{II}(Y,W),\xi)$ gives the result. □

Corollary 10.14 (Gauss's Theorema Egregium)**.** *For a surface $\Sigma^2 \subset \mathbb{R}^3$, the Gaussian curvature K is intrinsic:*

$$K = \kappa_1\kappa_2 = \det(A_\nu),$$

where κ_1, κ_2 are the principal curvatures.

Theorem 10.15 (Codazzi equation)**.** *For all fields X, Y, Z tangent to Σ and ξ normal:*

$$g(R(X,Y)Z,\xi) = g\big((\bar{\nabla}_X\mathrm{II})(Y,Z) - (\bar{\nabla}_Y\mathrm{II})(X,Z),\, \xi\big),$$

where $(\bar{\nabla}_X\mathrm{II})(Y,Z) = \nabla^{\perp}_X(\mathrm{II}(Y,Z)) - \mathrm{II}(\bar{\nabla}_X Y, Z) - \mathrm{II}(Y, \bar{\nabla}_X Z)$.

Remark 10.16. When $M = \mathbb{R}^n$ (zero curvature), the Codazzi equation simplifies to: $(\bar{\nabla}_X\mathrm{II})(Y,Z) = (\bar{\nabla}_Y\mathrm{II})(X,Z)$.

10.5 Hypersurfaces

Definition 10.17 (Hypersurface)**.** A *hypersurface* is a submanifold of codimension 1. The normal bundle has rank 1, generated by a unit field ν.

For a hypersurface, the scalar second fundamental form is:

$$h(X,Y) = g(\mathrm{II}(X,Y),\nu) = g(A_\nu(X),Y).$$

Definition 10.18 (Principal curvatures)**.** The *principal curvatures* $\kappa_1, \ldots, \kappa_{n-1}$ are the eigenvalues of A_ν. The eigenvectors are the *principal directions*.

Proposition 10.19. For a hypersurface:

$$\begin{aligned}H &= \frac{1}{n-1}(\kappa_1 + \cdots + \kappa_{n-1}) = \frac{1}{n-1}\mathrm{tr}(A_\nu),\\ K_{\mathrm{Gauss}} &= \kappa_1 \cdots \kappa_{n-1} = \det(A_\nu).\end{aligned}$$

10.6 Totally geodesic and umbilical submanifolds

Definition 10.20 (Totally geodesic)**.** Σ is *totally geodesic* if $\mathrm{II} \equiv 0$.

Example 10.21. Great circles of S^2 and equatorial spheres $S^k \subset S^n$ are totally geodesic.

Definition 10.22 (Totally umbilical)**.** Σ is *totally umbilical* if $\mathrm{II}(X,Y) = g(X,Y)\,\vec{H}$.

Example 10.23. The spheres $S^{n-1}(r) \subset \mathbb{R}^n$ are totally umbilical, with $\kappa_1 = \cdots = \kappa_{n-1} = 1/r$.

Fundamental formulas for submanifolds

$$
\begin{aligned}
\text{Gauss:} \quad & \nabla_X Y = \bar{\nabla}_X Y + \mathrm{I\!I}(X, Y) \\
\text{Weingarten:} \quad & \nabla_X \xi = -A_\xi(X) + \nabla^\perp_X \xi \\
\text{Gauss (curv.):} \quad & \bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(\mathrm{I\!I}(X, Z), \mathrm{I\!I}(Y, W)) \\
& \qquad - g(\mathrm{I\!I}(Y, Z), \mathrm{I\!I}(X, W)) \\
\text{Mean curv.:} \quad & \vec{H} = \frac{1}{k}\mathrm{tr}(\mathrm{I\!I})
\end{aligned}
$$

10.7 Exercises

Exercise 10.1. Compute the second fundamental form of the torus of revolution $\Phi(\theta, \phi) = ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$ in $\mathbb{R}^3$. Determine the principal curvatures and mean curvature.

Exercise 10.2. Show that for a hypersurface $\Sigma^{n-1} \subset \mathbb{R}^n$, the Gauss equation gives $\bar{K}(\sigma) = \kappa_i \kappa_j$ for the plane σ spanned by principal directions e_i and e_j.

Exercise 10.3. Let Σ be a minimal surface in $\mathbb{R}^3$. Show that $\kappa_1 = -\kappa_2$ at every point, and that $K = -\kappa_1^2 \leq 0$.

Exercise 10.4. Show that the helicoid $\Phi(u, v) = (v\cos u, v\sin u, au)$ is a minimal surface in $\mathbb{R}^3$.

Exercise 10.5. Let Σ be a compact boundaryless hypersurface of $\mathbb{R}^n$. Show that there exists a point $p \in \Sigma$ where all principal curvatures are strictly positive. *Hint:* consider the point farthest from the origin.

Exercise 10.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and $\Sigma = \{(x, y, f(x, y))\}$ the graph of f in $\mathbb{R}^3$. Compute the second fundamental form, principal curvatures, and mean curvature of Σ in terms of f and its derivatives.

Chapter 11

Introduction to Ricci Flow

11.1 Motivation and definition

The Ricci flow, introduced by R. Hamilton in 1982, is a fundamental tool in differential geometry. The idea is to deform a Riemannian metric in the direction of its Ricci curvature, thereby smoothing the geometry over time.

Intuition

The Ricci flow is the geometric analogue of the heat equation: just as heat diffuses to equalize temperature, the Ricci flow "diffuses" curvature to make it more uniform. Regions of large positive curvature contract, while regions of negative curvature expand.

Definition 11.1 (Ricci flow)**.** Let (M, g_0) be a compact Riemannian manifold. The *Ricci flow* is the evolution equation:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g), \qquad g(0) = g_0.$$

Remark 11.2. The factor -2 is a convention. The negative sign ensures that regions of positive Ricci curvature contract (as physically expected). The equation is a quasilinear parabolic system for the metric g_{ij}.

Definition 11.3 (Normalized Ricci flow)**.** The *normalized Ricci flow* preserves the volume:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) + \frac{2}{n}\bar{R}\,g,$$

where $\bar{R} = \frac{\int_M R\,dV_g}{\int_M dV_g}$ is the average scalar curvature and $n = \dim M$.

11.2 Short-time existence and uniqueness

Theorem 11.4 (Hamilton, 1982)**.** *Let (M, g_0) be a compact Riemannian manifold. There exists $T > 0$ and a unique smooth family of metrics $(g(t))_{t\in[0,T)}$ satisfying the Ricci flow with initial condition $g(0) = g_0$.*

Remark 11.5. Hamilton's original proof uses the Nash-Moser theorem. DeTurck (1983) subsequently gave a simplified proof by introducing a modification of the flow that makes it strictly parabolic.

Definition 11.6 (DeTurck trick)**.** Let $\tilde{g}$ be a fixed reference metric on M. The *Ricci-DeTurck flow* is:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) + \mathcal{L}_{W(g,\tilde{g})}g,$$

where W is a vector field depending on g and $\tilde{g}$ (defined via the difference of Christoffel symbols). This flow is strictly parabolic.

Proposition 11.7. The Ricci-DeTurck flow is equivalent to the Ricci flow modulo diffeomorphisms: if $g(t)$ solves the Ricci-DeTurck flow, there exists a family of diffeomorphisms $\varphi_t : M \to M$ such that $\varphi_t^* g(t)$ solves the Ricci flow.

11.3 Evolution of curvature

Theorem 11.8 (Evolution of scalar curvature)**.** *Under the Ricci flow, the scalar curvature evolves by:*

$$\frac{\partial R}{\partial t} = \Delta R + 2\left|\mathrm{Ric}\right|^2.$$

Proof. In local coordinates, $R = g^{ij}R_{ij}$. Differentiating:

$$\frac{\partial R}{\partial t} = g^{ij}\frac{\partial R_{ij}}{\partial t} + \frac{\partial g^{ij}}{\partial t}R_{ij}.$$

Since $\frac{\partial g^{ij}}{\partial t} = 2R^{ij}$, and using the evolution formula for the Ricci tensor:

$$\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 2R_{ikjl}R^{kl} - 2R_{ik}R_j^{\ k},$$

contraction yields the stated result. □

Corollary 11.9 (Maximum principle for R)**.** *If the initial scalar curvature satisfies $R(g_0) \geq R_{\min}$, then for all $t \in [0, T)$:*

$$R(g(t)) \geq \frac{R_{\min}}{1 - \frac{2R_{\min}}{n}t}.$$

In particular, if $R_{\min} > 0$, the flow develops a singularity in finite time $T \leq \frac{n}{2R_{\min}}$.

Theorem 11.10 (Evolution of the Ricci tensor)**.** *Under the Ricci flow:*

$$\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 2R_{ikjl}R^{kl} - 2R_{ik}R_j^{\ k}.$$

Theorem 11.11 (Evolution of the Riemann tensor)**.** *Under the Ricci flow:*

$$\frac{\partial R_{ijkl}}{\partial t} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}),$$

where $B_{ijkl} = R_{ipjq}R^{p\ \ q}_{\ k\ \ l}$ (summing over repeated indices with the Einstein convention).

11.4 Hamilton's maximum principle

Theorem 11.12 (Scalar maximum principle). *Let $u : M \times [0, T) \to \mathbb{R}$ satisfy:*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + F(u),$$

where $g(t)$ solves the Ricci flow and F is locally Lipschitz. If $u(x, 0) \geq c$ for all $x \in M$, then $u(x, t) \geq \phi(t)$ where ϕ solves the ODE $\phi' = F(\phi)$, $\phi(0) = c$.

Proof sketch. One uses the parabolic comparison principle. Let $x_0(t)$ be a point where $u(\cdot, t)$ attains its spatial minimum. At this point $\Delta_{g(t)} u \geq 0$ (since it is a minimum), so:

$$\left. \frac{\partial u}{\partial t} \right|_{x_0(t)} \geq F(u(x_0(t), t)).$$

It follows (via a regularization argument, since $x_0(t)$ may jump) that $u_{\min}(t) = \min_M u(\cdot, t)$ satisfies $\frac{d}{dt} u_{\min} \geq F(u_{\min})$ in the barrier sense. Comparison with the ODE $\phi' = F(\phi)$, $\phi(0) = c$ yields $u_{\min}(t) \geq \phi(t)$. The typical application is $u = R$ (scalar curvature), $F(u) = \frac{2}{n} u^2$, giving the bound $R \geq \frac{R_{\min}}{1 - \frac{2R_{\min}}{n} t}$. □

The metric depends on time

The Laplacian $\Delta_{g(t)}$ depends on the evolving metric. The maximum principle therefore requires careful treatment, since metric balls and norms change with time.

Theorem 11.13 (Hamilton's tensor maximum principle). *Let T_{ij} be a symmetric tensor evolving by:*

$$\frac{\partial T_{ij}}{\partial t} = \Delta T_{ij} + N_{ij}(T, g),$$

where N_{ij} satisfies: if $T_{ij} \geq 0$ and $T_{ij} v^j = 0$ at a point, then $N_{ij} v^i v^j \geq 0$. If $T_{ij}(0) \geq 0$, then $T_{ij}(t) \geq 0$ for all $t \in [0, T)$.

Proof sketch. Consider the smallest eigenvalue $\lambda_{\min}(t)$ of T_{ij} over M. If $\lambda_{\min}(t_0) = 0$ at a point x_0 with eigenvector v, then $T_{ij} v^j = 0$ at that point, so $N_{ij} v^i v^j \geq 0$ by hypothesis. A maximum principle argument (applied to the quantity $T_{ij} v^i v^j$ along curves of parallel vectors) shows that $\lambda_{\min}$ cannot cross zero from above. The technical difficulty is that the eigenvector depends on the point; this is handled by working on the orthonormal frame bundle. See Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), Theorem 9.1. □

Corollary 11.14 (Preservation of nonnegative Ricci). *If* $\mathrm{Ric}(g_0) \geq 0$*, then* $\mathrm{Ric}(g(t)) \geq 0$ *for all t in the interval of existence.*

11.5 Fundamental examples

Example 11.15 (Sphere S^n). On the sphere (S^n, g_0) with the round metric of sectional curvature 1, we have $\mathrm{Ric}(g_0) = (n-1) g_0$. The Ricci flow has the solution:

$$g(t) = (1 - 2(n-1)t)\, g_0.$$

The metric shrinks uniformly and vanishes at $T = \frac{1}{2(n-1)}$. The sphere contracts to a point in finite time.

Example 11.16 (Hyperbolic space)**.** On $(\mathbb{H}^n, g_0)$ with $\mathrm{Ric}(g_0) = -(n-1)g_0$:

$$g(t) = (1 + 2(n-1)t)\, g_0.$$

The metric expands uniformly. The flow exists for all $t > 0$.

Example 11.17 (Ricci solitons)**.** A *Ricci soliton* is a solution that evolves only by diffeomorphisms and rescaling:

$$\mathrm{Ric}(g) + \mathcal{L}_X g + \lambda g = 0,$$

where X is a vector field and $\lambda \in \mathbb{R}$. The soliton is called shrinking ($\lambda > 0$), steady ($\lambda = 0$), or expanding ($\lambda < 0$).

Example 11.18 (Flow on surfaces)**.** In dimension 2, $\mathrm{Ric} = \frac{R}{2} g$ and the Ricci flow becomes:

$$\frac{\partial g}{\partial t} = -Rg.$$

The normalized Ricci flow is $\frac{\partial g}{\partial t} = (\bar{R} - R)g$. Hamilton (1988) and Chow (1991) showed that on any compact surface, the normalized flow converges to a metric of constant curvature.

11.6 Convergence in dimension 3

Theorem 11.19 (Hamilton, 1982)**.** *Let (M^3, g_0) be a compact 3-dimensional Riemannian manifold with strictly positive Ricci curvature. The normalized Ricci flow converges to a metric of constant positive sectional curvature. In particular, M is diffeomorphic to a quotient S^3/Γ.*

Remark 11.20. This foundational result motivated the Hamilton-Perelman program, culminating in Perelman's proof of the Poincaré conjecture (2002–2003) in dimension 3.

Evolution equations under the Ricci flow

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= -2R_{ij} \\
\frac{\partial}{\partial t}\Gamma^k_{ij} &= -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) \\
\frac{\partial R}{\partial t} &= \Delta R + 2\,|\mathrm{Ric}|^2 \\
\frac{\partial R_{ij}}{\partial t} &= \Delta R_{ij} + 2R_{ikjl}R^{kl} - 2R_{ik}R_j^{\ k} \\
\frac{\partial}{\partial t} dV_g &= -R\, dV_g
\end{aligned}$$

11.7 Exercises

Exercise 11.1. Verify that $g(t) = (1 - 2(n-1)t)g_0$ solves the Ricci flow on (S^n, g_0). Compute the extinction time and $\mathrm{diam}(S^n, g(t))$ as a function of t.

Exercise 11.2. Show that under the Ricci flow, the volume evolves by: $\frac{d}{dt}\,\mathrm{Vol}(M, g(t)) = -\int_M R\,dV_g$. Deduce that the volume decreases when $R > 0$.

Exercise 11.3. Let (M^2, g_0) be a compact surface. Show that the normalized Ricci flow preserves total area and that the Euler characteristic $\chi(M)$ determines the sign of $\bar{R}$ via the Gauss-Bonnet theorem.

Exercise 11.4 (Hamilton's cigar soliton)**.** Show that the metric on $\mathbb{R}^2$:

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

is a steady Ricci soliton (the "cigar soliton"). Compute its Gaussian curvature.

Exercise 11.5. Let $g(t) = \varphi(t)g_0$ be a solution of the Ricci flow with $\varphi(0) = 1$. Show that this is only possible if (M, g_0) is an Einstein manifold: $\mathrm{Ric}(g_0) = \lambda g_0$. Determine $\varphi(t)$ as a function of λ.

Exercise 11.6. Using the maximum principle, show that if $R(g_0) \geq R_{\min} > 0$ on a compact manifold, then $R(g(t)) \to +\infty$ as t approaches the maximal existence time.

CHAPTER 11. INTRODUCTION TO RICCI FLOW

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