

Partial Differential Equations

Lecture Notes

Master M1 — 2025–2026

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*“Mathematical physics is the domain where we
know the most things and understand the least.”*

— Henri Poincaré

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Preface

Partial differential equations (PDEs) form one of the fundamental pillars of applied mathematics and mathematical physics. Since the foundational work of Euler, d'Alembert, Fourier, and Laplace in the 18th century, they have played a central role in modelling natural phenomena: heat diffusion, wave propagation, fluid flow, material elasticity, electromagnetism, and more recently in finance, biology, and image processing.

Course objectives

This course is aimed at third-year undergraduate students (L3 level) in mathematics. It assumes the following prerequisites:

- Real analysis: Lebesgue integration, L^p spaces, modes of convergence.
- Elementary functional analysis: Banach spaces, Hilbert spaces.
- Ordinary differential equations (ODEs): Cauchy–Lipschitz theorem.
- Linear algebra: diagonalization, bilinear forms.

The main objective is to provide a rigorous yet accessible introduction to classical and modern methods for solving PDEs. More precisely, upon completion of this course, the student will be able to:

1. Classify second-order PDEs (elliptic, parabolic, hyperbolic) and understand the physical implications of each type.
2. Solve classical PDEs explicitly using the method of characteristics, separation of variables, and Fourier series.
3. State and prove fundamental qualitative properties: maximum principle, regularity, asymptotic behaviour.
4. Use the Fourier transform to solve PDEs on \mathbb{R}^n .
5. Understand the modern functional framework: Sobolev spaces and variational formulations.

Course organization

The course is organized into ten chapters, following a natural pedagogical progression:

- Chapter 1: Introduction and classification.** Basic vocabulary (order, linearity, boundary conditions) and classification of second-order PDEs.
- Chapter 2: Transport equation.** The first PDE studied in detail, introducing the method of characteristics in a simple setting.
- Chapter 3: Separation of variables and Fourier series.** Development of the central tool of separation of variables, supported by the theory of Fourier series.
- Chapter 4: Heat equation.** Complete study: fundamental solution, maximum principle, regularity, asymptotic behaviour.
- Chapter 5: Wave equation.** d'Alembert's formula, Huygens' principle, energy and finite speed of propagation.
- Chapter 6: Laplace equation and harmonic functions.** Mean value properties, regularity, Harnack inequality.
- Chapter 7: Maximum principle.** Weak and strong versions for elliptic and parabolic operators.
- Chapter 8: Fourier transform and applications.** The Fourier transform as a tool for solving PDEs on \mathbb{R}^n .
- Chapter 9: Introduction to Sobolev spaces.** Weak derivatives, H^1 and H_0^1 spaces, Sobolev and Poincaré inequalities.
- Chapter 10: Variational formulation.** Lax–Milgram theorem, elliptic problems, introduction to finite elements.

Conventions and notation

Throughout this course, we use the following notation:

- Ω denotes a bounded open subset of \mathbb{R}^n with boundary $\partial\Omega$ sufficiently regular (at least piecewise C^1 , unless otherwise stated).
- $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$.
- $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian of u .
- $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)$ is the gradient.
- Du denotes the Jacobian matrix, D^2u the Hessian matrix.
- $C^k(\Omega)$, $C^\infty(\Omega)$, $C_c^\infty(\Omega)$: spaces of C^k , smooth, and compactly supported smooth functions.
- $L^p(\Omega)$: Lebesgue space, $1 \leq p \leq \infty$.
- $H^s(\Omega) = W^{s,2}(\Omega)$: Sobolev space.
- $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2 or in a Hilbert space.

- $\|\cdot\|$ denotes a norm (context specifies which one).
- Multi-indices: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

How to use this course

Each chapter contains:

- **Definitions** and **theorems** with detailed proofs.
- **Examples** worked out completely to illustrate the methods.
- **Intuition** boxes to develop geometric and physical understanding.
- **Warning** boxes to highlight common pitfalls.
- **Exercises** of increasing difficulty, some with hints.

Proofs marked with an asterisk (*) are beyond the syllabus and are offered for students wishing to deepen their understanding.

Physical motivation

PDEs arise naturally whenever one models a phenomenon depending on several independent variables. Let us consider some fundamental examples.

Heat diffusion. If $u(x, t)$ represents the temperature at point x at time t in a conducting body, Fourier's law states that the heat flux is proportional to the temperature gradient: $\vec{q} = -k\nabla u$. Combined with energy conservation, one obtains the heat equation:

$$\frac{\partial u}{\partial t} = k\Delta u.$$

Wave propagation. Vibrations of a string, a membrane, or acoustic waves are described by the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u,$$

where c is the propagation speed.

Electrostatics. The electric potential u in a charge-free region satisfies Laplace's equation:

$$\Delta u = 0.$$

In the presence of a charge density ρ , one obtains Poisson's equation $\Delta u = -\rho/\varepsilon_0$.

Fluid mechanics. The Navier–Stokes equations, describing the motion of a viscous incompressible fluid, form a system of nonlinear PDEs:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \vec{v}, \quad \nabla \cdot \vec{v} = 0.$$

The existence and regularity of solutions in three dimensions remains one of the Millennium Prize Problems.

Quantum mechanics. The Schrödinger equation describes the evolution of a quantum particle:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V\psi.$$

These examples illustrate the richness and universality of PDEs. This course will provide the mathematical tools needed to understand and solve them.

References

This course draws from several standard references:

- EVANS, L.C., *Partial Differential Equations*, AMS, 2010.
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Happy reading and good work!

The author

Chapter 1

Introduction to PDEs

Partial differential equations are the language in which nature writes its laws. The heat equation describes how temperature diffuses through metal. The wave equation governs a vibrating guitar string, the propagation of sound, and the gravitational waves predicted by Einstein. Laplace's equation reigns over electrostatics, Newtonian gravity, and incompressible fluid flow. Schrödinger's equation determines the evolution of every quantum system.

These equations, discovered over three centuries by Euler, d'Alembert, Fourier, Laplace, and many others, share a common characteristic: they relate an unknown function of several variables to its partial derivatives. They are richer — and harder — than ordinary differential equations, because the multiplicity of space and time variables engenders a variety of behaviors: diffusion, propagation, equilibrium. This first chapter maps out this territory.

1.1 What is a PDE?

Let us begin with the most general definition, before specializing.

Definition 1.1 (Partial differential equation). A **partial differential equation** (PDE) is an equation involving an unknown function u of several independent variables (x_1, \dots, x_n) and its partial derivatives. The general form is:

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \dots) = 0.$$

Definition 1.2 (Order of a PDE). The **order** of a PDE is the highest order of partial differentiation appearing in the equation.

Example 1.3. Some classical PDEs:

1. **Transport equation** (order 1): $u_t + c u_x = 0$.
2. **Heat equation** (order 2): $u_t = k u_{xx}$.
3. **Wave equation** (order 2): $u_{tt} = c^2 u_{xx}$.
4. **Laplace equation** (order 2): $\Delta u = 0$.
5. **Burgers' equation** (order 1, nonlinear): $u_t + u u_x = 0$.
6. **Korteweg–de Vries equation** (order 3): $u_t + 6u u_x + u_{xxx} = 0$.

1.2 Linearity and homogeneity

Definition 1.4 (Linear PDE). A PDE is **linear** if the operator L defined by the left-hand side is linear, i.e.,

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) \quad \forall \alpha, \beta \in \mathbb{R}.$$

The PDE is **homogeneous** if $Lu = 0$, and **inhomogeneous** if $Lu = f$ with $f \neq 0$.

Remark 1.5. The **superposition principle** applies to linear homogeneous PDEs: if u_1 and u_2 are solutions, then $\alpha u_1 + \beta u_2$ is also a solution for all $\alpha, \beta \in \mathbb{R}$. This principle is fundamental for the method of separation of variables.

Definition 1.6 (Semilinear, quasilinear, fully nonlinear).

- **Semilinear:** linear in the highest-order derivatives, with coefficients depending only on x : $\Delta u = f(x, u, \nabla u)$.
- **Quasilinear:** linear in the highest-order derivatives, with coefficients depending on u and ∇u : $\sum a_{ij}(x, u, \nabla u) u_{x_i x_j} = f(x, u, \nabla u)$.
- **Fully nonlinear:** nonlinear in the highest-order derivatives: $\det(D^2 u) = f$ (Monge–Ampère equation).

1.3 Classification of second-order PDEs

Consider a second-order PDE in two independent variables:

$$a(x, y) u_{xx} + 2b(x, y) u_{xy} + c(x, y) u_{yy} + d(x, y) u_x + e(x, y) u_y + f(x, y) u = g(x, y). \quad (1.1)$$

Definition 1.7 (Classification by discriminant). The **discriminant** of the PDE (1.1) is $\Delta_{\text{disc}} = b^2 - ac$. The PDE is said to be:

- **Elliptic** if $b^2 - ac < 0$ at every point of the domain.
- **Parabolic** if $b^2 - ac = 0$ at every point of the domain.
- **Hyperbolic** if $b^2 - ac > 0$ at every point of the domain.

Intuition

This classification has deep physical meaning:

- **Elliptic** PDEs (Laplace type) model equilibrium states: the solution at a point depends on the solution everywhere around it.
- **Parabolic** PDEs (heat type) model diffusion processes: information propagates at infinite speed but attenuates.
- **Hyperbolic** PDEs (wave type) model propagation phenomena: information travels at finite speed along characteristics.

Example 1.8.

1. $u_{xx} + u_{yy} = 0$ (Laplace): $a = 1, b = 0, c = 1$, so $b^2 - ac = -1 < 0$: **elliptic**.
2. $u_t - u_{xx} = 0$ (heat): $a = -1, b = 0, c = 0$, so $b^2 - ac = 0$: **parabolic**.
3. $u_{tt} - c^2 u_{xx} = 0$ (wave): $a = -c^2, b = 0, c_{\text{coeff}} = 1$, so $b^2 - ac = c^2 > 0$: **hyperbolic**.

1.3.1 Classification in higher dimensions

For a second-order PDE in n variables:

$$\sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \text{lower-order terms} = 0,$$

the classification is based on the **coefficient matrix** $A = (a_{ij})$, assumed symmetric:

- **Elliptic**: all eigenvalues of A have the same sign.
- **Hyperbolic**: one eigenvalue has opposite sign to the others.
- **Parabolic**: one zero eigenvalue, the rest having the same sign.

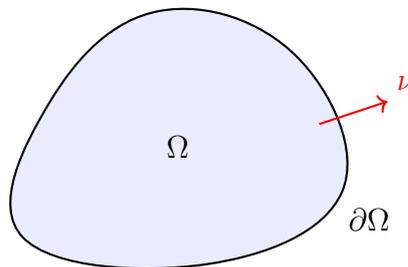
Theorem 1.9 (Canonical form — case $n = 2$). *By a suitable change of variables $(\xi, \eta) = (\xi(x, y), \eta(x, y))$, every linear second-order PDE in two variables can be reduced to one of the following canonical forms:*

- **Elliptic**: $u_{\xi\xi} + u_{\eta\eta} + \dots = 0$.
- **Parabolic**: $u_{\xi\xi} + \dots = 0$.
- **Hyperbolic**: $u_{\xi\xi} - u_{\eta\eta} + \dots = 0$ or equivalently $u_{\alpha\beta} + \dots = 0$ with $\alpha = \xi + \eta$, $\beta = \xi - \eta$.

1.4 Boundary conditions and initial conditions

Definition 1.10 (Boundary value problem). A **boundary value problem** consists of a PDE posed on a domain $\Omega \subset \mathbb{R}^n$ supplemented with conditions on the boundary $\partial\Omega$. The main types are:

1. **Dirichlet**: $u = g$ on $\partial\Omega$.
2. **Neumann**: $\frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$, where ν is the outward unit normal.
3. **Robin** (mixed): $\alpha u + \beta \frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$.



Domain Ω with outward normal ν

Definition 1.11 (Well-posed problem in the sense of Hadamard). A problem is said to be **well-posed** in the sense of Hadamard if it satisfies the following three conditions:

1. **Existence**: there exists at least one solution.
2. **Uniqueness**: there exists at most one solution.
3. **Stability**: the solution depends continuously on the data.

Importance of well-posedness

An ill-posed problem is not necessarily without mathematical or physical interest. Hadamard’s classical example shows that the Cauchy problem for Laplace’s equation is ill-posed, but regularization techniques allow one to address it.

Example 1.12 (Dirichlet problem for the Laplacian on the disk). Find u harmonic in the unit disk $D = \{(x, y) : x^2 + y^2 < 1\}$ with $u = g$ on $\partial D = \{x^2 + y^2 = 1\}$. In polar coordinates:

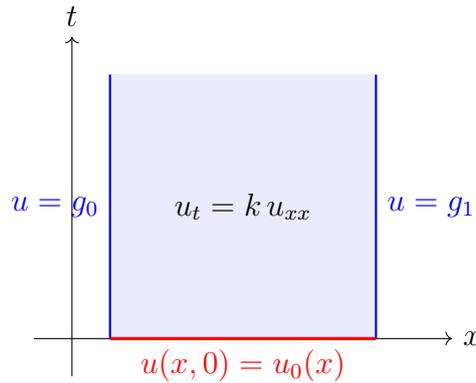
$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < 1, \\ u(1, \theta) = g(\theta). \end{cases}$$

This problem is well-posed (existence, uniqueness, stability).

1.5 Evolution problems

Definition 1.13 (Cauchy problem). For an evolution PDE (parabolic or hyperbolic), the **Cauchy problem** consists of finding $u(x, t)$ satisfying the PDE for $t > 0$ together with **initial conditions** at $t = 0$:

- For the heat equation: $u(x, 0) = u_0(x)$.
- For the wave equation: $u(x, 0) = u_0(x)$ and $u_t(x, 0) = v_0(x)$.



Mixed problem for the heat equation

1.6 Fundamental modelling examples

1.6.1 Derivation of the heat equation

Consider a thin rod of length L . Let $u(x, t)$ be the temperature at point $x \in [0, L]$ at time t . Energy conservation on a segment $[a, b]$ gives:

$$\frac{d}{dt} \int_a^b \rho c_p u(x, t) dx = -q(a, t) + q(b, t) + \int_a^b f(x, t) dx,$$

where ρ is the density, c_p the heat capacity, q the heat flux, and f a source term. Fourier’s law gives $q(x, t) = -k u_x(x, t)$. Differentiating under the integral sign:

$$\int_a^b \rho c_p u_t dx = \int_a^b k u_{xx} dx + \int_a^b f dx.$$

Since this holds for all $[a, b]$, we obtain:

$$\rho c_p u_t = k u_{xx} + f,$$

and setting $\kappa = k/(\rho c_p)$:

$$u_t = \kappa u_{xx} + \tilde{f}. \quad (1.2)$$

1.6.2 Derivation of the wave equation

Consider a vibrating string of linear density ρ under tension T . For small transverse displacements $u(x, t)$, Newton's second law applied to an element $[x, x + dx]$ gives:

$$\rho dx u_{tt} = T \sin \theta(x + dx) - T \sin \theta(x) \approx T(u_x(x + dx) - u_x(x)) = T u_{xx} dx.$$

Hence:

$$u_{tt} = c^2 u_{xx}, \quad c = \sqrt{T/\rho}. \quad (1.3)$$

1.7 Examples of explicit solutions

Example 1.14 (Solutions of Laplace's equation in 2D). The following harmonic functions solve $\Delta u = 0$ in \mathbb{R}^2 :

1. $u(x, y) = ax + by + c$ (affine functions).
2. $u(x, y) = x^2 - y^2$ (check: $u_{xx} + u_{yy} = 2 - 2 = 0$).
3. $u(x, y) = e^x \cos y$ (check: $u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$).
4. $u(r, \theta) = r^n \cos(n\theta)$ in polar coordinates, for all $n \in \mathbb{N}$.

Example 1.15 (Travelling waves). The function $u(x, t) = f(x - ct)$ solves the wave equation $u_{tt} = c^2 u_{xx}$ for any C^2 function f . Indeed:

$$u_t = -cf'(x - ct), \quad u_{tt} = c^2 f''(x - ct), \quad u_x = f'(x - ct), \quad u_{xx} = f''(x - ct).$$

So $u_{tt} = c^2 u_{xx}$. Similarly, $g(x + ct)$ is a solution (wave propagating to the left).

1.8 Notion of solution

Definition 1.16 (Classical solution). A **classical solution** of a PDE of order k is a function $u \in C^k(\Omega)$ that satisfies the PDE at every point of Ω .

Definition 1.17 (Weak solution). A **weak solution** is a function u belonging to a suitable function space (e.g., a Sobolev space) that satisfies the PDE in the sense of distributions, i.e.,

$$\int_{\Omega} u L^* \varphi dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in C_c^\infty(\Omega),$$

where L^* is the formal adjoint of the operator L .

Remark 1.18. The notion of weak solution is essential for the following reasons:

1. Some PDEs do not admit classical solutions (e.g., hyperbolic conservation laws develop shocks).
2. The weak framework allows the use of functional analysis tools (Lax–Milgram theorem, variational methods).
3. Under regularity hypotheses, one can often show that a weak solution is in fact classical.

1.9 Exercises

Exercise 1.1. Classify the following PDEs (elliptic, parabolic, hyperbolic):

1. $u_{xx} + 2u_{xy} + u_{yy} = 0$.
2. $u_{xx} + 2u_{xy} + 5u_{yy} = 0$.
3. $3u_{xx} - 4u_{xy} + u_{yy} = 0$.
4. $u_{xx} + x^2u_{yy} = 0$ (does the type depend on x ?).

Exercise 1.2. Show that the superposition $u = \sum_{k=1}^N c_k u_k$ of solutions of a linear homogeneous PDE is again a solution. Does this result extend to the inhomogeneous case?

Exercise 1.3. Verify that $u(x, y) = \ln(x^2 + y^2)$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$. Compute ∇u and give a physical interpretation of this solution.

Exercise 1.4. Derive the heat equation in dimension n :

$$u_t = \kappa \Delta u + f$$

using energy conservation and Fourier’s law $\vec{q} = -k\nabla u$ in an arbitrary domain $\Omega \subset \mathbb{R}^n$.

Exercise 1.5 (Hadamard’s example). Consider the Cauchy problem for Laplace’s equation:

$$u_{xx} + u_{yy} = 0, \quad u(x, 0) = 0, \quad u_y(x, 0) = \frac{1}{n} \sin(nx).$$

Show that the solution is $u(x, y) = \frac{1}{n^2} \sin(nx) \sinh(ny)$. Deduce that as $n \rightarrow \infty$, the initial data converge to 0 uniformly but the solution does not converge to 0 for $y \neq 0$. Conclude on ill-posedness.

Exercise 1.6. Consider Tricomi’s equation: $u_{xx} + y u_{yy} = 0$. Determine the regions of the plane where this PDE is elliptic, parabolic, or hyperbolic. Sketch these regions.

Chapter 2

Transport Equation

Picture a river carrying a patch of dye downstream. No mixing, no diffusion—just pure displacement. The concentration profile glides along unchanged, as if riding an invisible conveyor belt. This deceptively simple scenario is the subject of the transport equation, the most elementary PDE one can write, yet one that already reveals the deep structure underlying all of partial differential equations.

The transport equation was studied as early as the eighteenth century by Euler and d’Alembert, who recognised that wave propagation and fluid flow both reduce, in their simplest form, to tracking quantities along characteristic curves. What makes this equation so instructive is that it can be solved exactly, by a geometric argument rather than an algebraic trick: one simply follows the flow. This method of characteristics, as we shall see, is the key that unlocks not only transport but a wide family of first-order PDEs.

2.1 Introduction and physical motivation

The transport equation is the simplest PDE. It models the displacement of a quantity (concentration, density, pollutant) carried by a velocity field without diffusion or reaction.

Intuition

Imagine dye injected into a river with a uniform current. The dye is carried along without mixing: its concentration profile moves rigidly at the current speed. This is exactly what the transport equation describes.

2.2 Constant-coefficient transport equation

2.2.1 The Cauchy problem

Definition 2.1 (Homogeneous transport equation). The **transport equation** with constant velocity $c \in \mathbb{R}$ is:

$$\begin{cases} u_t + c u_x = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (2.1)$$

Theorem 2.2 (Solution of the Cauchy problem). *If $u_0 \in C^1(\mathbb{R})$, then problem (2.1) admits a unique solution $u \in C^1(\mathbb{R} \times [0, +\infty))$ given by:*

$$\boxed{u(x, t) = u_0(x - ct)}. \quad (2.2)$$

Proof. Method of characteristics. We seek curves $(x(t), t)$ in the (x, t) -plane along which the solution u is constant. If u is a solution, set $z(t) = u(x(t), t)$. Then:

$$z'(t) = u_t(x(t), t) + x'(t) u_x(x(t), t).$$

For $z'(t) = 0$ (constant along the curve), it suffices to choose $x'(t) = c$, giving the **characteristics**:

$$x(t) = x_0 + ct, \quad x_0 \in \mathbb{R}.$$

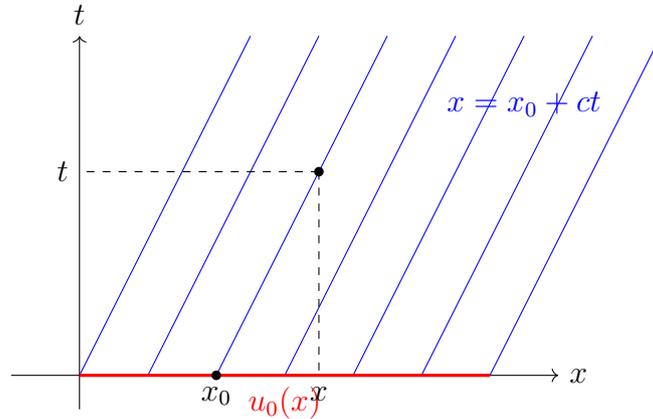
Along each characteristic, $u(x(t), t) = u(x_0, 0) = u_0(x_0)$. Expressing $x_0 = x - ct$, we obtain $u(x, t) = u_0(x - ct)$.

Verification. If $u_0 \in C^1$, then $u(x, t) = u_0(x - ct)$ is C^1 and:

$$u_t = -c u'_0(x - ct), \quad u_x = u'_0(x - ct),$$

so $u_t + c u_x = 0$ and $u(x, 0) = u_0(x)$.

Uniqueness. If v is another C^1 solution, then $w = u - v$ satisfies $w_t + c w_x = 0$ with $w(x, 0) = 0$. Along every characteristic, w is constant equal to 0, so $w \equiv 0$. \square



Characteristics of $u_t + cu_x = 0$

Transport equation — key formulas

- Solution: $u(x, t) = u_0(x - ct)$ (travelling wave).
- Characteristics: lines $x - ct = \text{const}$.
- The solution propagates at speed c without distortion.

2.2.2 Inhomogeneous equation

Theorem 2.3 (Transport with source term). *The problem:*

$$\begin{cases} u_t + c u_x = f(x, t), & x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (2.3)$$

has solution:

$$u(x, t) = u_0(x - ct) + \int_0^t f(x - c(t - s), s) ds. \quad (2.4)$$

Proof. Along the characteristic $x(s) = x - c(t - s)$, we have:

$$\frac{d}{ds}u(x(s), s) = u_t + c u_x = f(x(s), s).$$

Integrating from $s = 0$ to $s = t$:

$$u(x, t) - u(x - ct, 0) = \int_0^t f(x - c(t - s), s) ds.$$

□

2.3 Method of characteristics — general case

2.3.1 First-order quasilinear PDEs

Consider the general first-order PDE:

$$a(x, y, u) u_x + b(x, y, u) u_y = c(x, y, u). \quad (2.5)$$

Definition 2.4 (Characteristic equations). The **characteristic equations** associated with (2.5) form the ODE system:

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}. \quad (2.6)$$

Equivalently, parametrizing by s :

$$\frac{dx}{ds} = a, \quad \frac{dy}{ds} = b, \quad \frac{du}{ds} = c.$$

Theorem 2.5 (Local existence). *Let Γ be an initial curve parametrized by $(x_0(\tau), y_0(\tau), u_0(\tau))$ such that:*

$$\left. \frac{\partial(x, y)}{\partial(s, \tau)} \right|_{s=0} = a y'_0(\tau) - b x'_0(\tau) \neq 0 \quad (\text{transversality condition}).$$

Then there exists locally a unique solution $u(x, y)$ near Γ .

Transversality condition

If the initial curve is tangent to the characteristics at a point, the method fails: either there is no solution or infinitely many. This is analogous to the Cauchy problem for ODEs when the initial condition is posed at a singular point.

Example 2.6 (Linear PDE with variable coefficients). Solve:

$$x u_x + y u_y = u, \quad u(x, 1) = g(x).$$

The characteristic equations are:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}.$$

From $\frac{dx}{x} = \frac{dy}{y}$, we get $\frac{x}{y} = C_1$. From $\frac{dy}{y} = \frac{du}{u}$, we get $\frac{u}{y} = C_2$. So the general solution is:

$$u = y \Phi\left(\frac{x}{y}\right)$$

for an arbitrary function Φ . The initial condition $u(x, 1) = g(x)$ gives $\Phi(x) = g(x)$, hence:

$$u(x, y) = y g\left(\frac{x}{y}\right).$$

2.3.2 Nonlinear case: Burgers' equation

Definition 2.7 (Inviscid Burgers' equation). **Burgers' equation** is the nonlinear PDE:

$$u_t + u u_x = 0, \quad u(x, 0) = u_0(x). \quad (2.7)$$

Theorem 2.8 (Implicit solution of Burgers' equation). *The characteristics of (2.7) are the lines:*

$$x = x_0 + u_0(x_0)t.$$

Along each characteristic, u is constant: $u = u_0(x_0)$. The solution is defined implicitly by:

$$u(x, t) = u_0(x - u(x, t)t). \quad (2.8)$$

Proof. The characteristic equations are:

$$\frac{dx}{ds} = u, \quad \frac{dt}{ds} = 1, \quad \frac{du}{ds} = 0.$$

So u is constant along characteristics, and $x(t) = x_0 + u_0(x_0)t$. □

2.3.3 Shock formation

Theorem 2.9 (Shock formation time). *If $u_0 \in C^1(\mathbb{R})$ and u'_0 has a negative minimum, then the characteristics cross at time:*

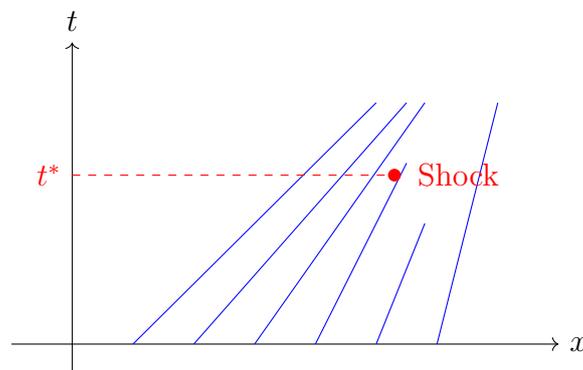
$$t^* = \frac{-1}{\min_{x \in \mathbb{R}} u'_0(x)} > 0. \quad (2.9)$$

For $t < t^$, the solution is C^1 . For $t \geq t^*$, a **shock** (discontinuity) forms.*

Proof. Two characteristics starting from x_0 and $x_1 > x_0$ cross if:

$$x_0 + u_0(x_0)t = x_1 + u_0(x_1)t,$$

i.e., $t = \frac{x_1 - x_0}{u_0(x_0) - u_0(x_1)} = \frac{-1}{(u_0(x_1) - u_0(x_0))/(x_1 - x_0)}$. Taking $x_1 \rightarrow x_0$, the first crossing occurs at $t^* = -1/\min u'_0$. □



Shock formation: crossing characteristics

Example 2.10 (Shock for Burgers' equation). Let $u_0(x) = 1 - x$ for $x \in [0, 1]$, $u_0(x) = 1$ for $x < 0$, $u_0(x) = 0$ for $x > 1$. Then $\min u'_0 = -1$ (on $(0, 1)$), so $t^* = 1$. The characteristics from $[0, 1]$ are $x = x_0 + (1 - x_0)t$ and they all meet at the point $(1, 1)$.

2.4 Transport in higher dimensions

Definition 2.11 (Transport equation in \mathbb{R}^n). Let $\vec{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ be a constant velocity vector. The transport equation reads:

$$u_t + \vec{b} \cdot \nabla u = 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n. \quad (2.10)$$

The solution is $u(x, t) = u_0(x - \vec{b}t)$.

For a variable velocity field $\vec{b}(x, t)$, the equation $u_t + \vec{b}(x, t) \cdot \nabla u = 0$ is still solved by characteristics, which are solutions of:

$$\frac{d\vec{X}}{dt} = \vec{b}(\vec{X}(t), t), \quad \vec{X}(0) = x_0.$$

2.5 Conservation and first integrals

Proposition 2.12 (Conservation of L^p norms). If u solves $u_t + cu_x = 0$ with $u_0 \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, then:

$$\|u(\cdot, t)\|_{L^p(\mathbb{R})} = \|u_0\|_{L^p(\mathbb{R})} \quad \forall t \geq 0.$$

Proof. For $1 \leq p < \infty$, the change of variable $y = x - ct$ gives:

$$\int_{\mathbb{R}} |u(x, t)|^p dx = \int_{\mathbb{R}} |u_0(x - ct)|^p dx = \int_{\mathbb{R}} |u_0(y)|^p dy.$$

The case $p = \infty$ is immediate. □

Theorem 2.13 (Conservative form and Rankine–Hugoniot condition). *The equation $u_t + (f(u))_x = 0$ (conservation law) admits discontinuous weak solutions. Across a discontinuity propagating at speed σ , the **Rankine–Hugoniot condition** reads:*

$$\sigma [u] = [f(u)], \quad (2.11)$$

where $[u] = u^+ - u^-$ denotes the jump.

2.6 Weak solutions and entropy

Definition 2.14 (Weak solution). A function $u \in L^\infty(\mathbb{R} \times [0, \infty))$ is a **weak solution** of $u_t + (f(u))_x = 0$ with initial data u_0 if:

$$\int_0^\infty \int_{\mathbb{R}} [u \varphi_t + f(u) \varphi_x] dx dt + \int_{\mathbb{R}} u_0(x) \varphi(x, 0) dx = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty))$.

Non-uniqueness of weak solutions

Weak solutions are generally not unique! One must impose an **entropy condition**

to select the physically admissible solution. Lax's condition requires:

$$f'(u^-) > \sigma > f'(u^+).$$

2.7 Exercises

Exercise 2.1. Solve $u_t + 3u_x = 0$, $u(x, 0) = e^{-x^2}$. Sketch $u(x, t)$ for $t = 0, 1, 2$.

Exercise 2.2. Solve $u_t + 2u_x = t \sin x$, $u(x, 0) = 0$.

Exercise 2.3. Solve by characteristics: $y u_x - x u_y = 0$, $u(x, 0) = x^2$ for $x > 0$. Interpret the characteristics geometrically.

Exercise 2.4. Consider Burgers' equation $u_t + u u_x = 0$ with:

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0, \\ 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x > 1. \end{cases}$$

1. Draw the characteristics.
2. Determine the shock formation time t^* .
3. Write the solution for $0 < t < t^*$.

Exercise 2.5. Solve the nonlinear transport equation: $u_t + u^2 u_x = 0$, $u(x, 0) = \frac{1}{1+x^2}$. Determine the time of formation of the first shock.

Exercise 2.6. Show that if u solves $u_t + c u_x = 0$, then $E(t) = \frac{1}{2} \int_{\mathbb{R}} u^2(x, t) dx$ is conserved. Generalize to $u_t + c(x) u_x = 0$ with $c_x = 0$. What happens if $c_x \neq 0$?

Exercise 2.7 (Rarefaction waves). Consider Burgers' equation with:

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

1. Show that no classical solution exists for $t > 0$.
2. Construct a **rarefaction wave** $u(x, t) = v(x/t)$ that is a continuous weak solution.
3. Verify that a shock solution (with speed $\sigma = 1/2$) is also a weak solution, but does not satisfy Lax's entropy condition.

Chapter 3

Separation of Variables and Fourier Series

In 1807, Joseph Fourier presented to the French Academy of Sciences a bold memoir: any periodic function, however irregular, can be written as a sum of sines and cosines. The claim was met with scepticism—Lagrange deemed it impossible—but it would prove to be one of the most fruitful ideas in all of mathematics. Fourier’s method, born from the study of heat propagation, offers a remarkable strategy for solving linear PDEs: decompose the problem into eigenmodes, solve each mode separately, then recombine. This principle of *separation of variables* transforms a PDE into a family of ordinary ODEs—a considerable gain.

3.1 Principle of separation of variables

The method of separation of variables (or Fourier’s method) is one of the most powerful techniques for solving linear PDEs on simple domains with homogeneous boundary conditions.

Intuition

The idea is to look for the solution as a product of functions, each depending on only one variable. The PDE then decomposes into several ODEs, which are much easier to solve.

3.1.1 General method

Consider a PDE of the form $L[u] = 0$ posed on a domain $\Omega = (0, L) \times (0, T)$ with homogeneous boundary conditions.

Step 1: Look for solutions of the form $u(x, t) = X(x)T(t)$.

Step 2: Substitute into the PDE and separate variables to obtain two ODEs coupled by a separation constant λ .

Step 3: Solve the Sturm–Liouville problem for X (with boundary conditions), which determines the eigenvalues λ_n .

Step 4: Solve the ODE for T with each λ_n .

Step 5: Form the general solution by superposition: $u = \sum_n c_n X_n(x) T_n(t)$.

Step 6: Determine the coefficients c_n from the initial condition using Fourier series.

Example 3.1 (Heat equation on a segment). Consider:

$$\begin{cases} u_t = k u_{xx}, & 0 < x < L, t > 0, \\ u(0, t) = u(L, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & 0 < x < L. \end{cases}$$

Setting $u(x, t) = X(x)T(t)$, the PDE gives $X T' = k X'' T$, i.e.:

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda \quad (\text{constant}).$$

This yields two ODEs:

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0,$$

$$T' + k\lambda T = 0.$$

The problem for X is a Sturm–Liouville problem whose nontrivial solutions are:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n \geq 1.$$

The ODE for T gives $T_n(t) = e^{-k\lambda_n t}$. The general solution is:

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t},$$

where the b_n are the Fourier sine coefficients of u_0 .

3.2 Fourier series

3.2.1 Definitions and coefficients

Definition 3.2 (Fourier series). Let $f : [-\pi, \pi] \rightarrow \mathbb{R}$ be an integrable function. The **Fourier series** of f is:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], \quad (3.1)$$

where the **Fourier coefficients** are:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx. \quad (3.2)$$

Definition 3.3 (Complex Fourier series). Equivalently, one can write:

$$f(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Fourier coefficients on $[0, L]$

For a function f on $[0, L]$:

- Sine coefficients: $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$.
- Cosine coefficients: $a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$.

3.2.2 Orthogonality

Proposition 3.4 (Orthogonality of trigonometric functions). The system $\{1, \cos(nx), \sin(nx)\}_{n \geq 1}$ is orthogonal in $L^2(-\pi, \pi)$:

$$\begin{aligned} \langle \cos(mx), \cos(nx) \rangle &= \int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \pi \delta_{mn}, \\ \langle \sin(mx), \sin(nx) \rangle &= \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = \pi \delta_{mn}, \\ \langle \cos(mx), \sin(nx) \rangle &= 0 \quad \forall m, n. \end{aligned}$$

3.2.3 Convergence of Fourier series

Theorem 3.5 (Pointwise convergence — Dirichlet). If f is 2π -periodic, piecewise continuous, and piecewise C^1 , then the Fourier series converges at every point:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = \frac{f(x^+) + f(x^-)}{2}.$$

In particular, at points of continuity the series converges to $f(x)$.

Theorem 3.6 (Uniform convergence). If f is 2π -periodic, continuous, and piecewise C^1 , then the Fourier series converges uniformly to f .

Theorem 3.7 (Convergence in L^2 — Parseval). If $f \in L^2(-\pi, \pi)$, the Fourier series converges to f in L^2 and **Parseval's identity** holds:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \tag{3.3}$$

Remark 3.8. Parseval's identity expresses that the trigonometric system forms a **Hilbert basis** of $L^2(-\pi, \pi)$. This is a special case of the completeness theorem.

Example 3.9. Let us compute the Fourier series of $f(x) = x$ on $[-\pi, \pi]$. By oddness, $a_n = 0$ for all $n \geq 0$. For the b_n :

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{2(-1)^{n+1}}{n}.$$

Therefore:

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx), \quad x \in (-\pi, \pi).$$

Parseval's identity gives $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

3.3 Sturm–Liouville problems

Definition 3.10 (Sturm–Liouville problem). A **regular Sturm–Liouville problem** is an eigenvalue problem of the form:

$$-(p(x)y')' + q(x)y = \lambda w(x)y, \quad x \in (a, b), \quad (3.4)$$

with separated boundary conditions, where $p > 0$, $w > 0$, and q are continuous on $[a, b]$.

Theorem 3.11 (Properties of Sturm–Liouville problems). *Under the above hypotheses:*

1. *There exists an infinite sequence of real eigenvalues $\lambda_1 < \lambda_2 < \dots \rightarrow +\infty$.*
2. *The corresponding eigenfunctions ϕ_n are orthogonal in $L_w^2(a, b)$: $\int_a^b \phi_m(x) \phi_n(x) w(x) dx = 0$ for $m \neq n$.*
3. *The eigenfunction ϕ_n has exactly $n - 1$ zeros in (a, b) .*
4. *The system $\{\phi_n\}$ forms a Hilbert basis of $L_w^2(a, b)$.*

Example 3.12 (Standard problems).

1. $X'' + \lambda X = 0$, $X(0) = X(L) = 0$ (Dirichlet):
 $\lambda_n = (n\pi/L)^2$, $X_n = \sin(n\pi x/L)$.
2. $X'' + \lambda X = 0$, $X'(0) = X'(L) = 0$ (Neumann):
 $\lambda_n = (n\pi/L)^2$ for $n \geq 0$, $X_0 = 1$, $X_n = \cos(n\pi x/L)$.
3. $X'' + \lambda X = 0$, $X(0) = 0$, $X'(L) = 0$ (mixed):
 $\lambda_n = ((2n - 1)\pi/(2L))^2$, $X_n = \sin((2n - 1)\pi x/(2L))$.

3.4 Applications to PDEs

3.4.1 Heat equation with Neumann conditions

Example 3.13. Solve:

$$\begin{cases} u_t = k u_{xx}, & 0 < x < L, t > 0, \\ u_x(0, t) = u_x(L, t) = 0, & t > 0, \\ u(x, 0) = u_0(x). \end{cases}$$

Separation gives $X'' + \lambda X = 0$ with $X'(0) = X'(L) = 0$ (Neumann). The eigenvalues are $\lambda_n = (n\pi/L)^2$, $n \geq 0$, and:

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t},$$

where $a_n = \frac{2}{L} \int_0^L u_0(x) \cos(n\pi x/L) dx$.

As $t \rightarrow \infty$, $u(x, t) \rightarrow a_0/2 = \frac{1}{L} \int_0^L u_0(x) dx$, the initial mean temperature.

3.4.2 Wave equation

Example 3.14. Solve:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0, \\ u(0, t) = u(L, t) = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x). \end{cases}$$

Setting $u(x, t) = X(x)T(t)$ gives:

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0 \quad \text{and} \quad T'' + c^2 \lambda T = 0.$$

The solutions are $X_n = \sin(n\pi x/L)$, $\omega_n = cn\pi/L$, and:

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)],$$

with:

$$A_n = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad B_n = \frac{2}{L\omega_n} \int_0^L v_0(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

3.4.3 Laplace equation on a rectangle

Example 3.15. Solve:

$$\begin{cases} \Delta u = 0, & 0 < x < a, 0 < y < b, \\ u(0, y) = u(a, y) = 0, \\ u(x, 0) = 0, \quad u(x, b) = f(x). \end{cases}$$

Setting $u = X(x)Y(y)$:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

With $X(0) = X(a) = 0$: $\lambda_n = (n\pi/a)^2$, $X_n = \sin(n\pi x/a)$. The equation for Y gives $Y'' - \lambda_n Y = 0$ with $Y(0) = 0$:

$$Y_n(y) = \sinh\left(\frac{n\pi y}{a}\right).$$

The solution is:

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right),$$

with $c_n = \frac{2}{a \sinh(n\pi b/a)} \int_0^a f(x) \sin(n\pi x/a) dx$.

3.5 Separation in polar coordinates

Example 3.16 (Laplace in the disk). Solve $\Delta u = 0$ in the disk $r < R$ with $u(R, \theta) = f(\theta)$. In polar coordinates, $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. Setting $u = \mathcal{R}(r)\Theta(\theta)$:

$$r^2 \frac{\mathcal{R}''}{\mathcal{R}} + r \frac{\mathcal{R}'}{\mathcal{R}} = -\frac{\Theta''}{\Theta} = \lambda.$$

Periodicity in θ forces $\lambda_n = n^2$, $n \geq 0$. The solutions are:

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

where a_n, b_n are the Fourier coefficients of f . This gives the **Poisson integral formula**:

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi) + r^2} f(\phi) d\phi.$$

3.6 Exercises

Exercise 3.1. Compute the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$. Deduce that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

Exercise 3.2. Solve the heat equation $u_t = u_{xx}$ on $(0, \pi)$ with $u(0, t) = u(\pi, t) = 0$ and $u(x, 0) = \sin(x) + 3 \sin(2x)$.

Exercise 3.3. Solve the wave equation $u_{tt} = 4u_{xx}$ on $(0, 1)$ with $u(0, t) = u(1, t) = 0$, $u(x, 0) = x(1-x)$, $u_t(x, 0) = 0$.

Exercise 3.4. Solve the Laplace problem on the rectangle $(0, \pi) \times (0, 1)$ with $u(0, y) = u(\pi, y) = 0$, $u(x, 0) = 0$, $u(x, 1) = \sin(3x)$.

Exercise 3.5. Show that the Fourier coefficients of f satisfy $a_n, b_n \rightarrow 0$ as $n \rightarrow \infty$ (Riemann–Lebesgue lemma).

Exercise 3.6. Solve the Dirichlet problem in the disk $r < 1$ with $u(1, \theta) = \cos^2 \theta$. Write the solution in closed form.

Exercise 3.7 (Singular Sturm–Liouville problem). Find the eigenvalues and eigenfunctions of:

$$-\frac{d}{dx} \left(x \frac{dy}{dx} \right) = \frac{\lambda}{x} y, \quad 1 < x < e, \quad y(1) = y(e) = 0.$$

Hint: set $x = e^s$.

Chapter 4

Heat Equation

It was to understand the propagation of heat that Joseph Fourier developed the series that bear his name. His *Théorie analytique de la chaleur* (1822) is one of the founding texts of mathematical physics. The heat equation—the simplest parabolic PDE—describes an irreversible phenomenon: an initial temperature profile smooths over time, peaks flatten, gradients fade. Unlike the wave equation, information propagates at infinite speed, and unlike the transport equation, solutions become immediately smooth. These surprising, counter-intuitive properties make the heat equation the prototype of all diffusion equations.

4.1 Physical motivation and formulation

The heat equation describes thermal diffusion in a conducting medium. If $u(x, t)$ represents the temperature, Fourier's law ($\vec{q} = -k\nabla u$) combined with energy conservation gives:

Definition 4.1 (Heat equation). The **heat equation** in dimension n is:

$$u_t - \kappa \Delta u = f(x, t), \quad x \in \Omega \subset \mathbb{R}^n, \quad t > 0, \quad (4.1)$$

where $\kappa > 0$ is the thermal diffusivity. The case $f = 0$ is called **homogeneous**.

Intuition

The heat equation smooths out irregularities: even if the initial temperature is discontinuous, the solution becomes instantaneously C^∞ for $t > 0$. This is the **smoothing effect** of diffusion. However, information propagates at infinite speed (no wavefront).

4.2 Fundamental solution

Definition 4.2 (Heat kernel). The **heat kernel** (or fundamental solution) in dimension n is:

$$\Phi(x, t) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp\left(-\frac{|x|^2}{4\kappa t}\right), \quad t > 0. \quad (4.2)$$

Theorem 4.3 (Properties of the heat kernel). *The kernel Φ satisfies:*

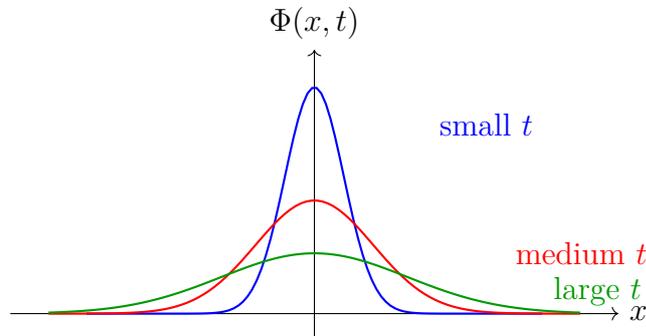
1. $\Phi_t = \kappa \Delta \Phi$ for $t > 0$, $x \neq 0$.
2. $\Phi(x, t) > 0$ for all $x \in \mathbb{R}^n$, $t > 0$.
3. $\int_{\mathbb{R}^n} \Phi(x, t) dx = 1$ for all $t > 0$.
4. $\Phi(\cdot, t) \rightarrow \delta_0$ in the sense of distributions as $t \rightarrow 0^+$.

Proof. (1) Direct verification in dimension $n = 1$. We have $\Phi(x, t) = (4\pi\kappa t)^{-1/2} e^{-x^2/(4\kappa t)}$. Computing:

$$\Phi_t = \Phi \left(-\frac{1}{2t} + \frac{x^2}{4\kappa t^2} \right), \quad \Phi_x = -\frac{x}{2\kappa t} \Phi, \quad \Phi_{xx} = \left(\frac{x^2}{4\kappa^2 t^2} - \frac{1}{2\kappa t} \right) \Phi.$$

So $\Phi_t = \kappa \Phi_{xx}$.

(3) By the substitution $y = x/\sqrt{4\kappa t}$: $\int_{\mathbb{R}^n} \Phi dx = \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|y|^2} dy = 1$. □



Evolution of the heat kernel

4.3 Cauchy problem on \mathbb{R}^n

Theorem 4.4 (Solution by convolution). *If $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, the solution of the Cauchy problem:*

$$u_t = \kappa \Delta u, \quad u(x, 0) = u_0(x),$$

is given by convolution:

$$u(x, t) = (\Phi(\cdot, t) * u_0)(x) = \int_{\mathbb{R}^n} \Phi(x - y, t) u_0(y) dy. \quad (4.3)$$

Moreover, $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$.

Proof sketch. One can differentiate under the integral sign because Φ is C^∞ for $t > 0$ and decays exponentially. The initial condition follows from property (4) of the kernel: $\Phi(\cdot, t) * u_0 \rightarrow u_0$ as $t \rightarrow 0^+$. □

Heat equation — key formulas

- Kernel ($n = 1$): $\Phi(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-x^2/(4\kappa t)}$.
- Cauchy solution: $u = \Phi * u_0$.

- With source: $u = \Phi * u_0 + \int_0^t \Phi(\cdot, t-s) * f(\cdot, s) ds$ (Duhamel's formula).

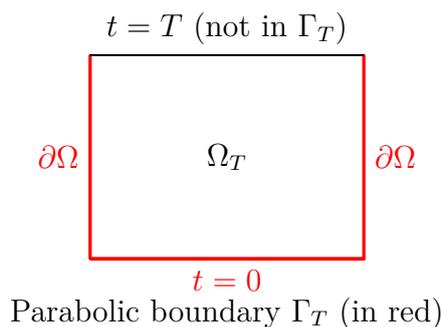
4.4 Maximum principle for the heat equation

Theorem 4.5 (Weak maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\Omega_T = \Omega \times (0, T]$. If $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $u_t - \kappa\Delta u \leq 0$ in Ω_T , then:*

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u, \quad (4.4)$$

where $\Gamma_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0, T])$ is the **parabolic boundary**.

Proof. Suppose by contradiction that the maximum is attained at an interior point $(x_0, t_0) \in \Omega_T$ with $t_0 > 0$. Then $\nabla u(x_0, t_0) = 0$, $D^2u(x_0, t_0) \leq 0$ (negative semi-definite), so $\Delta u(x_0, t_0) \leq 0$. Moreover, $u_t(x_0, t_0) \geq 0$ (since the maximum is attained at t_0). Therefore $u_t - \kappa\Delta u \geq 0$ at (x_0, t_0) , contradicting $u_t - \kappa\Delta u \leq 0$ (unless equality). One concludes by perturbation, considering $v = u - \varepsilon t$. \square



Corollary 4.6 (Uniqueness). *The Cauchy–Dirichlet problem for the heat equation (on a bounded domain) admits at most one classical solution.*

Proof. If u and v are two solutions, $w = u - v$ satisfies $w_t = \kappa\Delta w$ with $w = 0$ on Γ_T . By the maximum principle, $w \leq 0$ in $\overline{\Omega_T}$. Applying the same argument to $-w$, we get $w \geq 0$, so $w = 0$. \square

4.5 Regularity and smoothing effect

Theorem 4.7 (Infinite regularity). *If u solves $u_t = \kappa\Delta u$ in Ω_T with $u_0 \in L^2(\Omega)$, then $u \in C^\infty(\Omega \times (0, T])$.*

Irreversibility

The heat equation is irreversible in time: one cannot, in general, go backward in time. The problem $u_t = -\kappa\Delta u$ with final data is ill-posed in the sense of Hadamard.

4.6 Asymptotic behaviour

Theorem 4.8 (Decay in time). *Let u be the solution of $u_t = \kappa \Delta u$ on \mathbb{R}^n with $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then:*

1. $\|u(\cdot, t)\|_{L^\infty} \leq \frac{C}{t^{n/2}} \|u_0\|_{L^1}$.
2. $\|u(\cdot, t)\|_{L^2}$ is decreasing in t .
3. $\int_{\mathbb{R}^n} u(x, t) dx = \int_{\mathbb{R}^n} u_0(x) dx$ (mass conservation).

Proof. (2) We compute:

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^n} u^2 dx = \int_{\mathbb{R}^n} u u_t dx = \kappa \int_{\mathbb{R}^n} u \Delta u dx = -\kappa \int_{\mathbb{R}^n} |\nabla u|^2 dx \leq 0.$$

(3) Integrating the equation and using the divergence theorem: $\frac{d}{dt} \int u dx = \kappa \int \Delta u dx = 0$ (provided u decays sufficiently at infinity). \square

4.7 Duhamel's formula

Theorem 4.9 (Duhamel's principle). *The solution of the inhomogeneous problem:*

$$u_t - \kappa \Delta u = f(x, t), \quad u(x, 0) = u_0(x),$$

on \mathbb{R}^n is:

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) dy ds. \quad (4.5)$$

Intuition

Duhamel's principle is the PDE analogue of the variation of constants method for ODEs. The source term f is treated as a continuous superposition of initial conditions applied at each instant s .

4.8 Heat equation on a bounded interval

Example 4.10. Solve completely:

$$\begin{cases} u_t = u_{xx}, & 0 < x < \pi, t > 0, \\ u(0, t) = u(\pi, t) = 0, \\ u(x, 0) = x(\pi - x). \end{cases}$$

By separation of variables (cf. Chapter 3), the eigenmodes are $X_n = \sin(nx)$, $\lambda_n = n^2$, and:

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}.$$

The Fourier sine coefficients of $f(x) = x(\pi - x)$ are:

$$b_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin(nx) dx = \begin{cases} \frac{8}{n^3\pi} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Therefore:

$$u(x, t) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{(2k+1)^3} e^{-(2k+1)^2 t}.$$

For large t , $u(x, t) \approx \frac{8}{\pi} \sin(x) e^{-t}$: only the first mode survives.

4.9 Inhomogeneous boundary conditions

To handle $u(0, t) = g_0(t)$, $u(L, t) = g_1(t)$, we set $v(x, t) = u(x, t) - w(x, t)$ where w is a **lifting** of the boundary conditions:

$$w(x, t) = g_0(t) + \frac{x}{L}(g_1(t) - g_0(t)).$$

Then v satisfies the heat equation with homogeneous conditions and a modified source term.

4.10 Exercises

Exercise 4.1. Verify that $\Phi(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-x^2/(4\kappa t)}$ satisfies $\Phi_t = \kappa \Phi_{xx}$ for $t > 0$.

Exercise 4.2. Solve $u_t = u_{xx}$ on \mathbb{R} with $u(x, 0) = e^{-|x|}$.

Exercise 4.3. Show that the energy $E(t) = \frac{1}{2} \int_0^L u^2(x, t) dx$ is decreasing for the heat equation with homogeneous Dirichlet conditions. What is the rate of decay?

Exercise 4.4. Solve $u_t = u_{xx} + \sin(x)$ on $(0, \pi)$ with $u(0, t) = u(\pi, t) = 0$ and $u(x, 0) = 0$. *Hint:* look for a stationary solution, then use Duhamel.

Exercise 4.5. Let u solve $u_t = u_{xx}$ on $(0, 1)$ with $u(0, t) = 0$, $u(1, t) = 1$, $u(x, 0) = x$. Find the stationary solution and show that u converges to it.

Exercise 4.6 (Self-similarity). Look for a solution of $u_t = u_{xx}$ of the form $u(x, t) = t^{-\alpha} f(\xi)$ with $\xi = x/\sqrt{t}$. Determine α and the ODE satisfied by f .

Exercise 4.7. Show that the maximum temperature decreases over time for the heat equation on a bounded domain with homogeneous Dirichlet conditions. *Hint:* use the maximum principle.

Chapter 5

Wave Equation

Pluck a guitar string. The perturbation travels in both directions, bounces off the endpoints, and produces a sound whose frequency depends on the length, tension, and linear density of the string. The wave equation, formulated by d'Alembert in 1747 for the vibrating string, is the first PDE ever written. Unlike the heat equation, it is time-reversible: waves do not dissipate, they propagate at finite speed and preserve their shape. D'Alembert's solution in one dimension shows that every solution is a superposition of two waves travelling in opposite directions—an observation of remarkable simplicity and power.

5.1 Physical motivation

The wave equation describes the propagation of perturbations through a medium: vibrations of a string or membrane, sound waves, electromagnetic waves. Unlike the heat equation, propagation occurs at **finite speed** and without dissipation.

Definition 5.1 (Wave equation). The **wave equation** in dimension n is:

$$u_{tt} - c^2 \Delta u = f(x, t), \quad x \in \mathbb{R}^n, \quad t > 0, \quad (5.1)$$

where $c > 0$ is the propagation speed.

5.2 Wave equation in dimension 1

5.2.1 d'Alembert's formula

Theorem 5.2 (d'Alembert's formula). *The solution of the Cauchy problem:*

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x), \end{cases} \quad (5.2)$$

is given by d'Alembert's formula:

$$\boxed{u(x, t) = \frac{g(x + ct) + g(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} h(s) ds.} \quad (5.3)$$

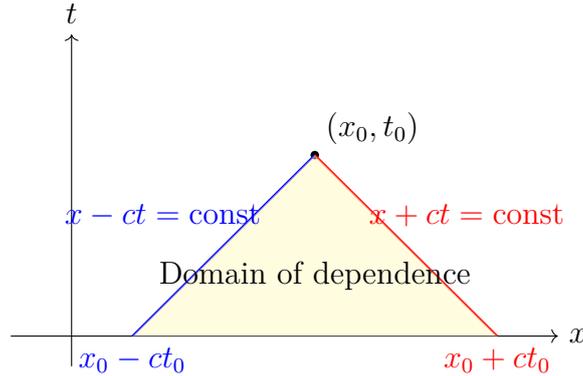
Proof. We introduce **characteristic coordinates** $\xi = x + ct$, $\eta = x - ct$. The equation $u_{tt} = c^2 u_{xx}$ becomes $u_{\xi\eta} = 0$, whose general solution is:

$$u(\xi, \eta) = F(\xi) + G(\eta) = F(x + ct) + G(x - ct),$$

where F and G are arbitrary functions. The initial conditions give:

$$F(x) + G(x) = g(x), \quad cF'(x) - cG'(x) = h(x).$$

Integrating the second equation and solving the system yields d'Alembert's formula. \square



1D wave equation — key formulas

- d'Alembert: $u = \frac{1}{2}[g(x + ct) + g(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(s) ds$.
- Characteristics: $x \pm ct = \text{const}$.
- General solution: $u = F(x + ct) + G(x - ct)$.

5.2.2 Domain of dependence and influence

Definition 5.3 (Domain of dependence). The **domain of dependence** of the point (x_0, t_0) is the interval $[x_0 - ct_0, x_0 + ct_0]$ on the axis $t = 0$. The solution at (x_0, t_0) depends only on the initial data on this interval.

Definition 5.4 (Domain of influence). The **domain of influence** of the point x_0 on the axis $t = 0$ is the cone $\{(x, t) : |x - x_0| \leq ct, t > 0\}$.

Theorem 5.5 (Finite speed of propagation). *If the initial data g and h are supported in $[a, b]$, then $u(x, t) = 0$ for $x \notin [a - ct, b + ct]$. Information propagates at most at speed c .*

5.3 Energy conservation

Theorem 5.6 (Energy conservation). *If u solves $u_{tt} = c^2 \Delta u$ in \mathbb{R}^n , the **energy**:*

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} (u_t^2 + c^2 |\nabla u|^2) dx \quad (5.4)$$

is conserved: $E(t) = E(0)$ for all $t \geq 0$.

Proof. We compute:

$$\begin{aligned} E'(t) &= \int_{\mathbb{R}^n} (u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t) dx \\ &= \int_{\mathbb{R}^n} u_t (u_{tt} - c^2 \Delta u) dx = 0. \end{aligned}$$

The last equality uses integration by parts ($\int \nabla u \cdot \nabla u_t = -\int u_t \Delta u$). \square

Corollary 5.7 (Uniqueness). *The Cauchy problem for the wave equation admits at most one solution (in a suitable regularity class).*

5.4 Vibrating string with fixed endpoints

Example 5.8. Solve:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0, \\ u(0, t) = u(L, t) = 0, \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x). \end{cases}$$

By separation of variables:

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)],$$

with $\omega_n = \frac{cn\pi}{L}$ (natural frequencies). The coefficients are:

$$A_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad B_n = \frac{2}{L\omega_n} \int_0^L h(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

The solution is a superposition of **normal modes** vibrating at frequencies $\omega_n = cn\pi/L$. The fundamental frequency is $\omega_1 = c\pi/L$; the others are **harmonics** $\omega_n = n\omega_1$.

Intuition

Each normal mode $\sin(n\pi x/L) \cos(\omega_n t)$ represents a **standing wave**: the spatial shape remains fixed and only the amplitude oscillates. The superposition of modes produces the complex shapes of a vibrating string.

5.5 Wave equation in higher dimensions

5.5.1 Dimension 3: Kirchhoff's formula

Theorem 5.9 (Kirchhoff's formula). *The solution of the Cauchy problem in dimension 3:*

$$u_{tt} = c^2 \Delta u, \quad u(x, 0) = g(x), \quad u_t(x, 0) = h(x),$$

is:

$$u(x, t) = \frac{1}{4\pi c^2 t} \iint_{S(x, ct)} h(y) dS(y) + \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \iint_{S(x, ct)} g(y) dS(y) \right], \quad (5.5)$$

where $S(x, ct)$ is the sphere centred at x with radius ct .

Theorem 5.10 (Strong Huygens' principle). *In dimension $n = 3$ (and more generally for odd $n \geq 3$), the **strong Huygens' principle** holds: the solution at (x, t) depends only on the data on the sphere $|y - x| = ct$, not in the ball. Physically, a point signal produces a sharp wavefront that passes and vanishes.*

5.5.2 Dimension 2: Poisson's formula

Theorem 5.11 (Poisson's formula — dimension 2). *In dimension 2, the solution is:*

$$u(x, t) = \frac{1}{2\pi c} \iint_{B(x, ct)} \frac{h(y)}{\sqrt{c^2 t^2 - |y - x|^2}} dy + \frac{\partial}{\partial t} \left[\frac{1}{2\pi c} \iint_{B(x, ct)} \frac{g(y)}{\sqrt{c^2 t^2 - |y - x|^2}} dy \right]. \quad (5.6)$$

No Huygens' principle in even dimensions

In dimension 2, the solution depends on data in the entire disk $B(x, ct)$, not just on the circle. This is why waves on water leave a wake behind them (unlike sound waves in 3D).

5.6 Damped wave equation

The damped wave equation reads:

$$u_{tt} + 2\gamma u_t = c^2 \Delta u, \quad \gamma > 0.$$

The term $2\gamma u_t$ models friction proportional to velocity.

Proposition 5.12 (Energy decay). For the damped equation, the energy decreases:

$$E'(t) = -2\gamma \int_{\mathbb{R}^n} u_t^2 dx \leq 0.$$

5.7 Exercises

Exercise 5.1. Use d'Alembert's formula to solve $u_{tt} = u_{xx}$ with $u(x, 0) = e^{-x^2}$ and $u_t(x, 0) = 0$. Sketch the solution for $t = 0, 1, 2$.

Exercise 5.2. Solve $u_{tt} = 4u_{xx}$ with $u(x, 0) = 0$ and $u_t(x, 0) = \mathbf{1}_{[-1, 1]}(x)$. Determine the domain of dependence and sketch the solution.

Exercise 5.3. Vibrating string: solve $u_{tt} = u_{xx}$ on $(0, \pi)$ with $u(0, t) = u(\pi, t) = 0$, $u(x, 0) = \sin(x) \sin(2x)$, $u_t(x, 0) = 0$.

Exercise 5.4. Show that energy conservation implies uniqueness for the Cauchy problem.

Exercise 5.5 (Reflection at a fixed boundary). Let $u_{tt} = c^2 u_{xx}$ for $x > 0$, $t > 0$ with $u(0, t) = 0$ and initial data compactly supported in $(0, \infty)$. Use the method of images (odd extension) to construct the solution.

Exercise 5.6. For the damped wave equation $u_{tt} + 2\gamma u_t = c^2 u_{xx}$, show that $E(t) \leq E(0) e^{-2\gamma t}$.

Exercise 5.7. Verify Kirchhoff's formula in 3D for the case $g = 0$, $h(x) = 1$ (constant data). Find $u(0, t)$ explicitly.

Exercise 5.8 (Duhamel's principle for the wave equation). Show that the solution of $u_{tt} - c^2 u_{xx} = f(x, t)$ with $u(x, 0) = u_t(x, 0) = 0$ is:

$$u(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds.$$

Chapter 6

Laplace Equation and Harmonic Functions

Laplace's equation $\Delta u = 0$ is the queen of elliptic PDEs. Its solutions, *harmonic functions*, model equilibrium states: steady-state temperature, electrostatic potential, irrotational flow. Pierre-Simon Laplace studied it as early as 1782 in the context of Newtonian gravitation, and its properties—maximum principle, infinite regularity, Poisson representation formula—make it one of the best-understood equations in all of mathematical physics.

6.1 Introduction and motivation

Laplace's equation $\Delta u = 0$ models equilibrium states: steady-state temperature, electrostatic potential in the absence of charges, potential flow. Its study reveals remarkable regularity, mean value, and maximum properties.

Definition 6.1 (Harmonic function). A function $u \in C^2(\Omega)$ is **harmonic** in an open set $\Omega \subset \mathbb{R}^n$ if:

$$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0 \quad \text{in } \Omega. \quad (6.1)$$

Definition 6.2 (Poisson's equation). **Poisson's equation** is the inhomogeneous equation:

$$-\Delta u = f \quad \text{in } \Omega. \quad (6.2)$$

6.2 Fundamental solution

Theorem 6.3 (Fundamental solution of the Laplacian). *The **fundamental solution** of Laplace's equation in dimension n is:*

$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| & \text{if } n = 2, \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \end{cases} \quad (6.3)$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ is the volume of the unit ball in \mathbb{R}^n . It satisfies $-\Delta \Gamma = \delta_0$ in the sense of distributions.

6.3 Mean value property

Theorem 6.4 (Mean value property). *If u is harmonic in Ω and $B(x, r) \subset \Omega$, then:*

$$u(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y) dS(y) \quad (\text{spherical mean}), \quad (6.4)$$

$$u(x) = \frac{1}{\omega_n r^n} \int_{B(x,r)} u(y) dy \quad (\text{volume mean}). \quad (6.5)$$

Proof. Set $\phi(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u dS$. By the substitution $y = x + r\sigma$, $\sigma \in \partial B(0, 1)$:

$$\phi(r) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(x + r\sigma) dS(\sigma).$$

Differentiating:

$$\phi'(r) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} \nabla u(x + r\sigma) \cdot \sigma dS(\sigma) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu} dS.$$

By the divergence theorem and $\Delta u = 0$:

$$\phi'(r) = \frac{1}{n\omega_n r^{n-1}} \int_{B(x,r)} \Delta u dy = 0.$$

So $\phi(r) = \phi(0^+) = u(x)$ by continuity. The volume formula follows by integrating in r . \square

Intuition

The value of a harmonic function at a point equals the average of its values on any sphere centred at that point. This is the continuous analogue of the fact that at thermal equilibrium, the temperature at a point is the average of the surrounding temperatures.

Theorem 6.5 (Converse of the mean value property). *If $u \in C(\Omega)$ satisfies the mean value property (6.4) for every ball $B(x, r) \subset \Omega$, then u is harmonic in Ω (and in particular C^∞).*

6.4 Maximum principle

Theorem 6.6 (Strong maximum principle for harmonic functions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be harmonic in Ω . Then:*

1. $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$ (maximum principle).
2. If the maximum is attained at an interior point, then u is constant in Ω (strong maximum principle).

The same holds for the minimum.

Proof. Suppose u attains its maximum M at $x_0 \in \Omega$. Let $r > 0$ be such that $B(x_0, r) \subset \Omega$. By the mean value property:

$$M = u(x_0) = \frac{1}{\omega_n r^n} \int_{B(x_0, r)} u(y) dy.$$

Since $u(y) \leq M$ everywhere and $u(x_0) = M$, the mean can equal M only if $u \equiv M$ in $B(x_0, r)$. The set $\{x : u(x) = M\}$ is therefore open (and closed in Ω by continuity). By connectedness, $u \equiv M$ in Ω . \square

Corollary 6.7 (Uniqueness of the Dirichlet problem). *The Dirichlet problem $\Delta u = 0$ in Ω , $u = g$ on $\partial\Omega$, admits at most one solution.*

Corollary 6.8 (Stability). *If u and v are harmonic with $u = g_1$, $v = g_2$ on $\partial\Omega$, then:*

$$\max_{\Omega} |u - v| \leq \max_{\partial\Omega} |g_1 - g_2|.$$

6.5 Regularity of harmonic functions

Theorem 6.9 (C^∞ regularity). *Every harmonic function is C^∞ . More precisely, if u is harmonic in Ω , then $u \in C^\infty(\Omega)$ and all its partial derivatives are harmonic.*

Theorem 6.10 (Analyticity). *Every harmonic function is real-analytic: it equals its Taylor series at every point of its domain.*

Theorem 6.11 (Gradient estimates). *If u is harmonic in $B(x_0, R)$, then:*

$$|\nabla u(x_0)| \leq \frac{n}{R} \max_{\partial B(x_0, R)} |u|. \quad (6.6)$$

More generally, for every multi-index α :

$$|D^\alpha u(x_0)| \leq \frac{C_{n, |\alpha|}}{R^{|\alpha|}} \max_{\partial B(x_0, R)} |u|.$$

6.6 Harnack's inequality

Theorem 6.12 (Harnack's inequality). *Let u be harmonic and **nonnegative** in $B(x_0, R) \subset \mathbb{R}^n$. For any $x \in B(x_0, R)$ with $|x - x_0| = r < R$:*

$$\frac{R^{n-2}(R-r)}{(R+r)^{n-1}} u(x_0) \leq u(x) \leq \frac{R^{n-2}(R+r)}{(R-r)^{n-1}} u(x_0). \quad (6.7)$$

In particular, on any compact $K \Subset \Omega$, there exists $C = C(K, \Omega)$ such that:

$$\max_K u \leq C \min_K u.$$

Remark 6.13. Harnack's inequality asserts that nonnegative harmonic functions cannot vary too rapidly. It is fundamental in regularity theory.

6.7 Green's representation formula

Definition 6.14 (Green's function). The **Green's function** of the domain Ω is $G(x, y) = \Gamma(x - y) - h^y(x)$, where h^y is harmonic in Ω with $h^y = \Gamma(\cdot - y)$ on $\partial\Omega$.

Theorem 6.15 (Representation formula). *If $u \in C^2(\bar{\Omega})$ is harmonic in Ω , then for every $x \in \Omega$:*

$$u(x) = - \int_{\partial\Omega} u(y) \frac{\partial G}{\partial \nu_y}(x, y) dS(y). \quad (6.8)$$

For Poisson's equation $-\Delta u = f$:

$$u(x) = \int_{\Omega} G(x, y) f(y) dy - \int_{\partial\Omega} u(y) \frac{\partial G}{\partial \nu_y}(x, y) dS(y). \quad (6.9)$$

Example 6.16 (Green's function for the disk). For the disk $B(0, R) \subset \mathbb{R}^2$, the Green's function is:

$$G(x, y) = -\frac{1}{2\pi} \ln|x - y| + \frac{1}{2\pi} \ln\left(\frac{|x||y|}{R} \left| \frac{x}{|x|^2} \cdot R^2 - y \right|\right).$$

The representation formula recovers the **Poisson kernel**.

6.8 Liouville's theorem

Theorem 6.17 (Liouville's theorem). *Every bounded harmonic function on \mathbb{R}^n is constant.*

Proof. Let u be bounded harmonic on \mathbb{R}^n , $|u| \leq M$. For any $x_0 \in \mathbb{R}^n$ and $R > 0$, the gradient estimate gives:

$$|\nabla u(x_0)| \leq \frac{n}{R} M.$$

Letting $R \rightarrow \infty$, we obtain $\nabla u(x_0) = 0$ for all x_0 , so u is constant. \square

6.9 Exercises

Exercise 6.1. Show that if u is harmonic in $B(0, R) \setminus \{0\} \subset \mathbb{R}^2$ and bounded, then the singularity at 0 is removable (removable singularity theorem).

Exercise 6.2. Use the Poisson formula to solve the Dirichlet problem in the disk $B(0, 1)$ with $u(1, \theta) = \cos(2\theta) + 3 \sin(\theta)$.

Exercise 6.3. Show that if u is harmonic and nonnegative in \mathbb{R}^n , then u is constant (consequence of Harnack).

Exercise 6.4. Compute the Green's function of the half-space $\{x_n > 0\}$ by the method of images.

Exercise 6.5. Show that the mean value of a harmonic function on a sphere equals its value at the centre. What happens for subharmonic functions ($\Delta u \geq 0$)?

Exercise 6.6. Let u be harmonic in $\Omega = B(0, 1) \setminus \overline{B(0, 1/2)} \subset \mathbb{R}^3$, with $u = 0$ on $|x| = 1/2$ and $u = 1$ on $|x| = 1$. Find u in the form $u(r) = a + b/r$.

Exercise 6.7 (Schwarz reflection principle). Let u be harmonic in the half-disk $\{r < 1, 0 < \theta < \pi\}$ with $u = 0$ on the diameter $[-1, 1]$. Show that u extends to a harmonic function on the full disk by odd reflection.

Chapter 7

Maximum Principle

7.1 Introduction

Imagine a metal plate whose edges are heated. Where is the hottest point inside? Physical intuition is unequivocal: in the absence of an internal heat source, the temperature maximum must be attained on the boundary. This observation, elevated to the status of a mathematical principle by Eberhard Hopf in the 1920s–1930s, has become one of the most powerful tools in PDE theory. The maximum principle requires no explicit knowledge of the solution: it provides a priori bounds, uniqueness results, and comparison theorems through the sheer force of the equation itself. Hopf's lemma, which refines the principle by showing that the normal derivative is strictly negative at the maximum point, is a gem of analysis whose applications irrigate the entire elliptic and parabolic theory.

Intuition

Physically, the maximum principle expresses the fact that in the absence of an internal heat source, the maximum temperature in a body is attained on the boundary. More generally, an equilibrium state cannot create a new extremum in the interior.

7.2 Maximum principle for elliptic operators

7.2.1 General elliptic operators

Definition 7.1 (Elliptic operator). A second-order differential operator:

$$Lu = - \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + c(x) u \quad (7.1)$$

is called **elliptic** in Ω if the matrix $(a_{ij}(x))$ is positive definite for all $x \in \Omega$, i.e., there exists $\theta > 0$ such that:

$$\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n.$$

7.2.2 Weak maximum principle

Theorem 7.2 (Weak maximum principle — case $c = 0$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and L an elliptic operator with $c = 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \leq 0$ in Ω ,*

then:

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u. \quad (7.2)$$

Proof. **Case** $Lu < 0$. Suppose u attains its maximum at $x_0 \in \Omega$. Then $\nabla u(x_0) = 0$ and $D^2u(x_0) \leq 0$ (negative semi-definite Hessian). Therefore:

$$Lu(x_0) = - \sum_{i,j} a_{ij}(x_0) u_{x_i x_j}(x_0) \geq 0,$$

since $\sum a_{ij} \xi_i \xi_j \geq 0$ for all ξ implies $\text{tr}(A \cdot D^2u) \leq 0$, hence $-\text{tr}(A \cdot D^2u) \geq 0$. This contradicts $Lu < 0$.

Case $Lu \leq 0$. Consider $v_\varepsilon(x) = u(x) + \varepsilon e^{\alpha x_1}$ for α large enough (so that $L(e^{\alpha x_1}) < 0$). Then $Lv_\varepsilon < 0$ and the previous case applies. Conclude by letting $\varepsilon \rightarrow 0$. \square

Theorem 7.3 (Weak maximum principle — case $c \geq 0$). *If $c \geq 0$ and $Lu \leq 0$ in Ω , then:*

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+, \quad (7.3)$$

where $u^+ = \max(u, 0)$.

7.2.3 Strong maximum principle

Theorem 7.4 (Strong maximum principle — Hopf). *Let Ω be a bounded connected open set and L elliptic with $c = 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \leq 0$ in Ω and attains its maximum at an interior point $x_0 \in \Omega$, then u is constant in Ω .*

Theorem 7.5 (Hopf's lemma). *Under the hypotheses of the previous theorem, if the maximum M is attained at a point $x_0 \in \partial\Omega$ (and not in the interior), and if Ω satisfies the **interior sphere condition** at x_0 , then:*

$$\frac{\partial u}{\partial \nu}(x_0) > 0, \quad (7.4)$$

where ν is the outward normal.

Proof sketch of Hopf's lemma. One constructs a **barrier** $w(x) = e^{-\alpha|x-y|^2} - e^{-\alpha R^2}$ in the annulus $B(y, R) \setminus B(y, \rho)$, where $B(y, R)$ is the interior sphere tangent at x_0 . For α large enough, $Lw < 0$. The comparison principle gives $u - M \leq -\varepsilon w$, whence $\frac{\partial u}{\partial \nu}(x_0) \geq \varepsilon \frac{\partial w}{\partial \nu}(x_0) > 0$. \square

7.3 Applications of the maximum principle

7.3.1 Uniqueness

Corollary 7.6 (Uniqueness for the Dirichlet problem). *The problem $Lu = f$ in Ω , $u = g$ on $\partial\Omega$ (with $c \geq 0$ and L elliptic) admits at most one classical solution.*

7.3.2 A priori estimates

Theorem 7.7 (A priori estimate). *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solve $Lu = f$ in Ω , $u = g$ on $\partial\Omega$, with L elliptic and $c = 0$. Then:*

$$\max_{\bar{\Omega}} |u| \leq \max_{\partial\Omega} |g| + C \max_{\bar{\Omega}} |f|, \quad (7.5)$$

where C depends on Ω , the coefficients of L , and θ .

7.3.3 Comparison principle

Theorem 7.8 (Comparison principle). *If $Lu \leq Lv$ in Ω and $u \leq v$ on $\partial\Omega$ (with $c \geq 0$), then $u \leq v$ in $\bar{\Omega}$.*

Proof. Set $w = u - v$. Then $Lw \leq 0$ and $w \leq 0$ on $\partial\Omega$. The maximum principle gives $w \leq 0$ in $\bar{\Omega}$. \square

7.4 Maximum principle for parabolic equations

Definition 7.9 (Parabolic operator). The operator $\mathcal{L}u = u_t - Lu$ where L is elliptic is called **parabolic**.

Theorem 7.10 (Weak parabolic maximum principle). *Let $\Omega_T = \Omega \times (0, T]$ and Γ_T the parabolic boundary. If $u \in C^{2,1}(\Omega_T) \cap C(\bar{\Omega}_T)$ satisfies $\mathcal{L}u \leq 0$ in Ω_T (with $c = 0$), then:*

$$\max_{\bar{\Omega}_T} u = \max_{\Gamma_T} u. \quad (7.6)$$

Theorem 7.11 (Strong parabolic maximum principle). *Under the above hypotheses, if u attains its maximum at a point $(x_0, t_0) \in \Omega_T$ with $t_0 > 0$, then u is constant in $\Omega \times [0, t_0]$.*

Remark 7.12. The fundamental difference from the elliptic case is that the parabolic maximum can only be attained on the parabolic boundary Γ_T (which does not include $t = T$). Time plays an asymmetric role: information propagates only into the future.

7.5 Sub- and supersolution method

Definition 7.13 (Subsolution, supersolution). A function $\underline{u} \in C^2(\Omega) \cap C(\bar{\Omega})$ is a **subsolution** of $Lu = f$ if $L\underline{u} \leq f$ in Ω and $\underline{u} \leq g$ on $\partial\Omega$. A **supersolution** \bar{u} satisfies the reversed inequalities.

Theorem 7.14 (Enclosure by sub/supersolutions). *If \underline{u} is a subsolution and \bar{u} a supersolution, then $\underline{u} \leq u \leq \bar{u}$ in $\bar{\Omega}$. If existence is established, the solution u is trapped between \underline{u} and \bar{u} .*

7.6 Maximum principle for nonlinear equations

Theorem 7.15 (Comparison principle for $-\Delta u = f(u)$). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be locally Lipschitz. If $-\Delta u \leq f(u)$ and $-\Delta v \geq f(v)$ in Ω , and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω , provided f is decreasing (or one has an a priori L^∞ estimate).*

7.7 Exercises

Exercise 7.1. Let u be harmonic in $B(0, 1) \subset \mathbb{R}^2$ with $u = x^2$ on $\partial B(0, 1)$. Show that $|u(x)| \leq 1$ in $B(0, 1)$. Determine u .

Exercise 7.2. Let $Lu = -\Delta u + u$ ($c = 1 > 0$). Show that if $Lu \leq 0$ and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω .

Exercise 7.3. Use the maximum principle to show that the solution of $-\Delta u = 1$ in $B(0, R)$ with $u = 0$ on ∂B satisfies $0 \leq u \leq R^2/(2n)$.

Exercise 7.4 (Hopf's lemma). Let u be harmonic in $B(0, 1)$ with $u < 1$ in $B(0, 1)$ and $u(e_1) = 1$ where $e_1 = (1, 0, \dots, 0)$. Show that $\frac{\partial u}{\partial r}(e_1) > 0$.

Exercise 7.5. Let u solve $u_t = u_{xx} + u$ on $(0, \pi) \times (0, T)$ with $u(0, t) = u(\pi, t) = 0$. Does the maximum principle apply directly? How can it be adapted? *Hint:* set $v = e^{-t}u$.

Exercise 7.6. Show by the comparison principle that the solution of $u_t = u_{xx}$, $u(0, t) = u(1, t) = 0$, $u(x, 0) = \sin(\pi x)$ satisfies $0 \leq u(x, t) \leq e^{-\pi^2 t}$ for all $t > 0$.

Exercise 7.7. Prove Liouville's theorem for harmonic functions using the maximum principle (without the gradient estimate). *Hint:* consider $v_R(x) = u(x) - u(0) - \varepsilon(R^2 - |x|^2)$.

Exercise 7.8. Let Ω be a bounded open subset of \mathbb{R}^n and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $-\Delta u + c(x)u = f$ with $c \geq 0$, $f \leq 0$, and $u = 0$ on $\partial\Omega$. Show that $u \leq 0$ in Ω .

Chapter 8

Fourier Transform and PDEs

In 1807, Joseph Fourier claimed that every function can be written as a sum of sinusoids. Lagrange was sceptical, Poisson doubtful, but the idea was too powerful to be ignored. The *Fourier transform*, the continuous generalization of Fourier series, converts a function of the spatial variable into a function of frequency. For PDEs with constant coefficients, this transformation is miraculous: derivatives become multiplications, partial differential equations become algebraic equations. The heat equation, the wave equation, the Schrödinger equation — all are solved elegantly by Fourier analysis on \mathbb{R}^n . This chapter develops the Fourier transform on L^1 and L^2 , establishes the inversion formula and Plancherel's theorem, then systematically applies them to classical PDEs.

8.1 Fourier transform on $L^1(\mathbb{R}^n)$

Definition 8.1 (Fourier transform). Let $f \in L^1(\mathbb{R}^n)$. The *Fourier transform* of f is the function $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$ defined by:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

Remark 8.2. There are several conventions for the Fourier transform. We use the symmetric convention with 2π in the exponential, which eliminates factors of $(2\pi)^{n/2}$ from the inversion formula.

Proposition 8.3 (Basic properties). Let $f, g \in L^1(\mathbb{R}^n)$:

1. **Linearity:** $\widehat{\alpha f + \beta g} = \alpha \hat{f} + \beta \hat{g}$.
2. **Translation:** $\widehat{f(\cdot - a)}(\xi) = e^{-2\pi i \langle a, \xi \rangle} \hat{f}(\xi)$.
3. **Modulation:** $\widehat{e^{2\pi i \langle a, \cdot \rangle} f}(\xi) = \hat{f}(\xi - a)$.
4. **Dilation:** $\widehat{f(\lambda \cdot)}(\xi) = \frac{1}{|\lambda|^n} \hat{f}(\xi/\lambda)$ for $\lambda \neq 0$.
5. **Convolution:** $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

Theorem 8.4 (Riemann-Lebesgue lemma). If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$ (continuous and vanishing at infinity): $\hat{f}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$.

Proof. For f the indicator of a rectangle, the result is a direct computation. By density of step functions in L^1 and continuity of \mathcal{F} (since $|\hat{f}(\xi)| \leq \|f\|_{L^1}$), the result extends to all $f \in L^1$. \square

Theorem 8.5 (Transform of derivatives). *If f and $\partial^\alpha f$ are in $L^1(\mathbb{R}^n)$, then:*

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi),$$

where α is a multi-index and $(2\pi i \xi)^\alpha = \prod_{j=1}^n (2\pi i \xi_j)^{\alpha_j}$.

Intuition

The Fourier transform converts derivatives into multiplication by polynomials. This is the fundamental reason for its effectiveness in solving linear PDEs with constant coefficients: a PDE becomes an algebraic equation in Fourier space.

8.2 Fourier transform on $L^2(\mathbb{R}^n)$

Theorem 8.6 (Plancherel). *The Fourier transform extends to an isometric isomorphism of $L^2(\mathbb{R}^n)$:*

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^n).$$

More generally, $\langle \hat{f}, \hat{g} \rangle_{L^2} = \langle f, g \rangle_{L^2}$ (Parseval's identity).

Proof sketch. One first proves the result for $f \in L^1 \cap L^2$ (a dense subset of L^2) by computing $\int |\hat{f}|^2$ via the convolution $f * \overline{f(-\cdot)}$. The result then extends by density and continuity. \square

Theorem 8.7 (Inversion formula). *If $f, \hat{f} \in L^1(\mathbb{R}^n)$, then:*

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \quad a.e.$$

8.3 Tempered distributions

Definition 8.8 (Schwartz space). The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ is the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ whose derivatives all decay faster than any polynomial:

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}^n.$$

Definition 8.9 (Tempered distribution). The space $\mathcal{S}'(\mathbb{R}^n)$ of *tempered distributions* is the topological dual of $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform extends to \mathcal{S}' by duality:

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \quad \forall T \in \mathcal{S}', \varphi \in \mathcal{S}.$$

Example 8.10 (Fourier transform of the Dirac mass). $\hat{\delta} = 1$, since $\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int \varphi(\xi) d\xi = \langle 1, \varphi \rangle$.

Example 8.11 (Fourier transform of the Gaussian). For $f(x) = e^{-\pi|x|^2}$, we have $\hat{f}(\xi) = e^{-\pi|\xi|^2}$. The Gaussian is a fixed point of the Fourier transform.

8.4 Solving the heat equation

Consider the Cauchy problem for the heat equation:

$$u_t = \kappa \Delta u, \quad x \in \mathbb{R}^n, \quad t > 0, \quad u(x, 0) = u_0(x). \quad (8.1)$$

Theorem 8.12 (Solution via Fourier transform). *The solution of (8.1) is given by:*

$$u(x, t) = (G_t * u_0)(x) = \int_{\mathbb{R}^n} G_t(x - y) u_0(y) dy,$$

where G_t is the heat kernel:

$$G_t(x) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp\left(-\frac{|x|^2}{4\kappa t}\right).$$

Proof. Applying the Fourier transform in x :

$$\hat{u}_t(\xi, t) = -4\pi^2\kappa |\xi|^2 \hat{u}(\xi, t).$$

This is an ODE in t , with solution $\hat{u}(\xi, t) = \hat{u}_0(\xi) e^{-4\pi^2\kappa|\xi|^2 t}$. Recognizing $e^{-4\pi^2\kappa|\xi|^2 t} = \hat{G}_t(\xi)$, we obtain $u = G_t * u_0$ by the convolution property. \square

Instantaneous propagation

The heat kernel satisfies $G_t(x) > 0$ for all x and all $t > 0$. Thus, even if u_0 has compact support, $u(x, t) > 0$ for all x as soon as $t > 0$: heat propagates at infinite speed.

8.5 Solving the wave equation

Theorem 8.13 (Solution of the wave equation in dimension n). *The Cauchy problem:*

$$u_{tt} = c^2 \Delta u, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x),$$

has the solution (in Fourier space):

$$\hat{u}(\xi, t) = \hat{u}_0(\xi) \cos(2\pi c |\xi| t) + \hat{v}_0(\xi) \frac{\sin(2\pi c |\xi| t)}{2\pi c |\xi|}.$$

Proof. The Fourier transform in x gives $\hat{u}_{tt} = -(2\pi c)^2 |\xi|^2 \hat{u}$. For each fixed ξ , this is a harmonic oscillator equation with frequency $\omega = 2\pi c |\xi|$. The general solution is $\hat{u}(\xi, t) = A(\xi) \cos(\omega t) + B(\xi) \sin(\omega t)$. Initial conditions give $A = \hat{u}_0$ and $B = \hat{v}_0/\omega$. \square

Example 8.14 (d'Alembert's formula ($n = 1$)). In dimension 1, inverting gives:

$$u(x, t) = \frac{u_0(x + ct) + u_0(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(s) ds.$$

8.6 Fundamental solutions

Definition 8.15 (Fundamental solution). The *fundamental solution* (or Green's function) of a linear operator L is the distribution E such that $LE = \delta$.

Example 8.16 (Laplacian in dimension $n \geq 3$). The fundamental solution of $-\Delta$ in \mathbb{R}^n ($n \geq 3$) is:

$$E(x) = \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}},$$

where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of S^{n-1} .

Fundamental solutions of classical PDEs

$$\text{Heat: } G_t(x) = \frac{1}{(4\pi\kappa t)^{n/2}} e^{-|x|^2/(4\kappa t)}, \quad t > 0$$

$$\text{Laplacian } (n = 3): \quad E(x) = \frac{1}{4\pi|x|}$$

$$\text{Laplacian } (n = 2): \quad E(x) = -\frac{1}{2\pi} \ln|x|$$

$$\text{Helmholtz } (n = 3): \quad E(x) = \frac{e^{ik|x|}}{4\pi|x|}$$

8.7 Exercises

Exercise 8.1. Compute the Fourier transform of $f(x) = e^{-a|x|}$ for $a > 0$ and $x \in \mathbb{R}$.

Exercise 8.2. Prove the convolution property: if $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

Exercise 8.3. Verify that the heat kernel G_t satisfies: (a) $\int_{\mathbb{R}^n} G_t(x) dx = 1$ for all $t > 0$; (b) $(\partial_t - \kappa\Delta)G_t = 0$ for $t > 0$; (c) $G_t \rightarrow \delta$ in the sense of distributions as $t \rightarrow 0^+$.

Exercise 8.4. Use the Fourier transform to solve the free Schrödinger equation: $i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi$, $\psi(x, 0) = \psi_0(x)$.

Exercise 8.5. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Show that if \hat{f} has compact support, then f extends to an entire function on \mathbb{C}^n (Paley-Wiener theorem).

Exercise 8.6. Solve the heat equation on \mathbb{R} with initial data $u_0(x) = \mathcal{K}_{[0,1]}(x)$ (indicator function). Express the solution using the error function erf.

Exercise 8.7. Show that the fundamental solution of the Laplacian in \mathbb{R}^n ($n \geq 3$) satisfies $-\Delta E = \delta$ in the sense of distributions, by computing ΔE for $x \neq 0$ and using the divergence theorem.

Chapter 9

Sobolev Spaces

Classical partial differential equations demand that solutions be smooth enough for derivatives to make pointwise sense. But many physical problems — vibrating membranes with corners, turbulent flows, composite materials — produce solutions that are not differentiable in the classical sense. Sergei Sobolev, in the 1930s, resolved this dilemma by introducing *weak derivatives*: a notion of differentiation that does not require pointwise regularity but only that the “integration by parts formula” hold in the distributional sense. The *Sobolev spaces* $W^{k,p}(\Omega)$, equipped with natural norms, are the function spaces where weak solutions of PDEs live. Today, all of modern PDE theory — variational methods, finite elements, regularity theory — rests on this framework.

9.1 Weak derivatives

Definition 9.1 (Weak derivative). Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1_{\text{loc}}(\Omega)$. We say that $v \in L^1_{\text{loc}}(\Omega)$ is the *weak derivative* $\partial^\alpha u$ (for a multi-index α) if:

$$\int_{\Omega} v \varphi \, dx = (-1)^{|\alpha|} \int_{\Omega} u \partial^\alpha \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

Intuition

The weak derivative generalizes the classical derivative to functions that are not necessarily differentiable in the usual sense. The formula is based on integration by parts: if u is C^k , the weak derivative coincides with the classical one. The key point is that a weak derivative can exist even if u has singularities.

Example 9.2 (Absolute value). The function $u(x) = |x|$ on \mathbb{R} has weak derivative $u'(x) = \text{sgn}(x)$ (the sign function). Indeed, for all $\varphi \in C_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} |x| \varphi'(x) \, dx = - \int_{\mathbb{R}} \text{sgn}(x) \varphi(x) \, dx.$$

Example 9.3 (Heaviside function). The Heaviside function $H(x) = \mathcal{H}_{(0,\infty)}(x)$ does not have a weak derivative in $L^1_{\text{loc}}(\mathbb{R})$: its distributional derivative is the Dirac mass δ_0 , which is not a function.

9.2 Sobolev spaces $W^{k,p}$

Definition 9.4 (Sobolev space $W^{k,p}(\Omega)$). Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ is:

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k\},$$

equipped with the norm:

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p}^p \right)^{1/p} \quad (1 \leq p < \infty).$$

For $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$.

Theorem 9.5 (Completeness). *The space $W^{k,p}(\Omega)$ is a Banach space. For $p = 2$, $H^k(\Omega)$ is a Hilbert space with inner product:*

$$\langle u, v \rangle_{H^k} = \sum_{|\alpha| \leq k} \int_{\Omega} \partial^\alpha u \cdot \overline{\partial^\alpha v} \, dx.$$

Definition 9.6 (Space $W_0^{k,p}(\Omega)$). The space $W_0^{k,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Functions in $W_0^{k,p}$ are those that “vanish on the boundary” in the sense of traces. We write $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

9.3 Density of smooth functions

Theorem 9.7 (Meyers-Serrin). *The space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for $1 \leq p < \infty$.*

Proof sketch. One uses a partition of unity $\{\rho_j\}$ and convolution regularizations $(\rho_j u) * \eta_{\varepsilon_j}$ with mollifiers η_ε . By choosing ε_j small enough for each j , one approximates u in $W^{k,p}$. \square

Theorem 9.8 (Density of $C^\infty(\overline{\Omega})$). *If Ω is a bounded open set with C^1 boundary, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for $1 \leq p < \infty$.*

9.4 Sobolev embedding theorems

Theorem 9.9 (Sobolev embedding). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $k \in \mathbb{N}^*$ and $1 \leq p < \infty$.*

1. If $kp < n$: $W^{k,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $p^* = \frac{np}{n-kp}$ (Sobolev exponent).
2. If $kp = n$: $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [p, \infty)$.
3. If $kp > n$: $W^{k,p}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$ with $\gamma = k - n/p$ if $k - n/p < 1$.

The critical exponent

The embedding $W^{1,p} \hookrightarrow L^{p^*}$ is *continuous* but *not compact*. The Rellich-Kondrachov theorem gives compactness for exponents strictly less than p^* .

Theorem 9.10 (Rellich-Kondrachov). *Let Ω be a bounded open set with Lipschitz boundary.*

1. *If $kp < n$: $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < p^*$ (compact embedding).*
2. *If $kp > n$: $W^{k,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ (compact embedding).*

Example 9.11 (Case $n = 1$). In dimension 1, for $p = 2$ and $k = 1$: $kp = 2 > n = 1$, so $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$. Every H^1 function in dimension 1 is continuous (after modification on a null set).

Example 9.12 (Case $n = 3$, $k = 1$, $p = 2$). $kp = 2 < n = 3$, so $p^* = \frac{3 \cdot 2}{3-2} = 6$. We have $H^1(\Omega) \hookrightarrow L^6(\Omega)$ (continuous) and $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $q < 6$ (compact).

9.5 Trace theorem

Theorem 9.13 (Trace). *Let Ω be a bounded open set with C^1 boundary. The restriction operator $\gamma_0 : C^\infty(\bar{\Omega}) \rightarrow C^\infty(\partial\Omega)$, $\gamma_0(u) = u|_{\partial\Omega}$, extends to a continuous surjective linear operator:*

$$\gamma_0 : W^{1,p}(\Omega) \rightarrow W^{1-1/p,p}(\partial\Omega), \quad 1 < p < \infty.$$

Moreover, $\ker \gamma_0 = W_0^{1,p}(\Omega)$.

Remark 9.14. This theorem justifies the Dirichlet condition $u = g$ on $\partial\Omega$ in the Sobolev framework: one requires $\gamma_0(u) = g$ in the trace sense, even if u is not continuous.

9.6 Poincaré inequality

Theorem 9.15 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 \leq p < \infty$. There exists a constant $C = C(\Omega, p, n) > 0$ such that:*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega).$$

Proof. By contradiction. Suppose there exists a sequence (u_k) in $W_0^{1,p}(\Omega)$ with $\|u_k\|_{L^p} = 1$ and $\|\nabla u_k\|_{L^p} \rightarrow 0$. By Rellich-Kondrachov, (u_k) has a subsequence converging in L^p . The limit u satisfies $\nabla u = 0$ a.e., so u is constant. Since $u \in W_0^{1,p}$, we have $u = 0$, contradicting $\|u\|_{L^p} = 1$. \square

Corollary 9.16. *On $W_0^{1,p}(\Omega)$ with Ω bounded, the seminorm $\|\nabla u\|_{L^p}$ is equivalent to the full norm $\|u\|_{W^{1,p}}$.*

Theorem 9.17 (Poincaré-Wirtinger inequality). *Let Ω be a bounded connected open set with Lipschitz boundary. For $1 \leq p < \infty$:*

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega),$$

where $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u \, dx$ is the mean of u .

Summary of Sobolev embeddings ($k = 1$)

Condition	Embedding	Type
$p < n$	$W^{1,p} \hookrightarrow L^{np/(n-p)}$	continuous
$p < n$	$W^{1,p} \hookrightarrow L^q, q < \frac{np}{n-p}$	compact
$p = n$	$W^{1,p} \hookrightarrow L^q, \forall q < \infty$	continuous
$p > n$	$W^{1,p} \hookrightarrow C^{0,1-n/p}$	continuous
$p > n$	$W^{1,p} \hookrightarrow C(\overline{\Omega})$	compact

9.7 Exercises

Exercise 9.1. Show that $u(x) = |x|^{-\alpha}$ belongs to $W^{1,p}(B(0,1))$ (unit ball in \mathbb{R}^n) if and only if $(\alpha + 1)p < n$.

Exercise 9.2. Let $\Omega = (0,1)$ and $u(x) = x^\beta$ for $\beta > 0$. Determine for which values of β and k we have $u \in H^k(\Omega)$.

Exercise 9.3. Prove the Sobolev inequality in dimension 1: for $u \in W^{1,1}(\mathbb{R})$, $\|u\|_{L^\infty} \leq \frac{1}{2} \|u'\|_{L^1}$.

Exercise 9.4. Show that $H^1(\mathbb{R}^n)$ does not embed into $L^\infty(\mathbb{R}^n)$ for $n \geq 2$. *Hint:* construct a logarithmic counterexample.

Exercise 9.5. Let Ω be a bounded open subset of \mathbb{R}^n with smooth boundary. Show that the Poincaré constant C_P satisfies $C_P \leq \frac{\text{diam}(\Omega)}{2}$ when $p = 2$.

Exercise 9.6. Show that if $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$ satisfy $-\Delta u = f$ in the weak sense, then $\|u\|_{H_0^1} \leq C \|f\|_{L^2}$ with a constant depending only on Ω .

Chapter 10

Variational Methods

10.1 Weak formulation of elliptic problems

Definition 10.1 (Associated bilinear form). Consider the Dirichlet problem for a second-order elliptic PDE:

$$-\sum_{i,j=1}^n \partial_j(a_{ij}(x)\partial_i u) + c(x)u = f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (10.1)$$

The *weak formulation* consists of finding $u \in H_0^1(\Omega)$ such that:

$$a(u, v) = \ell(v) \quad \forall v \in H_0^1(\Omega),$$

where the bilinear form a and the linear form ℓ are defined by:

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j} a_{ij} \partial_i u \partial_j v + c u v \right) dx,$$
$$\ell(v) = \int_{\Omega} f v dx.$$

Intuition

The weak formulation is obtained by multiplying the PDE by a test function $v \in H_0^1(\Omega)$ and integrating by parts. The advantage is twofold: the order of differentiation required on u is reduced (from 2 to 1), and one obtains a functional framework where theorems of functional analysis apply directly.

Definition 10.2 (Coercivity and continuity). Let V be a Hilbert space. A bilinear form $a : V \times V \rightarrow \mathbb{R}$ is:

- **continuous** if there exists $M > 0$ such that $|a(u, v)| \leq M \|u\|_V \|v\|_V$;
- **coercive** if there exists $\alpha > 0$ such that $a(u, u) \geq \alpha \|u\|_V^2$.

10.2 The Lax-Milgram theorem

Theorem 10.3 (Lax-Milgram). Let V be a Hilbert space, $a : V \times V \rightarrow \mathbb{R}$ a continuous and coercive bilinear form, and $\ell : V \rightarrow \mathbb{R}$ a continuous linear form. Then there exists a

unique $u \in V$ such that:

$$a(u, v) = \ell(v) \quad \forall v \in V.$$

Moreover, $\|u\|_V \leq \frac{1}{\alpha} \|\ell\|_{V'}$.

Proof. Step 1: Riesz representation. By the Riesz representation theorem, for each fixed $u \in V$, there exists a unique $Au \in V$ such that $a(u, v) = \langle Au, v \rangle_V$ for all v . The map $A : V \rightarrow V$ is linear and continuous with $\|A\| \leq M$.

Similarly, there exists $F \in V$ such that $\ell(v) = \langle F, v \rangle_V$.

The problem reduces to: find u such that $Au = F$.

Step 2: A is invertible. Coercivity gives $\langle Au, u \rangle = a(u, u) \geq \alpha \|u\|^2$, so A is injective and $\|Au\| \geq \alpha \|u\|$. The image $\text{Im}(A)$ is closed (since A is bounded below). If $w \perp \text{Im}(A)$, then $\langle Aw, w \rangle = 0$, hence $w = 0$ by coercivity. Thus $\text{Im}(A) = V$ and A is bijective. The unique solution is $u = A^{-1}F$ with $\|u\| \leq \frac{1}{\alpha} \|F\| = \frac{1}{\alpha} \|\ell\|_{V'}$. \square

Symmetry not required

The Lax-Milgram theorem does not require a to be symmetric. This is an advantage over the Riesz theorem, which would require symmetry.

10.3 Application to the Dirichlet problem

Theorem 10.4 (Existence and uniqueness for the Dirichlet problem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f \in L^2(\Omega)$, and suppose the coefficients $a_{ij} \in L^\infty(\Omega)$ satisfy the uniform ellipticity condition:*

$$\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega,$$

and $c(x) \geq 0$ a.e. Then problem (10.1) has a unique weak solution $u \in H_0^1(\Omega)$.

Proof. We verify the hypotheses of Lax-Milgram with $V = H_0^1(\Omega)$.

Continuity of a : $|a(u, v)| \leq \|a_{ij}\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \leq M \|u\|_{H^1} \|v\|_{H^1}$.

Coercivity: $a(u, u) \geq \lambda \|\nabla u\|_{L^2}^2 \geq \frac{\lambda}{1+C_P^2} \|u\|_{H^1}^2$ by the Poincaré inequality (where C_P is the Poincaré constant).

Continuity of ℓ : $|\ell(v)| = |\int f v| \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} \|v\|_{H^1}$.

By Lax-Milgram, there exists a unique solution $u \in H_0^1(\Omega)$. \square

Example 10.5 (Laplacian with source). For $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$: $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ and $\ell(v) = \int_\Omega f v \, dx$. The hypotheses are satisfied with $\lambda = 1$ and $c = 0$.

10.4 The Ritz-Galerkin method

Definition 10.6 (Galerkin approximation). Let $V_h \subset V$ be a finite-dimensional subspace. The *Galerkin approximation* of u is the unique $u_h \in V_h$ such that:

$$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h.$$

Theorem 10.7 (Céa's lemma). *If a is continuous (constant M) and coercive (constant α), then:*

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V.$$

Proof. By coercivity and the Galerkin equation (orthogonality of the error):

$$\begin{aligned} \alpha \|u - u_h\|^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\ &= a(u - u_h, u - v_h) \quad (\text{since } a(u - u_h, w_h) = 0 \text{ for } w_h \in V_h) \\ &\leq M \|u - u_h\| \|u - v_h\|. \end{aligned}$$

Dividing by $\|u - u_h\|$ and taking the infimum over v_h gives the result. \square

Remark 10.8. Céa's lemma shows that the Galerkin approximation is quasi-optimal: the error is at most M/α times the best possible approximation in V_h .

10.5 Variational principle (symmetric case)

Theorem 10.9 (Variational equivalence). *If a is symmetric, continuous and coercive, then u solves $a(u, v) = \ell(v)$ for all $v \in V$ if and only if u minimizes the energy functional:*

$$J(v) = \frac{1}{2}a(v, v) - \ell(v).$$

Proof. For any $v \in V$: $J(v) = J(u) + \frac{1}{2}a(v - u, v - u) + a(u, v - u) - \ell(v - u)$. If u is a solution, the last two terms cancel, giving $J(v) = J(u) + \frac{1}{2}a(v - u, v - u) \geq J(u)$ by coercivity. Conversely, if u minimizes J , then $\frac{d}{dt}J(u + tv)|_{t=0} = 0$ gives $a(u, v) = \ell(v)$. \square

Example 10.10 (Dirichlet principle). For the problem $-\Delta u = f$ with $u \in H_0^1(\Omega)$, the energy functional is:

$$J(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} f v dx.$$

Minimizing J is equivalent to solving $-\Delta u = f$.

10.6 Regularity of weak solutions

Theorem 10.11 (H^2 regularity). *Let Ω be a bounded open set with C^2 boundary and $u \in H_0^1(\Omega)$ the weak solution of $-\Delta u = f$ with $f \in L^2(\Omega)$. Then $u \in H^2(\Omega)$ and:*

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Theorem 10.12 (C^∞ regularity). *If $f \in C^\infty(\bar{\Omega})$ and $\partial\Omega$ is C^∞ , then the weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ satisfies $u \in C^\infty(\bar{\Omega})$.*

Remark 10.13. Regularity depends on the smoothness of the boundary and the right-hand side. If $\partial\Omega$ is not smooth enough (e.g., a domain with corners), H^2 regularity may fail.

Summary of the variational method

1. Write the weak formulation: $a(u, v) = \ell(v), \forall v \in V$.
2. Verify continuity of a and ℓ .
3. Verify coercivity of a (often via Poincaré).
4. Apply Lax-Milgram \Rightarrow existence and uniqueness.
5. Study regularity: $u \in H^2?$ $u \in C^\infty?$

10.7 Exercises

Exercise 10.1. Write the weak formulation of the problem: $-\operatorname{div}(A(x)\nabla u) + u = f$ in Ω , $u = 0$ on $\partial\Omega$, where $A(x)$ is a uniformly positive definite symmetric matrix. Verify the hypotheses of Lax-Milgram.

Exercise 10.2. Show that for the Neumann problem: $-\Delta u = f$ in Ω , $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, the compatibility condition $\int_{\Omega} f \, dx = 0$ is necessary. Formulate the weak problem in $H^1(\Omega)$ and apply Lax-Milgram in the quotient H^1/\mathbb{R} .

Exercise 10.3. Consider the Robin problem: $-\Delta u + u = f$ in Ω , $\frac{\partial u}{\partial \nu} + \alpha u = 0$ on $\partial\Omega$ with $\alpha > 0$. Write the weak formulation in $H^1(\Omega)$ and prove existence and uniqueness.

Exercise 10.4 (Céa's lemma in practice). Let $\Omega = (0, 1)$, $-u'' = f$ with $u(0) = u(1) = 0$. Take V_h to be the space of piecewise linear functions on a uniform mesh of size h . Show that $\|u - u_h\|_{H^1} \leq Ch \|u\|_{H^2}$.

Exercise 10.5. Show that if a is symmetric, the Galerkin approximation u_h minimizes $J(v) = \frac{1}{2}a(v, v) - \ell(v)$ over V_h . Deduce that the energy error satisfies $a(u - u_h, u - u_h) \leq a(u - v_h, u - v_h)$ for all $v_h \in V_h$.

Exercise 10.6. Let Ω be an L-shaped domain in \mathbb{R}^2 . Give an example of data $f \in L^2(\Omega)$ for which the solution of $-\Delta u = f$ is not in $H^2(\Omega)$. Explain the role of the corner singularity.

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