Ordinary Differential Equations

Lecture Notes

Licence L3 — 2025–2026

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"Nature is written in the language of mathematics."
— Galileo Galilei

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Preface

Ordinary differential equations are among the most fundamental and widely applied tools in all of mathematics. They arise naturally whenever a quantity changes at a rate that depends on its current state: the swing of a pendulum, the growth of a population, the discharge of a capacitor, the trajectory of a planet. Understanding how to formulate, solve, and analyse such equations is essential for students of mathematics, physics, engineering, biology, and economics alike.

This textbook is intended for a one-semester or two-semester course at the advanced undergraduate level. The only prerequisites are a solid grounding in single-variable and multivariable calculus, and a first course in linear algebra. Some familiarity with basic real analysis is helpful but not strictly required; all essential analytic arguments are presented in full.

The book is structured as follows. Chapters 1 and 2 introduce the vocabulary of the subject and develop systematic techniques for solving first-order equations. Chapters 3 and 4 treat higher-order linear equations and the Laplace transform. Chapters 5 and 6 address systems of equations and qualitative theory. Later chapters cover power series methods, boundary value problems, and an introduction to partial differential equations.

Each chapter contains:

- Precise definitions and clearly stated theorems with complete proofs.
- Numerous worked examples illustrating every technique.
- Graphical illustrations produced with TikZ and `pgfplots`.
- Exercises graded by difficulty: one star for routine practice, two stars for intermediate problems, and three stars for challenging or theoretical questions.
- A chapter summary for efficient review.

We have striven to balance rigour with accessibility. Every major existence and uniqueness theorem is proved, but the emphasis throughout is on understanding *why* the methods work, not merely on applying recipes.

The authors, 2026

Contents

Chapter 1

Introduction and Modelling — What Is an ODE?

1.1 Why Differential Equations?

Many of the fundamental laws of nature are not expressed as direct relationships between quantities, but as relationships between quantities and their *rates of change.* Newton's second law, for instance, does not tell us the position of a particle directly; it tells us how the acceleration (the second derivative of position) relates to the forces acting on the particle. To find the actual motion, we must solve a differential equation.

In this introductory chapter we survey several real-world models that lead to ordinary differential equations, establish the basic terminology and classification of such equations, describe what it means to "solve" an ODE, and introduce the geometric viewpoint of direction fields. By the end of the chapter the reader should have a clear picture of the landscape we shall explore in the rest of the book.

1.2 Real-World Models

1.2.1 The Simple Pendulum

Consider a point mass m suspended from a rigid, massless rod of length ℓ, free to swing in a vertical plane. Let $\theta(t)$ denote the angle the rod makes with the downward vertical at time t. Balancing the tangential component of gravity against the tangential acceleration gives

$$m\ell\ddot{\theta} = -mg\sin\theta, \tag{1.1}$$

or equivalently,

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0. \tag{1.2}$$

This is a *second-order, nonlinear* ordinary differential equation in the unknown function $\theta(t)$. For small angles one may approximate $\sin\theta \approx \theta$, yielding the linearised equation

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0, \tag{1.3}$$

whose solutions are sinusoidal oscillations of period $T = 2\pi\sqrt{\ell/g}$.

1.2.2 Population Growth

Let $P(t)$ be the size of a population at time t. The simplest model assumes that the rate of growth is proportional to the current population:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP, \qquad P(0) = P_0, \tag{1.4}$$

where $r > 0$ is the intrinsic growth rate. This is the *Malthusian* (exponential) growth model, with solution $P(t) = P_0 e^{rt}$.

A more realistic model accounts for limited resources by introducing a carrying capacity K:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right). \tag{1.5}$$

This is the *logistic equation*, a first-order, nonlinear ODE. Its solution is

$$P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right) e^{-rt}}.$$

1.2.3 Radioactive Decay

A sample of a radioactive isotope decays at a rate proportional to the amount present:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N, \qquad N(0) = N_0, \tag{1.6}$$

where $\lambda > 0$ is the decay constant. The solution $N(t) = N_0 e^{-\lambda t}$ gives the well-known exponential decay law. The *half-life* is $t_{1/2} = \ln 2/\lambda$.

1.2.4 RC Circuits

Consider a resistor R in series with a capacitor C and an applied voltage $E(t)$. By Kirchhoff's voltage law, the charge $q(t)$ on the capacitor satisfies

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = E(t). \tag{1.7}$$

This is a *first-order, linear* ODE. If $E(t) = E_0$ is constant and $q(0) = 0$, the solution is

$$q(t) = CE_0\left(1 - e^{-t/(RC)}\right).$$

The product $\tau = RC$ is the *time constant* of the circuit.

1.3 Basic Definitions

Definition 1.1 (Ordinary Differential Equation)**.** An *ordinary differential equation* (ODE) is an equation involving an unknown function $y = y(t)$ of a single independent variable t and one or more of its derivatives $y', y'', \ldots, y^{(n)}$. The general form is

$$F\left(t, y, y', y'', \ldots, y^{(n)}\right) = 0, \tag{1.8}$$

where F is a given function.

Definition 1.2 (Order of an ODE)**.** The *order* of an ODE is the order of the highest derivative that appears in the equation. Equation (1.8) is of order n if F depends non-trivially on $y^{(n)}$.

Example 1.3. The equation $y' + 2y = e^t$ is first-order. The pendulum equation (1.2) is second-order. The equation $y''' - 3y'' + y = 0$ is third-order.

Definition 1.4 (Degree of an ODE)**.** When an ODE can be written as a polynomial in the highest-order derivative, the *degree* of the ODE is the power of that highest derivative. For instance, $(y'')^3 + y' = t$ has order 2 and degree 3.

Definition 1.5 (Linear ODE)**.** An ODE of order n is *linear* if it can be written in the form

$$a_n(t)\, y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0(t)\, y = g(t), \tag{1.9}$$

where the coefficients $a_0, a_1, \ldots, a_n$ and the right-hand side g are given functions of t alone. An ODE that is not linear is called *nonlinear*.

Remark 1.6. Linearity requires that the unknown y and all its derivatives appear to the first power and are not multiplied together. Thus $y'' + y^2 = 0$ is nonlinear (because of y^2), and $yy' = 1$ is nonlinear (because of the product yy').

Definition 1.7 (Autonomous ODE)**.** An ODE is *autonomous* if the independent variable t does not appear explicitly. In first-order form: $y' = f(y)$. The logistic equation (1.5) and the pendulum equation (1.2) are both autonomous.

Definition 1.8 (Homogeneous Linear ODE)**.** A linear ODE (1.9) is *homogeneous* if $g(t) \equiv 0$; otherwise it is *non-homogeneous*.

1.4 Solution Concepts

Definition 1.9 (Solution of an ODE)**.** A *solution* of the ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ on an interval $I \subset \mathbb{R}$ is a function $\varphi\colon I \to \mathbb{R}$ that is n times differentiable and satisfies

$$F\Big(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n)}(t)\Big) = 0 \quad \text{for all } t \in I.$$

Definition 1.10 (General Solution)**.** The *general solution* of an n-th order ODE is a family of solutions containing n arbitrary constants $c_1, c_2, \ldots, c_n$, such that (under appropriate conditions) every solution can be obtained by choosing particular values of these constants.

Definition 1.11 (Particular Solution)**.** A *particular solution* is any single solution obtained from the general solution by assigning specific values to the arbitrary constants.

Definition 1.12 (Singular Solution)**.** A *singular solution* is a solution that cannot be obtained from the general solution for any choice of the constants. Singular solutions typically arise as envelopes of the family of curves defined by the general solution.

Example 1.13. The ODE $(y')^2 = 4y$ has the general solution $y = (t-c)^2$ for $c \in \mathbb{R}$, representing a family of parabolas. The function $y \equiv 0$ is also a solution, but it is not a member of this family for any finite c. It is a singular solution and is, geometrically, the envelope of the family of parabolas.

1.5 Initial Value Problems

Definition 1.14 (Initial Value Problem)**.** An *initial value problem* (IVP) for an n-th order ODE consists of the equation together with n initial conditions:

$$\begin{cases} y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}), \\ y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \ldots, \quad y^{(n-1)}(t_0) = y_{n-1}. \end{cases} \tag{1.10}$$

The point $(t_0, y_0, y_1, \ldots, y_{n-1})$ specifies the initial state of the system.

Example 1.15. The radioactive decay problem $N' = -\lambda N$, $N(0) = N_0$ is an IVP. Given the initial amount N_0, there is exactly one solution: $N(t) = N_0 e^{-\lambda t}$.

We state without proof the central existence and uniqueness theorem; a complete proof will be given in Chapter 5 when we have the tools of the contraction mapping principle at our disposal.

Theorem 1.16 (Picard–Lindelöf)**.** *Consider the IVP*

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

If f is continuous on an open rectangle $R = \{(t, y) : |t - t_0| < a,\ |y - y_0| < b\}$ and satisfies a Lipschitz condition in y, i.e., there exists $L > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \le L\,|y_1 - y_2| \quad \textit{for all } (t, y_1), (t, y_2) \in R,$$

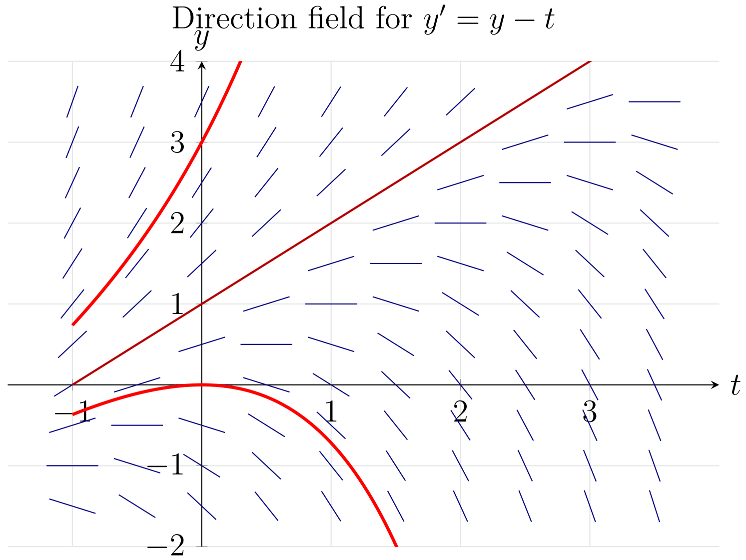
then there exists $\delta > 0$ and a unique function $\varphi\colon (t_0 - \delta,\ t_0 + \delta) \to \mathbb{R}$ satisfying the IVP.

Remark 1.17. If $\partial f/\partial y$ exists and is continuous on R, then f is automatically Lipschitz in y on any compact subset, so the theorem applies. The condition $|\partial f/\partial y| \leq L$ on R suffices.

1.6 Direction Fields and Integral Curves

For a first-order ODE $y' = f(t, y)$, the function f assigns a slope to every point (t, y) in its domain. By drawing short line segments of slope $f(t, y)$ at a grid of points, we obtain a *direction field* (also called a *slope field*). A solution curve, called an *integral curve*, is a curve $y = \varphi(t)$ that is tangent to the direction field at every point.

Direction field for $y' = y - t$



The direction field provides qualitative insight into the behaviour of solutions even when an explicit formula is unavailable. In the figure above, the red curves are integral curves of $y' = y - t$; note how they follow the pattern set by the short blue line segments.

Remark 1.18. By the Picard–Lindelöf theorem, if f satisfies the Lipschitz condition, integral curves cannot cross. Two distinct solution curves may approach one another but never intersect.

1.7 Classification of ODEs

We summarise the main classification criteria in the following table.

Criterion	**Types**	**Example**
Order	First, second, ..., n-th	$y' = 2y$ (first), $y'' + y = 0$ (second)
Linearity	Linear / Nonlinear	$y'' + y = t$ (linear), $y'' + y^2 = 0$ (nonlinear)
Autonomy	Autonomous / Non-autonomous	$y' = y^2$ (auto.), $y' = ty$ (non-auto.)
Homogeneity	Homogeneous / Non-homogeneous	$y'' + y = 0$ (hom.), $y'' + y = \sin t$ (non-hom.)
Coefficients	Constant / Variable	$y'' + 3y' + 2y = 0$ (const.), $ty'' + y = 0$ (var.)

1.8 Historical Context

The study of differential equations began in the late seventeenth century, immediately after the independent invention of calculus by Newton and Leibniz. Key milestones include:

- **1690s**: The Bernoulli brothers, Jakob and Johann, solved several families of first-order equations, including separable and what are now called Bernoulli equations.
- **1739**: Euler developed systematic methods for linear equations with constant coefficients and introduced the exponential ansatz.
- **1760s–1780s**: Lagrange introduced variation of parameters and studied singular solutions.
- **1820s–1830s**: Cauchy gave the first rigorous existence proofs using the method of successive approximations.
- **1890**: Picard refined Cauchy's method into the iteration scheme that bears his name, and Lindelöf sharpened the uniqueness conditions.
- **1880s–1910s**: Poincaré and Lyapunov created the qualitative theory of ODEs, shifting focus from finding explicit formulas to understanding the geometric and stability properties of solutions.

1.9 Exercises

Exercise 1.1 (★). Classify each of the following ODEs by order, linearity, and whether they are autonomous:

(a) $y' = 3y + \sin t$

(b) $y'' + y' + y^3 = 0$

(c) $(y')^2 + y = t$

(d) $y''' - 2y'' + y' = e^t$

(e) $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \sin\theta = 0$

Exercise 1.2 (★). Verify that $y(t) = Ce^{-2t} + \frac{1}{3}e^t$ is the general solution of $y' + 2y = e^t$.

Exercise 1.3 (★). Solve the IVP $y' = 3y$, $y(0) = 5$, and determine the interval of existence.

Exercise 1.4 (★★). Show that $y \equiv 0$ is a singular solution of $(y')^2 = 4y$. Find the general solution and verify that $y = 0$ is the envelope of the family.

Exercise 1.5 (★★). An RC circuit has $R = 100\,\Omega$, $C = 10^{-3}\,\mathrm{F}$, and is connected to a constant voltage $E_0 = 12\,\mathrm{V}$ at time $t = 0$ with no initial charge. Write the IVP for $q(t)$, solve it, and find the time at which the charge reaches 90% of its final value.

Exercise 1.6 (★★). A population satisfies the logistic equation with $r = 0.5$, $K = 1000$, and $P(0) = 50$. Find $P(t)$ explicitly and determine the time at which $P = 500$.

Exercise 1.7 (★★). Sketch (by hand or using software) the direction field for $y' = t^2 - y$ on $[-2, 2] \times [-2, 4]$. Draw at least three integral curves starting from different initial conditions. What long-term behaviour do you observe?

Exercise 1.8 (★★★). Let $f(t, y) = \sqrt{|y|}$ on $\mathbb{R}^2$. Show that f does not satisfy a Lipschitz condition in y near $y = 0$. Find two distinct solutions of the IVP $y' = \sqrt{|y|}$, $y(0) = 0$, and explain why this does not contradict the Picard–Lindelöf theorem.

Exercise 1.9 (★★★). For the nonlinear pendulum equation $\ddot{\theta} + \omega^2 \sin\theta = 0$ (with $\omega^2 = g/\ell$), define the energy

$$E(\theta, \dot{\theta}) = \tfrac{1}{2}\dot{\theta}^2 - \omega^2 \cos\theta.$$

Show that E is constant along solutions. Sketch the level curves of E in the $(\theta, \dot{\theta})$-plane and identify the equilibria and their stability.

Chapter Summary

- An **ODE** relates an unknown function of one variable to its derivatives.
- ODEs are classified by **order**, **linearity**, **autonomy**, and **homogeneity**.
- The **general solution** of an n-th order ODE contains n arbitrary constants; a **particular solution** results from fixing these constants; a **singular solution** lies outside the general family.
- An **initial value problem** specifies n conditions at a single point and, under the hypotheses of the **Picard–Lindelöf theorem**, has a unique local solution.
- **Direction fields** give qualitative insight into the behaviour of solutions.
- ODEs arise naturally in physics (mechanics, circuits), biology (population dynamics), chemistry (reaction rates), and many other fields.

Chapter 2

First-Order Equations

In this chapter we develop systematic methods for solving the most important classes of first-order ordinary differential equations. The general first-order ODE has the form

$$y' = f(t, y), \tag{2.1}$$

or equivalently $M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0$. While no single technique covers all cases, the methods presented here handle a very large proportion of the equations that arise in applications.

2.1 Separable Equations

Definition 2.1 (Separable Equation)**.** A first-order ODE is *separable* if it can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)\,h(y), \tag{2.2}$$

where g depends only on t and h depends only on y.

Method 2.2 (Separation of Variables). If $h(y) \neq 0$, we may rewrite the equation as

$$\frac{\mathrm{d}y}{h(y)} = g(t)\,\mathrm{d}t,$$

and integrate both sides:

$$\int \frac{\mathrm{d}y}{h(y)} = \int g(t)\,\mathrm{d}t + C.$$

The resulting implicit or explicit relation defines the solution.

Remark 2.3. Any constant y_0 satisfying $h(y_0) = 0$ gives an equilibrium solution $y(t) \equiv y_0$. Such solutions may be singular; they must always be checked separately.

Example 2.4. Solve $y' = ty^2$.
Solution. Separating variables (for $y \neq 0$):

$$\frac{\mathrm{d}y}{y^2} = t\,\mathrm{d}t \qquad \Longrightarrow \qquad -\frac{1}{y} = \frac{t^2}{2} + C \qquad \Longrightarrow \qquad y = \frac{-2}{t^2 + C_1},$$

where $C_1 = 2C$. Additionally, $y \equiv 0$ is a constant solution.

Example 2.5. Solve the logistic equation $P' = rP(1 - P/K)$, $P(0) = P_0$, with $0 < P_0 < K$.
Solution. Separating variables:

$$\frac{\mathrm{d}P}{P(1-P/K)} = r\,\mathrm{d}t.$$

Partial fractions give

$$\frac{1}{P(1-P/K)} = \frac{1}{P} + \frac{1/K}{1-P/K} = \frac{1}{P} + \frac{1}{K-P}.$$

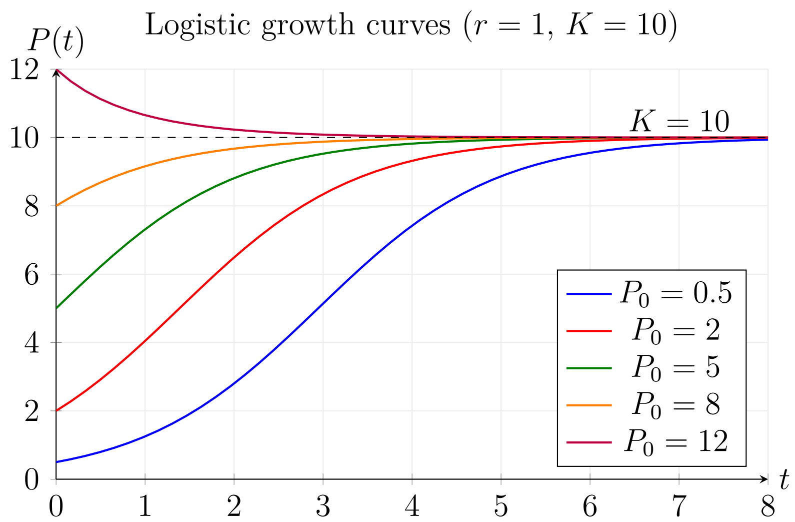
Integrating:

$$\ln|P| - \ln|K-P| = rt + C_0.$$

Applying $P(0) = P_0$ yields $C_0 = \ln\left(\frac{P_0}{K-P_0}\right)$. Solving for P:

$$P(t) = \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right)e^{-rt}}. \tag{2.3}$$

As $t \to \infty$, $P(t) \to K$: the population approaches the carrying capacity.



2.2 Linear First-Order Equations

Definition 2.6 (Linear First-Order ODE)**.** A *linear first-order ODE* has the standard form

$$y' + p(t)\,y = q(t), \tag{2.4}$$

where p and q are continuous functions on an interval I.

Method 2.7 (Integrating Factor). Define the *integrating factor*

$$\mu(t) = e^{\int p(t)\,\mathrm{d}t}. \tag{2.5}$$

Multiplying both sides of (2.4) by $\mu(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\mu(t)\,y\big] = \mu(t)\,q(t).$$

Integrating:

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\,q(t)\,\mathrm{d}t + C\right]. \tag{2.6}$$

Theorem 2.8 (Existence and Uniqueness for Linear First-Order ODEs)**.** *If p and q are continuous on an open interval I containing t_0, then the IVP*

$$y' + p(t)y = q(t), \qquad y(t_0) = y_0,$$

has a unique solution defined on the entire interval I.

Proof. The integrating factor $\mu(t) = e^{\int_{t_0}^{t} p(s)\,\mathrm{d}s}$ is well-defined, positive, and differentiable on I. Multiplying the equation by μ and integrating from t_0 to t gives

$$\mu(t)y(t) - y_0 = \int_{t_0}^{t} \mu(s)\,q(s)\,\mathrm{d}s,$$

so

$$y(t) = \frac{1}{\mu(t)}\left[y_0 + \int_{t_0}^{t} \mu(s)\,q(s)\,\mathrm{d}s\right].$$

This formula defines a unique continuous function on all of I, and direct differentiation confirms it satisfies the ODE. Uniqueness follows because if y_1 and y_2 are two solutions with the same initial condition, then $w = y_1 - y_2$ satisfies $w' + p(t)w = 0$, $w(t_0) = 0$, whence $w(t) = 0 \cdot e^{-\int_{t_0}^{t} p(s)\,\mathrm{d}s} = 0$. □

Example 2.9. Solve $y' + 2ty = t$, $y(0) = 1$.
Solution. Here $p(t) = 2t$, $q(t) = t$. The integrating factor is

$$\mu(t) = e^{\int 2t\,\mathrm{d}t} = e^{t^2}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[e^{t^2} y\big] = t\,e^{t^2},$$

so

$$e^{t^2} y = \int t\,e^{t^2}\,\mathrm{d}t = \tfrac{1}{2}e^{t^2} + C.$$

Hence $y = \frac{1}{2} + Ce^{-t^2}$. Applying $y(0) = 1$: $C = \frac{1}{2}$. The solution is

$$y(t) = \frac{1}{2}\left(1 + e^{-t^2}\right).$$

Example 2.10. Solve the RC circuit equation $Rq' + q/C = E_0$ with $q(0) = 0$.
Solution. Dividing by R: $q' + \frac{1}{RC}\,q = \frac{E_0}{R}$. With $\tau = RC$, the integrating factor is $\mu(t) = e^{t/\tau}$. Then

$$e^{t/\tau} q = \frac{E_0}{R} \int e^{t/\tau}\,\mathrm{d}t = E_0 C\, e^{t/\tau} + K.$$

From $q(0) = 0$: $K = -E_0 C$. Therefore

$$q(t) = E_0 C\left(1 - e^{-t/\tau}\right).$$

2.3 Bernoulli Equations

Definition 2.11 (Bernoulli Equation)**.** A *Bernoulli equation* has the form

$$y' + p(t)\,y = q(t)\,y^n, \tag{2.7}$$

where $n \in \mathbb{R}$, $n \neq 0, 1$ (the cases $n = 0$ and $n = 1$ are linear).

Method 2.12 (Bernoulli Substitution). Setting $v = y^{1-n}$ and differentiating: $v' = (1 - n)\,y^{-n}\,y'$. Dividing the Bernoulli equation by y^n and multiplying by $(1 - n)$:

$$v' + (1-n)\,p(t)\,v = (1-n)\,q(t). \tag{2.8}$$

This is a linear first-order ODE in v, solvable by the integrating factor method.

Example 2.13. Solve $y' + y = y^3$.
Solution. Here $n = 3$, so $v = y^{-2}$, $v' = -2y^{-3}y'$. Dividing by y^3: $y^{-3}y' + y^{-2} = 1$, so $-\frac{1}{2}v' + v = 1$, i.e.,

$$v' - 2v = -2.$$

Integrating factor: $\mu = e^{-2t}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-2t}v\right] = -2e^{-2t}, \qquad e^{-2t}v = e^{-2t} + C, \qquad v = 1 + Ce^{2t}.$$

Returning to y:

$$y^{-2} = 1 + Ce^{2t} \qquad \Longrightarrow \qquad y = \pm\frac{1}{\sqrt{1 + Ce^{2t}}}.$$

2.4 Exact Equations

Definition 2.14 (Exact Equation)**.** The equation

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{2.9}$$

is *exact* if there exists a function $F(t,y)$ such that

$$\frac{\partial F}{\partial t} = M \qquad \text{and} \qquad \frac{\partial F}{\partial y} = N.$$

In this case $F(t, y) = C$ defines the solution implicitly.

Theorem 2.15 (Exactness Criterion). *If M, N, $\partial M/\partial y$, and $\partial N/\partial t$ are continuous on a simply connected domain D, then equation (2.9) is exact if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \qquad \text{on } D. \tag{2.10}$$

Proof. (*Necessity*) If F exists with $F_t = M$ and $F_y = N$, then $M_y = F_{ty} = F_{yt} = N_t$ by equality of mixed partials.

(*Sufficiency*) Define

$$F(t, y) = \int_{t_0}^{t} M(s, y)\,\mathrm{d}s + g(y),$$

where g is to be determined. Then $F_t = M$. Now require $F_y = N$:

$$F_y = \int_{t_0}^{t} M_y(s, y)\,\mathrm{d}s + g'(y) = \int_{t_0}^{t} N_t(s, y)\,\mathrm{d}s + g'(y) = N(t, y) - N(t_0, y) + g'(y).$$

Setting $F_y = N$ gives $g'(y) = N(t_0, y)$, which depends only on y and can be integrated. □

Method 2.16 (Solving Exact Equations)**Step 1:** Check exactness: verify $M_y = N_t$.

Step 2: Compute $F(t, y) = \int M(t, y)\,\mathrm{d}t + g(y)$ (treating y as a constant in the integral).

Step 3: Determine $g(y)$ by differentiating F with respect to y and setting $F_y = N$.

Step 4: The solution is $F(t, y) = C$.

Example 2.17. Solve $(2ty + 3)\,\mathrm{d}t + (t^2 + 4y)\,\mathrm{d}y = 0$.
Solution. Here $M = 2ty + 3$ and $N = t^2 + 4y$. Check: $M_y = 2t$ and $N_t = 2t$. The equation is exact.
Integrate M with respect to t:

$$F(t, y) = \int (2ty + 3)\,\mathrm{d}t = t^2 y + 3t + g(y).$$

Now $F_y = t^2 + g'(y)$. Set equal to $N = t^2 + 4y$: $g'(y) = 4y$, so $g(y) = 2y^2$.
The solution is

$$t^2 y + 3t + 2y^2 = C.$$

2.4.1 Integrating Factors for Non-Exact Equations

When $M_y \neq N_t$, we seek a function μ such that $\mu M\,\mathrm{d}t + \mu N\,\mathrm{d}y = 0$ is exact. This requires $(\mu M)_y = (\mu N)_t$.

Proposition 2.18. *If $\dfrac{M_y - N_t}{N}$ depends only on t, then*

$$\mu(t) = \exp\left(\int \frac{M_y - N_t}{N}\,\mathrm{d}t\right)$$

is an integrating factor depending only on t.

Proposition 2.19. *If $\dfrac{N_t - M_y}{M}$ depends only on y, then*

$$\mu(y) = \exp\Big(\int \frac{N_t - M_y}{M}\,\mathrm{d}y\Big)$$

is an integrating factor depending only on y.

Example 2.20. Solve $(3y + 2t)\,\mathrm{d}t + t\,\mathrm{d}y = 0$.
Solution. Here $M = 3y + 2t$, $N = t$. Check: $M_y = 3$, $N_t = 1$, so $M_y \neq N_t$. Compute

$$\frac{M_y - N_t}{N} = \frac{3-1}{t} = \frac{2}{t},$$

which depends only on t. Thus $\mu(t) = e^{\int 2/t\,\mathrm{d}t} = t^2$. Multiplying:

$$(3t^2y + 2t^3)\,\mathrm{d}t + t^3\,\mathrm{d}y = 0.$$

Now $M^* = 3t^2y + 2t^3$, $N^* = t^3$: $M_y^* = 3t^2 = N_t^*$. Exact. Integrate:

$$F = \int t^3\,\mathrm{d}y = t^3y + h(t).$$

Then $F_t = 3t^2y + h'(t) = 3t^2y + 2t^3$, so $h'(t) = 2t^3$, $h(t) = \frac{t^4}{2}$.
The solution is $t^3y + \frac{t^4}{2} = C$.

2.5 Homogeneous Equations

Definition 2.21 (Homogeneous Function)**.** A function $f(t, y)$ is *homogeneous of degree k* if $f(\lambda t, \lambda y) = \lambda^k f(t, y)$ for all $\lambda > 0$.

Definition 2.22 (Homogeneous First-Order ODE)**.** A first-order ODE $M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0$ is *homogeneous* (in the sense of homogeneous functions) if M and N are both homogeneous of the same degree. Equivalently, the equation can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \Phi\Big(\frac{y}{t}\Big) \tag{2.11}$$

for some function Φ.

Method 2.23 (Substitution $y = ut$). Setting $y = ut$ (so $u = y/t$), we have $y' = u + tu'$. Equation (2.11) becomes

$$u + tu' = \Phi(u),$$

which is separable:

$$\frac{\mathrm{d}u}{\Phi(u) - u} = \frac{\mathrm{d}t}{t}.$$

Example 2.24. Solve $t\,y' = y + \sqrt{t^2 + y^2}$, $t > 0$.
Solution. Rewrite as $y' = \frac{y}{t} + \sqrt{1 + (y/t)^2}$, which has the form $y' = \Phi(y/t)$ with $\Phi(u) = u + \sqrt{1 + u^2}$. Setting $y = ut$:

$$u + tu' = u + \sqrt{1 + u^2},$$

so $tu' = \sqrt{1 + u^2}$. Separating:

$$\frac{\mathrm{d}u}{\sqrt{1 + u^2}} = \frac{\mathrm{d}t}{t}.$$

Integrating: $\sinh^{-1}(u) = \ln t + C_1$, i.e., $u = \sinh(\ln t + C_1)$. Finally,

$$y = t\,\sinh(\ln t + C_1).$$

2.6 Riccati Equations

Definition 2.25 (Riccati Equation)**.** A *Riccati equation* has the form

$$y' = P(t) + Q(t)\,y + R(t)\,y^2. \tag{2.12}$$

In general, Riccati equations cannot be solved in closed form. However, if one particular solution $y_1(t)$ is known, the substitution $y = y_1 + 1/v$ reduces the equation to a linear first-order ODE in v.

Method 2.26 (Riccati Reduction). Suppose $y_1(t)$ is a known particular solution of (2.12). Set $y = y_1 + \frac{1}{v}$. Then

$$y' = y_1' - \frac{v'}{v^2}.$$

Substituting into (2.12) and simplifying (using $y_1' = P + Qy_1 + Ry_1^2$):

$$v' + (Q + 2Ry_1)\,v = -R. \tag{2.13}$$

This is a linear ODE in v.

Example 2.27. Solve $y' = 1 + t^2 - 2ty + y^2$, given that $y_1(t) = t$ is a particular solution.
Solution. Check: $y_1' = 1$ and $1 + t^2 - 2t \cdot t + t^2 = 1$. Confirmed.
Here $P(t) = 1 + t^2$, $Q(t) = -2t$, $R(t) = 1$. Setting $y = t + 1/v$:

$$v' + (-2t + 2t)\,v = -1 \qquad \Longrightarrow \qquad v' = -1,$$

so $v = -t + C$. Therefore

$$y = t + \frac{1}{C - t}.$$

2.7 Clairaut Equations

Definition 2.28 (Clairaut Equation)**.** A *Clairaut equation* has the form

$$y = ty' + \psi(y'), \tag{2.14}$$

where ψ is a given function.

Method 2.29 (Solving Clairaut Equations). Differentiate (2.14) with respect to t:

$$y' = y' + (t + \psi'(y'))\, y''.$$

Hence either $y'' = 0$ or $t + \psi'(y') = 0$.

Case 1: $y'' = 0$ means $y' = c$ (constant), and the general solution is the family of straight lines

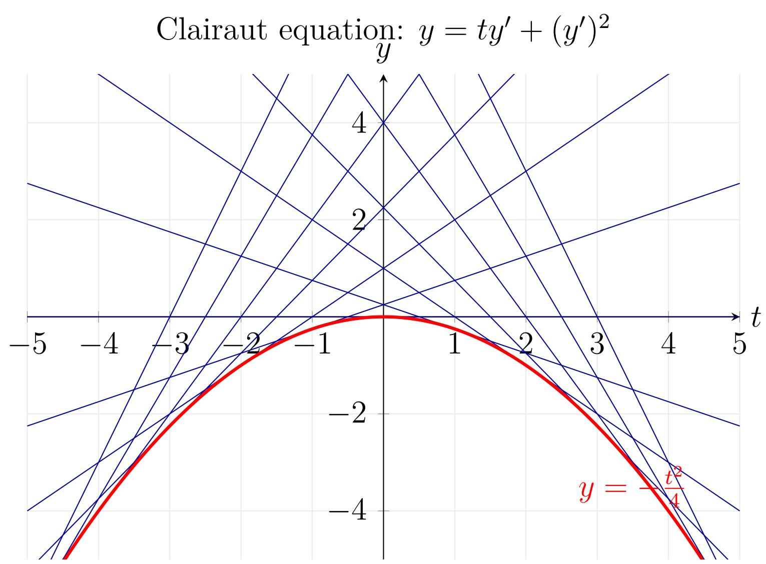
$$y = ct + \psi(c).$$

Case 2: $t = -\psi'(p)$ with $p = y'$, combined with $y = tp + \psi(p)$, gives a parametric representation of the *singular solution*, which is the envelope of the family.

Example 2.30. Solve $y = ty' + (y')^2$.
Solution. This is a Clairaut equation with $\psi(p) = p^2$.
General solution: $y = ct + c^2$ (a family of lines).
Singular solution: From $t + \psi'(p) = 0$: $t + 2p = 0$, so $p = -t/2$. Then

$$y = t\left(-\frac{t}{2}\right) + \left(-\frac{t}{2}\right)^2 = -\frac{t^2}{2} + \frac{t^2}{4} = -\frac{t^2}{4}.$$

The singular solution is the parabola $y = -t^2/4$, which is the envelope of the family of lines.



2.8 Lagrange Equations

Definition 2.31 (Lagrange Equation)**.** A *Lagrange equation* (also called a d'Alembert equation) has the form

$$y = t\,\varphi(y') + \psi(y'), \tag{2.15}$$

where φ and ψ are given functions. A Clairaut equation is the special case $\varphi(p) = p$.

Method 2.32 (Solving Lagrange Equations)*.* Set $p = y'$ and differentiate (2.15) with respect to t:

$$p = \varphi(p) + t\,\varphi'(p)\,p' + \psi'(p)\,p'.$$

If $\varphi(p) \neq p$, we can rearrange:

$$\frac{\mathrm{d}t}{\mathrm{d}p} = \frac{t\,\varphi'(p) + \psi'(p)}{p - \varphi(p)},$$

which is a linear first-order ODE in t as a function of p. Solving this and substituting back via (2.15) gives the solution parametrically in terms of p.

Example 2.33. Solve $y = 2ty' - (y')^2$.
Solution. Here $\varphi(p) = 2p$, $\psi(p) = -p^2$. Differentiating:

$$p = 2p + 2tp' - 2pp',$$

so

$$-p = (2t - 2p)\,p', \qquad \text{i.e.,} \qquad \frac{\mathrm{d}t}{\mathrm{d}p} = \frac{2t - 2p}{-p} = -\frac{2t}{p} + 2.$$

This is linear: $t' + \frac{2}{p}\,t = 2$. Integrating factor: $\mu = p^2$. Then

$$\frac{\mathrm{d}}{\mathrm{d}p}\big[p^2 t\big] = 2p^2, \qquad p^2 t = \frac{2p^3}{3} + C, \qquad t = \frac{2p}{3} + \frac{C}{p^2}.$$

From the original equation: $y = 2tp - p^2 = 2p\left(\frac{2p}{3} + \frac{C}{p^2}\right) - p^2 = \frac{p^2}{3} + \frac{2C}{p}$.
The solution is given parametrically:

$$\begin{cases} t = \dfrac{2p}{3} + \dfrac{C}{p^2}, \\ y = \dfrac{p^2}{3} + \dfrac{2C}{p}. \end{cases}$$

2.9 Table of Standard Forms

Type	Standard Form	Method
Separable	$y' = g(t)\,h(y)$	Separate and integrate
Linear	$y' + p(t)y = q(t)$	Integrating factor $\mu = e^{\int p\,\mathrm{d}t}$
Bernoulli	$y' + p(t)y = q(t)y^n$	$v = y^{1-n}$ reduces to linear
Exact	$M\,\mathrm{d}t + N\,\mathrm{d}y = 0,\ M_y = N_t$	Find F with $F_t = M$, $F_y = N$
Homogeneous	$y' = \Phi(y/t)$	$y = ut$ reduces to separable
Riccati	$y' = P + Qy + Ry^2$	Know y_1; set $y = y_1 + 1/v$
Clairaut	$y = ty' + \psi(y')$	General: $y = ct + \psi(c)$; singular: envelope
Lagrange	$y = t\varphi(y') + \psi(y')$	Differentiate; linear ODE in $t(p)$

2.10 Additional Worked Examples

Example 2.34. Solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{t^2}{1+y^2}$.
Solution. Separating: $(1+y^2)\,\mathrm{d}y = t^2\,\mathrm{d}t$. Integrating:

$$y + \frac{y^3}{3} = \frac{t^3}{3} + C.$$

This defines y implicitly as a function of t.

Example 2.35. Solve $y' - \frac{2y}{t} = t^2\cos t$, $t > 0$.
Solution. Here $p(t) = -2/t$, so

$$\mu(t) = e^{\int -2/t\,\mathrm{d}t} = e^{-2\ln t} = t^{-2}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{y}{t^2}\right] = \cos t,$$

so $\frac{y}{t^2} = \sin t + C$, giving $y = t^2(\sin t + C)$.

Example 2.36. Solve $y' + \frac{y}{t} = t\sqrt{y}$, $t > 0$, $y > 0$.
Solution. This is a Bernoulli equation with $n = 1/2$. Set $v = y^{1/2}$, so $v' = \frac{1}{2}y^{-1/2}y'$.

Dividing by $y^{1/2}$: $y^{-1/2}y' + \frac{y^{1/2}}{t} = t$, so $2v' + \frac{v}{t} = t$, i.e.,

$$v' + \frac{v}{2t} = \frac{t}{2}.$$

Integrating factor: $\mu = e^{\int 1/(2t)\,\mathrm{d}t} = t^{1/2}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[t^{1/2}\,v\Big] = \frac{t^{3/2}}{2}, \qquad t^{1/2}\,v = \frac{t^{5/2}}{5} + C.$$

So $v = \frac{t^2}{5} + Ct^{-1/2}$, and $y = v^2 = \left(\frac{t^2}{5} + Ct^{-1/2}\right)^2$.

Example 2.37. Solve $(e^y + 2ty)\,\mathrm{d}t + (te^y + t^2 + 1)\,\mathrm{d}y = 0$.
Solution. Here $M = e^y + 2ty$, $N = te^y + t^2 + 1$. Check:

$$M_y = e^y + 2t, \qquad N_t = e^y + 2t.$$

Exact. Integrate M with respect to t:

$$F = te^y + t^2y + g(y).$$

Then $F_y = te^y + t^2 + g'(y) = te^y + t^2 + 1$, so $g'(y) = 1$, $g(y) = y$. The solution is

$$te^y + t^2y + y = C.$$

Example 2.38. Solve $(t^2 + y^2)\,\mathrm{d}t - 2ty\,\mathrm{d}y = 0$.
Solution. Both $M = t^2 + y^2$ and $N = -2ty$ are homogeneous of degree 2. Set $y = ut$:

$$y' = \frac{t^2 + u^2t^2}{2t \cdot ut} = \frac{1 + u^2}{2u}.$$

Since $y' = u + tu'$:

$$tu' = \frac{1 + u^2}{2u} - u = \frac{1 - u^2}{2u}.$$

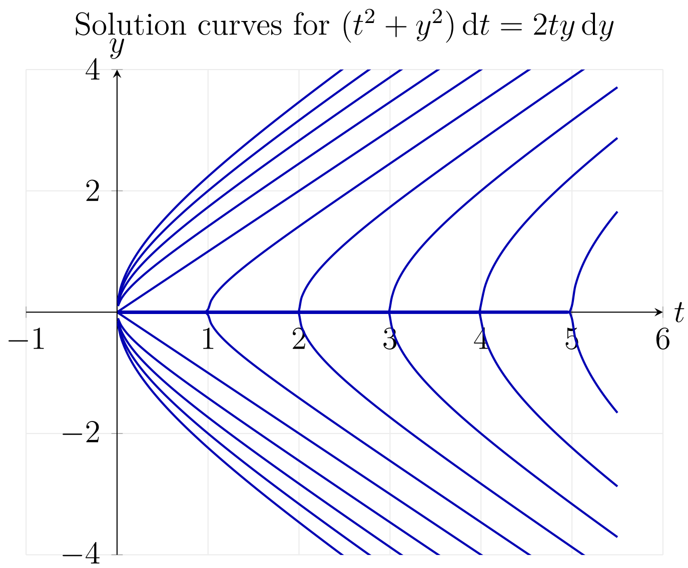
Separating:

$$\frac{2u\,\mathrm{d}u}{1 - u^2} = \frac{\mathrm{d}t}{t}.$$

Integrating: $-\ln|1 - u^2| = \ln|t| + C_0$, so $\frac{1}{|1-u^2|} = A\,|t|$ where $A = e^{C_0}$. Returning to y: $u = y/t$, so

$$\frac{t^2}{t^2 - y^2} = At, \qquad \text{i.e.,} \qquad t = A(t^2 - y^2).$$

Equivalently, $t^2 - y^2 = Ct$ for some constant C.



2.11 Exercises

Separable Equations

Exercise 2.1 (★). Solve $y' = \dfrac{t}{y}$, $y(1) = 2$.

Exercise 2.2 (★). Solve $y' = y \sin t$, $y(0) = 1$.

Exercise 2.3 (★★). Solve $\dfrac{dy}{dt} = \dfrac{t^2 y - y}{t+1}$ and determine all equilibrium solutions.

Linear Equations

Exercise 2.4 (★). Solve $y' + 3y = 6$, $y(0) = 1$.

Exercise 2.5 (★). Solve $ty' + 2y = t^3$, $t > 0$, $y(1) = 0$.

Exercise 2.6 (★★). Solve $y' + y \tan t = \sec t$, $-\pi/2 < t < \pi/2$, $y(0) = 0$.

Bernoulli Equations

Exercise 2.7 (★★). Solve $y' - y = -y^2 e^t$.

Exercise 2.8 (★★). Solve $y' + \dfrac{y}{t} = \dfrac{\ln t}{y}$, $t > 0$, $y > 0$.

Exact Equations

Exercise 2.9 (★). Solve $(2t + 3y)\,dt + (3t + 2y)\,dy = 0$.

Exercise 2.10 (★★). Show that $(y^2 + t)\,dt + 2ty\,dy = 0$ is not exact, find an integrating factor depending on t, and solve.

Exercise 2.11 (★★). Solve $\left(\dfrac{1}{t} + \dfrac{y}{t^2 + y^2}\right) dt + \left(\dfrac{t}{t^2 + y^2}\right) dy = 0$, $t > 0$.

Homogeneous Equations

Exercise 2.12 (★). Solve $y' = \dfrac{y+t}{t}$.

Exercise 2.13 (★★). Solve $(t^2 + 3y^2)\,\mathrm{d}t - 2ty\,\mathrm{d}y = 0$.

Riccati, Clairaut, Lagrange

Exercise 2.14 (★★). Solve $y' = -y^2 + \dfrac{2}{t^2}$, given that $y_1 = \dfrac{1}{t}$ is a particular solution.

Exercise 2.15 (★★). Solve the Clairaut equation $y = ty' - (y')^3$. Find both the general solution and the singular solution.

Exercise 2.16 (★★★). Solve $y = t(y')^2 - (y')^3$ parametrically.

Mixed and Challenging Problems

Exercise 2.17 (★★). For each equation below, identify the type and solve:

(a) $(3t^2y + y^3)\,\mathrm{d}t + (t^3 + 3ty^2)\,\mathrm{d}y = 0$

(b) $y' = \dfrac{y^2 - t^2}{2ty}$

(c) $y' + y\cot t = 2\cos t$

Exercise 2.18 (★★★). Consider the IVP $y' = 3y^{2/3}$, $y(0) = 0$.

(a) Show that $y \equiv 0$ is a solution.

(b) Show that for any $a \geq 0$, the function

$$y_a(t) = \begin{cases} 0 & \text{if } t \leq a, \\ (t-a)^3 & \text{if } t > a, \end{cases}$$

is also a solution.

(c) Explain why this does not contradict the Picard–Lindelöf theorem.

Exercise 2.19 (★★★). The family of curves $y = Ct^2$ fills the upper half-plane. Find the *orthogonal trajectories* of this family, i.e., the curves that intersect each member of the family at right angles. Sketch both families.

Chapter Summary

- A **separable** equation $y' = g(t)h(y)$ is solved by separating variables and integrating; always check for equilibrium (possibly singular) solutions $h(y_0) = 0$.
- A **linear** equation $y' + py = q$ is solved using the integrating factor $\mu = e^{\int p\,\mathrm{d}t}$. The solution exists and is unique on any interval where p and q are continuous.
- A **Bernoulli** equation $y' + py = qy^n$ is reduced to linear form by $v = y^{1-n}$.

- An equation $M\,\mathrm{d}t + N\,\mathrm{d}y = 0$ is **exact** when $M_y = N_t$; the solution is $F(t, y) = C$ where $F_t = M$, $F_y = N$. Non-exact equations may admit integrating factors $\mu(t)$ or $\mu(y)$.

- A **homogeneous** equation (equal-degree numerator and denominator) is reduced to separable form by $y = ut$.

- A **Riccati** equation requires one known solution; then a substitution reduces it to a linear equation.

- **Clairaut** and **Lagrange** equations are solved by differentiation: Clairaut yields a family of lines and a singular envelope; Lagrange leads to a linear ODE in $t(p)$.

- The table of standard forms (Section 2.9) is a reliable guide for identifying the type of a given first-order ODE.

Chapter 3

Existence and Uniqueness Theorems

The previous chapters showed us how to solve certain classes of ODEs explicitly. But explicit formulae are the exception rather than the rule: most differential equations cannot be solved in closed form. This raises fundamental questions. Given an initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

does a solution exist at all? If it does, is it unique? How far can it be extended? And does the solution depend continuously on the initial data? This chapter provides rigorous answers via the Picard–Lindelöf theorem, Peano's theorem, and Grönwall's lemma.

3.1 Motivating Examples

Before stating any theorems, let us see that the questions above are not vacuous.

Example 3.1 (Non-uniqueness)**.** Consider the initial value problem

$$y' = y^{2/3}, \qquad y(0) = 0.$$

One solution is $y(t) = 0$ for all t. Another is

$$y(t) = \begin{cases} 0 & t \le c, \\ \left(\dfrac{t-c}{3}\right)^3 & t > c, \end{cases}$$

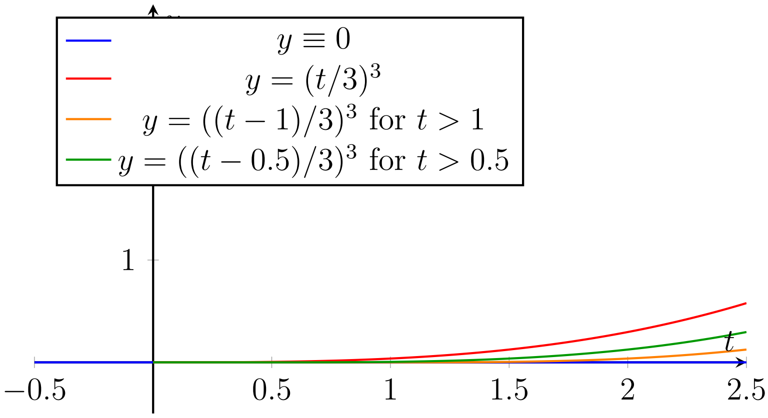
for any $c \ge 0$. Thus uniqueness fails: there are infinitely many solutions through the origin. The problem is that $f(t, y) = y^{2/3}$ is not Lipschitz in y near $y = 0$.

Example 3.2 (Finite-time blow-up)**.** Consider

$$y' = y^2, \qquad y(0) = 1.$$

Separation of variables gives $y(t) = \dfrac{1}{1-t}$, which blows up at $t = 1$. The solution exists on $(-\infty, 1)$ but cannot be extended to all of $\mathbb{R}$. Existence is only *local* in general.

Non-unique solutions of $y' = y^{2/3}$, $y(0) = 0$



3.2 The Lipschitz Condition

The key hypothesis that guarantees uniqueness is a Lipschitz condition on the right-hand side.

Definition 3.3 (Lipschitz condition)**.** Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be open. A function $f\colon D \to \mathbb{R}^n$ is **Lipschitz in** y (uniformly in t) on D if there exists a constant $L \geq 0$ such that

$$\|f(t, y_1) - f(t, y_2)\| \leq L\,\|y_1 - y_2\|$$

for all $(t, y_1), (t, y_2) \in D$. The constant L is called a **Lipschitz constant**.

Definition 3.4 (Locally Lipschitz)**.** The function f is **locally Lipschitz in** y on D if for each compact $K \subset D$, the restriction $f|_K$ satisfies a Lipschitz condition (with a constant L_K that may depend on K).

Proposition 3.5 (Sufficient condition for Lipschitz)**.** *If $f\colon D \to \mathbb{R}^n$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous on the open set D, then f is locally Lipschitz in y on D.*

Proof. Let $K \subset D$ be a compact convex set. Since $\partial f/\partial y$ is continuous on K, it is bounded: there exists $L \geq 0$ with $\|\partial f/\partial y(t, y)\| \leq L$ for all $(t, y) \in K$. For $(t, y_1), (t, y_2) \in K$, the mean value inequality gives

$$\|f(t, y_1) - f(t, y_2)\| \leq \sup_{s\in[0,1]} \|{*}\| \frac{\partial f}{\partial y}\big(t, y_2 + s(y_1 - y_2)\big) \cdot \|y_1 - y_2\| \leq L\,\|y_1 - y_2\|. \qquad \square$$

Example 3.6 (Checking the Lipschitz condition)**.** (a) $f(t, y) = ty + \sin y$. We have $\partial f/\partial y = t + \cos y$, which is continuous everywhere. Hence f is locally Lipschitz.

(b) $f(t, y) = y^{2/3}$. Here $\partial f/\partial y = \frac{2}{3}\,y^{-1/3}$, which is unbounded near $y = 0$. Indeed, f is *not* Lipschitz near $y = 0$.

(c) $f(t, y) = y^2$. Then $\partial f/\partial y = 2y$, continuous everywhere. So f is locally Lipschitz but not globally Lipschitz (the derivative is unbounded as $|y| \to \infty$).

3.3 The Picard–Lindelöf Theorem

3.3.1 Integral formulation

The starting point is to reformulate the initial value problem as an integral equation.

Proposition 3.7 (Integral equation equivalence)**.** *Let f be continuous on an open set $D \subset \mathbb{R} \times \mathbb{R}^n$ containing (t_0, y_0). A continuous function $y\colon I \to \mathbb{R}^n$ (where I is an interval containing t_0) is a solution of*

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

if and only if it satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f\big(s, y(s)\big)\,\mathrm{d}s \qquad \textit{for all } t \in I. \tag{3.1}$$

Proof. If y solves the IVP, integrate both sides of $y'(s) = f(s, y(s))$ from t_0 to t and use $y(t_0) = y_0$. Conversely, if (3.1) holds, differentiate with respect to t (using continuity of the integrand) to recover $y'(t) = f(t, y(t))$, and setting $t = t_0$ gives $y(t_0) = y_0$. □

3.3.2 Picard iteration

The idea is to define a sequence of successive approximations:

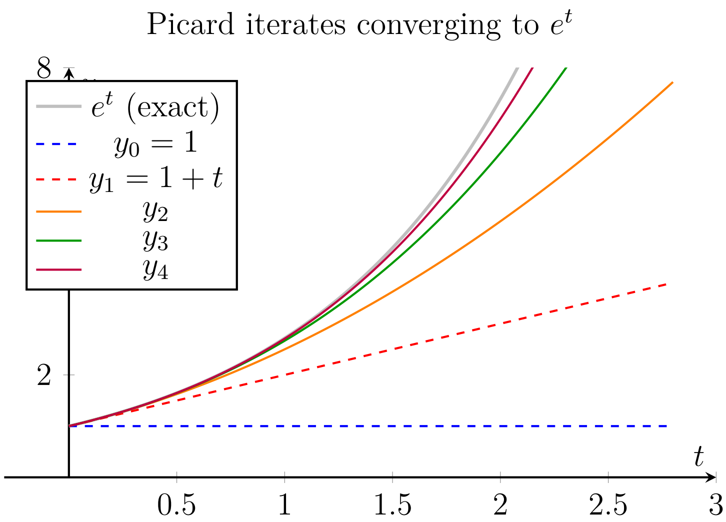
$$\begin{aligned} y_0(t) &= y_0, \\ y_{n+1}(t) &= y_0 + \int_{t_0}^{t} f\big(s, y_n(s)\big)\,\mathrm{d}s, \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{3.2}$$

If this sequence converges (uniformly), its limit $y(t) = \lim_{n\to\infty} y_n(t)$ will satisfy the integral equation (3.1) and hence solve the IVP.

Example 3.8 (Picard iterates for $y' = y$, $y(0) = 1$)**.** Here $f(t, y) = y$, $t_0 = 0$, $y_0 = 1$. We compute:

$$\begin{aligned} y_0(t) &= 1, \\ y_1(t) &= 1 + \int_0^t 1\,\mathrm{d}s = 1 + t, \\ y_2(t) &= 1 + \int_0^t (1 + s)\,\mathrm{d}s = 1 + t + \frac{t^2}{2}, \\ y_3(t) &= 1 + \int_0^t \left(1 + s + \frac{s^2}{2}\right)\mathrm{d}s = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}. \end{aligned}$$

By induction, $y_n(t) = \displaystyle\sum_{k=0}^{n} \frac{t^k}{k!}$, which converges to e^t as $n \to \infty$.



Example 3.9 (Picard iterates for $y' = t + y^2$, $y(0) = 0$)**.** We compute the first few iterates:

$$\begin{aligned}
y_0(t) &= 0, \\
y_1(t) &= \int_0^t (s + 0)\,\mathrm{d}s = \frac{t^2}{2}, \\
y_2(t) &= \int_0^t \left(s + \frac{s^4}{4}\right)\mathrm{d}s = \frac{t^2}{2} + \frac{t^5}{20}, \\
y_3(t) &= \int_0^t \left(s + \left(\frac{s^2}{2} + \frac{s^5}{20}\right)^2\right)\mathrm{d}s = \frac{t^2}{2} + \frac{t^5}{20} + \frac{t^8}{160} + \frac{t^{11}}{4400}.
\end{aligned}$$

Even without a closed form, the iterates give systematic polynomial approximations to the solution.

3.3.3 Statement and proof

Theorem 3.10 (Picard–Lindelöf)**.** *Let $f\colon D \to \mathbb{R}^n$ be continuous on the open set $D \subset \mathbb{R} \times \mathbb{R}^n$, and suppose f is Lipschitz in y with constant L on the closed rectangle*

$$R = \Big\{(t, y) \in \mathbb{R} \times \mathbb{R}^n : |t - t_0| \le a,\ \|y - y_0\| \le b\Big\} \subset D.$$

Let $M = \max_{(t,y)\in R} \|f(t, y)\|$ and $\alpha = \min\big(a,\ b/M\big)$. Then the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

*has a **unique** solution $y \in C^1\big([t_0 - \alpha,\ t_0 + \alpha];\ \mathbb{R}^n\big)$.*

Proof. We use the Banach fixed point theorem applied to the Picard operator on an appropriate complete metric space.

Step 1: Setup. Let $I = [t_0 - \alpha,\ t_0 + \alpha]$ and define the space

$$X = \Big\{\varphi \in C(I, \mathbb{R}^n) : \|\varphi(t) - y_0\| \le b \text{ for all } t \in I\Big\},$$

equipped with the weighted supremum norm

$$\|\varphi\|_\lambda = \sup_{t\in I} e^{-\lambda|t-t_0|} \|\varphi(t) - y_0\|,$$

where $\lambda > L$ will be chosen later. The space $(X, \|\cdot\|_\lambda)$ is a closed subset of $C(I, \mathbb{R}^n)$ with the $\|\cdot\|_\lambda$ norm, hence it is a complete metric space.

Step 2: The Picard operator maps X to X. Define $T\colon X \to C(I, \mathbb{R}^n)$ by

$$(T\varphi)(t) = y_0 + \int_{t_0}^{t} f\big(s, \varphi(s)\big)\, \mathrm{d}s.$$

For $\varphi \in X$ and $t \in I$, since $\big(s, \varphi(s)\big) \in R$ we have $\|f(s, \varphi(s))\| \le M$, so

$$\|(T\varphi)(t) - y_0\| \le \int_{t_0}^{t} \|f(s, \varphi(s))\|\, \mathrm{d}s \le M\,|t - t_0| \le M\alpha \le b.$$

Thus $T\varphi \in X$.

Step 3: T is a contraction. For $\varphi, \psi \in X$,

$$\begin{aligned}
\|(T\varphi)(t) - (T\psi)(t)\| &\le \left| \int_{t_0}^{t} \Big\| f\big(s, \varphi(s)\big) - f\big(s, \psi(s)\big) \Big\|\, \mathrm{d}s \right| \\
&\le L \left| \int_{t_0}^{t} \|\varphi(s) - \psi(s)\|\, \mathrm{d}s \right| \\
&= L \left| \int_{t_0}^{t} e^{\lambda|s-t_0|} e^{-\lambda|s-t_0|} \|\varphi(s) - \psi(s)\|\, \mathrm{d}s \right| \\
&\le L\, \|\varphi - \psi\|_\lambda \left| \int_{t_0}^{t} e^{\lambda|s-t_0|}\, \mathrm{d}s \right| \\
&\le L\, \|\varphi - \psi\|_\lambda \cdot \frac{e^{\lambda|t-t_0|}}{\lambda}.
\end{aligned}$$

Multiplying both sides by $e^{-\lambda|t-t_0|}$ and taking the supremum over $t \in I$,

$$\|T\varphi - T\psi\|_\lambda \le \frac{L}{\lambda}\, \|\varphi - \psi\|_\lambda.$$

Choosing $\lambda > L$ makes $q = L/\lambda < 1$, so T is a contraction on $(X, \|\cdot\|_\lambda)$.

Step 4: Conclusion. By the Banach fixed point theorem, T has a unique fixed point $y \in X$, i.e.,

$$y(t) = y_0 + \int_{t_0}^{t} f\big(s, y(s)\big)\, \mathrm{d}s.$$

By Proposition 3.7, y is the unique solution of the IVP on I. Since f is continuous and y satisfies the integral equation, y is in fact C^1. □

Remark 3.11. Many textbooks prove the Picard–Lindelöf theorem using the standard supremum norm $\|\cdot\|_\infty$ and the factor $\alpha^n L^n / n!$ to control the n-th iterate directly. The weighted-norm approach above avoids induction and gives the contraction property in one step.

3.4 Peano's Existence Theorem

If we drop the Lipschitz condition and only assume continuity, we lose uniqueness but retain existence.

Theorem 3.12 (Peano). *Let $f\colon D \to \mathbb{R}^n$ be continuous on an open set $D \subset \mathbb{R} \times \mathbb{R}^n$ with $(t_0, y_0) \in D$. Then the IVP $y' = f(t, y)$, $y(t_0) = y_0$ has at least one solution on some interval $[t_0 - \alpha,\, t_0 + \alpha]$ with $\alpha > 0$.*

Remark 3.13. The proof uses the Arzelà–Ascoli theorem and is beyond our scope here. The crucial point is that *continuity alone guarantees existence but not uniqueness*, as Example 3.1 demonstrates.

3.5 Maximal Solutions and Blow-Up

Definition 3.14 (Maximal solution). A solution $y\colon (\omega_-, \omega_+) \to \mathbb{R}^n$ is called **maximal** (or **non-extendable**) if it cannot be extended to a solution on any strictly larger interval. The interval (ω_-, ω_+) is the **maximal interval of existence**.

Theorem 3.15 (Blow-up alternative). *Let f be continuous and locally Lipschitz in y on an open set $D \subset \mathbb{R} \times \mathbb{R}^n$. Let $y\colon (\omega_-, \omega_+) \to \mathbb{R}^n$ be the unique maximal solution of $y' = f(t, y)$, $y(t_0) = y_0$. Then either $\omega_+ = +\infty$, or $\big(t, y(t)\big)$ leaves every compact subset of D as $t \to \omega_+^-$ (and similarly for ω_-).*

Remark 3.16. In practice, "leaving every compact set" means either:

(i) $\|y(t)\| \to \infty$ as $t \to \omega_+^-$ (blow-up in finite time), or

(ii) $\big(t, y(t)\big)$ approaches the boundary of D.

For $D = \mathbb{R} \times \mathbb{R}^n$, only case (i) is possible for finite ω_+.

Example 3.17 (Blow-up revisited). For $y' = y^2$, $y(0) = y_0 > 0$: the solution $y(t) = (y_0^{-1} - t)^{-1}$ has $\omega_+ = 1/y_0$. As $t \to (1/y_0)^-$, we get $y(t) \to +\infty$. The larger the initial condition, the sooner the blow-up.

3.6 Grönwall's Lemma

Grönwall's lemma is one of the most useful tools in the theory of ODEs. It turns a differential or integral inequality into a pointwise bound.

Lemma 3.18 (Grönwall, integral form). *Let $u, \beta\colon [a, b] \to \mathbb{R}$ be continuous and non-negative, and let $\alpha \geq 0$ be a constant. If*

$$u(t) \leq \alpha + \int_a^t \beta(s)\, u(s)\, \mathrm{d}s \qquad \text{for all } t \in [a, b], \tag{3.3}$$

then

$$u(t) \le \alpha \exp\!\left(\int_a^t \beta(s)\,\mathrm{d}s\right) \qquad \textit{for all } t \in [a,b]. \tag{3.4}$$

Proof. Define $U(t) = \alpha + \displaystyle\int_a^t \beta(s)\,u(s)\,\mathrm{d}s$, so that $U(a) = \alpha$ and $u(t) \le U(t)$. Then

$$U'(t) = \beta(t)\,u(t) \le \beta(t)\,U(t).$$

If $\alpha > 0$, then $U(t) > 0$ on $[a,b]$, and we may write

$$\frac{U'(t)}{U(t)} \le \beta(t).$$

Integrating from a to t:

$$\ln U(t) - \ln U(a) \le \int_a^t \beta(s)\,\mathrm{d}s,$$

so $U(t) \le U(a)\exp\!\left(\int_a^t \beta(s)\,\mathrm{d}s\right) = \alpha\exp\!\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$. Since $u(t) \le U(t)$, the result follows.

For $\alpha = 0$, apply the result with α replaced by $\varepsilon > 0$ (the hypothesis still holds since $u(t) \le \varepsilon + \int_a^t \beta(s)\,u(s)\,\mathrm{d}s$), obtaining $u(t) \le \varepsilon\exp\!\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$. Letting $\varepsilon \to 0$ yields $u(t) \le 0$, hence $u(t) = 0$. □

Corollary 3.19 (Grönwall, constant coefficient)**.** *If* $u(t) \le \alpha + L\displaystyle\int_a^t u(s)\,\mathrm{d}s$ *for all* $t \in [a,b]$ *with* $\alpha \ge 0$ *and* $L \ge 0$*, then*

$$u(t) \le \alpha\, e^{L(t-a)}.$$

In particular, if $\alpha = 0$*, then* $u \equiv 0$.

3.7 Continuous Dependence on Initial Data

Solutions of an ODE are not just unique under a Lipschitz condition; they also depend continuously on the initial values.

Theorem 3.20 (Continuous dependence)**.** *Let* f *be continuous and Lipschitz in* y *with constant* L *on* $D \subset \mathbb{R} \times \mathbb{R}^n$*. Let* y *and* $\tilde{y}$ *be solutions of*

$$y' = f(t,y), \quad y(t_0) = y_0 \qquad \textit{and} \qquad \tilde{y}' = f(t,\tilde{y}), \quad \tilde{y}(t_0) = \tilde{y}_0,$$

on a common interval $[t_0, t_0 + T]$*. Then*

$$\|y(t) - \tilde{y}(t)\| \le \|y_0 - \tilde{y}_0\|\; e^{L(t-t_0)} \qquad \textit{for all } t \in [t_0, t_0+T]. \tag{3.5}$$

Proof. Let $w(t) = \|y(t) - \tilde{y}(t)\|$. From the integral formulation,

$$w(t) \le \|y_0 - \tilde{y}_0\| + \int_{t_0}^t \|f(s,y(s)) - f(s,\tilde{y}(s))\|\,\mathrm{d}s \le \|y_0 - \tilde{y}_0\| + L\int_{t_0}^t w(s)\,\mathrm{d}s.$$

By Grönwall's lemma (Corollary 3.19) with $\alpha = \|y_0 - \tilde{y}_0\|$, we get the result. □

Remark 3.21. The bound (3.5) shows that perturbations in initial data grow at most exponentially. This exponential sensitivity is sharp in general (think of $y' = Ly$). For chaotic systems the effective Lipschitz constant is large, explaining rapid divergence of nearby trajectories.

3.8 Further Counterexamples

Example 3.22 (No global existence for $y' = 1 + y^2$)**.** The IVP $y' = 1 + y^2$, $y(0) = 0$ has the solution $y(t) = \tan t$, which blows up at $t = \pi/2$. Here $f(t, y) = 1 + y^2$ is locally Lipschitz but not globally Lipschitz, and no global solution exists.

Example 3.23 (Global existence with linear growth)**.** If $\|f(t, y)\| \leq A + B\|y\|$ for constants $A, B \geq 0$ and all $(t, y) \in \mathbb{R} \times \mathbb{R}^n$, then every maximal solution exists for all time. Indeed, Grönwall's lemma shows that $\|y(t)\|$ grows at most exponentially, preventing finite-time blow-up.

Exercise 3.1. Prove the claim of Example 3.23 in detail. *Hint:* Apply Grönwall's lemma to $u(t) = \|y(t)\|$ and use the blow-up alternative (Theorem 3.15).

3.9 Exercises

Exercise 3.2. Consider $y' = \cos y$, $y(0) = 0$. Compute the first three Picard iterates y_1, y_2, y_3.

Exercise 3.3. For the system $x' = y$, $y' = -x$ with $x(0) = 1$, $y(0) = 0$, compute the Picard iterates $(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ and identify the pattern. What is the exact solution?

Exercise 3.4. Determine whether the following functions are Lipschitz, locally Lipschitz, or neither in y on the given domain:

(a) $f(t, y) = e^{-t}\sin(y)$ on $\mathbb{R} \times \mathbb{R}$.

(b) $f(t, y) = \sqrt{|y|}$ on $\mathbb{R} \times \mathbb{R}$.

(c) $f(t, y) = t^2 y^3$ on $\mathbb{R} \times \mathbb{R}$.

(d) $f(t, y) = \dfrac{y}{1 + y^2}$ on $\mathbb{R} \times \mathbb{R}$.

Exercise 3.5. Let y solve $y' = f(t, y)$ on $[0, T]$ with $\|f(t, y)\| \leq 3\|y\| + 2$. Show that $\|y(t)\| \leq (\|y(0)\| + \frac{2}{3})\, e^{3t} - \frac{2}{3}$.

Exercise 3.6. Show that $y' = 3y^{2/3}$, $y(0) = 0$ has a two-parameter family of solutions. Find them explicitly.

Exercise 3.7. Find the blow-up time for $y' = y^p$ with $y(0) = y_0 > 0$ and $p > 1$. Express it in terms of p and y_0.

Exercise 3.8. Let $u \geq 0$ satisfy $u(t) \leq 3 + 2\int_0^t s\,u(s)\,\mathrm{d}s$ for $t \in [0, T]$. Show that $u(t) \leq 3e^{t^2}$.

Exercise 3.9. Two solutions of $y' = -y + \sin t$ start at $y_0 = 1$ and $\tilde{y}_0 = 1.01$. Give an upper bound on $|y(t) - \tilde{y}(t)|$ for $t \geq 0$ using continuous dependence. Can you improve the bound by exploiting the sign of the Lipschitz constant?

3.10 Chapter Summary

- The **Picard–Lindelöf theorem** guarantees local existence and uniqueness of solutions when f is continuous and Lipschitz in y.
- The proof constructs the solution as the limit of **Picard iterates** and uses the Banach fixed point theorem.
- **Peano's theorem**: continuity alone suffices for existence, but uniqueness may fail (e.g., $y' = y^{2/3}$).
- **Maximal solutions** either exist for all time or leave every compact subset of the domain (blow-up alternative).
- **Grönwall's lemma** converts integral inequalities into pointwise exponential bounds, and is the key tool for proving uniqueness, continuous dependence, and stability estimates.
- Solutions depend **continuously** on initial data, with at most exponential amplification of perturbations.

Chapter 4

Second-Order Linear ODEs with Constant Coefficients

Second-order linear ODEs with constant coefficients form one of the most important classes of differential equations in both pure and applied mathematics. They model mechanical vibrations, electrical circuits, and countless other phenomena. In this chapter we develop the complete theory: characteristic equation, fundamental systems, the Wronskian, particular solutions by undetermined coefficients, resonance, and the Euler–Cauchy equation.

4.1 Physical Motivation

4.1.1 Spring-mass system

Consider a mass m attached to a spring with stiffness $k > 0$, subject to a damping force proportional to velocity (coefficient $c \geq 0$) and an external driving force $F(t)$. Newton's second law gives

$$m\,y'' + c\,y' + k\,y = F(t), \tag{4.1}$$

where $y(t)$ denotes the displacement from equilibrium.

4.1.2 RLC circuit

An RLC series circuit with inductance L, resistance R, and capacitance C driven by a voltage source $V(t)$ satisfies

$$L\,q'' + R\,q' + \frac{1}{C}\,q = V(t), \tag{4.2}$$

where $q(t)$ is the charge on the capacitor. The analogy with (4.1) is exact: $L \leftrightarrow m$, $R \leftrightarrow c$, $1/C \leftrightarrow k$.

4.2 The Homogeneous Equation

We study

$$ay'' + by' + cy = 0, \tag{4.3}$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$.

4.2.1 The characteristic equation

Seeking solutions of the form $y(t) = e^{rt}$ and substituting into (4.3) gives

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \quad \Longrightarrow \quad ar^2 + br + c = 0.$$

Definition 4.1 (Characteristic equation)**.** The **characteristic equation** (or **auxiliary equation**) of $ay'' + by' + cy = 0$ is the quadratic

$$ar^2 + br + c = 0, \tag{4.4}$$

with discriminant $\Delta = b^2 - 4ac$.

The nature of the roots determines the form of the general solution. We treat the three cases separately.

4.2.2 Case 1: Two distinct real roots ($\Delta > 0$)

Theorem 4.2 (Distinct real roots)**.** *If $\Delta > 0$, the characteristic equation has two distinct real roots $r_1 \neq r_2$, and the general solution of (4.3) is*

$$y(t) = C_1e^{r_1t} + C_2e^{r_2t}, \tag{4.5}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants.

Proof. Both $y_1(t) = e^{r_1t}$ and $y_2(t) = e^{r_2t}$ are solutions by construction. Their Wronskian (see Section 6.1) is

$$W(t) = \begin{vmatrix} e^{r_1t} & e^{r_2t} \\ r_1e^{r_1t} & r_2e^{r_2t} \end{vmatrix} = (r_2 - r_1)\, e^{(r_1+r_2)t} \neq 0,$$

since $r_1 \neq r_2$. Hence $\{y_1, y_2\}$ is a fundamental system, and every solution is a linear combination of y_1 and y_2. □

Example 4.3 (Overdamped spring)**.** Solve $y''+5y'+6y = 0$, with $y(0) = 2$, $y'(0) = -3$. The characteristic equation $r^2 + 5r + 6 = (r + 2)(r + 3) = 0$ gives $r_1 = -2$, $r_2 = -3$. General solution: $y(t) = C_1e^{-2t} + C_2e^{-3t}$.
Applying initial conditions:

$$C_1 + C_2 = 2, \qquad -2C_1 - 3C_2 = -3.$$

Solving: $C_1 = 3$, $C_2 = -1$, so $y(t) = 3e^{-2t} - e^{-3t}$.

4.2.3 Case 2: Repeated real root ($\Delta = 0$)

Theorem 4.4 (Repeated root)**.** *If $\Delta = 0$, the characteristic equation has a double root $r = -b/(2a)$, and the general solution is*

$$y(t) = (C_1 + C_2\,t)\,e^{rt}. \tag{4.6}$$

Proof. We already know $y_1(t) = e^{rt}$ is a solution. We seek a second, linearly independent solution. Try $y_2(t) = t\,e^{rt}$. Then

$$y_2' = e^{rt}(1 + rt), \qquad y_2'' = e^{rt}(2r + r^2 t).$$

Substituting into $ay'' + by' + cy$:

$$\begin{aligned} a(2r + r^2 t)e^{rt} + b(1 + rt)e^{rt} + c\,t\,e^{rt} &= e^{rt}\Big[(ar^2 + br + c)t + 2ar + b\Big] \\ &= e^{rt}[0 \cdot t + 2ar + b]. \end{aligned}$$

Since $r = -b/(2a)$, we have $2ar + b = 0$. Thus $y_2 = te^{rt}$ is indeed a solution. Its Wronskian with y_1 is $W(t) = e^{2rt} \neq 0$, so $\{e^{rt}, te^{rt}\}$ is a fundamental system. □

Example 4.5 (Critically damped system)**.** Solve $y'' + 4y' + 4y = 0$, with $y(0) = 1$, $y'(0) = 0$.
Characteristic equation: $r^2 + 4r + 4 = (r + 2)^2 = 0$, giving $r = -2$. General solution: $y(t) = (C_1 + C_2 t)e^{-2t}$.
From $y(0) = C_1 = 1$ and $y'(0) = C_2 - 2C_1 = 0$ we get $C_2 = 2$. Thus $y(t) = (1 + 2t)\,e^{-2t}$.

4.2.4 Case 3: Complex conjugate roots ($\Delta < 0$)

Theorem 4.6 (Complex conjugate roots)**.** *If $\Delta < 0$, the characteristic equation has conjugate complex roots $r = \alpha \pm i\beta$ where $\alpha = -b/(2a)$ and $\beta = \sqrt{4ac - b^2}/(2a) > 0$. The general solution (real-valued) is*

$$y(t) = e^{\alpha t}\Big(C_1 \cos \beta t + C_2 \sin \beta t\Big). \tag{4.7}$$

Proof. The complex-valued functions $e^{(\alpha + i\beta)t}$ and $e^{(\alpha - i\beta)t}$ are solutions. Taking real and imaginary parts of $e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t)$ gives two real solutions:

$$y_1(t) = e^{\alpha t} \cos \beta t, \qquad y_2(t) = e^{\alpha t} \sin \beta t.$$

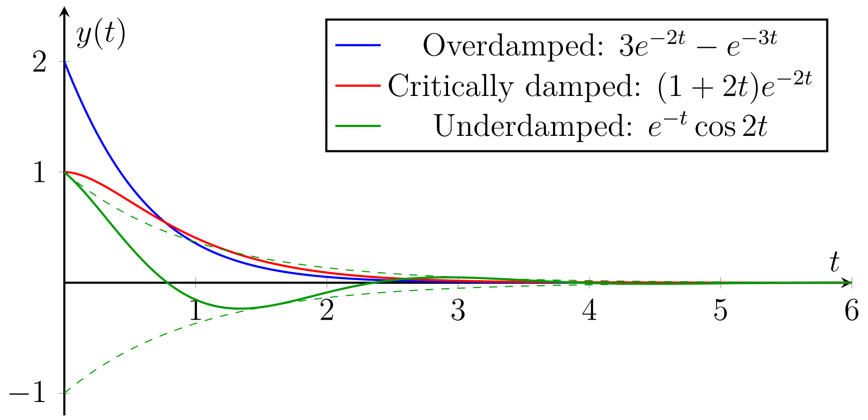
Their Wronskian is

$$W(t) = e^{2\alpha t} \begin{vmatrix} \cos \beta t & \sin \beta t \\ \alpha \cos \beta t - \beta \sin \beta t & \alpha \sin \beta t + \beta \cos \beta t \end{vmatrix} = \beta\, e^{2\alpha t} \neq 0.$$

Hence $\{y_1, y_2\}$ is a fundamental system. □

Example 4.7 (Underdamped oscillation)**.** Solve $y'' + 2y' + 5y = 0$, with $y(0) = 1$, $y'(0) = -1$.
Characteristic equation: $r^2 + 2r + 5 = 0$, giving $r = -1 \pm 2i$. So $\alpha = -1$, $\beta = 2$.
General solution: $y(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$.
From $y(0) = C_1 = 1$ and $y'(0) = -C_1 + 2C_2 = -1$, we get $C_2 = 0$. Thus $y(t) = e^{-t} \cos 2t$.

Three damping regimes



4.3 Fundamental Solutions and the Wronskian

Definition 4.8 (Wronskian)**.** Given two differentiable functions $y_1, y_2 \colon I \to \mathbb{R}$, their **Wronskian** is

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t). \tag{4.8}$$

Theorem 4.9 (Abel's identity)**.** *If y_1 and y_2 are solutions of $ay'' + by' + cy = 0$ on an interval I, then*

$$W(t) = W(t_0)\, \exp\!\left(-\frac{b}{a}(t - t_0)\right) \tag{4.9}$$

for any $t_0 \in I$. In particular, W is either identically zero or never zero on I.

Proof. Differentiate the Wronskian:

$$W' = y_1 y_2'' - y_2 y_1''.$$

From the ODE, $y_i'' = -(b/a)y_i' - (c/a)y_i$ for $i = 1, 2$. Thus

$$\begin{aligned} W' &= y_1\Big[-(b/a)y_2' - (c/a)y_2\Big] - y_2\Big[-(b/a)y_1' - (c/a)y_1\Big] \\ &= -(b/a)(y_1 y_2' - y_2 y_1') = -(b/a)\, W. \end{aligned}$$

This first-order linear ODE in W has the solution $W(t) = W(t_0)\, e^{-(b/a)(t-t_0)}$. ∎

Proposition 4.10 (Fundamental system criterion)**.** *Two solutions y_1, y_2 of $ay'' + by' + cy = 0$ form a fundamental system (i.e., every solution is a linear combination of y_1 and y_2) if and only if $W(y_1, y_2)(t_0) \neq 0$ for some (equivalently, every) $t_0 \in I$.*

4.4 Non-Homogeneous Equations: Undetermined Coefficients

We now consider

$$ay'' + by' + cy = g(t), \tag{4.10}$$

where $g(t)$ is a given forcing function.

Theorem 4.11 (Structure of general solution)**.** *The general solution of (4.10) is*

$$y(t) = y_h(t) + y_p(t),$$

where y_h is the general solution of the associated homogeneous equation and y_p is any particular solution of (4.10).

Proof. If y solves (4.10), then $y - y_p$ solves the homogeneous equation: $a(y-y_p)'' + b(y-y_p)' + c(y-y_p) = g - g = 0$. Hence $y - y_p = y_h$. □

4.4.1 Method of undetermined coefficients

Method 4.12 (Undetermined coefficients). When $g(t)$ is a polynomial, exponential, sine/cosine, or a product thereof, we guess the form of y_p and determine the unknown coefficients by substitution.

The key rule: if the trial form overlaps with a solution of the homogeneous equation, multiply by t (or t^2) until the overlap is removed.

Forcing term $g(t)$	**Trial** $y_p(t)$	**Modification**
$P_n(t)$ (polynomial of degree n)	$A_n t^n + \cdots + A_1 t + A_0$	$\times t^s$
$e^{\gamma t}$	$Ae^{\gamma t}$	$\times t^s$
$\cos \omega t$ or $\sin \omega t$	$A \cos \omega t + B \sin \omega t$	$\times t^s$
$e^{\gamma t} \cos \omega t$	$e^{\gamma t}(A \cos \omega t + B \sin \omega t)$	$\times t^s$
$t^n e^{\gamma t}$	$(A_n t^n + \cdots + A_0)\, e^{\gamma t}$	$\times t^s$
$t^n e^{\gamma t} \cos \omega t$	$e^{\gamma t}(A_n t^n + \cdots + A_0) \cos \omega t$ $+ e^{\gamma t}(B_n t^n + \cdots + B_0) \sin \omega t$	$\times t^s$

Table 4.1: Trial particular solutions. Here s is the multiplicity of γ (or $\gamma \pm i\omega$) as a root of the characteristic equation ($s = 0, 1$, or 2).

Example 4.13 (Polynomial forcing)**.** Solve $y'' - 3y' + 2y = 4t^2$.
Characteristic equation: $r^2 - 3r + 2 = (r-1)(r-2) = 0$, so $y_h = C_1 e^t + C_2 e^{2t}$.
Since $g(t) = 4t^2$ is a degree-2 polynomial and 0 is not a root of the characteristic equation ($s = 0$), we try $y_p = At^2 + Bt + C$. Then $y_p' = 2At + B$, $y_p'' = 2A$, so

$$2A - 3(2At + B) + 2(At^2 + Bt + C) = 2At^2 + (2B - 6A)t + (2A - 3B + 2C).$$

Matching coefficients with $4t^2 + 0t + 0$:

$$2A = 4, \quad 2B - 6A = 0, \quad 2A - 3B + 2C = 0.$$

So $A = 2$, $B = 6$, $C = 7$, and $y_p = 2t^2 + 6t + 7$.
General solution: $y = C_1 e^t + C_2 e^{2t} + 2t^2 + 6t + 7$.

Example 4.14 (Exponential forcing with overlap)**.** Solve $y'' - 2y' + y = e^t$.
Characteristic equation: $(r-1)^2 = 0$, so $r = 1$ is a double root and $y_h = (C_1 + C_2 t)e^t$.
Since e^t corresponds to $\gamma = 1$, which is a root of multiplicity $s = 2$, we try $y_p = At^2 e^t$.
Then

$$y_p = At^2 e^t, \quad y_p' = A(2t + t^2)e^t, \quad y_p'' = A(2 + 4t + t^2)e^t.$$

Substituting:

$$A(2 + 4t + t^2)e^t - 2A(2t + t^2)e^t + At^2 e^t = 2Ae^t.$$

Setting $2A = 1$ gives $A = 1/2$. Thus $y_p = \dfrac{t^2}{2}\, e^t$.
General solution: $y = (C_1 + C_2 t)e^t + \dfrac{t^2}{2}\, e^t = \left(C_1 + C_2 t + \tfrac{t^2}{2}\right)e^t$.

Example 4.15 (Sinusoidal forcing)**.** Solve $y'' + 4y = 3\sin t$.
Characteristic equation: $r^2 + 4 = 0$, so $r = \pm 2i$ and $y_h = C_1 \cos 2t + C_2 \sin 2t$.
Since $\omega = 1 \neq 2$, there is no overlap ($s = 0$). Try $y_p = A\cos t + B\sin t$. Then $y_p'' = -A\cos t - B\sin t$, and

$$(-A\cos t - B\sin t) + 4(A\cos t + B\sin t) = 3A\cos t + 3B\sin t.$$

Matching: $3A = 0$, $3B = 3$, so $A = 0$, $B = 1$ and $y_p = \sin t$.
General solution: $y = C_1 \cos 2t + C_2 \sin 2t + \sin t$.

4.5 Resonance

Resonance occurs when the forcing frequency matches a natural frequency of the system, causing the response to grow without bound (in the undamped case) or to reach a maximum amplitude (with damping).

Example 4.16 (Pure resonance)**.** Consider $y'' + \omega_0^2 y = \cos \omega_0 t$ (no damping, forcing at natural frequency ω_0).
Characteristic roots: $r = \pm i\omega_0$, so $y_h = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$.

Since $\cos\omega_0 t$ appears in y_h, we need $s = 1$: try $y_p = t(A\cos\omega_0 t + B\sin\omega_0 t)$. Computing:

$$y_p' = A\cos\omega_0 t + B\sin\omega_0 t + t(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t),$$
$$y_p'' = -2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t - t\omega_0^2(A\cos\omega_0 t + B\sin\omega_0 t).$$

Substituting into $y'' + \omega_0^2 y$:

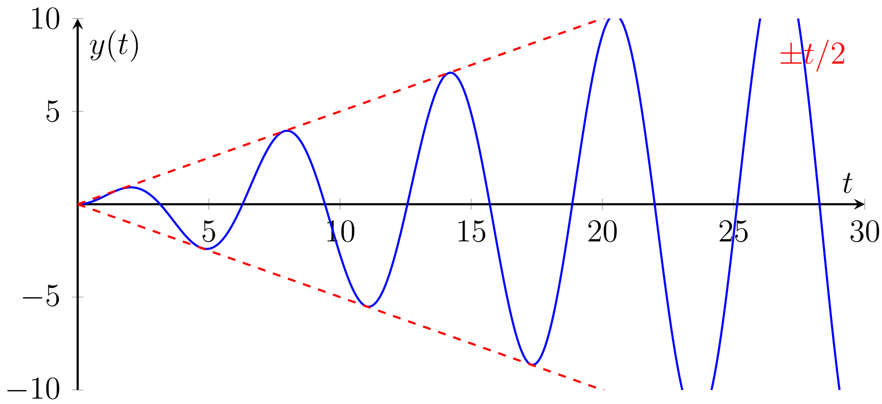
$$-2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t = \cos\omega_0 t.$$

So $A = 0$, $B = \dfrac{1}{2\omega_0}$, and

$$y_p(t) = \frac{t}{2\omega_0}\sin\omega_0 t.$$

The factor of t means the amplitude grows linearly—this is resonance.

Resonance: $y'' + y = \cos t$, $y(0) = y'(0) = 0$



Example 4.17 (Near-resonance and beats). Consider $y'' + \omega_0^2 y = \cos\omega t$ with $\omega \neq \omega_0$ but $\omega \approx \omega_0$, and $y(0) = y'(0) = 0$. The solution is

$$y(t) = \frac{1}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t) = \frac{2}{\omega_0^2 - \omega^2}\sin\!\left(\frac{\omega_0 - \omega}{2}\,t\right)\sin\!\left(\frac{\omega_0 + \omega}{2}\,t\right).$$

When $\omega_0 \approx \omega$, the first sine factor produces a slowly varying envelope (the "beat" phenomenon), while the second gives rapid oscillations.

4.6 The Euler–Cauchy Equation

Definition 4.18 (Euler–Cauchy equation). An **Euler–Cauchy equation** (or **equidimensional equation**) is an ODE of the form

$$at^2y'' + bty' + cy = 0, \qquad t > 0, \tag{4.11}$$

where a, b, c are real constants with $a \neq 0$.

Method 4.19 (Solving Euler–Cauchy equations). Seek solutions of the form $y = t^r$ (for $t > 0$). Then $y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$, and substitution gives

$$ar(r-1) + br + c = 0 \qquad \text{(indicial equation)}.$$

The three cases mirror those of constant-coefficient equations:

Theorem 4.20 (Euler–Cauchy: general solution)**.** *Let r_1, r_2 be the roots of $ar(r-1) + br + c = 0$.*

(i) ***Distinct real roots*** *($r_1 \neq r_2$ real): $y(t) = C_1 t^{r_1} + C_2 t^{r_2}$.*

(ii) ***Repeated root*** *($r_1 = r_2 = r$): $y(t) = (C_1 + C_2 \ln t)\, t^r$.*

(iii) ***Complex roots*** *($r = \alpha \pm i\beta$, $\beta > 0$): $y(t) = t^\alpha\Big(C_1 \cos(\beta \ln t) + C_2 \sin(\beta \ln t)\Big)$.*

Proof. Case (i) is immediate. For (ii), with r a double root, $y_1 = t^r$ is one solution; a second is found by reduction of order (Section 4.7) or by the substitution $\tau = \ln t$, giving $y_2 = t^r \ln t$. For (iii), write $t^r = t^{\alpha + i\beta} = t^\alpha e^{i\beta \ln t} = t^\alpha(\cos(\beta \ln t) + i \sin(\beta \ln t))$, and take real and imaginary parts. □

Example 4.21. Solve $t^2 y'' - 2y = 0$ for $t > 0$.
Indicial equation: $r(r-1) - 2 = r^2 - r - 2 = (r-2)(r+1) = 0$, so $r_1 = 2$, $r_2 = -1$.
General solution: $y(t) = C_1 t^2 + C_2 t^{-1}$.

Example 4.22. Solve $t^2 y'' + ty' + y = 0$ for $t > 0$.
Indicial equation: $r(r-1) + r + 1 = r^2 + 1 = 0$, so $r = \pm i$. Thus $\alpha = 0$, $\beta = 1$, and $y(t) = C_1 \cos(\ln t) + C_2 \sin(\ln t)$.

Remark 4.23. The substitution $\tau = \ln t$ (so $t = e^\tau$) transforms the Euler–Cauchy equation (4.11) into a constant-coefficient equation in τ. This is often the quickest approach.

4.7 Reduction of Order

If one solution y_1 of a second-order linear ODE is known, a second linearly independent solution can be found by a standard technique.

Method 4.24 (Reduction of order). Given $y'' + p(t)y' + q(t)y = 0$ with a known solution $y_1(t) \neq 0$, set $y = v(t)\, y_1(t)$. Then v satisfies a first-order ODE in $w = v'$:

$$w' + \left(\frac{2y_1'}{y_1} + p(t)\right) w = 0, \tag{4.12}$$

which is separable. Solving for w and integrating gives v, hence $y_2 = v\, y_1$.

Derivation. Set $y = vy_1$. Then $y' = v'y_1 + vy_1'$, $y'' = v''y_1 + 2v'y_1' + vy_1''$. Substituting:

$$\begin{aligned}(v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + q(vy_1) &= 0\\ v''y_1 + v'(2y_1' + py_1) + v\underbrace{(y_1'' + py_1' + qy_1)}_{=0} &= 0.\end{aligned}$$

Setting $w = v'$ yields (4.12). □

Example 4.25 (Reduction of order)**.** Given that $y_1(t) = e^t$ solves $y'' - 2y' + y = 0$, find a second solution.
Here $p(t) = -2$. Set $w = v'$:

$$w' + \left(\frac{2e^t}{e^t} - 2\right)w = w' + (2-2)w = w' = 0.$$

So $w = C$ (constant), $v = Ct$, and $y_2 = te^t$, confirming the result from the repeated-root case.

Example 4.26 (Reduction of order for Euler–Cauchy)**.** Given that $y_1 = t$ solves $t^2y'' - 2ty' + 2y = 0$ for $t > 0$, find y_2.
Rewrite in standard form: $y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0$, so $p(t) = -2/t$. With $y_1 = t$, $y_1' = 1$:

$$w' + \left(\frac{2}{t} - \frac{2}{t}\right)w = 0 \quad\Longrightarrow\quad w = C.$$

Thus $v = Ct$, and $y_2 = t \cdot t = t^2$.

4.8 Worked Examples

Example 4.27 (Complete IVP)**.** Solve $y'' + y = t\cos t$, with $y(0) = 0$, $y'(0) = 0$.
Homogeneous solution. $r^2 + 1 = 0$, $r = \pm i$, so $y_h = C_1\cos t + C_2\sin t$.
Particular solution. The forcing $g(t) = t\cos t$ has the form $t\,e^{0\cdot t}\cos t$, corresponding to $\gamma = 0$, $\omega = 1$, $n = 1$. Since $0 + i\cdot 1 = i$ is a root of the characteristic equation with multiplicity $s = 1$, we multiply by t:

$$y_p = t\Big[(At + B)\cos t + (Ct + D)\sin t\Big].$$

After computing y_p'' and substituting (a lengthy but routine calculation), one finds

$$y_p = \frac{t^2}{4}\sin t - \frac{t}{4}\cos t.$$

Initial conditions. $y(0) = C_1 + 0 = 0$, so $C_1 = 0$. $y'(0) = C_2 + (-1/4) = 0$, so $C_2 = 1/4$.
Solution: $y(t) = \dfrac{1}{4}\sin t + \dfrac{t^2}{4}\sin t - \dfrac{t}{4}\cos t = \dfrac{(1+t^2)\sin t - t\cos t}{4}.$

Example 4.28 (Exponential forcing with complex roots)**.** Solve $y'' + 2y' + 5y = 4e^{-t}\cos 2t$.
Characteristic equation: $r^2 + 2r + 5 = 0$, giving $r = -1 \pm 2i$. So $y_h = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$.
The forcing is $g(t) = 4e^{-t}\cos 2t$, corresponding to $\gamma + i\omega = -1 + 2i$, which is a root with multiplicity $s = 1$. Try $y_p = te^{-t}(A\cos 2t + B\sin 2t)$.
After substitution and simplification: $-4A\sin 2t + 4B\cos 2t = 4\cos 2t$, giving $A = 0$, $B = 1$. So $y_p = te^{-t}\sin 2t$.
General solution: $y = e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + te^{-t}\sin 2t$.

Example 4.29 (Superposition)**.** Solve $y'' + 4y = 3\sin t + 5e^{2t}$.
By superposition, find particular solutions for each term separately.
For $g_1 = 3\sin t$: from Example 4.15, $y_{p_1} = \sin t$.
For $g_2 = 5e^{2t}$: try $y_{p_2} = Ae^{2t}$. Then $4Ae^{2t} + 4Ae^{2t} = 8Ae^{2t} = 5e^{2t}$, so $A = 5/8$.
General solution: $y = C_1 \cos 2t + C_2 \sin 2t + \sin t + \frac{5}{8}e^{2t}$.

4.9 Summary of Solution Methods

Roots of char. eq.	**Form**	**General solution**
$r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$	Distinct real	$C_1e^{r_1t} + C_2e^{r_2t}$
$r_1 = r_2 = r \in \mathbb{R}$	Repeated	$(C_1 + C_2t)e^{rt}$
$r = \alpha \pm i\beta$	Complex	$e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

Table 4.2: Homogeneous solutions for $ay'' + by' + cy = 0$.

Damping	**Condition**	**Behavior**
Overdamped	$b^2 > 4ac$	Exponential decay, no oscillation
Critically damped	$b^2 = 4ac$	Fastest non-oscillatory decay
Underdamped	$b^2 < 4ac$	Oscillatory decay
Undamped	$b = 0$	Perpetual oscillation

Table 4.3: Damping regimes for $ay'' + by' + cy = 0$ with $a, c > 0$, $b \geq 0$.

4.10 Exercises

Exercise 4.1. Solve the following homogeneous equations. Find the general solution and classify the damping regime.

(a) $y'' + 6y' + 8y = 0$

(b) $y'' + 6y' + 9y = 0$

(c) $y'' + 2y' + 10y = 0$

(d) $4y'' + y = 0$

Exercise 4.2. Solve the IVPs:

(a) $y'' - y = 0$, $y(0) = 3$, $y'(0) = -1$.

(b) $y'' + 4y' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$.

(c) $y'' + 9y = 0$, $y(\pi/3) = 0$, $y'(\pi/3) = 3$.

Exercise 4.3. Find the general solution using undetermined coefficients:

(a) $y'' + y' - 2y = 4e^{3t}$

(b) $y'' + 4y = 8\cos 2t$

(c) $y'' - 4y' + 4y = 6e^{2t}$

(d) $y'' + y = t^2 + 1$

(e) $y'' + 2y' + y = 3te^{-t}$

Exercise 4.4. An undamped oscillator $y'' + 9y = A\cos 3t$ is driven at its natural frequency.

(a) Find the general solution with $y(0) = 0$, $y'(0) = 0$.

(b) Determine the amplitude of oscillation at time $t = 10$.

(c) Sketch the solution for $A = 1$ on $[0, 20]$.

Exercise 4.5. Solve the Euler–Cauchy equations for $t > 0$:

(a) $t^2y'' + 5ty' + 4y = 0$

(b) $t^2y'' - ty' + y = 0$

(c) $t^2y'' + ty' + 4y = 0$

Exercise 4.6. Use reduction of order to find a second solution, given y_1:

(a) $y'' - 4y' + 4y = 0$, $y_1 = e^{2t}$.

(b) $t^2y'' - 3ty' + 4y = 0$ $(t > 0)$, $y_1 = t^2$.

(c) $(1 - t)y'' + ty' - y = 0$ $(0 < t < 1)$, $y_1 = e^t$.

Exercise 4.7. Compute the Wronskian and verify Abel's identity for the solutions $y_1 = e^{-t}\cos 2t$ and $y_2 = e^{-t}\sin 2t$ of $y'' + 2y' + 5y = 0$.

Exercise 4.8. A 2 kg mass on a spring ($k = 50\,\mathrm{N/m}$) with damping ($c = 4\,\mathrm{N\cdot s/m}$) is released from rest at a displacement of 0.1 m.

(a) Write and solve the IVP.

(b) Is the system underdamped, critically damped, or overdamped?

(c) Find the quasi-frequency and quasi-period.

(d) Determine the time at which the displacement first returns to zero.

Exercise 4.9. Solve $y'' + 25y = \cos 4t$ with $y(0) = 0$, $y'(0) = 0$. Identify the beat frequency and the carrier frequency. Sketch the solution for $t \in [0, 10\pi]$.

4.11 Chapter Summary

- The general solution of $ay'' + by' + cy = 0$ is determined by the **characteristic equation** $ar^2 + br + c = 0$ and its three cases (distinct real, repeated, complex conjugate roots).
- The **Wronskian** determines linear independence of solutions. Abel's identity shows it is either always zero or never zero.
- For **non-homogeneous** equations $ay''+by'+cy = g(t)$: the general solution is y_h+y_p, where y_p can be found by undetermined coefficients when g involves polynomials, exponentials, and trigonometric functions.
- **Resonance** occurs when the forcing frequency matches a natural frequency, causing unbounded growth (undamped) or maximal amplitude (damped).
- The **Euler–Cauchy equation** $at^2y'' + bty' + cy = 0$ is solved by the ansatz $y = t^r$, leading to an indicial equation.
- **Reduction of order**: knowing one solution y_1, a second independent solution can always be constructed.

Chapter 5

Linear Differential Systems

5.1 Motivation: Coupled Dynamical Models

Many phenomena in physics, biology, and engineering are naturally described not by a single differential equation but by a *system* of coupled equations. We present two classical examples.

Example 5.1 (Coupled oscillators)**.** Consider two masses m_1, m_2 connected by springs with constants k_1, k_2, k_3 arranged in series along a line. Letting $x_1(t), x_2(t)$ denote the displacements from equilibrium, Newton's second law gives

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1), \qquad m_2 x_2'' = -k_3 x_2 - k_2(x_2 - x_1).$$

Setting $X = (x_1, x_1', x_2, x_2')^T$ this becomes a first-order system $X' = AX$ with a 4×4 constant matrix A.

Example 5.2 (Lotka–Volterra predator-prey model)**.** Let $x(t)$ be the prey population and $y(t)$ the predator population. The classical model reads

$$x' = \alpha x - \beta xy, \qquad y' = \delta xy - \gamma y,$$

where $\alpha, \beta, \gamma, \delta > 0$. This is a *nonlinear* system; however, linearizing around the coexistence equilibrium $(\gamma/\delta,\ \alpha/\beta)$ produces a linear system that governs the local dynamics.

These examples illustrate that first-order linear systems arise both directly and as linearizations of nonlinear models. The general theory developed in this chapter applies to both settings.

5.2 General Form and Basic Properties

Definition 5.3 (Linear differential system)**.** A **linear differential system** on an interval $I \subset \mathbb{R}$ is an equation of the form

$$X'(t) = A(t)\,X(t) + B(t), \tag{5.1}$$

where $A : I \to \mathcal{M}_n(\mathbb{R})$ and $B : I \to \mathbb{R}^n$ are continuous, $X : I \to \mathbb{R}^n$ is the unknown. When $B \equiv 0$ the system is **homogeneous**:

$$X'(t) = A(t)\,X(t). \tag{5.2}$$

Theorem 5.4 (Existence and uniqueness for linear systems)**.** *Let A and B be continuous on an open interval I and let $t_0 \in I$, $X_0 \in \mathbb{R}^n$. Then the initial value problem*

$$X' = A(t)X + B(t), \qquad X(t_0) = X_0$$

has a unique *solution defined on the* entire *interval I.*

Proof. The right-hand side $f(t, X) = A(t)X + B(t)$ is globally Lipschitz in X (with Lipschitz constant $\|A(t)\|$), so the Picard–Lindelöf theorem yields a unique local solution. Since the Lipschitz constant is locally bounded and the solution cannot blow up in finite time (Grönwall's inequality gives $\|X(t)\| \le Ce^{\int \|A(s)\| \mathrm{d}s}$), the solution extends to all of I. □

Proposition 5.5 (Superposition principle)**.** *The set of solutions of the homogeneous system* (5.2) *is a vector space of dimension n.*

Proof. Linearity of the equation implies that the solution set is a subspace of $\mathcal{C}^1(I, \mathbb{R}^n)$. To see that its dimension is n, consider the linear map $\Phi : X \mapsto X(t_0)$ from the solution space to $\mathbb{R}^n$. By Theorem 5.4, Φ is a bijection. □

5.3 Homogeneous Systems with Constant Coefficients

Throughout this section, $A \in \mathcal{M}_n(\mathbb{R})$ is a constant matrix and we study

$$X' = AX. \tag{5.3}$$

5.3.1 The Matrix Exponential

Definition 5.6 (Matrix exponential)**.** For $A \in \mathcal{M}_n(\mathbb{R})$ the **matrix exponential** is

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{5.4}$$

The series converges absolutely in any matrix norm.

Theorem 5.7 (Properties of the matrix exponential). *Let $A, B \in \mathcal{M}_n(\mathbb{R})$ and $t, s \in \mathbb{R}$.*

(i) $e^0 = I_n$.

(ii) $e^{(t+s)A} = e^{tA}\, e^{sA}$.

(iii) If $AB = BA$, *then* $e^{A+B} = e^A\, e^B$.

(iv) e^A *is invertible with* $(e^A)^{-1} = e^{-A}$.

(v) $\dfrac{\mathrm{d}}{\mathrm{d}t} e^{tA} = A\, e^{tA} = e^{tA}\, A$.

(vi) If P is invertible, then $e^{PAP^{-1}} = P\, e^A\, P^{-1}$.

Proof. Properties (i)–(iv) follow from the series definition and absolute convergence. For (v), differentiating term-by-term:

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=1}^{\infty} \frac{k\, t^{k-1} A^k}{k!} = A \sum_{k=1}^{\infty} \frac{(tA)^{k-1}}{(k-1)!} = A\, e^{tA}.$$

For (vi), note $(PAP^{-1})^k = PA^kP^{-1}$ and sum. □

Theorem 5.8 (Solution via matrix exponential). *The unique solution of $X' = AX$, $X(0) = X_0$ is*

$$X(t) = e^{tA}\, X_0. \tag{5.5}$$

Proof. By property (v) of Theorem 5.7, $\frac{\mathrm{d}}{\mathrm{d}t}(e^{tA}X_0) = Ae^{tA}X_0 = A\,X(t)$. At $t = 0$, $X(0) = e^0 X_0 = X_0$. □

Remark 5.9. Computing e^{tA} directly from the series is rarely practical. The standard strategies are:

- Diagonalization: if $A = PDP^{-1}$ with $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, then $e^{tA} = P\, \mathrm{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_n})\, P^{-1}$.
- Jordan form: $A = PJP^{-1}$ gives $e^{tA} = Pe^{tJ}P^{-1}$.
- Cayley–Hamilton: express e^{tA} as a polynomial of degree $\leq n-1$ in A.

5.3.2 Distinct Real Eigenvalues

Method 5.10 (Distinct real eigenvalues). Suppose $A \in \mathcal{M}_n(\mathbb{R})$ has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ with eigenvectors $v_1, \dots, v_n$. The general solution of $X' = AX$ is

$$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n, \qquad c_i \in \mathbb{R}. \tag{5.6}$$

Example 5.11 (A 2×2 system with distinct eigenvalues)**.** Solve $X' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} X$.

The characteristic polynomial is $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$.

- $\lambda_1 = 4$: eigenvector $v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
- $\lambda_2 = -1$: eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

General solution:

$$X(t) = c_1 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

5.3.3 Complex Eigenvalues

When $A \in \mathcal{M}_n(\mathbb{R})$ has a complex eigenvalue $\lambda = \alpha + i\beta$ (with $\beta \neq 0$) and eigenvector $w = u + iv$ ($u, v \in \mathbb{R}^n$), the conjugate $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue with eigenvector $\bar{w} = u - iv$.

Method 5.12 (Complex eigenvalues – real form)*.* The pair of conjugate eigenvalues $\alpha \pm i\beta$ contributes two real-valued solutions:

$$\begin{aligned} X_1(t) &= e^{\alpha t}\Big(\cos(\beta t)\, u - \sin(\beta t)\, v\Big), \\ X_2(t) &= e^{\alpha t}\Big(\sin(\beta t)\, u + \cos(\beta t)\, v\Big). \end{aligned} \tag{5.7}$$

Example 5.13 (Complex eigenvalues)**.** Solve $X' = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} X$.

Eigenvalues: $\lambda = \pm 2i$. For $\lambda = 2i$: $w = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, so $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. General solution:

$$X(t) = c_1 \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + c_2 \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}.$$

The trajectories are circles centered at the origin (a *center*).

5.3.4 Repeated Eigenvalues and Jordan Form

When A does not have n linearly independent eigenvectors, we need generalized eigenvectors.

Definition 5.14 (Generalized eigenvector)**.** A vector $w \in \mathbb{R}^n$ is a **generalized eigenvector** of A of rank k for eigenvalue λ if

$$(A - \lambda I)^k w = 0 \quad \text{and} \quad (A - \lambda I)^{k-1} w \neq 0.$$

Method 5.15 (Jordan block contribution)*.* For a Jordan block of size m associated to eigenvalue λ, with generalized eigenvectors $w_1, \ldots, w_m$ satisfying $(A - \lambda I)w_1 = 0$, $(A -$

$\lambda I)w_j = w_{j-1}$ for $j \geq 2$, the corresponding m linearly independent solutions are

$$X_j(t) = e^{\lambda t} \sum_{i=0}^{j-1} \frac{t^i}{i!} w_{j-i}, \qquad j = 1, \ldots, m. \tag{5.8}$$

Example 5.16 (Repeated eigenvalue, 2×2)**.** Solve $X' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} X$.

The eigenvalue $\lambda = 3$ has algebraic multiplicity 2 but geometric multiplicity 1. Eigenvector: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Generalized eigenvector from $(A - 3I)v_2 = v_1$: $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. General solution:

$$X(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$

Remark 5.17. For a Jordan block $J = \lambda I + N$ where N is the nilpotent part,

$$e^{tJ} = e^{\lambda t} e^{tN} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2 & \cdots & t^{m-1}/(m-1)! \\ 0 & 1 & t & \cdots & t^{m-2}/(m-2)! \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & t \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

5.3.5 Complete Classification of 2×2 Linear Systems

Let $A \in \mathcal{M}_2(\mathbb{R})$ with trace $\tau = \mathrm{tr}(A)$ and determinant $\Delta = \det(A)$. The eigenvalues satisfy $\lambda^2 - \tau\lambda + \Delta = 0$ with discriminant $D = \tau^2 - 4\Delta$.

Theorem 5.18 (Classification of 2×2 linear systems)**.** *Assuming $\Delta \neq 0$ (so the origin is the only equilibrium):*

(i) $D > 0$, $\Delta > 0$, $\tau < 0$: *stable node (two negative real eigenvalues).*

(ii) $D > 0$, $\Delta > 0$, $\tau > 0$: *unstable node (two positive real eigenvalues).*

(iii) $\Delta < 0$: *saddle point (one positive, one negative eigenvalue).*

(iv) $D < 0$, $\tau < 0$: *stable spiral (focus) (complex eigenvalues with negative real part).*

(v) $D < 0$, $\tau > 0$: *unstable spiral (focus) (complex eigenvalues with positive real part).*

(vi) $D < 0$, $\tau = 0$: *center (purely imaginary eigenvalues).*

(vii) $D = 0$, $\tau \neq 0$: *degenerate (star) node or defective node depending on geometric multiplicity.*

5.4 Phase Portraits for 2D Linear Systems

We illustrate the main types with phase portraits.

Stable node: $\lambda_1 = -3,\ \lambda_2 = -1$ Unstable node: $\lambda_1 = 1,\ \lambda_2 = 2$

Figure 5.1: Stable node (left) and unstable node (right).

Saddle: $\lambda_1 = -2,\ \lambda_2 = 1$ Center: $\lambda = \pm 2i$

Figure 5.2: Saddle point (left) and center (right).

5.5 Fundamental Matrix and Wronskian

Definition 5.19 (Fundamental matrix)**.** A **fundamental matrix** for $X' = A(t)X$ is an $n \times n$ matrix $\Phi(t)$ whose columns form a basis of the solution space. Equivalently, $\Phi'(t) = A(t)\Phi(t)$ and $\det \Phi(t) \neq 0$ for all $t \in I$.

Definition 5.20 (Wronskian of a system)**.** If $X_1, \dots, X_n$ are n solutions of $X' = A(t)X$, their **Wronskian** is $W(t) = \det\Big(X_1(t) \mid \cdots \mid X_n(t)\Big)$.

Theorem 5.21 (Abel–Liouville formula for systems)**.** *If $\Phi(t)$ is a fundamental matrix*

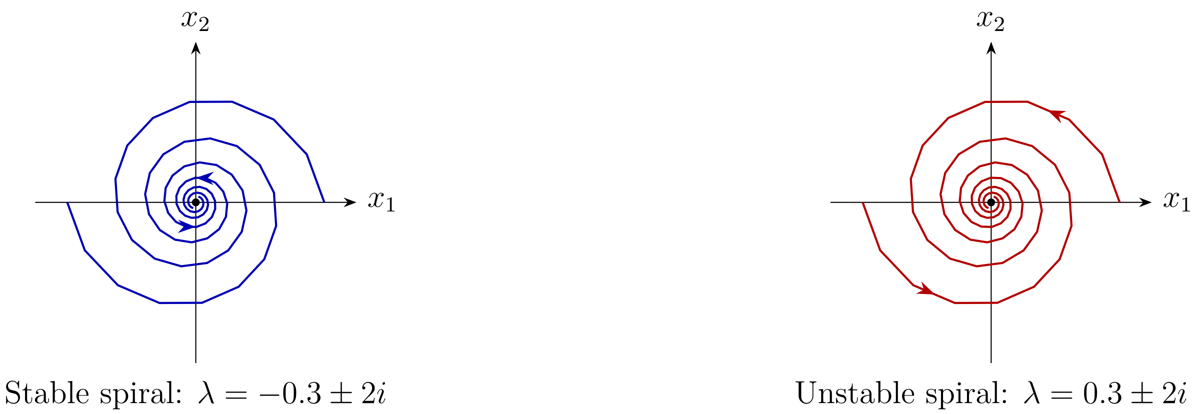


Figure 5.3: Stable spiral (left) and unstable spiral (right).

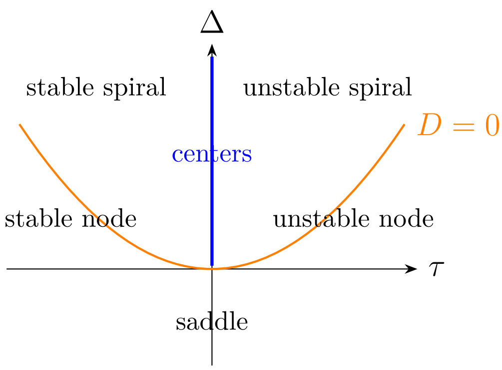


Figure 5.4: Classification in the trace-determinant plane.

for $X' = A(t)X$, then

$$W(t) = \det\Phi(t) = W(t_0)\,\exp\Big(\int_{t_0}^{t} \operatorname{tr} A(s)\,\mathrm{d}s\Big). \tag{5.9}$$

Proof. Write $W(t) = \det\Phi(t)$. Using the formula $\frac{\mathrm{d}}{\mathrm{d}t}\det\Phi = \det\Phi\cdot\operatorname{tr}(\Phi^{-1}\Phi')$ and noting $\Phi^{-1}\Phi' = \Phi^{-1}A\Phi$, so $\operatorname{tr}(\Phi^{-1}\Phi') = \operatorname{tr}(\Phi^{-1}A\Phi) = \operatorname{tr}(A)$, we get

$$W'(t) = \operatorname{tr} A(t)\cdot W(t).$$

This is a scalar linear ODE whose solution is (5.9). □

Corollary 5.22. *The Wronskian $W(t)$ is either identically zero or never zero on I.*

Remark 5.23. The **principal fundamental matrix** at t_0 is the unique fundamental matrix Φ satisfying $\Phi(t_0) = I_n$. For constant coefficients, $\Phi(t) = e^{(t-t_0)A}$.

5.6 Non-Homogeneous Systems: Variation of Parameters

Theorem 5.24 (Variation of parameters for systems)**.** *Let $\Phi(t)$ be a fundamental matrix of $X' = A(t)X$. The general solution of $X' = A(t)X + B(t)$ is*

$$X(t) = \Phi(t)\,C + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\,B(s)\,\mathrm{d}s, \tag{5.10}$$

where $C \in \mathbb{R}^n$ is an arbitrary constant vector.

Proof. We seek a particular solution of the form $X_p(t) = \Phi(t)\,U(t)$ with $U : I \to \mathbb{R}^n$ to be determined. Differentiating:

$$X_p' = \Phi' U + \Phi U' = A\Phi U + \Phi U'.$$

We need $X_p' = AX_p + B = A\Phi U + B$, hence $\Phi U' = B$, i.e., $U'(t) = \Phi^{-1}(t)B(t)$. Integrating gives the result. □

Example 5.25 (Non-homogeneous 2×2 system)**.** Solve $X' = \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix} X + \begin{pmatrix}0\\ e^{2t}\end{pmatrix}$, $X(0) = \begin{pmatrix}0\\ 0\end{pmatrix}$.

The homogeneous system has a repeated eigenvalue $\lambda = 2$. A fundamental matrix is

$$\Phi(t) = e^{2t}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, \qquad \Phi^{-1}(t) = e^{-2t}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}.$$

Then $\Phi^{-1}(t)B(t) = e^{-2t}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ e^{2t}\end{pmatrix} = \begin{pmatrix}-t\\ 1\end{pmatrix}$.

Integrating from 0 to t: $\int_0^t \begin{pmatrix} -s \\ 1 \end{pmatrix} \mathrm{d}s = \begin{pmatrix} -t^2/2 \\ t \end{pmatrix}$.

So $X_p(t) = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -t^2/2 \\ t \end{pmatrix} = e^{2t} \begin{pmatrix} -t^2/2 + t^2 \\ t \end{pmatrix} = e^{2t} \begin{pmatrix} t^2/2 \\ t \end{pmatrix}$.

With $X(0) = 0$, the solution is $X(t) = e^{2t} \begin{pmatrix} t^2/2 \\ t \end{pmatrix}$.

5.7 Exercises

Exercise 5.1. Solve the system $X' = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} X$ with $X(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Exercise 5.2. Compute e^{tA} for $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Exercise 5.3. Show that if A is antisymmetric ($A^T = -A$) then e^{tA} is orthogonal for every $t \in \mathbb{R}$.

Exercise 5.4. Classify the equilibrium at the origin and sketch the phase portrait for:

(a) $X' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} X$.

(b) $X' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} X$.

(c) $X' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X$.

Exercise 5.5. Solve $X' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$ with $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Exercise 5.6. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Compute e^{tA} and solve $X' = AX$ with $X(0) = (1, 0, 0)^T$.

Exercise 5.7. Verify the Abel–Liouville formula for the system $X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$ by computing $W(t)$ directly and via (5.9).

Exercise 5.8. Consider the 3×3 system $X' = AX$ with $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Find the general solution and describe the long-term behavior.

5.8 Chapter Summary

- A linear system $X' = A(t)X + B(t)$ has a unique solution on the entire interval where A, B are continuous.
- The solution space of the homogeneous system is an n-dimensional vector space.
- For constant coefficients, the solution is $X(t) = e^{tA}X_0$, and e^{tA} can be computed via diagonalization, Jordan form, or Cayley–Hamilton.
- For 2×2 systems, the type of equilibrium (node, saddle, spiral, center) is determined by the trace and determinant of A.
- The Wronskian satisfies $W' = \operatorname{tr}(A)\, W$ (Abel–Liouville), so it is either always zero or never zero.
- Variation of parameters: $X_p(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)B(s)\,\mathrm{d}s$.

Chapter 6

Explicit Solution Methods

This chapter develops powerful analytic techniques for solving linear ODEs and systems: the Wronskian, variation of parameters for higher-order equations, Green's functions, power series methods, and reduction of order.

6.1 The Wronskian: Definition, Properties, and Abel's Theorem

Definition 6.1 (Wronskian of n functions)**.** Given n functions $y_1, \dots, y_n \in \mathcal{C}^{n-1}(I)$, their **Wronskian** is

$$W(y_1, \dots, y_n)(t) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} (t). \tag{6.1}$$

Proposition 6.2 (Wronskian and linear independence)**.** *Let $y_1, \dots, y_n$ be solutions of the nth-order linear ODE*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0 \tag{6.2}$$

with continuous coefficients on I. Then $y_1, \dots, y_n$ are linearly independent if and only if $W(y_1, \dots, y_n)(t) \neq 0$ for some (equivalently, all) $t \in I$.

Theorem 6.3 (Abel's theorem)**.** *Let $y_1, \dots, y_n$ be solutions of (6.2). Then*

$$W(t) = W(t_0) \exp\left(- \int_{t_0}^{t} p_{n-1}(s)\, \mathrm{d}s \right). \tag{6.3}$$

Proof. We give the full proof for general n. Write the equation in system form by setting

$X = (y, y', \ldots, y^{(n-1)})^T$. Then $X' = A(t)X$ where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{n-1} \end{pmatrix}.$$

The trace of this companion matrix is $\operatorname{tr} A(t) = -p_{n-1}(t)$. By the Abel–Liouville formula (Theorem 5.21),

$$W(t) = W(t_0) \exp\Big(\int_{t_0}^t \operatorname{tr} A(s)\,\mathrm{d}s\Big) = W(t_0) \exp\Big(-\int_{t_0}^t p_{n-1}(s)\,\mathrm{d}s\Big). \qquad \square$$

Example 6.4 (Abel's theorem for second-order equations)**.** For $y'' + p(t)y' + q(t)y = 0$ with solutions y_1, y_2:

$$W(y_1, y_2)(t) = W(t_0)\, e^{-\int_{t_0}^t p(s)\,\mathrm{d}s}.$$

In particular, for $y'' - y = 0$ ($p \equiv 0$), the solutions $\cosh t, \sinh t$ have constant Wronskian $W = 1$.

Proposition 6.5 (Properties of the Wronskian)**.** *Let $y_1, \ldots, y_n$ be solutions of (6.2).*

(i) W is either identically zero or never zero on I.

(ii) $W \neq 0$ if and only if $\{y_1, \ldots, y_n\}$ is a fundamental system of solutions.

(iii) W satisfies the first-order ODE $W' + p_{n-1}(t)W = 0$.

(iv) If the equation has constant coefficients (p_i constant), then $W(t) = W(0)\, e^{-p_{n-1}t}$.

6.2 Variation of Parameters

6.2.1 For nth-Order Linear ODEs

Consider the non-homogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_0(t)y = g(t). \tag{6.4}$$

Theorem 6.6 (Variation of parameters – nth-order)**.** *Let $y_1, \ldots, y_n$ be a fundamental system of solutions of the homogeneous equation (6.2) with Wronskian W. A particular solution of (6.4) is*

$$y_p(t) = \sum_{k=1}^n y_k(t) \int_{t_0}^t \frac{W_k(s)}{W(s)}\, g(s)\,\mathrm{d}s, \tag{6.5}$$

where W_k is obtained from W by replacing the kth column with $(0, \ldots, 0, 1)^T$.

Proof. We seek $y_p = \sum_{k=1}^n u_k(t)\, y_k(t)$ with the $n - 1$ constraints

$$\sum_{k=1}^n u_k'(t)\, y_k^{(j)}(t) = 0, \qquad j = 0, 1, \ldots, n-2. \tag{6.6}$$

Under these constraints, $y_p^{(j)} = \sum_k u_k y_k^{(j)}$ for $j = 0, \dots, n-1$, and

$$y_p^{(n)} = \sum_{k=1}^{n} u_k y_k^{(n)} + \sum_{k=1}^{n} u_k' y_k^{(n-1)}.$$

Substituting into (6.4) and using the fact that each y_k solves the homogeneous equation:

$$\sum_{k=1}^{n} u_k' y_k^{(n-1)} = g(t).$$

Together with (6.6), we have the system

$$\begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{pmatrix}. \tag{6.7}$$

By Cramer's rule, $u_k'(t) = W_k(t)\, g(t)/W(t)$. Integrating gives (6.5). □

Example 6.7 (Second-order variation of parameters)**.** Solve $y'' - y = \dfrac{1}{\cosh t}$.
Fundamental system: $y_1 = e^t$, $y_2 = e^{-t}$, with $W = -2$.

$$W_1 = \det \begin{pmatrix} 0 & e^{-t} \\ 1/\cosh t & -e^{-t} \end{pmatrix} = -\frac{e^{-t}}{\cosh t},$$
$$W_2 = \det \begin{pmatrix} e^t & 0 \\ e^t & 1/\cosh t \end{pmatrix} = \frac{e^t}{\cosh t}.$$

Thus

$$u_1'(t) = \frac{W_1}{W} \cdot \frac{1}{1} = \frac{e^{-t}}{2\cosh t} = \frac{1}{1+e^{2t}},$$
$$u_2'(t) = \frac{W_2}{W} \cdot \frac{1}{1} = -\frac{e^{t}}{2\cosh t} = -\frac{1}{1+e^{-2t}}.$$

Integrating: $u_1(t) = \arctan(e^t)$ and $u_2(t) = \arctan(e^{-t})$. A particular solution is

$$y_p(t) = e^t \arctan(e^t) + e^{-t} \arctan(e^{-t}).$$

Example 6.8 (Third-order variation of parameters)**.** Solve $y''' + y' = \sec t$ on $(-\pi/2, \pi/2)$.
The characteristic equation $r^3 + r = r(r^2+1) = 0$ gives $r = 0, \pm i$. Fundamental system: $y_1 = 1$, $y_2 = \cos t$, $y_3 = \sin t$. The Wronskian is

$$W = \det \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix} = 1.$$

By Cramer's rule:

$$\begin{aligned} u_1' &= \sec t \cdot \det \begin{pmatrix} 0 & \cos t \\ 0 & -\sin t \end{pmatrix} \Big/ 1 \text{ (with appropriate expansion)} \\ &= \sec t. \end{aligned}$$

Working through all the cofactors yields $u_1' = \sec t$, $u_2' = 0$, $u_3' = -1$, giving $u_1 = \ln|\sec t + \tan t|$, $u_2 = 0$, $u_3 = -t$.
Particular solution: $y_p = \ln|\sec t + \tan t| - t \sin t$.

6.2.2 For Systems

The variation of parameters formula for systems was already established in Theorem 5.24. We restate it here with the initial condition formulation.

Proposition 6.9 (Duhamel formula)**.** *The solution of $X' = A(t)X + B(t)$, $X(t_0) = X_0$ is*

$$X(t) = \Phi(t, t_0)\, X_0 + \int_{t_0}^{t} \Phi(t, s)\, B(s)\, \mathrm{d}s, \tag{6.8}$$

where $\Phi(t, s)$ is the state transition matrix, i.e., $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$ with Φ any fundamental matrix. For constant A, $\Phi(t, s) = e^{(t-s)A}$.

6.3 Green's Function

The variation of parameters formula can be recast using an integral kernel known as the *Green's function.*

Definition 6.10 (Green's function for an IVP)**.** For the nth-order equation (6.4) with initial conditions $y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$, the **Green's function** $G(t, s)$ is defined for $t_0 \le s \le t$ by

$$G(t, s) = \sum_{k=1}^{n} \frac{W_k(s)}{W(s)}\, y_k(t), \tag{6.9}$$

where $y_1, \ldots, y_n$ and W_k are as in Theorem 6.6.

Theorem 6.11 (Solution via Green's function)**.** *The solution of (6.4) with zero initial conditions at t_0 is*

$$y(t) = \int_{t_0}^{t} G(t, s)\, g(s)\, \mathrm{d}s. \tag{6.10}$$

Proposition 6.12 (Properties of the Green's function)**.** *The Green's function $G(t, s)$ satisfies:*

(i) As a function of t (with s fixed), $G(\cdot, s)$ solves the homogeneous equation for $t > s$.

(ii) $G(s,s) = G'(s,s) = \cdots = G^{(n-2)}(s,s) = 0$ *and* $G^{(n-1)}(s,s) = 1$*, where derivatives are with respect to* t*.*

(iii) $G(t,s) = 0$ *for* $t < s$ *(causality).*

Example 6.13 (Green's function for $y'' + y = g(t)$)**.** Fundamental system: $y_1 = \cos t$, $y_2 = \sin t$, $W = 1$.

$$G(t,s) = \frac{-\sin s \cdot \cos t + \cos s \cdot \sin t}{1} = \sin(t-s).$$

The solution with $y(0) = y'(0) = 0$ is $y(t) = \int_0^t \sin(t-s)\, g(s)\, ds$. This is a *convolution* when g is defined on $[0,\infty)$.

Example 6.14 (Green's function for a boundary value problem)**.** Consider $y''+y = g(t)$ on $[0,\pi/2]$ with $y(0) = 0$, $y(\pi/2) = 0$.
The homogeneous solutions satisfying one boundary condition each are: $\phi_1(t) = \sin t$ (satisfies $\phi_1(0) = 0$) and $\phi_2(t) = \cos t$ (satisfies $\phi_2(\pi/2) = 0$). The Wronskian of ϕ_1, ϕ_2 is $W = -1$.
The Green's function is

$$G(t,s) = \begin{cases} -\sin(s)\cos(t) & \text{if } t \geq s, \\ -\sin(t)\cos(s) & \text{if } t \leq s. \end{cases}$$

The solution is $y(t) = \int_0^{\pi/2} G(t,s)\, g(s)\, ds$.

6.4 Power Series Solutions

6.4.1 Ordinary Points

Consider the second-order equation

$$y'' + P(t)y' + Q(t)y = 0. \tag{6.11}$$

Definition 6.15 (Ordinary and singular points)**.** A point t_0 is an **ordinary point** of (6.11) if P and Q are analytic at t_0. Otherwise t_0 is a **singular point**.

Theorem 6.16 (Power series at an ordinary point)**.** *If* t_0 *is an ordinary point of* (6.11)*, then every solution can be represented as a power series*

$$y(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n \tag{6.12}$$

convergent at least on $|t - t_0| < R$*, where* R *is the distance from* t_0 *to the nearest singular point (in* $\mathbb{C}$*).*

Method 6.17 (Finding power series solutions). **Step 1:** Substitute $y = \sum_{n=0}^{\infty} a_n(t-t_0)^n$ into the ODE.

Step 2: Compute $y' = \sum_{n=1}^{\infty} n a_n (t-t_0)^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n(t-t_0)^{n-2}$.

Step 3: Multiply through by the coefficient functions and re-index so all series have the same power of $(t-t_0)$.

Step 4: Set all coefficients to zero to obtain a recurrence relation for the a_n.

Step 5: Solve the recurrence; identify the two linearly independent solutions from free parameters a_0 and a_1.

Example 6.18 (Airy equation)**.** Solve $y'' - ty = 0$ near $t_0 = 0$.
Substituting $y = \sum a_n t^n$:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} a_n t^{n+1} = 0.$$

Shifting indices: let $m = n-2$ in the first sum and $m = n+1$ in the second:

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}\, t^m - \sum_{m=1}^{\infty} a_{m-1}\, t^m = 0.$$

For $m = 0$: $2a_2 = 0$, so $a_2 = 0$. For $m \geq 1$:

$$a_{m+2} = \frac{a_{m-1}}{(m+2)(m+1)}. \tag{6.13}$$

The recurrence produces two independent solutions:

$$y_1(t) = a_0\left(1 + \frac{t^3}{2\cdot 3} + \frac{t^6}{2\cdot 3\cdot 5\cdot 6} + \cdots\right),$$
$$y_2(t) = a_1\left(t + \frac{t^4}{3\cdot 4} + \frac{t^7}{3\cdot 4\cdot 6\cdot 7} + \cdots\right).$$

Both series converge for all $t \in \mathbb{R}$ (no finite singular points). The functions $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$ are specific linear combinations of y_1 and y_2.

Example 6.19 (Hermite equation)**.** Solve $y'' - 2ty' + 2\alpha y = 0$ ($\alpha \in \mathbb{R}$) near $t_0 = 0$.
With $y = \sum a_n t^n$, we obtain the recurrence

$$a_{n+2} = \frac{2(n-\alpha)}{(n+1)(n+2)}\, a_n, \qquad n \geq 0.$$

The even coefficients depend on a_0, the odd on a_1. When $\alpha = m \in \mathbb{N}$, one of the two series terminates, producing the *Hermite polynomials* $H_m(t)$.

6.4.2 Regular Singular Points and the Frobenius Method

Definition 6.20 (Regular singular point)**.** A singular point t_0 of $y''+P(t)y'+Q(t)y=0$ is a **regular singular point** if $(t-t_0)P(t)$ and $(t-t_0)^2Q(t)$ are analytic at t_0. Otherwise it is an **irregular singular point**.

Theorem 6.21 (Frobenius method)**.** *If t_0 is a regular singular point, there exists at least one solution of the form*

$$y(t) = (t-t_0)^r \sum_{n=0}^{\infty} a_n (t-t_0)^n, \qquad a_0 \neq 0, \tag{6.14}$$

*where r satisfies the **indicial equation**.*

Method 6.22 (Frobenius procedure)*.* Write the ODE in the form

$$t^2y'' + t\Big[tp_0 + tp_1t + \cdots\Big]y' + \Big[q_0 + q_1t + \cdots\Big]y = 0$$

(assuming $t_0 = 0$ for simplicity).

Step 1: Write the **indicial equation**: $r(r-1) + p_0\,r + q_0 = 0$.

Step 2: Find the two roots $r_1 \geq r_2$.

Step 3: **Case 1** ($r_1 - r_2 \notin \mathbb{Z}$): two Frobenius series.

Step 4: **Case 2** ($r_1 = r_2$): second solution involves a logarithmic term.

Step 5: **Case 3** ($r_1 - r_2 \in \mathbb{Z}_{>0}$): the larger root always gives a Frobenius series; the second solution may involve a logarithm.

Example 6.23 (Bessel equation of order zero)**.** Consider $t^2y'' + ty' + t^2y = 0$ (regular singular point at $t_0 = 0$).
The indicial equation is $r^2 = 0$, giving a double root $r = 0$. The Frobenius series is

$$y_1(t) = J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{t}{2}\right)^{2n} = 1 - \frac{t^2}{4} + \frac{t^4}{64} - \cdots$$

The second solution involves a logarithmic term:

$$y_2(t) = J_0(t)\ln t + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}H_n}{(n!)^2}\left(\frac{t}{2}\right)^{2n},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the nth harmonic number.

Example 6.24 (Euler–Cauchy equation)**.** The equation $t^2y'' + \alpha ty' + \beta y = 0$ has $t_0 = 0$ as a regular singular point. The indicial equation $r(r-1) + \alpha r + \beta = 0$ recovers the standard Euler–Cauchy approach:

- Distinct real roots $r_1 \neq r_2$: $y = c_1 |t|^{r_1} + c_2 |t|^{r_2}$.
- Double root $r_1 = r_2 = r$: $y = |t|^r (c_1 + c_2 \ln |t|)$.
- Complex roots $r = \alpha' \pm i\beta'$: $y = |t|^{\alpha'} [c_1 \cos(\beta' \ln |t|) + c_2 \sin(\beta' \ln |t|)]$.

6.5 Reduction of Order

Theorem 6.25 (Reduction of order)**.** *If y_1 is a nonzero solution of $y'' + P(t)y' + Q(t)y = 0$, then a second linearly independent solution is*

$$y_2(t) = y_1(t) \int \frac{e^{-\int P(t)\,\mathrm{d}t}}{y_1(t)^2}\,\mathrm{d}t. \tag{6.15}$$

Proof. Set $y_2 = v(t)\,y_1(t)$. Substituting into the equation:

$$v''y_1 + 2v'y_1' + vy_1'' + P(v'y_1 + vy_1') + Qvy_1 = 0.$$

Since $y_1'' + Py_1' + Qy_1 = 0$, this simplifies to

$$v''y_1 + v'(2y_1' + Py_1) = 0.$$

Setting $w = v'$: $w'y_1 + w(2y_1' + Py_1) = 0$, i.e., $\frac{w'}{w} = -\frac{2y_1'}{y_1} - P$. Integrating: $\ln |w| = -2 \ln |y_1| - \int P\,\mathrm{d}t$, hence $w = \frac{e^{-\int P\,\mathrm{d}t}}{y_1^2}$. One more integration gives v, and thus $y_2 = vy_1$. □

Remark 6.26. Note the connection to Abel's theorem: the numerator $e^{-\int P\,\mathrm{d}t}$ in (6.15) is proportional to $W(t)/y_1^2$, consistent with $W = y_1y_2' - y_1'y_2$.

Example 6.27 (Reduction of order)**.** Given that $y_1 = t$ solves $t^2y'' - 2y = 0$ for $t > 0$ (rewritten as $y'' - \frac{2}{t^2}y = 0$, so $P = 0$), find a second solution.

$$y_2 = t \int \frac{e^0}{t^2}\,\mathrm{d}t = t \int t^{-2}\,\mathrm{d}t = t \cdot (-t^{-1}) = -1.$$

Hmm, a constant is trivial. Let us redo more carefully: the equation is $t^2y'' - 2y = 0$, i.e., $y'' - \frac{2}{t^2}y = 0$. We have $P(t) = 0$, $Q(t) = -2/t^2$. Formula:

$$y_2 = t \int \frac{1}{t^2}\,\mathrm{d}t = t \cdot \left(-\frac{1}{t}\right) = -1.$$

So $y_2 = -1$, or equivalently $y_2 = 1$. But $y_2 = 1$ does not solve the equation: $0 - 2 \neq 0$. The issue is that we need to verify: the actual equation $t^2y'' - 2y = 0$ requires $P = 0$ and $Q = -2/t^2$ only if we divide by t^2. Let us try $y_1 = t^2$.
Check: $y_1 = t^2$, $y_1' = 2t$, $y_1'' = 2$. Then $t^2 \cdot 2 - 2 \cdot t^2 = 0$. So $y_1 = t^2$ is a solution.

$$y_2 = t^2 \int \frac{1}{t^4}\,\mathrm{d}t = t^2 \cdot \left(-\frac{1}{3t^3}\right) = -\frac{1}{3t}.$$

So a second independent solution is $y_2 = t^{-1}$. General solution: $y = c_1t^2 + c_2t^{-1}$.

Example 6.28 (From a known solution of a variable-coefficient equation)**.** Verify that $y_1 = e^t$ solves $y'' - \frac{2}{t}y' + \left(1 + \frac{2}{t^2}\right) y = 0$ (for $t > 0$ – note this is not a standard form and requires care) and find a second solution.
We have $P(t) = -2/t$. By the reduction of order formula:

$$y_2 = e^t \int \frac{\exp\left(\int \frac{2}{t}\,\mathrm{d}t\right)}{e^{2t}}\,\mathrm{d}t = e^t \int \frac{t^2}{e^{2t}}\,\mathrm{d}t.$$

The integral $\int t^2 e^{-2t}\,\mathrm{d}t$ can be computed by integration by parts (twice), yielding

$$\int t^2 e^{-2t}\,\mathrm{d}t = -\frac{e^{-2t}}{2}\left(t^2 + t + \frac{1}{2}\right).$$

Hence $y_2 = -\frac{1}{2}\,e^{-t}\left(t^2 + t + \frac{1}{2}\right)$, or equivalently $y_2 = e^{-t}(2t^2 + 2t + 1)$ (up to a constant multiple).

6.6 Method Summary

Table 6.1: Summary of explicit solution methods for linear ODEs.

Method	Applicable When	Key Formula/Idea
Constant coeff. (char. equation)	$a_n y^{(n)} + \cdots + a_0 y = 0$, a_i constant	Solve $a_n r^n + \cdots + a_0 = 0$
Variation of parameters	Any linear ODE $Ly = g(t)$, given fund. system	$u_k' = W_k g/W$; integrate
Green's function	Linear ODE with homogeneous BCs or ICs	$y(t) = \int G(t,s)g(s)\,\mathrm{d}s$
Power series	Ordinary point; analytic coefficients	Substitute $y = \sum a_n t^n$; recurrence for a_n
Frobenius method	Regular singular point	$y = t^r \sum a_n t^n$; indicial equation
Reduction of order	One solution y_1 known	$y_2 = y_1 \int e^{-\int P}/y_1^2$
Matrix exponential	Systems $X' = AX$, A constant	$X(t) = e^{tA}X_0$

6.7 Additional Worked Examples

Example 6.29 (Power series: Legendre equation)**.** The Legendre equation $(1-t^2)y''-2ty'+\ell(\ell+1)y=0$ has $t_0=0$ as an ordinary point. With $y=\sum a_n t^n$, the recurrence is

$$a_{n+2}=\frac{n(n+1)-\ell(\ell+1)}{(n+1)(n+2)}\,a_n.$$

When $\ell\in\mathbb{N}$, one series terminates: the resulting polynomial (suitably normalized) is the Legendre polynomial $P_\ell(t)$. For $\ell=2$: $a_2=-3a_0$, $a_4=0$, so $P_2(t)=\frac{1}{2}(3t^2-1)$ (with normalization $P_2(1)=1$).

Example 6.30 (Frobenius: Bessel equation of order ν)**.** The equation $t^2y''+ty'+(t^2-\nu^2)y=0$ has indicial equation $r^2-\nu^2=0$, so $r=\pm\nu$. For $r=\nu$ with $\nu\geq 0$:

$$J_\nu(t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{t}{2}\right)^{2n+\nu}.$$

When $2\nu\notin\mathbb{Z}$, a second solution is $J_{-\nu}(t)$. When $\nu\in\mathbb{Z}$, the second solution requires a logarithmic term (the Bessel function of the second kind Y_ν).

Example 6.31 (Variation of parameters for a system)**.** Solve $X'=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}X+\begin{pmatrix}0\\ \sec t\end{pmatrix}$, $X(0)=0$.

The homogeneous solution gives $\Phi(t)=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$, $\Phi^{-1}(t)=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$. Then

$$\Phi^{-1}(t)\begin{pmatrix}0\\ \sec t\end{pmatrix}=\begin{pmatrix}-\tan t\\ 1\end{pmatrix}.$$

Integrating from 0 to t: $\int_0^t\begin{pmatrix}-\tan s\\ 1\end{pmatrix}\mathrm{d}s=\begin{pmatrix}\ln\cos t\\ t\end{pmatrix}$.

So

$$X(t)=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}\ln\cos t\\ t\end{pmatrix}=\begin{pmatrix}\cos t\ln\cos t+t\sin t\\ -\sin t\ln\cos t+t\cos t\end{pmatrix}.$$

Example 6.32 (Combining reduction of order with variation of parameters)**.** Solve $y''-\frac{2}{t}y'+\frac{2}{t^2}y=t$ for $t>0$.
Step 1. The homogeneous equation $t^2y''-2ty'+2y=0$ is Euler–Cauchy with indicial equation $r(r-1)-2r+2=r^2-3r+2=(r-1)(r-2)=0$. So $y_1=t$, $y_2=t^2$.
Step 2. Wronskian: $W=ty_2'-y_2\cdot 1=2t^2-t^2=t^2$.

Step 3. Variation of parameters for $y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t$.

$$u_1' = -\frac{y_2 \cdot g}{W} = -\frac{t^2 \cdot t}{t^2} = -t, \qquad u_1 = -\frac{t^2}{2},$$
$$u_2' = \frac{y_1 \cdot g}{W} = \frac{t \cdot t}{t^2} = 1, \qquad u_2 = t.$$

Particular solution: $y_p = u_1 y_1 + u_2 y_2 = -\frac{t^2}{2} \cdot t + t \cdot t^2 = \frac{t^3}{2}$.
General solution: $y = c_1 t + c_2 t^2 + \frac{t^3}{2}$.

6.8 Exercises

Exercise 6.1. Compute the Wronskian of e^t, te^t, t^2e^t and verify Abel's theorem for the ODE they satisfy.

Exercise 6.2. Use variation of parameters to solve $y'' + 4y = \tan(2t)$.

Exercise 6.3. Find the Green's function for $y'' - y = g(t)$ with $y(0) = y'(0) = 0$ and use it to express the solution as an integral.

Exercise 6.4. Find the power series solution of $(1 + t^2)y'' + 2ty' - 2y = 0$ about $t_0 = 0$. Determine the radius of convergence.

Exercise 6.5. Use the Frobenius method to find one solution of $ty'' + y' + ty = 0$ (Bessel equation of order zero, written differently).

Exercise 6.6. Given that $y_1 = \sin t/t$ solves $t^2y'' + ty' + (t^2 - 1)y = 0$ for $t > 0$ (Bessel of order 1, related form), use reduction of order to find a second linearly independent solution.

Exercise 6.7. Solve $y'' + \frac{1}{t}y' - \frac{1}{t^2}y = \frac{1}{t}$ for $t > 0$, given that $y_1 = t$ is a homogeneous solution.

Exercise 6.8. Find the first four nonzero terms of two linearly independent power series solutions of $(1 - t)y'' + y = 0$ about $t_0 = 0$.

Exercise 6.9. Show that the Wronskian of any two solutions of $y'' + (\sin t)y' + (\cos t)y = 0$ satisfies $W(t) = W(0)\, e^{\cos t - 1}$.

Exercise 6.10. Use the Duhamel formula to solve $X' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 1 \\ t \end{pmatrix}$, $X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Exercise 6.11. Classify the singular points of the equation $t^2(t-1)^3y'' + 3t(t-1)y' + 5y = 0$ as regular or irregular.

Exercise 6.12. Find the Green's function for the boundary value problem $y'' + y = g(t)$, $y(0) = 0$, $y(\pi) = 0$. When does this problem have a solution?

6.9 Chapter Summary

- The **Wronskian** of n solutions of a linear ODE measures their linear independence. Abel's theorem provides an explicit formula: $W(t) = W(t_0)\exp(-\int_{t_0}^{t} p_{n-1})$.
- **Variation of parameters** produces a particular solution of any non-homogeneous linear ODE given a fundamental system. The key system is $\sum u_k' y_k^{(j)} = 0$ for $j < n-1$ and $\sum u_k' y_k^{(n-1)} = g$.
- The **Green's function** $G(t, s)$ is an integral kernel that encodes the response of the system to a point source. It satisfies the homogeneous equation with specific jump conditions.
- **Power series** solutions work at ordinary points; the radius of convergence is at least the distance to the nearest singular point.
- The **Frobenius method** handles regular singular points with solutions of the form $t^r \sum a_n t^n$. The indicial equation determines the exponent r.
- **Reduction of order**: knowing one solution y_1 of a second-order equation, the formula $y_2 = y_1 \int e^{-\int P}/y_1^2$ gives a second independent solution.
- All these methods are complementary; the method summary table (Table 6.1) provides a quick reference for choosing the appropriate technique.

Chapter 7

Laplace Transform and Applications

7.1 Motivating Example

Consider an RLC series circuit with resistance R, inductance L, and capacitance C, driven by a voltage source $E(t)$ that switches on at $t = 0$ and off at $t = T$:

$$E(t) = \begin{cases} E_0 & \text{if } 0 \le t < T, \\ 0 & \text{if } t \ge T. \end{cases}$$

The charge $q(t)$ on the capacitor satisfies

$$L\,q'' + R\,q' + \frac{1}{C}\,q = E(t), \qquad q(0) = 0,\; q'(0) = 0. \tag{7.1}$$

The forcing function $E(t)$ is discontinuous, so the usual power series or undetermined-coefficients methods do not apply directly. The *Laplace transform* converts this problem into an algebraic equation in a new variable s, handles discontinuities effortlessly, and incorporates initial conditions automatically.

7.2 Definition and Existence

Definition 7.1 (Laplace transform)**.** Let $f\colon [0,\infty) \to \mathbb{R}$ be piecewise continuous. The **Laplace transform** of f is

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t, \tag{7.2}$$

whenever the improper integral converges.

Definition 7.2 (Exponential order)**.** A function f is of **exponential order** α if there exist constants $M > 0$ and $T \ge 0$ such that

$$|f(t)| \le M\,e^{\alpha t} \quad \text{for all } t \ge T.$$

Theorem 7.3 (Existence of the Laplace transform). *If f is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for all $s > \alpha$, and moreover $|F(s)| \leq M/(s - \alpha)$ for $s > \alpha$.*

Proof. For $s > \alpha$ and $t \geq T$, we have

$$\left| e^{-st} f(t) \right| \leq M\, e^{-st}\, e^{\alpha t} = M\, e^{-(s-\alpha)t}.$$

The integral $\int_T^\infty M\, e^{-(s-\alpha)t}\,\mathrm{d}t = M/(s-\alpha)$ converges. Since f is piecewise continuous, $\int_0^T e^{-st} f(t)\,\mathrm{d}t$ is finite, so the full integral converges absolutely and satisfies the stated bound. □

7.3 Table of Fundamental Transforms

We derive the most important transforms from the definition.

Example 7.4 (Transform of 1). For $f(t) = 1$ and $s > 0$:

$$\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \left[-\frac{e^{-st}}{s} \right]_0^\infty = \frac{1}{s}.$$

Example 7.5 (Transform of e^{at}). For $s > a$:

$$\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a}.$$

Example 7.6 (Transform of t^n). We claim $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$ for $n \in \mathbb{N}$ and $s > 0$. For $n = 0$ this is $1/s$. Integration by parts gives

$$\mathcal{L}\{t^n\} = \left[-\frac{t^n e^{-st}}{s} \right]_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st}\,\mathrm{d}t = \frac{n}{s}\,\mathcal{L}\{t^{n-1}\},$$

and induction yields $\mathcal{L}\{t^n\} = n!/s^{n+1}$.

Example 7.7 (Transform of $\sin(bt)$ and $\cos(bt)$). Since $e^{ibt} = \cos(bt) + i\sin(bt)$ and $\mathcal{L}\{e^{ibt}\} = 1/(s - ib)$, we compute

$$\frac{1}{s - ib} = \frac{s + ib}{s^2 + b^2}.$$

Taking real and imaginary parts:

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2}, \qquad \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2}.$$

7.4 Properties of the Laplace Transform

Theorem 7.8 (Linearity)**.** *For constants a, b and functions f, g with Laplace transforms:*

$$\mathcal{L}\{af + bg\} = a\,\mathcal{L}\{f\} + b\,\mathcal{L}\{g\}.$$

Proof. Immediate from linearity of the integral. □

Theorem 7.9 (First shifting theorem (s-shift))**.** *If $\mathcal{L}\{f\}(s) = F(s)$ for $s > \alpha$, then*

$$\mathcal{L}\{e^{at} f(t)\}(s) = F(s-a) \quad \text{for } s > \alpha + a. \tag{7.3}$$

Proof.

$$\mathcal{L}\{e^{at} f(t)\}(s) = \int_0^\infty e^{-st}\, e^{at}\, f(t)\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t} f(t)\,\mathrm{d}t = F(s-a).$$ □

Example 7.10 (Application of the s-shift)**.**

$$\mathcal{L}\{e^{-3t}\sin(2t)\} = \frac{2}{(s+3)^2+4}.$$

Theorem 7.11 (Second shifting theorem (t-shift))**.** *Let $u_c(t)$ denote the Heaviside step function (see Definition 7.16). If $\mathcal{L}\{f\} = F(s)$, then*

$$\mathcal{L}\{u_c(t)\, f(t-c)\}(s) = e^{-cs}\, F(s) \quad (c \geq 0). \tag{7.4}$$

Proof. Substituting $\tau = t - c$:

$$\int_0^\infty e^{-st}\, u_c(t)\, f(t-c)\,\mathrm{d}t = \int_c^\infty e^{-st}\, f(t-c)\,\mathrm{d}t = \int_0^\infty e^{-s(\tau+c)} f(\tau)\,\mathrm{d}\tau = e^{-cs}\, F(s).$$ □

Theorem 7.12 (Laplace transform of derivatives)**.** *Let f be continuous, f' piecewise continuous, both of exponential order. Then*

$$\mathcal{L}\{f'\}(s) = s\,F(s) - f(0). \tag{7.5}$$

More generally, if $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous, all of exponential order, then

$$\mathcal{L}\{f^{(n)}\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \tag{7.6}$$

Proof. For the first derivative, integration by parts gives

$$\int_0^\infty e^{-st} f'(t)\,\mathrm{d}t = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = -f(0) + s\,F(s).$$

The boundary term at ∞ vanishes by the exponential order assumption. The general formula follows by induction. □

Theorem 7.13 (Multiplication by t^n). *If $\mathcal{L}\{f\} = F(s)$, then*

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s). \tag{7.7}$$

Proof. Differentiating under the integral sign:

$$F'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty (-t)\,e^{-st} f(t)\,\mathrm{d}t = -\mathcal{L}\{tf(t)\}.$$

The general formula follows by iterating. □

Theorem 7.14 (Convolution theorem). *Define the* ***convolution*** *of f and g by*

$$(f * g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,\mathrm{d}\tau. \tag{7.8}$$

If $\mathcal{L}\{f\} = F$ and $\mathcal{L}\{g\} = G$, then

$$\mathcal{L}\{f * g\}(s) = F(s)\,G(s). \tag{7.9}$$

Proof. We write

$$F(s)\,G(s) = \int_0^\infty e^{-s\sigma} f(\sigma)\,\mathrm{d}\sigma \int_0^\infty e^{-s\tau} g(\tau)\,\mathrm{d}\tau = \int_0^\infty\int_0^\infty e^{-s(\sigma+\tau)} f(\sigma)\,g(\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Substituting $t = \sigma + \tau$ (so $\sigma = t - \tau$) and changing the order of integration (justified by absolute convergence), the region $\sigma \geq 0,\ \tau \geq 0$ becomes $t \geq 0,\ 0 \leq \tau \leq t$:

$$F(s)\,G(s) = \int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t = \mathcal{L}\{f * g\}(s).$$ □

Remark 7.15. Convolution is commutative ($f * g = g * f$), associative, and distributes over addition, but $f * 1 \neq f$ in general (indeed $(f * 1)(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$).

7.5 Heaviside and Dirac Delta Functions

Definition 7.16 (Heaviside step function). For $c \geq 0$, the **Heaviside step function** is

$$u_c(t) = \begin{cases} 0 & \text{if } t < c, \\ 1 & \text{if } t \geq c. \end{cases}$$

Proposition 7.17 (Transform of u_c). *For $c \geq 0$ and $s > 0$:*

$$\mathcal{L}\{u_c(t)\}(s) = \frac{e^{-cs}}{s}.$$

Proof. $\int_c^\infty e^{-st}\,\mathrm{d}t = e^{-cs}/s.$ □

Remark 7.18 (Piecewise functions via Heaviside). Any piecewise-defined function can be written using Heaviside functions. For example, the forcing $E(t)$ from (7.1) is $E(t) = E_0\big(u_0(t) - u_T(t)\big)$, so

$$\mathcal{L}\{E\}(s) = E_0\left(\frac{1}{s} - \frac{e^{-Ts}}{s}\right) = \frac{E_0(1 - e^{-Ts})}{s}.$$

Definition 7.19 (Dirac delta function)**.** The **Dirac delta function** $\delta(t-c)$ (for $c \geq 0$) is informally a "function" satisfying $\delta(t-c) = 0$ for $t \neq c$ and $\int_{-\infty}^{\infty} \delta(t-c)\,\varphi(t)\,\mathrm{d}t = \varphi(c)$ for any continuous φ. Rigorously, it is a distribution.

Proposition 7.20 (Transform of $\delta(t-c)$)**.** *For $c \geq 0$:*

$$\mathcal{L}\{\delta(t-c)\}(s) = e^{-cs}.$$

In particular, $\mathcal{L}\{\delta(t)\} = 1$.

7.6 Inverse Laplace Transform and Partial Fractions

The inverse Laplace transform $\mathcal{L}^{-1}\{F\} = f$ is determined (for continuous functions) by uniqueness. In practice, we decompose rational functions of s using partial fractions and read off inverse transforms from a table.

Method 7.21 (Partial fraction decomposition). Given $F(s) = P(s)/Q(s)$ with $\deg P < \deg Q$:

(i) Factor $Q(s)$ into linear and irreducible quadratic factors.

(ii) For each simple real root $s = a$: term $A/(s-a)$.

(iii) For a repeated root $s = a$ of multiplicity m: terms $A_1/(s-a) + A_2/(s-a)^2 + \cdots + A_m/(s-a)^m$.

(iv) For an irreducible quadratic $(s-\alpha)^2 + \beta^2$: term $(As + B)/\big((s-\alpha)^2 + \beta^2\big)$.

(v) Solve for coefficients by equating numerators.

Example 7.22 (Inverse Laplace by partial fractions)**.** Find $\mathcal{L}^{-1}\left\{\dfrac{3s+7}{(s+1)(s^2+4)}\right\}$.

Solution. Write

$$\frac{3s+7}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

Multiplying through: $3s + 7 = A(s^2+4) + (Bs+C)(s+1)$. Setting $s = -1$: $4 = 5A$, so $A = 4/5$. Comparing s^2: $0 = A + B$, so $B = -4/5$. Comparing constant terms:

$7 = 4A + C$, so $C = 7 - 16/5 = 19/5$. Thus

$$\mathcal{L}^{-1}\left\{\frac{3s+7}{(s+1)(s^2+4)}\right\} = \frac{4}{5}\,e^{-t} - \frac{4}{5}\cos(2t) + \frac{19}{10}\sin(2t).$$

7.7 Solving ODEs with the Laplace Transform

Method 7.23 (Laplace transform method for IVPs). To solve an initial value problem:

Step 1. Apply $\mathcal{L}$ to both sides using linearity and the derivative rule (Theorem 7.12).

Step 2. Substitute initial conditions $y(0), y'(0), \ldots$

Step 3. Solve algebraically for $Y(s) = \mathcal{L}\{y\}(s)$.

Step 4. Find $y(t) = \mathcal{L}^{-1}\{Y\}$ using partial fractions and the table.

Example 7.24 (Second-order IVP)**.** Solve $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$.

Solution. Applying $\mathcal{L}$:

$$s^2Y - s\cdot 1 - 0 - 3(sY - 1) + 2Y = \frac{1}{s-3}.$$

Simplifying: $(s^2 - 3s + 2)Y = s - 3 + \frac{1}{s-3}$, i.e.,

$$(s-1)(s-2)\,Y = \frac{(s-3)^2+1}{s-3} = \frac{s^2-6s+10}{s-3}.$$

Partial fractions of $Y(s) = \dfrac{s^2-6s+10}{(s-1)(s-2)(s-3)}$:

$$\frac{s^2-6s+10}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}.$$

Cover-up: $A = \frac{1-6+10}{(-1)(-2)} = \frac{5}{2}$, $B = \frac{4-12+10}{(1)(-1)} = -2$, $C = \frac{9-18+10}{(2)(1)} = \frac{1}{2}$. Therefore

$$y(t) = \frac{5}{2}\,e^{t} - 2\,e^{2t} + \frac{1}{2}\,e^{3t}.$$

Example 7.25 (ODE with discontinuous forcing)**.** Solve $y'' + y = u_\pi(t)$, $y(0) = 0$, $y'(0) = 1$.

Solution. Taking transforms:

$$s^2Y - 1 + Y = \frac{e^{-\pi s}}{s}, \quad \text{so} \quad Y = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s(s^2+1)}.$$

Partial fractions: $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$. By the t-shift theorem:

$$y(t) = \sin t + u_\pi(t)\Big[1 - \cos(t-\pi)\Big] = \sin t + u_\pi(t)\Big[1 + \cos t\Big].$$

Example 7.26 (Impulse response via Dirac delta)**.** Solve $y''+4y=3\,\delta(t-2)$, $y(0)=0$, $y'(0)=0$.

Solution.

$$s^2Y+4Y=3e^{-2s}, \quad Y=\frac{3e^{-2s}}{s^2+4}.$$

Since $\mathcal{L}^{-1}\{1/(s^2+4)\}=\frac{1}{2}\sin(2t)$, the t-shift gives

$$y(t)=\frac{3}{2}\,u_2(t)\sin\big(2(t-2)\big).$$

7.8 Systems and Periodic Forcing

Example 7.27 (System of ODEs)**.** Solve the system

$$x'=2x+y, \quad y'=3x+4y, \qquad x(0)=1,\ y(0)=0.$$

Taking Laplace transforms:

$$\begin{aligned} sX-1&=2X+Y,\\ sY&=3X+4Y. \end{aligned}$$

From the second equation: $Y=3X/(s-4)$. Substituting into the first:

$$(s-2)X-1=\frac{3X}{s-4}, \quad \Longrightarrow \quad X\left[(s-2)-\frac{3}{s-4}\right]=1.$$

Thus $X=\frac{s-4}{s^2-6s+5}=\frac{s-4}{(s-1)(s-5)}$. Partial fractions:

$$X=\frac{3/4}{s-1}+\frac{1/4}{s-5}, \qquad Y=\frac{3X}{s-4}=\frac{-3/4}{s-1}+\frac{3/4}{s-5}.$$

Therefore $x(t)=\frac{3}{4}e^t+\frac{1}{4}e^{5t}$ and $y(t)=-\frac{3}{4}e^t+\frac{3}{4}e^{5t}$.

Theorem 7.28 (Transform of periodic functions)**.** *If f has period $T>0$ (i.e., $f(t+T)=f(t)$ for all $t\geq 0$), then*

$$\mathcal{L}\{f\}(s)=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,\mathrm{d}t. \tag{7.10}$$

Proof. Split the integral into intervals of length T:

$$F(s)=\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)\,\mathrm{d}t.$$

In each term, substitute $\tau=t-nT$, using $f(\tau+nT)=f(\tau)$:

$$F(s)=\sum_{n=0}^{\infty}e^{-nsT}\int_0^T e^{-s\tau}f(\tau)\,\mathrm{d}\tau=\frac{1}{1-e^{-sT}}\int_0^T e^{-s\tau}f(\tau)\,\mathrm{d}\tau.$$ □

Example 7.29 (Square wave)**.** The square wave of period $2a$ is $f(t) = 1$ for $0 \le t < a$ and $f(t) = -1$ for $a \le t < 2a$. Then

$$\int_0^{2a} e^{-st} f(t)\,\mathrm{d}t = \frac{1 - e^{-as}}{s} - \frac{e^{-as} - e^{-2as}}{s} = \frac{(1 - e^{-as})^2}{s}.$$

By Theorem 7.28:

$$\mathcal{L}\{f\}(s) = \frac{(1 - e^{-as})^2}{s(1 - e^{-2as})} = \frac{1 - e^{-as}}{s(1 + e^{-as})} = \frac{1}{s}\tanh\left(\frac{as}{2}\right).$$

7.9 Complete Reference Table

$f(t)$ $(t \ge 0)$	$F(s) = \mathcal{L}\{f\}(s)$	Domain
1	$1/s$	$s > 0$
t^n $(n \in \mathbb{N})$	$n!/s^{n+1}$	$s > 0$
e^{at}	$1/(s-a)$	$s > a$
$\sin(bt)$	$b/(s^2 + b^2)$	$s > 0$
$\cos(bt)$	$s/(s^2 + b^2)$	$s > 0$
$\sinh(bt)$	$b/(s^2 - b^2)$	$s > \lvert b\rvert$
$\cosh(bt)$	$s/(s^2 - b^2)$	$s > \lvert b\rvert$
$t^n e^{at}$	$n!/(s-a)^{n+1}$	$s > a$
$e^{at}\sin(bt)$	$b/\big((s-a)^2 + b^2\big)$	$s > a$
$e^{at}\cos(bt)$	$(s-a)/\big((s-a)^2 + b^2\big)$	$s > a$
$t\sin(bt)$	$2bs/(s^2 + b^2)^2$	$s > 0$
$t\cos(bt)$	$(s^2 - b^2)/(s^2 + b^2)^2$	$s > 0$
$u_c(t)$	e^{-cs}/s	$s > 0$
$\delta(t-c)$	e^{-cs}	all s
$u_c(t)\,f(t-c)$	$e^{-cs}F(s)$	—
$e^{at}f(t)$	$F(s-a)$	—
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	—
$(f * g)(t)$	$F(s)\,G(s)$	—
$f'(t)$	$sF(s) - f(0)$	—
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	—

7.10 Exercises

Exercise 7.1. Compute the Laplace transforms of: (a) $f(t) = 3t^2 - 2e^{-t} + 5\sin(4t)$; (b) $g(t) = t\,e^{2t}\cos(3t)$; (c) $h(t) = u_2(t)\,(t-2)^3\,e^{-(t-2)}$.

Exercise 7.2. Find the inverse Laplace transforms of: (a) $\dfrac{2s+5}{s^2+6s+13}$; (b) $\dfrac{s^2}{(s+1)^3}$; (c) $\dfrac{e^{-3s}}{s^2(s+2)}$.

Exercise 7.3. Use the Laplace transform to solve $y'' + 2y' + 5y = 4e^{-t}\sin(2t)$, $y(0) = 1$, $y'(0) = -1$.

Exercise 7.4. Solve $y'' - y = u_1(t) - u_3(t)$, $y(0) = 0$, $y'(0) = 0$. Express the solution in piecewise form.

Exercise 7.5. Use the convolution theorem to find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}$.

Exercise 7.6. Use Laplace transforms to solve the system

$$x' = x - 2y, \quad y' = 5x - y, \qquad x(0) = 2,\ y(0) = 1.$$

Exercise 7.7. Find the Laplace transform of the sawtooth wave $f(t) = t$ for $0 \le t < 1$ with period 1.

Exercise 7.8. A spring-mass system satisfies $y'' + \omega^2 y = \sum_{k=0}^{N} \delta(t - k\pi/\omega)$ with $y(0) = y'(0) = 0$. Show that $y(t) = \frac{1}{\omega}\sum_{k=0}^{N} u_{k\pi/\omega}(t)\sin\Big(\omega(t - k\pi/\omega)\Big)$ and discuss resonance when $N \to \infty$.

7.11 Chapter Summary

- The Laplace transform converts an ODE initial value problem into an algebraic equation in s, incorporating initial conditions automatically.
- Key properties—linearity, shifting, derivative rule, convolution—form a toolkit for transforming and inverting.
- The Heaviside function and Dirac delta allow systematic treatment of discontinuous and impulsive forcing.
- Partial fraction decomposition is the principal technique for computing inverse transforms.
- The method extends naturally to systems and to periodic forcing via the periodic transform formula.

Chapter 8

Stability of Solutions and Equilibria

8.1 Motivating Example: The Pendulum

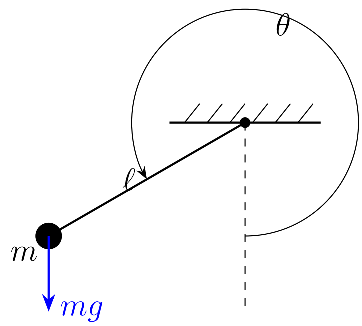
Consider a rigid pendulum of length ℓ and mass m with angular displacement θ from the downward vertical, subject to gravity and damping:

$$m\ell\,\theta'' + c\,\theta' + mg\sin\theta = 0, \tag{8.1}$$

or equivalently the system

$$\theta' = \omega, \qquad \omega' = -\frac{g}{\ell}\sin\theta - \frac{c}{m\ell}\,\omega.$$

This system has equilibria at $(\theta, \omega) = (n\pi, 0)$ for $n \in \mathbb{Z}$. Physical intuition tells us the hanging position $\theta = 0$ is *stable* while the inverted position $\theta = \pi$ is *unstable*. How can we make these notions precise, and prove them rigorously?



8.2 Autonomous Systems and Equilibrium Points

Throughout this chapter we consider the autonomous system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{8.2}$$

where $\mathbf{f}\colon D \to \mathbb{R}^n$ is continuously differentiable on an open set $D \subseteq \mathbb{R}^n$.

Definition 8.1 (Equilibrium point)**.** A point $\mathbf{x}^* \in D$ is an **equilibrium point** (or **critical point**, **fixed point**, **stationary point**) of (8.2) if $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.

Remark 8.2. If $\mathbf{x}^*$ is an equilibrium, then $\mathbf{x}(t) \equiv \mathbf{x}^*$ is a solution for all t. The question of stability concerns the behavior of *other* solutions that start near $\mathbf{x}^*$.

Definition 8.3 (Classification of equilibria for planar linear systems)**.** For the linear system $\mathbf{x}' = A\mathbf{x}$ with $A \in \mathbb{R}^{2\times 2}$ and eigenvalues λ_1, λ_2, the equilibrium at the origin is classified as:

(i) **Stable node**: λ_1, λ_2 real, both negative.

(ii) **Unstable node**: λ_1, λ_2 real, both positive.

(iii) **Saddle point**: λ_1, λ_2 real, opposite signs.

(iv) **Stable spiral** (focus): complex eigenvalues with negative real part.

(v) **Unstable spiral** (focus): complex eigenvalues with positive real part.

(vi) **Center**: purely imaginary eigenvalues.

(vii) **Degenerate cases**: repeated eigenvalues (star node or improper node).

8.3 Lyapunov Stability

Definition 8.4 (Lyapunov stability)**.** The equilibrium $\mathbf{x}^*$ of (8.2) is:

(i) **Stable (in the sense of Lyapunov)** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta \implies \|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon \quad \text{for all } t \geq 0.$$

(ii) **Asymptotically stable** if it is stable and moreover there exists $\delta_0 > 0$ such that
$$\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta_0 \implies \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*.$$

(iii) **Unstable** if it is not stable.

Remark 8.5. Without loss of generality, we may assume $\mathbf{x}^* = \mathbf{0}$ by the change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$. We adopt this convention for the remainder of this chapter.

Example 8.6 (Linear stability)**.** For the linear system $\mathbf{x}' = A\mathbf{x}$, the origin is:

- asymptotically stable if and only if all eigenvalues of A have strictly negative real parts;
- stable if and only if all eigenvalues have non-positive real parts, and every eigenvalue with zero real part has equal algebraic and geometric multiplicity;
- unstable otherwise.

8.4 Lyapunov's Direct Method

Lyapunov's direct (or second) method establishes stability without explicitly solving the ODE. The idea is to find a generalized "energy" function that decreases along trajectories.

Definition 8.7 (Lyapunov function)**.** Let $\mathbf{0}$ be an equilibrium of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. A continuously differentiable function $V\colon D \to \mathbb{R}$ (with $\mathbf{0} \in D$) is a **Lyapunov function** if:

(i) $V(\mathbf{0}) = 0$;

(ii) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in D \setminus \{\mathbf{0}\}$ (i.e., V is *positive definite*);

(iii) $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in D$, where the **orbital derivative** is

$$\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(\mathbf{x}). \tag{8.3}$$

If moreover $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in D \setminus \{\mathbf{0}\}$, we say V is a **strict Lyapunov function**.

Remark 8.8. The orbital derivative $\dot{V}(\mathbf{x})$ gives the rate of change of V along solutions: if $\mathbf{x}(t)$ is a solution, then $\frac{\mathrm{d}}{\mathrm{d}t} V(\mathbf{x}(t)) = \dot{V}(\mathbf{x}(t))$ by the chain rule. Hence $\dot{V} \leq 0$ means V is non-increasing along trajectories.

Theorem 8.9 (Lyapunov stability theorem)**.** *Let $\mathbf{0}$ be an equilibrium of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. If there exists a Lyapunov function V on a neighborhood D of the origin (with $\dot{V} \leq 0$), then the origin is **stable**.*

Proof. Let $\varepsilon > 0$ be small enough that the closed ball $\overline{B}_\varepsilon = \{\mathbf{x} : \|\mathbf{x}\| \leq \varepsilon\} \subseteq D$. Since V is continuous, positive definite, and $V(\mathbf{0}) = 0$, define

$$m = \min_{\|\mathbf{x}\| = \varepsilon} V(\mathbf{x}) > 0.$$

By continuity of V at the origin, there exists $\delta \in (0, \varepsilon)$ such that

$$\|\mathbf{x}\| < \delta \implies V(\mathbf{x}) < m.$$

Now suppose $\|\mathbf{x}(0)\| < \delta$. Then $V(\mathbf{x}(0)) < m$. Since $\dot{V} \leq 0$ along trajectories, we have $V(\mathbf{x}(t)) \leq V(\mathbf{x}(0)) < m$ for all $t \geq 0$. But if $\mathbf{x}(t)$ ever reached the sphere $\|\mathbf{x}\| = \varepsilon$, we would have $V(\mathbf{x}(t)) \geq m$, a contradiction. Therefore $\|\mathbf{x}(t)\| < \varepsilon$ for all $t \geq 0$. ☐

Theorem 8.10 (Asymptotic stability theorem)**.** *Under the hypotheses of Theorem 8.9, if moreover $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in D \setminus \{\mathbf{0}\}$, then the origin is **asymptotically stable**.*

Proof. Stability is given by Theorem 8.9. Choose δ as in that proof, and let $\|\mathbf{x}(0)\| < \delta$. The function $t \mapsto V(\mathbf{x}(t))$ is non-increasing and bounded below by 0, so it converges to some limit $L \geq 0$. We claim $L = 0$.

Suppose $L > 0$. Then $\mathbf{x}(t)$ remains in the compact annular region $K = \{\mathbf{x} : L \leq V(\mathbf{x}) \leq V(\mathbf{x}(0))\} \subseteq \overline{B}_\varepsilon \setminus \{\mathbf{0}\}$. On this compact set, the continuous function $\dot{V}$ achieves a maximum $-\mu < 0$. Therefore

$$V(\mathbf{x}(t)) \leq V(\mathbf{x}(0)) - \mu t \to -\infty \quad \text{as } t \to \infty,$$

contradicting $V \geq 0$. Hence $L = 0$, and since V is positive definite, $\mathbf{x}(t) \to \mathbf{0}$. □

Theorem 8.11 (Lyapunov instability theorem)**.** *Let* $\mathbf{0}$ *be an equilibrium. Suppose there exists a continuously differentiable function* V *on a neighborhood of the origin with* $V(\mathbf{0}) = 0$ *and* $\dot{V}(\mathbf{x}) > 0$ *for* $\mathbf{x} \neq \mathbf{0}$*. If every neighborhood of the origin contains a point where* $V > 0$*, then the origin is* ***unstable****.*

Example 8.12 (Damped nonlinear pendulum)**.** For the pendulum (10.1) with $c > 0$, set $\omega_0^2 = g/\ell$ and $\gamma = c/(m\ell)$, and consider the energy

$$V(\theta, \omega) = \frac{1}{2}\omega^2 + \omega_0^2(1 - \cos\theta). \tag{8.4}$$

Then $V(0,0) = 0$, $V > 0$ for $(\theta, \omega) \neq (0,0)$ with $|\theta| < \pi$, and

$$\dot{V} = \omega\,\omega' + \omega_0^2 \sin\theta \cdot \theta' = \omega(-\omega_0^2 \sin\theta - \gamma\omega) + \omega_0^2\omega\sin\theta = -\gamma\,\omega^2 \leq 0.$$

By Theorem 8.9, the origin is stable. Since $\dot{V} = 0$ only when $\omega = 0$, we can in fact conclude asymptotic stability via LaSalle's principle (Theorem 8.14).

8.4.1 LaSalle's Invariance Principle

When $\dot{V} \leq 0$ but not strictly negative everywhere, the asymptotic stability theorem does not apply directly. LaSalle's principle fills this gap.

Definition 8.13 (Invariant set)**.** A set $M \subseteq D$ is **positively invariant** under $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ if every solution starting in M remains in M for all $t \geq 0$. It is **invariant** if solutions remain in M for all $t \in \mathbb{R}$.

Theorem 8.14 (LaSalle's invariance principle)**.** *Let* $\Omega \subseteq D$ *be a compact positively invariant set for* $\mathbf{x}' = \mathbf{f}(\mathbf{x})$*. Let* $V\colon D \to \mathbb{R}$ *be continuously differentiable with* $\dot{V}(\mathbf{x}) \leq 0$ *on* Ω*. Define*

$$E = \{\mathbf{x} \in \Omega : \dot{V}(\mathbf{x}) = 0\},$$

and let M *be the largest invariant subset of* E*. Then every solution starting in* Ω *converges to* M *as* $t \to \infty$*.*

Corollary 8.15. *If* V *is a Lyapunov function on a neighborhood of the origin,* $\dot{V} \leq 0$*, and the only invariant subset of* $\{x : \dot{V}(\mathbf{x}) = 0\}$ *is* $\{\mathbf{0}\}$*, then the origin is asymptotically stable.*

Example 8.16 (Pendulum revisited)**.** In Example 8.12, $\dot{V} = 0$ iff $\omega = 0$. On the set $\{\omega = 0\}$, the dynamics require $\omega' = -\omega_0^2 \sin\theta = 0$, hence $\theta = 0$. Thus $M = \{(0,0)\}$, and LaSalle's principle gives asymptotic stability of the hanging equilibrium.

8.5 Linearization and the Hartman–Grobman Theorem

Definition 8.17 (Jacobian matrix)**.** The **Jacobian matrix** of $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ at an equilibrium $\mathbf{x}^*$ is the $n \times n$ matrix

$$J = D\mathbf{f}(\mathbf{x}^*) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix}_{\mathbf{x}=\mathbf{x}^*}.$$

The **linearization** of the system at $\mathbf{x}^*$ is $\mathbf{y}' = J\,\mathbf{y}$, where $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$.

Definition 8.18 (Hyperbolic equilibrium)**.** An equilibrium $\mathbf{x}^*$ is **hyperbolic** if no eigenvalue of $J = D\mathbf{f}(\mathbf{x}^*)$ has zero real part.

Theorem 8.19 (Hartman–Grobman)**.** *Let $\mathbf{x}^*$ be a hyperbolic equilibrium of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with Jacobian J. Then there exists a homeomorphism of a neighborhood of $\mathbf{x}^*$ onto a neighborhood of the origin that maps orbits of the nonlinear system to orbits of the linear system $\mathbf{y}' = J\,\mathbf{y}$, preserving the direction of time.*

Remark 8.20. The Hartman–Grobman theorem guarantees that near a hyperbolic equilibrium, the nonlinear phase portrait is qualitatively the same as the linearized one. In particular:

- If all eigenvalues of J have negative real parts, the equilibrium is asymptotically stable.
- If any eigenvalue has positive real part, the equilibrium is unstable.

For non-hyperbolic equilibria (eigenvalues on the imaginary axis), the linearization is *inconclusive*, and one must resort to Lyapunov methods or center manifold theory.

Proposition 8.21 (Stability via linearization in $\mathbb{R}^2$)**.** *For a planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with equilibrium at the origin and Jacobian J, let $\tau = \operatorname{tr} J$ and $\Delta = \det J$. Then:*

(i) If $\Delta < 0$: saddle (unstable).

(ii) If $\Delta > 0$ and $\tau < 0$: asymptotically stable (node if $\tau^2 > 4\Delta$, spiral if $\tau^2 < 4\Delta$).

(iii) If $\Delta > 0$ and $\tau > 0$: unstable (node or spiral).

(iv) If $\Delta > 0$ and $\tau = 0$: center for the linearization, inconclusive for the nonlinear system.

8.6 Phase Plane Analysis

Phase plane analysis provides geometric insight into the behavior of two-dimensional autonomous systems. We sketch trajectories in the (x_1, x_2)-plane, marking equilibria, determining their types, and identifying features such as separatrices, limit cycles, and basins of attraction.

Method 8.22 (Phase plane analysis). For $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$:

Step 1. Find all equilibria by solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

Step 2. Compute the Jacobian at each equilibrium; determine eigenvalues.

Step 3. Classify each equilibrium (node, saddle, spiral, center).

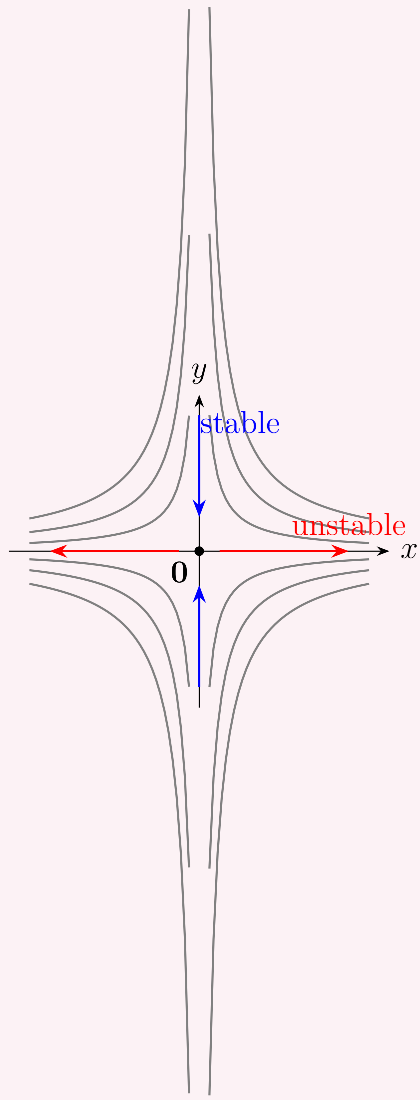
Step 4. Sketch nullclines: curves where $f_1 = 0$ or $f_2 = 0$.

Step 5. Determine the direction field and sketch representative trajectories.

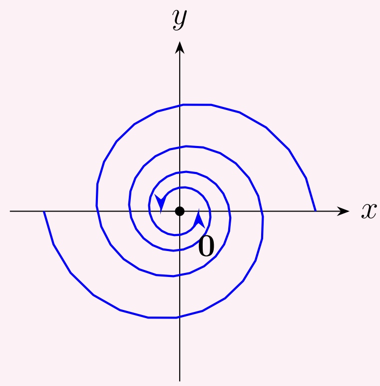
Step 6. Identify invariant sets, separatrices, limit cycles if present.

8.6.1 Phase Portraits

Example 8.23 (Saddle point)**.** The system $x' = x,\ y' = -y$ has eigenvalues $\lambda_1 = 1,\ \lambda_2 = -1$ at the origin: a saddle. Solutions are $x(t) = x_0 e^t,\ y(t) = y_0 e^{-t}$, and trajectories satisfy $xy = \text{const}$.



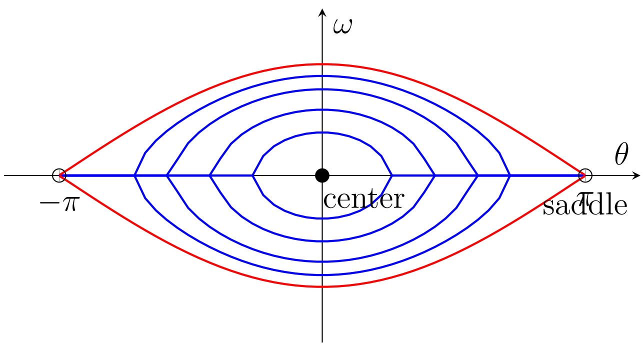
Example 8.24 (Stable spiral)**.** The system $x' = -x + 2y,\ y' = -2x - y$ has Jacobian with $\tau = -2,\ \Delta = 5$, and eigenvalues $-1 \pm 2i$: a stable spiral.



8.7 Examples: Nonlinear Systems

8.7.1 Nonlinear Pendulum

The undamped pendulum $\theta'' + (g/\ell)\sin\theta = 0$ has the phase portrait below. The separatrices (heteroclinic orbits connecting saddle points at $\pm\pi$) divide the phase plane into oscillatory regions (closed orbits) and rotational regions (unbounded trajectories).



8.7.2 Lotka–Volterra Predator-Prey Model

Example 8.25 (Lotka–Volterra system)**.** The classical predator-prey model is

$$x' = \alpha x - \beta xy, \qquad y' = -\gamma y + \delta xy, \tag{8.5}$$

where $x \geq 0$ is the prey population, $y \geq 0$ is the predator population, and $\alpha, \beta, \gamma, \delta > 0$.

Equilibria. Setting the right-hand side to zero gives two equilibria:

$$\mathbf{x}_1^* = (0, 0), \qquad \mathbf{x}_2^* = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right).$$

Linearization at $\mathbf{x}_2^*$. The Jacobian is

$$J = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{pmatrix}.$$

At $\mathbf{x}_2^* = (\gamma/\delta, \alpha/\beta)$:

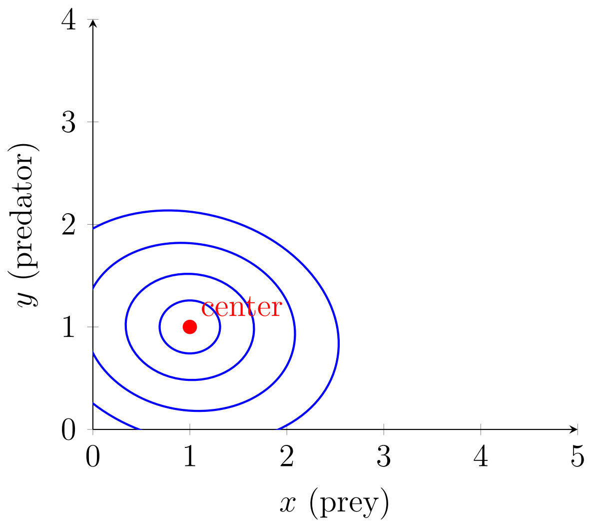
$$J = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \alpha\delta/\beta & 0 \end{pmatrix}, \qquad \lambda = \pm i\sqrt{\alpha\gamma}.$$

The eigenvalues are purely imaginary, so the linearization predicts a center. This is a non-hyperbolic case; the linearization is inconclusive.

Lyapunov analysis. Define the conserved quantity (first integral)

$$V(x, y) = \delta x - \gamma \ln x + \beta y - \alpha \ln y. \tag{8.6}$$

One verifies that $\dot{V} = 0$ along solutions, so trajectories lie on level curves of V. Since V is strictly convex near $\mathbf{x}_2^*$ with a minimum there, the level curves are closed, confirming that $\mathbf{x}_2^*$ is a center (stable but not asymptotically stable).



8.7.3 Van der Pol Oscillator

Example 8.26 (Van der Pol oscillator)**.** The van der Pol equation is

$$x'' - \mu(1 - x^2)x' + x = 0, \qquad \mu > 0, \tag{8.7}$$

or as a system:

$$x' = y, \qquad y' = \mu(1 - x^2)y - x.$$

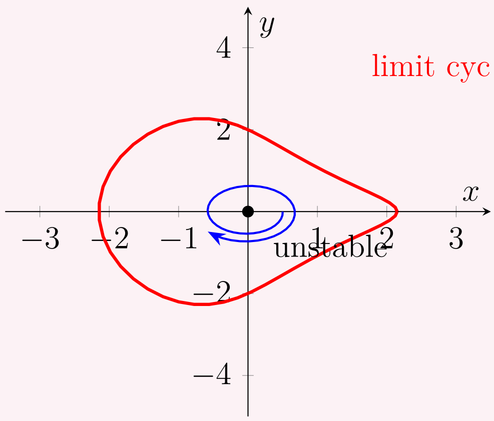
The unique equilibrium is the origin.

Linearization. The Jacobian at the origin is

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}, \qquad \tau = \mu > 0,\ \Delta = 1 > 0.$$

Since $\tau > 0$, the origin is an unstable spiral (for small μ) or unstable node (for $\mu \geq 2$).

Limit cycle. Despite the instability of the equilibrium, all trajectories are bounded (one can show this using Lyapunov-type arguments). By the Poincaré–Bendixson theorem, there must exist a stable limit cycle. This is a distinctive feature of relaxation oscillations: the system settles into a periodic orbit regardless of initial conditions.



8.8 Lyapunov Functions: Construction Techniques

Finding Lyapunov functions is generally an art. We collect useful strategies.

Method 8.27 (Energy-based Lyapunov functions)*.* For mechanical systems $q'' + g(q) = 0$ (possibly with damping), the total energy

$$V(q, p) = \frac{1}{2}p^2 + \int_0^q g(s)\,\mathrm{d}s \qquad (p = q')$$

is a natural candidate.

Method 8.28 (Quadratic Lyapunov functions)*.* For a linear system $\mathbf{x}' = A\mathbf{x}$ with A having all eigenvalues with negative real parts, the quadratic form $V(\mathbf{x}) = \mathbf{x}^T P\,\mathbf{x}$ is a Lyapunov function, where P is the unique positive definite solution of the **Lyapunov equation**

$$A^T P + PA = -Q \tag{8.8}$$

for any chosen positive definite matrix Q (commonly $Q = I$).

Example 8.29 (Quadratic Lyapunov function)**.** Consider $x' = -x + y,\ y' = -x - 3y$. The matrix $A = \left(\begin{smallmatrix} -1 & 1 \\ -1 & -3 \end{smallmatrix}\right)$ has eigenvalues $-2 \pm i$ (both with negative real part). Taking $Q = I$, the Lyapunov equation $A^T P + PA = -I$ yields

$$P = \begin{pmatrix} 5/8 & -1/8 \\ -1/8 & 3/8 \end{pmatrix},$$

and $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x} = \frac{1}{8}(5x^2 - 2xy + 3y^2)$ is a Lyapunov function with $\dot{V} = -x^2 - y^2$.

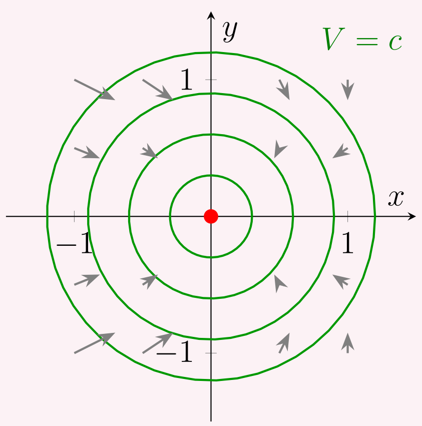
Method 8.30 (Variable gradient method). Seek V by postulating a gradient form $\nabla V = M(\mathbf{x})$ with undetermined coefficients, then imposing the integrability condition $\partial M_i / \partial x_j = \partial M_j / \partial x_i$ and the sign condition $\dot{V} = M \cdot \mathbf{f} \leq 0$.

8.9 Lyapunov Level Curves

Example 8.31 (Visualizing a Lyapunov function)**.** Consider the system $x' = -x + y^2,\ y' = -y$. The origin is an equilibrium. The function $V(x, y) = x^2 + y^2$ gives

$$\dot{V} = 2x(-x + y^2) + 2y(-y) = -2x^2 + 2xy^2 - 2y^2.$$

This is not sign-definite globally, but in a small neighborhood of the origin (say $|y| < 1$), we have $2xy^2 \leq 2\,|x|\,y^2 \leq x^2 + y^4 \leq x^2 + y^2$ for $|y| \leq 1$, so $\dot{V} \leq -x^2 - y^2 < 0$ for small $\|(x, y)\|$, confirming local asymptotic stability.



8.10 Summary of Stability Criteria

Method	Condition	Conclusion
Eigenvalues of J	All $\mathrm{Re}(\lambda_i) < 0$	Asymptotically stable
	Any $\mathrm{Re}(\lambda_i) > 0$	Unstable
	Some $\mathrm{Re}(\lambda_i) = 0$, rest < 0	Inconclusive (non-hyperbolic)
Lyapunov function	$V > 0$, $\dot{V} \leq 0$	Stable
	$V > 0$, $\dot{V} < 0$	Asymptotically stable
	LaSalle: $\dot{V} \leq 0$, largest invariant set in $\{\dot{V} = 0\}$ is $\{\mathbf{0}\}$	Asymptotically stable
Trace–determinant	$\Delta > 0$, $\tau < 0$	Asymptotically stable (2D)
($\mathbb{R}^2$ only)	$\Delta < 0$	Saddle (unstable)

8.11 Exercises

Exercise 8.1. Find and classify all equilibria of the system

$$x' = x(3 - x - 2y), \qquad y' = y(2 - x - y).$$

Determine the stability of each equilibrium using linearization.

Exercise 8.2. For the system $x' = -x^3$, $y' = -y$, show that $V(x, y) = x^4/4 + y^2/2$ is a strict Lyapunov function, and conclude that the origin is asymptotically stable.

Exercise 8.3. Consider $x'' + x' + x^3 = 0$. Write this as a first-order system, find a Lyapunov function, and use LaSalle's invariance principle to prove asymptotic stability of the origin.

Exercise 8.4. For the Lotka–Volterra system (9.3) with $\alpha = 1, \beta = 0.5, \gamma = 0.75, \delta = 0.25$:

(a) Find the non-trivial equilibrium.

(b) Verify that $V(x, y) = 0.25x - 0.75 \ln x + 0.5y - \ln y$ is a conserved quantity.

(c) Sketch several level curves of V in the first quadrant.

Exercise 8.5. Two species compete according to

$$x' = x(1 - x - \alpha y), \qquad y' = y(1 - \beta x - y).$$

Find all equilibria. For which values of $\alpha, \beta > 0$ is the coexistence equilibrium stable?

Exercise 8.6. For the van der Pol oscillator (9.2) with $\mu = 0.5$:

(a) Show that the origin is an unstable spiral.

(b) Show that the set $\{(x, y) : x^2 + y^2 \leq R^2\}$ is positively invariant for sufficiently large R.

(c) Conclude the existence of a limit cycle by the Poincaré–Bendixson theorem.

Exercise 8.7. For the matrix $A = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}$, solve the Lyapunov equation $A^T P + PA = -I$ and verify that $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ is a Lyapunov function for $\mathbf{x}' = A\mathbf{x}$.

Exercise 8.8. Consider the system

$$x' = -x + 2\arctan(y), \qquad y' = -y + 2\arctan(x).$$

(a) Show that the origin is the unique equilibrium.

(b) Show that the origin is locally asymptotically stable by linearization.

(c) Attempt to construct a Lyapunov function for global stability analysis.

(d) Does the system have other attractors? Justify your answer.

8.12 Chapter Summary

- An equilibrium $\mathbf{x}^*$ is Lyapunov stable if nearby solutions remain nearby, and asymptotically stable if they also converge to $\mathbf{x}^*$.
- **Linearization**: near a hyperbolic equilibrium, the nonlinear system behaves qualitatively like its linearization (Hartman–Grobman theorem). Non-hyperbolic cases require further analysis.
- **Lyapunov's direct method**: a positive definite function with non-positive orbital derivative proves stability; strict negativity gives asymptotic stability.
- **LaSalle's invariance principle** extends asymptotic stability conclusions to cases where $\dot{V} \leq 0$ but is not strictly negative everywhere.
- The trace–determinant plane gives a complete classification of linear planar systems.
- Phase plane analysis—nullclines, direction fields, separatrices—provides geometric understanding of two-dimensional dynamics.
- Classical examples (pendulum, Lotka–Volterra, van der Pol) illustrate the interplay between linearization, Lyapunov methods, and geometric reasoning.

Chapter 9

Introduction to Nonlinear Systems and Bifurcations

9.1 Motivation: Nonlinearity in Nature

Many phenomena in science and engineering are inherently nonlinear. Population dynamics, chemical reactions, fluid mechanics, and climate models all give rise to differential equations that cannot be reduced to linear form. In this chapter we develop qualitative tools for analysing autonomous planar systems

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y), \tag{9.1}$$

where $f, g \colon U \to \mathbb{R}$ are smooth functions defined on an open set $U \subseteq \mathbb{R}^2$.

Example 9.1 (Population growth with saturation)**.** The logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

models a population $N(t)$ with growth rate $r > 0$ and carrying capacity $K > 0$. The nonlinear term $-rN^2/K$ prevents unbounded exponential growth and produces a sigmoidal solution curve.

Example 9.2 (Chemical kinetics)**.** Consider two reacting species with concentrations x and y governed by

$$\dot{x} = a - x - \frac{4xy}{1 + x^2}, \qquad \dot{y} = bx\left(1 - \frac{y}{1 + x^2}\right),$$

where $a, b > 0$ are parameters. The rational nonlinearities preclude any closed-form solution; qualitative methods are essential.

9.2 Phase Plane Analysis

9.2.1 Nullclines and Direction Fields

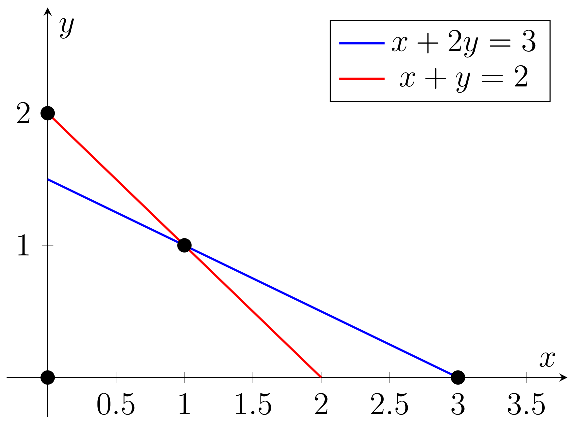
Definition 9.3 (Nullclines)**.** For the system (9.1), the x-*nullcline* is the set $\{(x, y) \in U : f(x, y) = 0\}$, and the y-*nullcline* is $\{(x, y) \in U : g(x, y) = 0\}$. Equilibrium points lie at the intersections of the two nullclines.

On the x-nullcline the vector field is vertical ($\dot{x} = 0$), and on the y-nullcline it is horizontal ($\dot{y} = 0$). The nullclines therefore divide the phase plane into regions in which the signs of $\dot{x}$ and $\dot{y}$ are constant.

Example 9.4 (Nullclines of a competing-species model)**.** Consider

$$\dot{x} = x(3 - x - 2y), \qquad \dot{y} = y(2 - y - x).$$

The x-nullcline consists of the lines $x = 0$ and $x + 2y = 3$; the y-nullcline consists of $y = 0$ and $x + y = 2$. The equilibrium points are $(0, 0)$, $(3, 0)$, $(0, 2)$, and $(1, 1)$.



9.2.2 Equilibrium Classification in Nonlinear Systems

Theorem 9.5 (Hartman–Grobman)**.** *Let $\mathbf{x}^*$ be a hyperbolic equilibrium of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, i.e. the Jacobian $D\mathbf{F}(\mathbf{x}^*)$ has no eigenvalue with zero real part. Then, near $\mathbf{x}^*$, the nonlinear flow is topologically conjugate to the linearised flow $\dot{\mathbf{u}} = D\mathbf{F}(\mathbf{x}^*)\,\mathbf{u}$.*

In practice, one computes the Jacobian

$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}\Bigg|_{(x^*, y^*)}$$

and classifies the equilibrium via the eigenvalues of J, exactly as for linear systems, provided $\mathbf{x}^*$ is hyperbolic.

9.3 The Poincaré–Bendixson Theorem

Theorem 9.6 (Poincaré–Bendixson)**.** *Let* $\mathbf{F} \in C^1(\mathbb{R}^2)$*. Suppose that a forward orbit* γ^+ *is contained in a compact set* $K \subset \mathbb{R}^2$ *that contains no equilibrium points. Then* γ^+ *approaches a periodic orbit (a* limit cycle*).*

Remark 9.7. The Poincaré–Bendixson theorem is specific to $\mathbb{R}^2$. In $\mathbb{R}^3$ and higher dimensions, bounded trajectories can exhibit chaotic behaviour (cf. the Lorenz system).

Corollary 9.8 (Trapping region criterion)**.** *If one can exhibit an annular region* A *such that the vector field points inward on both boundary components and* A *contains no equilibrium, then* A *contains at least one limit cycle.*

9.4 Limit Cycles

Definition 9.9 (Limit cycle)**.** A *limit cycle* is an isolated periodic orbit Γ of the planar system (9.1). It is *stable* (attracting) if nearby orbits spiral towards Γ as $t \to +\infty$, *unstable* if they spiral away, and *semi-stable* if the behaviour differs on each side.

9.4.1 The van der Pol Oscillator

The van der Pol equation

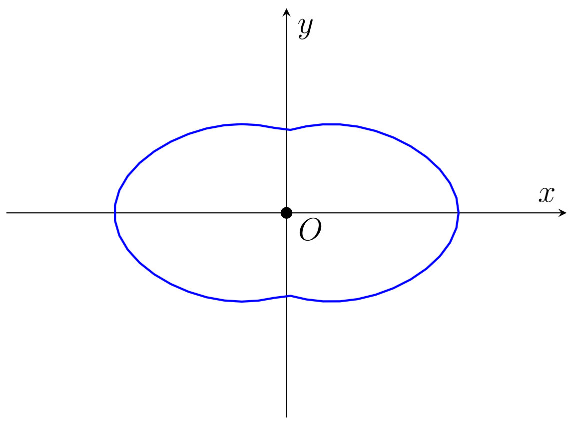
$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu > 0, \tag{9.2}$$

can be rewritten as the system

$$\dot{x} = y, \qquad \dot{y} = -x - \mu(x^2 - 1)y.$$

The origin is the unique equilibrium. For any $\mu > 0$, the Jacobian there has eigenvalues with positive real part, so it is an unstable spiral. By constructing a suitable trapping region (a large circle on which the flow points inward), the Poincaré–Bendixson theorem guarantees the existence of a stable limit cycle.

Van der Pol limit cycle ($\mu = 1$)



9.4.2 Dulac's Criterion

Dulac's criterion provides a sufficient condition for the *non-existence* of periodic orbits.

Theorem 9.10 (Dulac's criterion)**.** *Let $D \subseteq \mathbb{R}^2$ be a simply connected open set and $B\colon D \to \mathbb{R}$ a C^1 function such that $\nabla \cdot (B\mathbf{F})$ does not change sign in D and is not identically zero on any open subset. Then the system (9.1) has no periodic orbit lying entirely in D.*

Proof. Suppose Γ is a periodic orbit enclosing a region $R \subset D$. By Green's theorem,

$$\oint_\Gamma \big(-Bg\,\mathrm{d}x + Bf\,\mathrm{d}y\big) = \iint_R \left(\frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

The left side equals $\oint_\Gamma B(f\,\mathrm{d}y - g\,\mathrm{d}x)$; along Γ we have $\mathrm{d}y/\mathrm{d}t = g$ and $\mathrm{d}x/\mathrm{d}t = f$, so the integrand becomes $B(fg - gf) = 0$, giving 0 on the left. But the right side has a fixed sign, a contradiction. □

Example 9.11 (Applying Dulac's criterion)**.** For the system $\dot{x} = y$, $\dot{y} = -x - y^3$, choose $B = 1$. Then

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 + (-3y^2) = -3y^2 \leq 0,$$

and $-3y^2 = 0$ only on the x-axis (a set of measure zero). By Dulac's criterion there is no periodic orbit in $\mathbb{R}^2$.

9.5 Bifurcation Theory

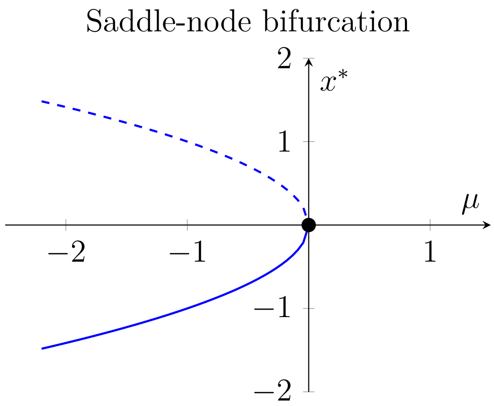
A *bifurcation* occurs when a small change in a parameter μ causes a qualitative change in the phase portrait of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x};\mu)$.

9.5.1 Saddle-Node Bifurcation

Definition 9.12 (Saddle-node bifurcation)**.** The prototypical saddle-node (or fold) bifurcation is given by

$$\dot{x} = \mu + x^2.$$

For $\mu < 0$ there are two equilibria $x^* = \pm\sqrt{-\mu}$ (one stable, one unstable); for $\mu > 0$ there are none.

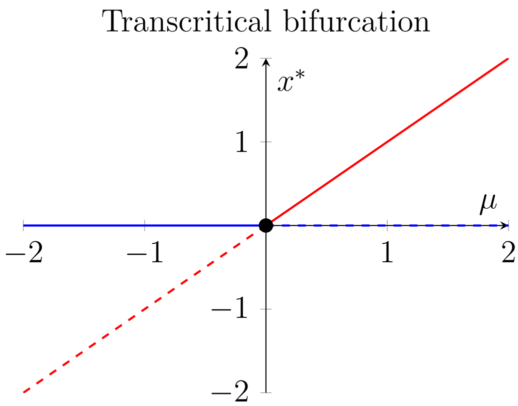


9.5.2 Transcritical Bifurcation

Definition 9.13 (Transcritical bifurcation)**.** The normal form is

$$\dot{x} = \mu x - x^2.$$

There are always two equilibria, $x^* = 0$ and $x^* = \mu$, which exchange stability as μ passes through 0.

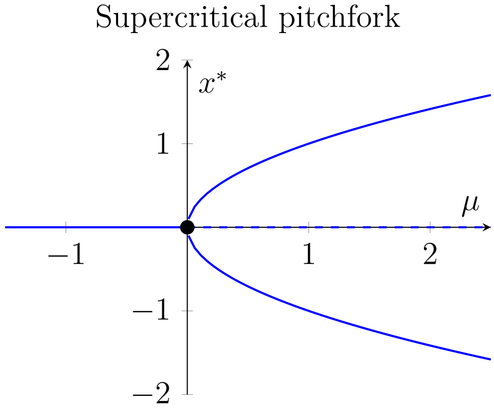


9.5.3 Pitchfork Bifurcation

Definition 9.14 (Pitchfork bifurcation)**.** The *supercritical* pitchfork has normal form

$$\dot{x} = \mu x - x^3.$$

For $\mu \leq 0$, $x^* = 0$ is the unique (stable) equilibrium. For $\mu > 0$, the origin becomes unstable and two stable branches $x^* = \pm\sqrt{\mu}$ appear.



9.5.4 Hopf Bifurcation

Theorem 9.15 (Hopf bifurcation)**.** *Consider a planar system* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x};\mu)$ *with an equilibrium* $\mathbf{x}^*(\mu)$ *whose Jacobian has eigenvalues* $\lambda(\mu) = \alpha(\mu) \pm i\beta(\mu)$*. Suppose that at* $\mu = \mu_0$*:*

(i) $\alpha(\mu_0) = 0$ *and* $\beta(\mu_0) \neq 0$ *(a pair of purely imaginary eigenvalues);*

(ii) $\alpha'(\mu_0) \neq 0$ *(the eigenvalues cross the imaginary axis with non-zero speed).*

Then a family of periodic orbits is born from the equilibrium at $\mu = \mu_0$*. In the* supercritical *case a stable limit cycle appears for* $\mu > \mu_0$*; in the* subcritical *case an unstable limit cycle exists for* $\mu < \mu_0$*.*

Example 9.16 (Supercritical Hopf bifurcation)**.** The system in polar coordinates

$$\dot{r} = \mu r - r^3, \qquad \dot{\theta} = 1,$$

has equilibrium $r = 0$ (the origin). For $\mu > 0$, the stable limit cycle $r = \sqrt{\mu}$ appears, illustrating a supercritical Hopf bifurcation at $\mu = 0$.

9.6 Index Theory

Definition 9.17 (Index of a curve)**.** Let $\mathcal{C}$ be a simple closed curve in the phase plane that does not pass through any equilibrium. The *index* of $\mathcal{C}$ (with respect to the vector field $\mathbf{F}$) is the number of complete counter-clockwise rotations made by the vector $\mathbf{F}$ as one traverses $\mathcal{C}$ once counter-clockwise:

$$I_{\mathcal{C}} = \frac{1}{2\pi} \oint_{\mathcal{C}} \mathrm{d}\left(\arctan \frac{g}{f}\right).$$

Proposition 9.18 (Properties of the index)**.** *(i) The index of a node, spiral, or centre is* $+1$*.*

(ii) The index of a saddle is -1.

(iii) If $\mathcal{C}$ *encloses no equilibrium, then* $I_{\mathcal{C}} = 0$.

(iv) If $\mathcal{C}$ *encloses equilibria* $\mathbf{x}_1^*, \ldots, \mathbf{x}_k^*$, *then* $I_{\mathcal{C}} = \sum_{j=1}^{k} I_j$.

(v) A limit cycle must have index $+1$, *so it must enclose equilibria whose indices sum to* $+1$.

Corollary 9.19. *Every limit cycle must enclose at least one equilibrium point.*

9.7 Worked Examples

9.7.1 The Lotka–Volterra Predator–Prey Model

Example 9.20 (Lotka–Volterra system)**.** The classical predator–prey model reads

$$\dot{x} = \alpha x - \beta xy, \qquad \dot{y} = \delta xy - \gamma y, \tag{9.3}$$

with $\alpha, \beta, \gamma, \delta > 0$. Here x denotes prey and y denotes predators.
Equilibria. Setting the right-hand sides to zero gives $(0, 0)$ and $(x^*, y^*) = (\gamma/\delta,\ \alpha/\beta)$.
Jacobian. We compute

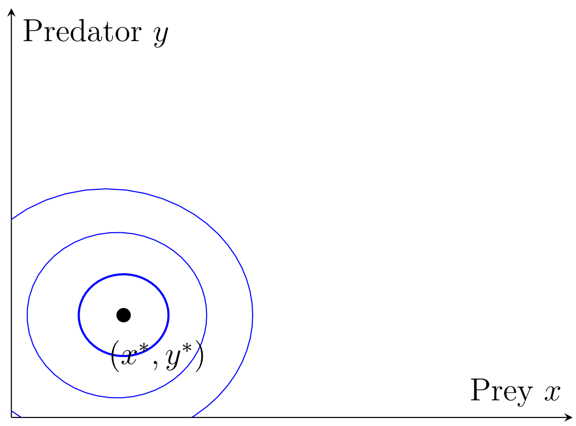
$$J = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}.$$

At $(0, 0)$: eigenvalues $\alpha > 0$ and $-\gamma < 0$ — a saddle.
At $(\gamma/\delta, \alpha/\beta)$: the trace is 0 and the determinant is $\alpha\gamma > 0$, so the eigenvalues are purely imaginary. The equilibrium is a centre for the linearised system, and the Lotka–Volterra system possesses the conserved quantity

$$H(x, y) = \delta x - \gamma \ln x + \beta y - \alpha \ln y,$$

confirming that every orbit in the positive quadrant is a closed curve (a true nonlinear centre).

Lotka–Volterra phase portrait



9.7.2 The Van der Pol Oscillator Revisited

Example 9.21 (Liénard form and relaxation oscillations)**.** Setting $y = \dot{x} + \mu(x^3/3 - x)$ transforms the van der Pol equation (9.2) into the Liénard form

$$\dot{x} = y - \mu\left(\frac{x^3}{3} - x\right), \qquad \dot{y} = -x.$$

For large μ, the limit cycle approaches the cubic nullcline and the solution exhibits *relaxation oscillations*: slow drift along the nullcline branches alternating with fast jumps between them.

9.7.3 The SIR Epidemic Model

Example 9.22 (SIR dynamics)**.** The Kermack–McKendrick SIR model is

$$\dot{S} = -\beta SI, \qquad \dot{I} = \beta SI - \gamma I, \qquad \dot{R} = \gamma I,$$

with $S + I + R = N$ constant. Since $R = N - S - I$, we reduce to the planar system

$$\dot{S} = -\beta SI, \qquad \dot{I} = (\beta S - \gamma)I.$$

The *basic reproduction number* $\mathcal{R}_0 = \beta S(0)/\gamma$ determines whether an epidemic occurs: if $\mathcal{R}_0 > 1$, then I initially increases.
Phase portrait. All trajectories in the (S, I)-plane with $S, I > 0$ satisfy

$$\frac{\mathrm{d}I}{\mathrm{d}S} = -1 + \frac{\gamma}{\beta S},$$

which integrates to $I = N - S + (\gamma/\beta)\ln(S/S_0)$. The disease-free line $I = 0$ is an invariant set, and every orbit approaches a point $(S_\infty, 0)$ with $S_\infty > 0$.

9.8 Exercises

Exercise 9.1. For $\dot{x} = x(2 - x - y)$, $\dot{y} = y(3 - 2y - x)$, find all equilibria and sketch the nullclines and direction field in the first quadrant. Classify each equilibrium using linearisation.

Exercise 9.2. Show that the system $\dot{x} = 2xy + x$, $\dot{y} = x^2 - y^2$ has no periodic orbit in the half-plane $\{y > 0\}$. *Hint:* try $B(x, y) = 1/y^2$.

Exercise 9.3. Consider $\dot{r} = r(1 - r^2) + \mu r$, $\dot{\theta} = 1$ in polar coordinates. For which values of μ does a limit cycle exist? What is its radius?

Exercise 9.4. For the system

$$\dot{x} = \mu x - y - x(x^2 + y^2), \quad \dot{y} = x + \mu y - y(x^2 + y^2),$$

show that a Hopf bifurcation occurs at $\mu = 0$. Is it supercritical or subcritical? Find the radius of the limit cycle for $\mu > 0$.

Exercise 9.5. Sketch the bifurcation diagram for $\dot{x} = \mu - x^2 + x^3$. Identify the type of each bifurcation.

Exercise 9.6. Verify that $H(x, y) = \delta x - \gamma \ln x + \beta y - \alpha \ln y$ is a conserved quantity for the Lotka–Volterra system (9.3). Use this to prove that non-trivial orbits in the positive quadrant are closed curves.

Exercise 9.7. A planar system has exactly three equilibria: a stable node, an unstable spiral, and a saddle. Can it have a limit cycle? If so, which equilibria must the cycle enclose? Justify using index theory.

Exercise 9.8. In the SIR model of Example 9.22, prove that the maximum of $I(t)$ is attained when $S = \gamma/\beta$. Express the peak infection level I_{max} in terms of $S(0)$, $I(0)$, γ, and β.

Chapter Summary

- Nullclines and direction fields provide geometric information about the phase portrait of a planar system.
- The Hartman–Grobman theorem justifies linearisation near hyperbolic equilibria.
- The Poincaré–Bendixson theorem guarantees the existence of limit cycles in bounded, equilibrium-free planar regions.
- Dulac's criterion rules out the existence of periodic orbits when a suitable multiplier function can be found.
- Bifurcations (saddle-node, transcritical, pitchfork, Hopf) describe qualitative changes as parameters vary.
- Index theory constrains which equilibria a limit cycle can enclose.

Chapter 10

Applications — Mechanics, Circuits, Biology, Demography

10.1 Mechanical Oscillations

10.1.1 The Simple Pendulum

A rigid pendulum of length ℓ and mass m in a gravitational field g satisfies

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0. \tag{10.1}$$

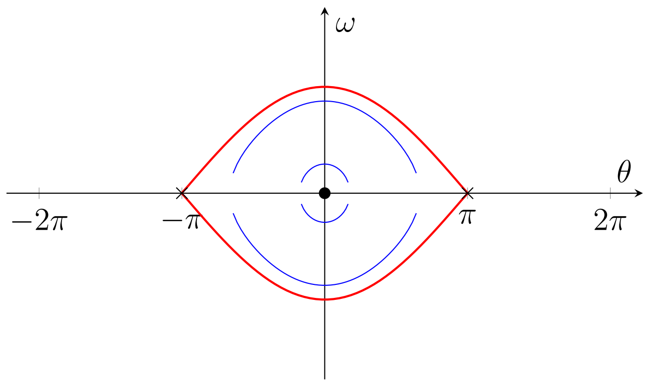
Writing $\omega = \dot{\theta}$, the phase portrait in the (θ, ω)-plane consists of:

- *Centres* at $\theta = 2k\pi$ (stable equilibria: hanging vertically) surrounded by closed orbits (oscillations).
- *Saddle points* at $\theta = (2k+1)\pi$ (inverted equilibrium), connected by heteroclinic orbits (separatrices).
- *Rotational orbits* outside the separatrices (the pendulum goes over the top).

The energy integral is

$$E = \tfrac{1}{2}\omega^2 - \frac{g}{\ell}\cos\theta.$$

Pendulum phase portrait



Example 10.1 (Small-angle approximation)**.** For small angles, $\sin\theta \approx \theta$, and equation (10.1) reduces to the simple harmonic oscillator $\ddot{\theta} + \omega_0^2\theta = 0$ with $\omega_0 = \sqrt{g/\ell}$ and period $T = 2\pi/\omega_0$.

10.1.2 Coupled Oscillators

Example 10.2 (Two masses connected by springs)**.** Consider two unit masses connected in series by springs of stiffness k, attached to fixed walls. The equations of motion are

$$\ddot{x}_1 = -2kx_1 + kx_2, \qquad \ddot{x}_2 = kx_1 - 2kx_2.$$

Introducing $u = x_1 + x_2$ and $v = x_1 - x_2$ decouples the system:

$$\ddot{u} = -ku, \qquad \ddot{v} = -3kv.$$

The normal-mode frequencies are $\omega_1 = \sqrt{k}$ (in-phase mode) and $\omega_2 = \sqrt{3k}$ (out-of-phase mode).

10.1.3 Resonance

Example 10.3 (Forced, damped harmonic oscillator)**.** The equation

$$m\ddot{x} + c\dot{x} + kx = F_0\cos(\omega t)$$

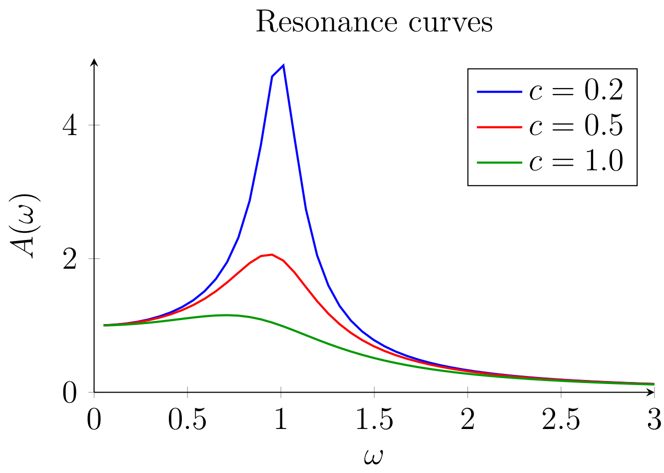
has a particular solution with amplitude

$$A(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}.$$

The amplitude is maximal at the resonance frequency

$$\omega_r = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}},$$

provided $c < \sqrt{2km}$. As $c \to 0$, $\omega_r \to \omega_0$ and the peak amplitude $A(\omega_r) \to \infty$.



10.2 Electrical Circuits

By Kirchhoff's laws, the voltages and currents in a circuit satisfy ODEs whose form depends on the circuit elements.

10.2.1 RC Circuit

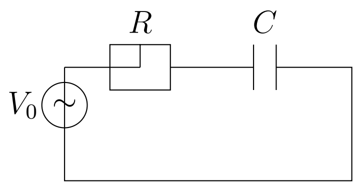
Example 10.4 (RC circuit with DC source)**.** A resistor R in series with a capacitor C driven by a constant voltage V_0 satisfies

$$RC\,\dot{V}_C + V_C = V_0,$$

where $V_C(t)$ is the voltage across the capacitor. The solution with $V_C(0) = 0$ is

$$V_C(t) = V_0\Big(1 - e^{-t/(RC)}\Big).$$

The time constant is $\tau = RC$.



10.2.2 RL Circuit

Example 10.5 (RL circuit)**.** A resistor R in series with an inductor L driven by V_0:

$$L\,\dot{I} + RI = V_0.$$

With $I(0) = 0$:

$$I(t) = \frac{V_0}{R}\Big(1 - e^{-Rt/L}\Big).$$

The time constant is $\tau = L/R$.

10.2.3 RLC Circuit

Example 10.6 (Series RLC circuit)**.** The charge $q(t)$ on the capacitor in a series RLC circuit satisfies

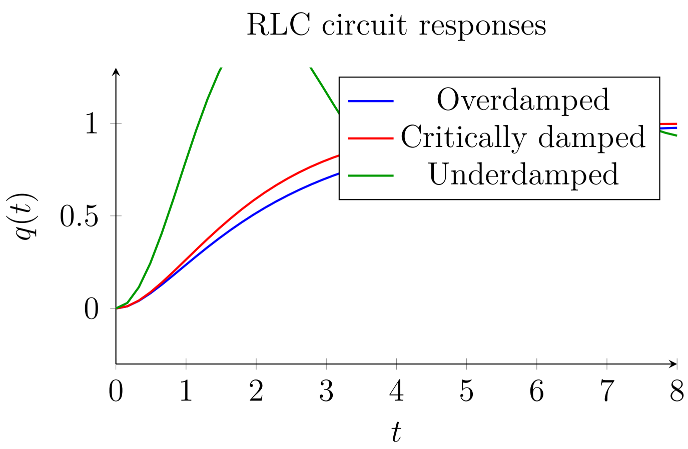
$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V(t). \tag{10.2}$$

This is a second-order linear ODE with constant coefficients. The characteristic equation is $L\lambda^2 + R\lambda + 1/C = 0$, giving

$$\lambda = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}.$$

Three regimes arise:

- **Overdamped** ($R^2 > 4L/C$): two negative real roots.
- **Critically damped** ($R^2 = 4L/C$): a repeated real root.
- **Underdamped** ($R^2 < 4L/C$): complex conjugate roots leading to damped oscillations.



10.3 Population Dynamics

10.3.1 Malthusian Growth

Example 10.7 (Malthus model)**.** The simplest population model is

$$\dot{N} = rN, \qquad N(0) = N_0,$$

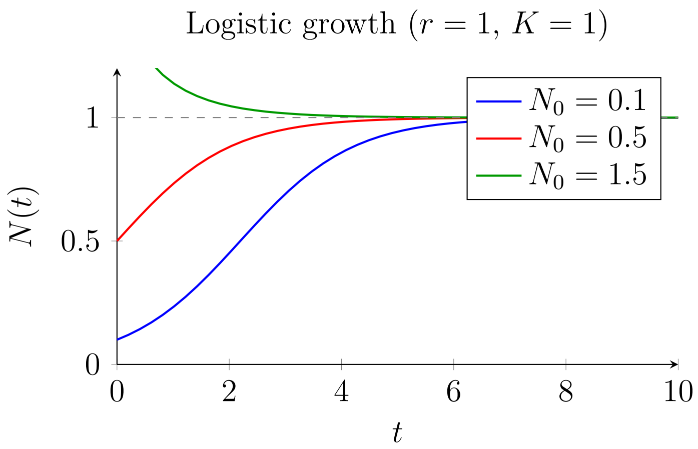
with solution $N(t) = N_0 e^{rt}$. If $r > 0$ the population grows exponentially; if $r < 0$ it decays. The model is unrealistic for large populations because it ignores resource limitations.

10.3.2 Logistic Growth

Example 10.8 (Logistic equation — complete solution)**.** The logistic equation $\dot{N} = rN(1 - N/K)$ is separable. The solution is

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}}.$$

As $t \to \infty$, $N(t) \to K$ regardless of $N_0 \in (0, \infty)$. The inflection point (maximum growth rate) occurs at $N = K/2$.



10.3.3 Lotka–Volterra Predator–Prey

Example 10.9 (Oscillatory coexistence)**.** With parameters $\alpha = \gamma = 1$ and $\beta = \delta = 0.5$, the Lotka–Volterra system (9.3) has co-existence equilibrium $(x^*, y^*) = (2, 2)$ and conserved quantity $H = 0.5x - \ln x + 0.5y - \ln y$. All orbits with $x, y > 0$ are closed, and both populations oscillate periodically with the predator lagging the prey by a quarter period.

10.3.4 The SIR Model

Example 10.10 (Epidemic threshold)**.** Consider a population of $N = 1000$ with $\beta = 0.3$ and $\gamma = 0.1$. If $I(0) = 1$ and $S(0) = 999$, then $\mathcal{R}_0 = \beta S(0)/\gamma = 2997$, so an epidemic occurs. The fraction of the population that ultimately remains susceptible, $s_\infty = S_\infty/N$, satisfies the transcendental equation

$$\ln s_\infty = \mathcal{R}_0(s_\infty - 1).$$

10.4 Chemical Kinetics

10.4.1 First-Order Reactions

Example 10.11 (Radioactive decay). A radioactive substance decays at a rate proportional to the amount present:

$$\dot{A} = -\lambda A, \qquad A(0) = A_0,$$

with solution $A(t) = A_0 e^{-\lambda t}$. The half-life is $t_{1/2} = \ln 2/\lambda$.

10.4.2 Second-Order and Reversible Reactions

Example 10.12 (Second-order reaction $A + B \to C$). If the initial concentrations are $[A]_0 = a$ and $[B]_0 = b$ with $a \neq b$, the rate law $\dot{x} = k(a-x)(b-x)$ (where x is the concentration of product) gives

$$x(t) = ab\,\frac{e^{k(a-b)t} - 1}{ae^{k(a-b)t} - b}.$$

10.4.3 Michaelis–Menten Kinetics

Example 10.13 (Enzyme kinetics). In the Michaelis–Menten mechanism, substrate S binds an enzyme E to form a complex C, which then yields product P:

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \xrightarrow{k_2} E + P.$$

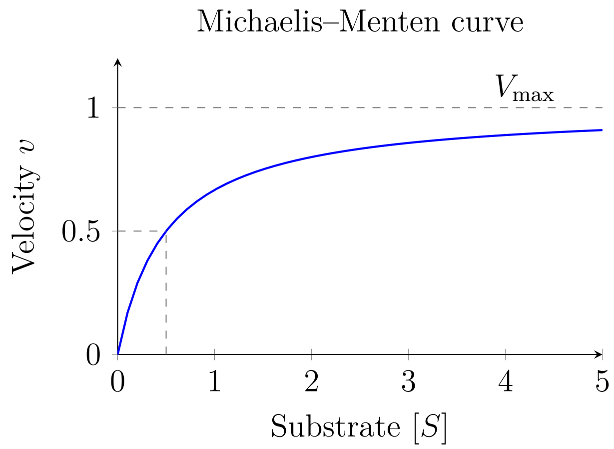
Applying the law of mass action with $e_0 = E + C$ constant yields

$$\begin{aligned}\dot{s} &= -k_1 e_0 s + (k_1 s + k_{-1})c,\\ \dot{c} &= k_1 e_0 s - (k_1 s + k_{-1} + k_2)c.\end{aligned}$$

Under the *quasi-steady-state approximation* ($\dot{c} \approx 0$), the reaction velocity is

$$v = k_2 c \approx \frac{V_{\max} s}{K_m + s},$$

where $V_{\max} = k_2 e_0$ and $K_m = (k_{-1} + k_2)/k_1$.



10.5 Other Applications

10.5.1 Pursuit Curves

Example 10.14 (Linear pursuit)**.** A pursuer at position (x, y) always moves directly towards a prey travelling along the y-axis at constant speed v. If the pursuer has speed w, the pursuit curve $y = y(x)$ satisfies

$$xy'' = \frac{v}{w}\sqrt{1 + (y')^2}.$$

Setting $p = y'$ reduces the order:

$$xp' = \frac{v}{w}\sqrt{1 + p^2},$$

which is separable. When $v = w$, the pursuit curve is a parabola. When $w > v$, the pursuer eventually catches the prey; when $w < v$, the pursuer approaches asymptotically but never reaches it.

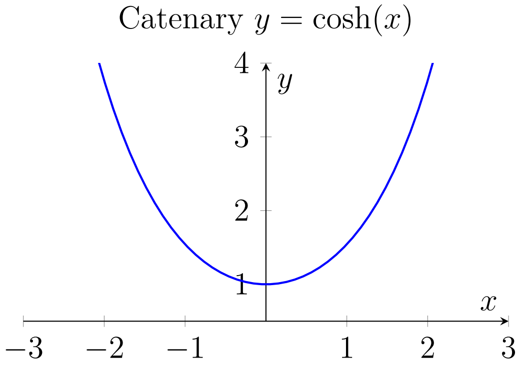
10.5.2 The Catenary

Example 10.15 (Hanging chain)**.** A uniform chain of linear density ρ hanging under gravity satisfies

$$y'' = \frac{\rho g}{T_0}\sqrt{1 + (y')^2},$$

where T_0 is the horizontal tension. Setting $a = T_0/(\rho g)$ and $p = y'$, we get the separable ODE $p' = \frac{1}{a}\sqrt{1 + p^2}$, whose solution is $p = \sinh(x/a)$, hence

$$y(x) = a\cosh\left(\frac{x}{a}\right) + C.$$



10.6 Modelling Methodology

The examples in this chapter illustrate a common workflow:

Method 10.16 (Mathematical modelling with ODEs)**Step 1. Identify variables and parameters.** Determine the dependent variables (state), independent variable (usually time), and physical parameters.

Step 2. Write balance/conservation laws. Apply Newton's law, Kirchhoff's laws, mass action, or conservation of mass/energy to derive the ODE(s).

Step 3. Non-dimensionalise. Introduce dimensionless variables to reduce the number of parameters and reveal the essential scales.

Step 4. Analyse. Solve explicitly if possible; otherwise, find equilibria, linearise, study stability, identify bifurcations, and sketch phase portraits.

Step 5. Interpret. Translate mathematical results back into the language of the application.

Example 10.17 (Non-dimensionalisation of the pendulum)**.** Let $\tau = t\sqrt{g/\ell}$. Then equation (10.1) becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \sin\theta = 0,$$

which is parameter-free. All simple pendulums of any length share the same dimensionless dynamics.

10.7 Extended Worked Examples

Example 10.18 (Damped pendulum with torque)**.** The equation

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = \Gamma$$

models a pendulum with damping $b > 0$ and constant applied torque Γ. Setting $\omega = \dot{\theta}$:

$$\dot{\theta} = \omega, \qquad \dot{\omega} = -b\omega - \sin\theta + \Gamma.$$

Equilibria: $\omega = 0$ and $\sin\theta = \Gamma$, which exist only for $|\Gamma| \le 1$. If $\Gamma < 1$, there is a stable node at $\theta_s = \arcsin\Gamma$ and a saddle at $\theta_u = \pi - \arcsin\Gamma$. At $\Gamma = 1$, these coalesce (a saddle-node bifurcation), and for $\Gamma > 1$ no equilibrium exists; the pendulum rotates continuously.

Example 10.19 (Coupled RLC and mechanical analogy)**.** The series RLC equation (10.2) is mathematically identical to $m\ddot{x} + c\dot{x} + kx = F(t)$ under the correspondence

$$q \leftrightarrow x, \quad L \leftrightarrow m, \quad R \leftrightarrow c, \quad 1/C \leftrightarrow k, \quad V(t) \leftrightarrow F(t).$$

This analogy allows one to transfer intuition freely between mechanical and electrical systems.

Example 10.20 (Lotka–Volterra with harvesting)**.** Adding a constant harvesting rate $h \geq 0$ to the prey equation:

$$\dot{x} = \alpha x - \beta xy - h, \qquad \dot{y} = \delta xy - \gamma y.$$

The co-existence equilibrium shifts to $y^* = (\alpha - h\delta/\gamma)/\beta$ (for sufficiently small h). When h exceeds a critical threshold, the prey equilibrium disappears via a saddle-node bifurcation, leading to extinction of both species.

Example 10.21 (Logistic growth with periodic harvesting)**.** Consider

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - h(1 + \varepsilon \sin \omega t),$$

where $h > 0$ is the mean harvest rate and $\varepsilon \in [0, 1)$ modulates seasonality. This is a non-autonomous ODE. For small ε, perturbation methods show that the average population oscillates near the equilibrium of the time-averaged equation, $N^* = K(1-h/(rK/4))/2$ (assuming the average equation has two positive roots).

10.8 Exercises

Exercise 10.1. For the simple pendulum (10.1), show that the energy $E = \frac{1}{2}\dot{\theta}^2 - (g/\ell)\cos\theta$ is constant along solutions. Classify the orbits in the phase plane according to the value of E.

Exercise 10.2. Find the steady-state (particular) solution of the RLC equation (10.2) with $V(t) = V_0 \cos(\omega t)$. For what ω is the amplitude of the charge q maximised?

Exercise 10.3. Solve the logistic equation $\dot{N} = 2N(1 - N/500)$ with $N(0) = 50$ by separation of variables. Find the time at which $N = 250$.

Exercise 10.4. Consider the Lotka–Volterra system with $\alpha = 2$, $\beta = 1$, $\gamma = 1$, $\delta = 1$. Find the co-existence equilibrium and linearise. Determine the period of oscillation predicted by the linearisation.

Exercise 10.5. In the SIR model, show that the peak number of infected individuals is

$$I_{\max} = I_0 + S_0 - \frac{\gamma}{\beta} - \frac{\gamma}{\beta}\ln\left(\frac{\beta S_0}{\gamma}\right).$$

Exercise 10.6. Derive the Michaelis–Menten approximation $v = V_{\max}s/(K_m + s)$ from the full enzyme system by applying the quasi-steady-state assumption $\dot{c} \approx 0$.

Exercise 10.7. Solve the pursuit curve ODE $xp' = (v/w)\sqrt{1+p^2}$ with $p(x_0) = 0$ in the case $v/w = 1$. Show that the solution is a parabola.

Exercise 10.8. A chain of length L hangs between two supports at equal height separated by a horizontal distance $d < L$. Show that the parameter a in $y = a\cosh(x/a)$ satisfies $\sinh(d/(2a)) = L/(2a)$ (after centering the catenary). Prove that this equation has a unique solution $a > 0$ when $d < L$.

Exercise 10.9. Non-dimensionalise the damped pendulum equation $\ddot{\theta} + b\dot{\theta} + \omega_0^2 \sin\theta = 0$ using the time scale $1/\omega_0$. Identify the single dimensionless parameter and discuss the phase portrait for small and large values of this parameter.

Exercise 10.10. An RC circuit is driven by an alternating source $V(t) = V_0 \sin(\omega t)$. Find the steady-state solution and compute the phase lag between the driving voltage and the capacitor voltage. For what value of ω is the lag $\pi/4$?

Exercise 10.11 (Modelling project)**.** Choose one of the following scenarios and set up, analyse, and interpret a model using ODEs:

(a) The spread of a rumour in a population of N people, where the rate of spread is proportional to the product of those who know it and those who do not.

(b) A parachutist falling under gravity with air resistance proportional to v^2, opening a parachute at a specified altitude.

(c) The temperature of a building with heating, insulation, and an outdoor temperature that varies sinusoidally over 24 hours.

Chapter Summary

- The simple pendulum, coupled oscillators, and resonance illustrate how mechanical systems naturally give rise to second-order ODEs.
- RC, RL, and RLC circuits provide first- and second-order linear ODEs; the mechanical–electrical analogy connects the two domains.
- Population models (Malthus, logistic, Lotka–Volterra, SIR) demonstrate both linear and nonlinear dynamics, including conservation laws and threshold phenomena.
- Michaelis–Menten kinetics illustrates quasi-steady-state reduction of a system.
- Pursuit curves and the catenary give elegant geometric applications of separable and second-order ODEs.
- A systematic modelling methodology — identifying variables, writing balance laws, non-dimensionalising, analysing, and interpreting — ties together all applications.

Quick Reference Card

Standard ODE Types and Solution Methods

ODE Type	**Form**	**Method**
Separable	$y' = f(x)g(y)$	$\displaystyle\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x$
Linear first-order	$y' + P(x)y = Q(x)$	Integrating factor $\mu = e^{\int P\,\mathrm{d}x}$
Exact	$M\,\mathrm{d}x + N\,\mathrm{d}y = 0$, $M_y = N_x$	Find F: $F_x = M$, $F_y = N$
Bernoulli	$y' + Py = Qy^n$	Substitution $v = y^{1-n}$
Homogeneous	$y' = \phi(y/x)$	Substitution $v = y/x$
Riccati	$y' = P + Qy + Ry^2$	One particular solution $\to$ Bernoulli
Second-order const. coeff.	$ay'' + by' + cy = f(x)$	Characteristic equation + undetermined coefficients or variation of parameters
Euler–Cauchy	$x^2y'' + bxy' + cy = 0$	Try $y = x^r$
System $\dot{\mathbf{x}} = A\mathbf{x}$	Constant matrix A	$\mathbf{x}(t) = e^{At}\mathbf{x}_0$ via eigenvalues

Key Theorems

Theorem	Statement (informal)
Picard–Lindelöf	If f is Lipschitz in y, the IVP $y' = f(x, y)$, $y(x_0) = y_0$ has a unique local solution.
Peano	If f is continuous, the IVP has at least one solution (uniqueness may fail).
Grönwall's inequality	Controls the growth of solutions; key tool for continuous dependence and uniqueness proofs.
Hartman–Grobman	Near a hyperbolic equilibrium, the nonlinear flow is topologically conjugate to the linearised flow.
Poincaré–Bendixson	A bounded orbit of a planar system that avoids equilibria must approach a periodic orbit.
Liouville's formula	$W(t) = W(t_0) \exp\Big(\int_{t_0}^{t} \operatorname{tr} A(s)\, \mathrm{d}s\Big)$.

Table of Laplace Transforms

$f(t) \quad (t \geq 0)$	$\mathcal{L}\{f\}(s) = F(s)$
1	$\frac{1}{s}, \quad s > 0$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$	$\frac{e^{-cs}}{s}$
$u_c(t)\, f(t-c)$	$e^{-cs}F(s)$
$\delta(t-c)$	e^{-cs}
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$(f * g)(t)$	$F(s) \cdot G(s)$

Stability Summary for $\dot{\mathbf{x}} = A\mathbf{x}$ in $\mathbb{R}^2$

Eigenvalues	**Type**	**Stability**
$\lambda_1 < \lambda_2 < 0$	Stable node	Asymp. stable
$0 < \lambda_1 < \lambda_2$	Unstable node	Unstable
$\lambda_1 < 0 < \lambda_2$	Saddle	Unstable
$\alpha \pm i\beta$, $\alpha < 0$	Stable spiral	Asymp. stable
$\alpha \pm i\beta$, $\alpha > 0$	Unstable spiral	Unstable
$\pm i\beta$	Centre	Stable (not asymp.)
$\lambda < 0$ repeated, 2 eigenvectors	Stable star node	Asymp. stable
$\lambda < 0$ repeated, 1 eigenvector	Stable improper node	Asymp. stable

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Index