Real Analysis II

Course Notes

From Integration to Multivariable Calculus

First Year

Contents

Preface

This volume constitutes the second semester of the first-year course in Real Analysis. It is a direct continuation of *Real Analysis I*, where we established the foundations of the real number system, studied sequences and series of real numbers, explored the theory of limits and continuity for functions of a single real variable, and developed the differential calculus.

In the present volume we pursue three broad themes.

(i) **Integration.** We give a rigorous construction of the Riemann integral via Darboux sums, establish the Fundamental Theorem of Calculus in both of its classical forms, and develop the standard techniques of integration—substitution and integration by parts. We also treat improper integrals and their convergence.

(ii) **Sequences and series of functions.** The central concept here is *uniform convergence*. We prove the classical interchange theorems (limit with continuity, limit with integration, limit with differentiation), introduce the Weierstrass M-test, and apply the theory to power series. The Weierstrass approximation theorem provides a beautiful conclusion to this circle of ideas.

(iii) **Multivariable calculus** (to appear in subsequent chapters). We extend the differential and integral calculus to functions of several real variables, studying partial derivatives, the total differential, the implicit and inverse function theorems, and multiple integrals.

Throughout, we maintain the same standard of rigour as in the first semester: every major result receives a complete proof. Numerous examples, counterexamples, and exercises complement the theoretical development.

The Authors

Chapter 1

Riemann Integration

Motivation

The problem of computing the area of a region bounded by a curve is one of the oldest in mathematics, going back at least to Archimedes' method of exhaustion. Given a non-negative continuous function $f\colon [a,b] \to \mathbb{R}$, we want to assign a precise numerical value to the "area under the curve $y = f(x)$ between $x = a$ and $x = b$."

The basic idea is to approximate the region by rectangles. If we divide the interval $[a,b]$ into small subintervals and on each one erect a rectangle whose height is the infimum (respectively the supremum) of f, we obtain a *lower sum* (respectively an *upper sum*). The integral, when it exists, is the unique real number that lies between every lower sum and every upper sum.

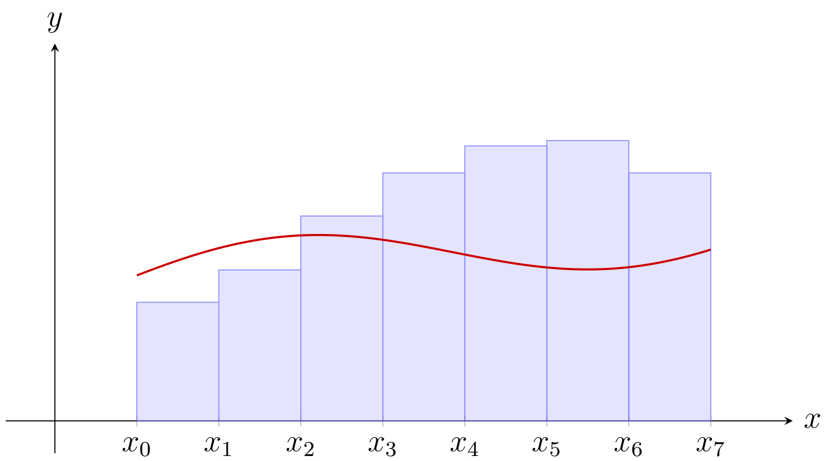


Figure 1.1: Lower Darboux sum: rectangles with heights $m_i = \inf_{[x_{i-1},x_i]} f$.

This chapter makes these ideas precise by developing the Riemann integral in its Darboux formulation.

1.1 Partitions and Darboux Sums

Definition 1.1 (Partition)**.** Let $[a, b]$ be a closed bounded interval. A *partition* of $[a, b]$ is a finite set

$$P = \{x_0, x_1, \dots, x_n\} \quad \text{with} \quad a = x_0 < x_1 < \cdots < x_n = b.$$

The intervals $[x_{i-1}, x_i]$, $i = 1, \dots, n$, are called the *subintervals* of P, and $\Delta x_i = x_i - x_{i-1}$ is the *width* of the i-th subinterval. The *mesh* (or *norm*) of P is $\|P\| = \max_{1 \le i \le n} \Delta x_i$.

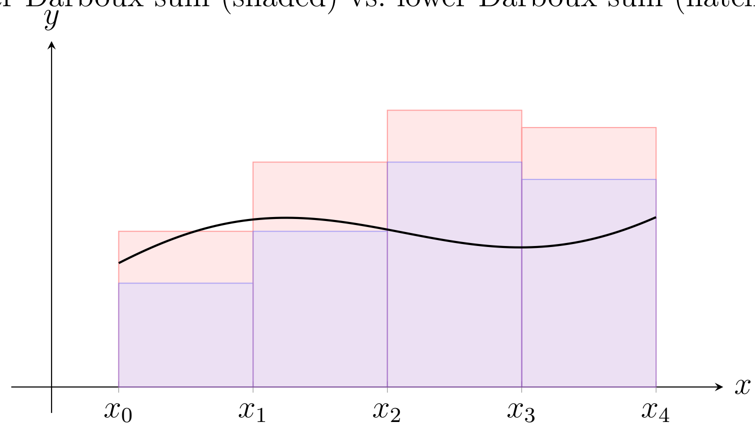
Definition 1.2 (Darboux sums)**.** Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. For each $i = 1, \dots, n$ set

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

The *lower Darboux sum* and *upper Darboux sum* of f with respect to P are

$$L(f, P) = \sum_{i=1}^{n} m_i \, \Delta x_i, \qquad U(f, P) = \sum_{i=1}^{n} M_i \, \Delta x_i.$$

Upper Darboux sum (shaded) vs. lower Darboux sum (hatched)



Definition 1.3 (Refinement)**.** A partition P' is called a *refinement* of a partition P if $P \subset P'$. Given two partitions P_1 and P_2, their *common refinement* is $P_1 \cup P_2$.

1.2 Refinement Inequalities

Theorem 1.1 (Refinement inequalities)**.** Let $f\colon [a, b] \to \mathbb{R}$ be bounded, and let P and P' be partitions of $[a, b]$ with $P \subset P'$ (i.e. P' is a refinement of P). Then

$$L(f, P) \le L(f, P') \le U(f, P') \le U(f, P).$$

Proof. It suffices to prove the result when P' is obtained from P by adding a single point; the general case follows by induction.

Let $P = \{x_0, \dots, x_n\}$ and let $P' = P \cup \{t\}$ where $x_{k-1} < t < x_k$ for some index k. All subintervals except $[x_{k-1}, x_k]$ are unchanged, so we need only compare the contributions of $[x_{k-1}, x_k]$ and its two halves $[x_{k-1}, t]$ and $[t, x_k]$.

Set

$$m_k = \inf_{[x_{k-1},x_k]} f, \quad m_k' = \inf_{[x_{k-1},t]} f, \quad m_k'' = \inf_{[t,x_k]} f.$$

Since $[x_{k-1}, t] \subset [x_{k-1}, x_k]$ and $[t, x_k] \subset [x_{k-1}, x_k]$, we have $m_k \le m_k'$ and $m_k \le m_k''$. Therefore

$$\begin{aligned} m_k'(t - x_{k-1}) + m_k''(x_k - t) &\ge m_k(t - x_{k-1}) + m_k(x_k - t) \\ &= m_k(x_k - x_{k-1}) = m_k\,\Delta x_k. \end{aligned}$$

Hence the contribution to the lower sum does not decrease: $L(f, P) \le L(f, P')$.

An entirely analogous argument, using sup and the inequality $M_k \ge M_k'$, $M_k \ge M_k''$, shows that $U(f, P') \le U(f, P)$.

Finally, $L(f, P') \le U(f, P')$ is immediate from $m_i \le M_i$ for every subinterval. □

Corollary 1.1 (Any lower sum $\le$ any upper sum)**.** For any two partitions P_1, P_2 of $[a, b]$,

$$L(f, P_1) \le U(f, P_2).$$

Proof. Let $P = P_1 \cup P_2$. Then P refines both P_1 and P_2, so by Theorem 1.1,

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2).$$ □

1.3 Upper and Lower Integrals; Riemann Integrability

Definition 1.4 (Upper and lower integrals)**.** Let $f\colon [a, b] \to \mathbb{R}$ be bounded. The *lower integral* and *upper integral* of f are

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup_P L(f, P), \qquad \overline{\int_a^b} f(x)\,\mathrm{d}x = \inf_P U(f, P),$$

where the supremum and infimum are taken over all partitions P of $[a, b]$.

By Corollary 1.1, every lower sum is a lower bound for the set of upper sums, so

$$\underline{\int_a^b} f\,\mathrm{d}x \;\le\; \overline{\int_a^b} f\,\mathrm{d}x.$$

Definition 1.5 (Riemann integrability)**.** A bounded function $f\colon [a,b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

In that case, the common value is called the *Riemann integral* of f on $[a,b]$ and is denoted

$$\int_a^b f(x)\,\mathrm{d}x.$$

We write $\mathcal{R}([a,b])$ for the set of Riemann-integrable functions on $[a,b]$.

1.4 Integrability Criterion

The following criterion is the workhorse for proving integrability.

Theorem 1.2 (Riemann integrability criterion)**.** A bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof. **(⇒).** Suppose $f \in \mathcal{R}([a,b])$ and let $I = \int_a^b f\,\mathrm{d}x$. Given $\varepsilon > 0$, by definition of supremum and infimum there exist partitions P_1, P_2 such that

$$I - \frac{\varepsilon}{2} < L(f,P_1) \quad \text{and} \quad U(f,P_2) < I + \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. By the refinement inequalities,

$$U(f,P) - L(f,P) \le U(f,P_2) - L(f,P_1) < \left(I + \frac{\varepsilon}{2}\right) - \left(I - \frac{\varepsilon}{2}\right) = \varepsilon.$$

(⇐). Conversely, suppose that for every $\varepsilon > 0$ there exists a partition P with $U(f,P) - L(f,P) < \varepsilon$. Since

$$0 \le \overline{\int_a^b} f\,\mathrm{d}x - \underline{\int_a^b} f\,\mathrm{d}x \le U(f,P) - L(f,P) < \varepsilon$$

for every $\varepsilon > 0$, we conclude that the upper and lower integrals are equal, so f is integrable.

□

1.5 Continuous Functions Are Integrable

Theorem 1.3 (Integrability of continuous functions)**.** Every continuous function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable.

Proof. Since $[a, b]$ is compact, f is *uniformly* continuous on $[a, b]$: given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$. On each subinterval $[x_{i-1}, x_i]$ the function f attains its supremum M_i and infimum m_i (by the extreme value theorem), say at points u_i and v_i respectively. Since $|u_i - v_i| \leq \Delta x_i < \delta$, we have

$$M_i - m_i = f(u_i) - f(v_i) < \frac{\varepsilon}{b - a}.$$

Therefore

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i < \frac{\varepsilon}{b - a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b - a} \cdot (b - a) = \varepsilon.$$

By Theorem 1.2, f is Riemann integrable. □

1.6 Monotone Functions Are Integrable

Theorem 1.4 (Integrability of monotone functions)**.** Every monotone function $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof. Assume f is non-decreasing (the non-increasing case is analogous). Then f is bounded, with $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$. On each subinterval $[x_{i-1}, x_i]$, we have $m_i = f(x_{i-1})$ and $M_i = f(x_i)$.

Given $\varepsilon > 0$, choose the uniform partition P with n subintervals of equal width $\Delta x_i = (b - a)/n$. Then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big) \frac{b - a}{n} \\ &= \frac{b - a}{n} \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big) \\ &= \frac{b - a}{n} \big(f(b) - f(a)\big). \end{aligned}$$

The last expression is a telescoping sum. Choosing n large enough so that

$$\frac{b - a}{n} \big(f(b) - f(a)\big) < \varepsilon,$$

i.e. $n > (b - a)(f(b) - f(a))/\varepsilon$, we conclude by the integrability criterion. □

1.7 Piecewise Continuous Functions

Definition 1.6 (Piecewise continuous function)**.** A function $f\colon [a,b] \to \mathbb{R}$ is *piecewise continuous* if there exists a partition $a = c_0 < c_1 < \cdots < c_m = b$ such that f is continuous on each open subinterval (c_{j-1}, c_j) and the one-sided limits $\lim_{x\to c_j^+} f(x)$ and $\lim_{x\to c_j^-} f(x)$ exist (and are finite) at each partition point.

Proposition 1.1 (Integrability of piecewise continuous functions)**.** Every piecewise continuous function on $[a,b]$ is Riemann integrable.

Proof. On each subinterval $[c_{j-1}, c_j]$, the function f is bounded (since the one-sided limits exist) and has at most finitely many discontinuities (at the endpoints). One shows, using the integrability criterion, that a bounded function with finitely many discontinuities is integrable. Indeed, near each discontinuity one can choose subintervals of total width less than $\varepsilon/(4M)$ (where $M = \sup|f|$), and on the remaining subintervals the uniform continuity argument from Theorem 1.3 applies. By the Chasles relation (Proposition 1.4), the integral over $[a,b]$ is the sum of the integrals over the subintervals. □

1.8 Properties of the Riemann Integral

Proposition 1.2 (Linearity)**.** If $f, g \in \mathcal{R}([a,b])$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in \mathcal{R}([a,b])$ and

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

Proof. We prove the result for αf and $f+g$ separately.

Scalar multiple. Suppose $\alpha \geq 0$ (the case $\alpha < 0$ is similar, exchanging sup and inf). For any partition P and any subinterval $[x_{i-1}, x_i]$,

$$\inf_{[x_{i-1},x_i]} (\alpha f) = \alpha \inf_{[x_{i-1},x_i]} f, \qquad \sup_{[x_{i-1},x_i]} (\alpha f) = \alpha \sup_{[x_{i-1},x_i]} f.$$

Hence $L(\alpha f, P) = \alpha L(f,P)$ and $U(\alpha f, P) = \alpha U(f,P)$. Taking supremum and infimum over all P gives the result.

Sum. For any subinterval,

$$\inf(f+g) \geq \inf f + \inf g \quad \text{and} \quad \sup(f+g) \leq \sup f + \sup g.$$

Therefore $L(f+g,P) \geq L(f,P) + L(g,P)$ and $U(f+g,P) \leq U(f,P) + U(g,P)$. It follows that

$$U(f+g,P) - L(f+g,P) \leq \big(U(f,P) - L(f,P)\big) + \big(U(g,P) - L(g,P)\big).$$

Given $\varepsilon > 0$, choose P_1 with $U(f,P_1) - L(f,P_1) < \varepsilon/2$ and P_2 with $U(g,P_2) - L(g,P_2) < \varepsilon/2$; set $P = P_1 \cup P_2$. Then

$$U(f+g,P) - L(f+g,P) < \varepsilon,$$

so $f+g$ is integrable. The value of the integral follows from

$$L(f,P)+L(g,P) \le L(f+g,P) \le \int_a^b (f+g) \le U(f+g,P) \le U(f,P)+U(g,P),$$

letting the partition refine and noting that the outer bounds converge to $\int f + \int g$. □

Proposition 1.3 (Positivity and monotonicity)**.** Let $f, g \in \mathcal{R}([a,b])$.

(a) **Positivity.** If $f(x) \ge 0$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \ge 0$.

(b) **Monotonicity.** If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$.

Proof. (a) If $f \ge 0$ then $m_i \ge 0$ for every subinterval, so $L(f,P) \ge 0$ for every partition P. Hence $\int_a^b f\,\mathrm{d}x = \sup_P L(f,P) \ge 0$.

(b) Apply (a) to $g - f \ge 0$ and use linearity. □

Proposition 1.4 (Chasles relation)**.** If $f \in \mathcal{R}([a,b])$ and $a < c < b$, then $f \in \mathcal{R}([a,c]) \cap \mathcal{R}([c,b])$ and

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Proof. Given $\varepsilon > 0$, choose a partition P of $[a,b]$ with $U(f,P) - L(f,P) < \varepsilon$. Let $P' = P \cup \{c\}$; then $U(f,P') - L(f,P') \le U(f,P) - L(f,P) < \varepsilon$ by the refinement inequality. Write $P' = P_1 \cup P_2$ where $P_1 = P' \cap [a,c]$ is a partition of $[a,c]$ and $P_2 = P' \cap [c,b]$ is a partition of $[c,b]$. Then

$$U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) = U(f,P') - L(f,P') < \varepsilon.$$

Since each term on the left is non-negative, both are less than ε. Hence f is integrable on $[a,c]$ and $[c,b]$. The additivity of the integral values follows from $L(f,P') = L(f,P_1) + L(f,P_2)$ and similarly for upper sums. □

1.9 The Triangle Inequality for Integrals

Theorem 1.5 (Mean value inequality)**.** If $f \in \mathcal{R}([a,b])$ then $|f| \in \mathcal{R}([a,b])$ and

$$|*| \int_a^b f(x)\,\mathrm{d}x \le \int_a^b |f(x)|\,\mathrm{d}x.$$

Proof. First we show $|f|$ is integrable. For any subinterval $[x_{i-1}, x_i]$,

$$\sup_{[x_{i-1},x_i]} |f| - \inf_{[x_{i-1},x_i]} |f| \le \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f,$$

since $\left||f(x)|-|f(y)|\right| \leq |f(x)-f(y)|$. Therefore $U(|f|,P)-L(|f|,P) \leq U(f,P)-L(f,P)$ for every partition P, and integrability of $|f|$ follows from the integrability criterion.

For the inequality, note that $-|f| \leq f \leq |f|$. By monotonicity of the integral,

$$-\int_a^b |f| \,\mathrm{d}x \leq \int_a^b f \,\mathrm{d}x \leq \int_a^b |f| \,\mathrm{d}x,$$

which is precisely $\left|\int_a^b f \,\mathrm{d}x\right| \leq \int_a^b |f| \,\mathrm{d}x$. □

1.10 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is one of the most important results in all of analysis. It establishes the deep connection between the differential and integral calculus.

1.10.1 First Fundamental Theorem

Theorem 1.6 (First Fundamental Theorem of Calculus)**.** Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable, and define the *area function*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \qquad x \in [a,b].$$

Then:

(a) F is continuous on $[a,b]$.

(b) If f is continuous at a point $c \in (a,b)$, then F is differentiable at c and $F'(c) = f(c)$.

In particular, if f is continuous on $[a,b]$, then F is a C^1 antiderivative of f on $[a,b]$.

Proof. **(a) Continuity of F.** Since f is Riemann integrable on $[a,b]$, it is bounded; let $M = \sup_{[a,b]} |f|$. For $x, y \in [a,b]$ with $x < y$,

$$|F(y) - F(x)| = |*| \int_x^y f(t)\,\mathrm{d}t \leq \int_x^y |f(t)| \,\mathrm{d}t \leq M(y-x).$$

Given $\varepsilon > 0$, choosing $\delta = \varepsilon/M$ (or $\delta = 1$ if $M = 0$) shows that $|y - x| < \delta$ implies $|F(y) - F(x)| < \varepsilon$.

(b) Differentiability at a point of continuity. Let $c \in (a,b)$ and suppose f is continuous at c. For $h \neq 0$ with $c + h \in [a,b]$,

$$\frac{F(c+h) - F(c)}{h} = \frac{1}{h} \int_c^{c+h} f(t)\,\mathrm{d}t.$$

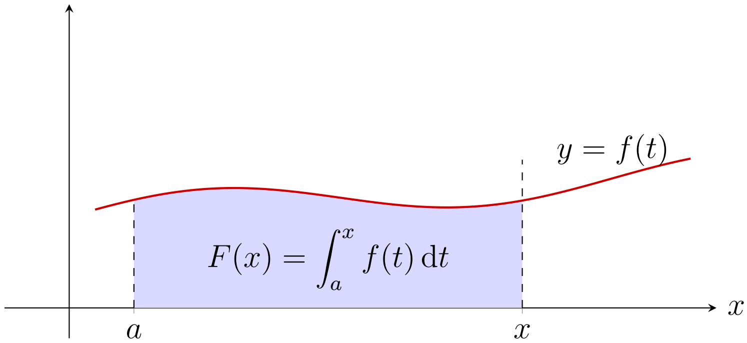
We must show this converges to $f(c)$ as $h \to 0$. Write

$$\frac{1}{h} \int_c^{c+h} f(t)\,\mathrm{d}t - f(c) = \frac{1}{h} \int_c^{c+h} \big(f(t) - f(c)\big)\,\mathrm{d}t.$$

Given $\varepsilon > 0$, by continuity of f at c there exists $\delta > 0$ such that $|t - c| < \delta$ implies $|f(t) - f(c)| < \varepsilon$. For $0 < |h| < \delta$, every t between c and $c + h$ satisfies $|t - c| \le |h| < \delta$, so

$$|*| \frac{1}{h} \int_c^{c+h} \big(f(t) - f(c)\big)\,\mathrm{d}t \le \frac{1}{|h|} \int_{\min(c,c+h)}^{\max(c,c+h)} |f(t) - f(c)|\,\mathrm{d}t < \frac{1}{|h|} \cdot \varepsilon \cdot |h| = \varepsilon.$$

Therefore $F'(c) = f(c)$. □



1.10.2 Second Fundamental Theorem (Newton–Leibniz formula)

Theorem 1.7 (Second Fundamental Theorem of Calculus)**.** Let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable. If $G\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and $G'(x) = f(x)$ for all $x \in (a, b)$, then

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a).$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem, for each i there exists $c_i \in (x_{i-1}, x_i)$ with

$$G(x_i) - G(x_{i-1}) = G'(c_i)\,\Delta x_i = f(c_i)\,\Delta x_i.$$

Summing over i and using the telescoping property,

$$G(b) - G(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

Since $m_i \le f(c_i) \le M_i$, we have

$$L(f, P) \le G(b) - G(a) \le U(f, P).$$

This holds for every partition P. Because f is integrable, $\sup_P L(f, P) = \inf_P U(f, P) = \int_a^b f\,\mathrm{d}x$, and the only number that lies between all lower and all upper sums is the integral. Therefore $G(b) - G(a) = \int_a^b f\,\mathrm{d}x$. □

Remark 1.1 (Notation for antiderivative evaluation)**.** We write $\big[G(x)\big]_a^b = G(b) - G(a)$. The Second Fundamental Theorem thus reads $\displaystyle\int_a^b G'(x)\,\mathrm{d}x = \big[G(x)\big]_a^b$.

1.11 Integration Techniques

1.11.1 Integration by Parts

Theorem 1.8 (Integration by parts)**.** Let $u, v\colon [a,b] \to \mathbb{R}$ be continuously differentiable. Then

$$\int_a^b u(x)\, v'(x)\, \mathrm{d}x = \Big[u(x)\, v(x)\Big]_a^b - \int_a^b u'(x)\, v(x)\, \mathrm{d}x.$$

Proof. By the product rule, $(uv)' = u'v + uv'$, so $uv' = (uv)' - u'v$. Since u and v are C^1, all three functions uv', $(uv)'$, and $u'v$ are continuous, hence integrable. Integrating from a to b and applying the Second Fundamental Theorem to $(uv)'$,

$$\int_a^b u\, v'\, \mathrm{d}x = \int_a^b (uv)'\, \mathrm{d}x - \int_a^b u'v\, \mathrm{d}x = \Big[uv\Big]_a^b - \int_a^b u'v\, \mathrm{d}x.$$ □

Example 1.1 (Integration by parts)**.** Compute $I = \displaystyle\int_0^1 x\, e^x\, \mathrm{d}x$.

Set $u(x) = x$, $v'(x) = e^x$, so $u'(x) = 1$, $v(x) = e^x$. Then

$$I = \Big[x\, e^x\Big]_0^1 - \int_0^1 e^x\, \mathrm{d}x = e - \Big[e^x\Big]_0^1 = e - (e - 1) = 1.$$

Example 1.2 (Reduction formula)**.** For $n \ge 1$, let $I_n = \int_0^{\pi/2} \sin^n x\, \mathrm{d}x$. Integration by parts with $u = \sin^{n-1} x$, $v' = \sin x$ gives

$$I_n = \frac{n-1}{n}\, I_{n-2} \qquad (n \ge 2),$$

with $I_0 = \pi/2$ and $I_1 = 1$.

1.11.2 Integration by Substitution

Theorem 1.9 (Change of variable)**.** Let $\varphi\colon [\alpha,\beta] \to \mathbb{R}$ be a C^1 function with $\varphi([\alpha,\beta]) \subset [a,b]$, and let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, \mathrm{d}x = \int_\alpha^\beta f\Big(\varphi(t)\Big)\, \varphi'(t)\, \mathrm{d}t.$$

Proof. Let F be an antiderivative of f (which exists by the First Fundamental Theorem, since f is continuous). Define $H(t) = F(\varphi(t))$. By the chain rule,

$$H'(t) = F'(\varphi(t))\, \varphi'(t) = f(\varphi(t))\, \varphi'(t).$$

Since H' is continuous, the Second Fundamental Theorem gives

$$\int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, \mathrm{d}t = H(\beta) - H(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, \mathrm{d}x.$$ □

Example 1.3 (Substitution example)**.** Compute $\displaystyle\int_0^1 \frac{2x}{1+x^2}\,\mathrm{d}x$.

Set $\varphi(t) = t$, $x = t$, or more directly, let $u = 1 + x^2$, so $\mathrm{d}u = 2x\,\mathrm{d}x$. When $x = 0$, $u = 1$; when $x = 1$, $u = 2$. Thus

$$\int_0^1 \frac{2x}{1+x^2}\,\mathrm{d}x = \int_1^2 \frac{\mathrm{d}u}{u} = \Big[\ln u\Big]_1^2 = \ln 2.$$

Example 1.4 (Trigonometric substitution)**.** Compute $\displaystyle\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$.

Set $x = \sin\theta$, $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \pi/2$. Then

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta = \frac{\pi}{4}.$$

This confirms the area of a quarter-disk of radius 1.

1.12 Improper Integrals

Definition 1.7 (Improper integral on $[a, +\infty)$)**.** Let $f\colon [a, +\infty) \to \mathbb{R}$ be Riemann integrable on $[a, T]$ for every $T > a$. If the limit

$$\lim_{T\to+\infty} \int_a^T f(x)\,\mathrm{d}x$$

exists and is finite, we say the improper integral $\displaystyle\int_a^{+\infty} f(x)\,\mathrm{d}x$ *converges* and we set

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{T\to+\infty} \int_a^T f(x)\,\mathrm{d}x.$$

Otherwise, we say the integral *diverges*.

Definition 1.8 (Improper integral at a singularity)**.** Let $f\colon (a, b] \to \mathbb{R}$ be Riemann integrable on $[a+\eta, b]$ for every $0 < \eta < b - a$. If $\displaystyle\lim_{\eta\to0^+} \int_{a+\eta}^b f(x)\,\mathrm{d}x$ exists and is finite, we set

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\eta\to0^+} \int_{a+\eta}^b f(x)\,\mathrm{d}x.$$

Example 1.5. The integral $\int_1^{+\infty} x^{-\alpha}\,\mathrm{d}x$riemann-alpha For $\alpha \neq 1$:

$$\int_1^T x^{-\alpha}\,\mathrm{d}x = \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_1^T = \frac{T^{1-\alpha} - 1}{1-\alpha}.$$

- If $\alpha > 1$: $T^{1-\alpha} \to 0$ as $T \to +\infty$, so the integral converges to $\dfrac{1}{\alpha - 1}$.
- If $\alpha < 1$: $T^{1-\alpha} \to +\infty$, so the integral diverges.

- If $\alpha = 1$: $\displaystyle\int_1^T \frac{\mathrm{d}x}{x} = \ln T \to +\infty$, so the integral diverges.

Conclusion: $\displaystyle\int_1^{+\infty} \frac{\mathrm{d}x}{x^\alpha}$ converges if and only if $\alpha > 1$.

Proposition 1.5 (Comparison test for improper integrals)**.** Let $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

(a) If $\int_a^{+\infty} g \,\mathrm{d}x$ converges, then so does $\int_a^{+\infty} f \,\mathrm{d}x$.

(b) If $\int_a^{+\infty} f \,\mathrm{d}x$ diverges, then so does $\int_a^{+\infty} g \,\mathrm{d}x$.

Proof. (a) The function $F(T) = \int_a^T f \,\mathrm{d}x$ is non-decreasing and bounded above by $\int_a^{+\infty} g \,\mathrm{d}x$, hence has a finite limit.

(b) Contrapositive of (a). □

Definition 1.9 (Absolute convergence)**.** An improper integral $\int_a^{+\infty} f \,\mathrm{d}x$ is *absolutely convergent* if $\int_a^{+\infty} |f| \,\mathrm{d}x$ converges.

Proposition 1.6 (Absolute convergence implies convergence)**.** If $\int_a^{+\infty} |f(x)| \,\mathrm{d}x$ converges, then $\int_a^{+\infty} f(x) \,\mathrm{d}x$ converges.

Proof. Write $f = f^+ - f^-$ where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Then $0 \leq f^\pm \leq |f|$, so both $\int f^+$ and $\int f^-$ converge by the comparison test. Hence $\int f = \int f^+ - \int f^-$ converges. □

1.13 Wallis and Stirling Formulas

Theorem 1.10 (Wallis formula)**.**

$$\lim_{n\to\infty} \frac{1}{2n+1} \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 = \frac{\pi}{2},$$

or equivalently,

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

Theorem 1.11 (Stirling formula)**.** As $n \to \infty$,

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n,$$

meaning $\displaystyle\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} = 1$.

These classical formulas connect the Wallis integrals $I_n = \int_0^{\pi/2} \sin^n x \, \mathrm{d}x$ with the constants π and e. Their proofs, which rely on the reduction formula of Example 1.2 and careful asymptotic analysis, can be found in more advanced treatments.

1.14 Exercises

Exercise 1.1 (Direct computation via Darboux sums)**.** Let $f(x) = x^2$ on $[0, 1]$. Using the uniform partition with n subintervals, compute $L(f, P_n)$ and $U(f, P_n)$ explicitly. Verify that both converge to $1/3$ as $n \to \infty$, and conclude that $\int_0^1 x^2 \, \mathrm{d}x = 1/3$.

Exercise 1.2 (Non-integrable function)**.** Show that the Dirichlet function $\mathbf{1}_{\mathbb{Q}}$ (equal to 1 on $\mathbb{Q}$ and 0 on $\mathbb{R} \setminus \mathbb{Q}$) is not Riemann integrable on any interval $[a, b]$ with $a < b$.

Exercise 1.3 (Integrability of f^2)**.** Prove that if $f \in \mathcal{R}([a, b])$, then $f^2 \in \mathcal{R}([a, b])$. *Hint:* use $M_i^2 - m_i^2 = (M_i + m_i)(M_i - m_i) \leq 2M(M_i - m_i)$.

Exercise 1.4 (Product of integrable functions)**.** Prove that if $f, g \in \mathcal{R}([a, b])$, then $fg \in \mathcal{R}([a, b])$. *Hint:* write $fg = \frac{1}{4}\big((f + g)^2 - (f - g)^2\big)$.

Exercise 1.5 (Mean Value Theorem for integrals)**.** Let $f \colon [a, b] \to \mathbb{R}$ be continuous. Prove that there exists $c \in [a, b]$ such that $\displaystyle\int_a^b f(x) \, \mathrm{d}x = f(c)(b - a)$.

Exercise 1.6 (Second Mean Value Theorem)**.** Let $f \colon [a, b] \to \mathbb{R}$ be continuous and $g \colon [a, b] \to \mathbb{R}$ be integrable with $g \geq 0$. Prove that there exists $c \in [a, b]$ such that $\displaystyle\int_a^b f(x)g(x) \, \mathrm{d}x = f(c) \int_a^b g(x) \, \mathrm{d}x$.

Exercise 1.7 (Integration by parts)**.** Compute the following integrals using integration by parts:

(a) $\displaystyle\int_0^1 x^2 e^x \, \mathrm{d}x$

(b) $\displaystyle\int_0^\pi x \sin x \, \mathrm{d}x$

(c) $\displaystyle\int_1^e (\ln x)^2 \, \mathrm{d}x$

Exercise 1.8 (Substitution practice)**.** Evaluate:

(a) $\displaystyle\int_0^{\pi/4} \tan x \, \mathrm{d}x$

(b) $\displaystyle\int_0^1 \frac{x}{(1 + x^2)^2} \, \mathrm{d}x$

(c) $\displaystyle\int_0^{+\infty} e^{-x} \sin x \, \mathrm{d}x$

Exercise 1.9 (Convergence of improper integrals)**.** Determine whether each integral converges or diverges:

(a) $\displaystyle\int_1^{+\infty} \frac{\mathrm{d}x}{x^2+1}$

(b) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}}$

(c) $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,\mathrm{d}x$

(d) $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}x}{1+x^3}$

Exercise 1.10 (Integral remainder for Taylor series)**.** Let $f \in C^{n+1}([a,b])$. Using integration by parts n times, derive the integral form of the Taylor remainder:

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{f^{(n+1)}(t)}{n!}(b-t)^n\,\mathrm{d}t.$$

Exercise 1.11 (Cauchy–Schwarz inequality for integrals)**.** Let $f, g \in \mathcal{R}([a,b])$. Prove that

$$\left(\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2 \le \int_a^b f(x)^2\,\mathrm{d}x \cdot \int_a^b g(x)^2\,\mathrm{d}x.$$

Hint: for $\lambda \in \mathbb{R}$, expand $\int_a^b (f+\lambda g)^2\,\mathrm{d}x \ge 0$.

Exercise 1.12 (An integral identity)**.** Let $f\colon [0,\pi] \to \mathbb{R}$ be continuous. Prove that

$$\int_0^\pi x\,f(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi f(\sin x)\,\mathrm{d}x.$$

Hint: use the substitution $u = \pi - x$.

Chapter 7 — Summary

- A bounded function $f\colon [a,b] \to \mathbb{R}$ is **Riemann integrable** when its upper and lower Darboux integrals coincide.
- The **integrability criterion**: f is integrable iff $\forall \varepsilon > 0$, $\exists P$ with $U(f,P) - L(f,P) < \varepsilon$.
- **Continuous** and **monotone** functions on $[a,b]$ are integrable.
- The integral is **linear**, **monotone**, and satisfies the **Chasles relation** and the **triangle inequality**.
- The **First FTC**: if f is continuous, then $F(x) = \int_a^x f$ is differentiable with $F' = f$.
- The **Second FTC** (Newton–Leibniz): if $G' = f$ on (a,b), then $\int_a^b f = G(b) - G(a)$.
- **Integration by parts** and **substitution** are the main computational tools.
- **Improper integrals** extend the theory to unbounded domains or unbounded integrands; convergence is tested by comparison.

Chapter 2

Sequences and Series of Functions

Motivation

In Analysis I we studied sequences and series of real numbers. Now we consider sequences (f_n) and series $\sum f_n$ whose terms are *functions*. The central question is:

Under what conditions can we interchange a limit with an integral, a derivative, or another limit?

The answer depends on the *mode of convergence*. Pointwise convergence alone is too weak to guarantee these interchanges; the correct notion turns out to be *uniform convergence*. This chapter develops the theory systematically, culminating in the Weierstrass M-test, power series, and the Weierstrass approximation theorem.

2.1 Pointwise and Uniform Convergence

Definition 2.1 (Pointwise convergence)**.** Let $(f_n)_{n\geq 1}$ be a sequence of functions $f_n\colon D \to \mathbb{R}$. We say (f_n) *converges pointwise* to $f\colon D \to \mathbb{R}$ if for every $x \in D$,

$$\lim_{n\to\infty} f_n(x) = f(x),$$

i.e., for every $x \in D$ and every $\varepsilon > 0$, there exists $N = N(x, \varepsilon) \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$.

The crucial point in pointwise convergence is that N may depend on x. When we require N to be *independent* of x, we obtain a stronger mode of convergence.

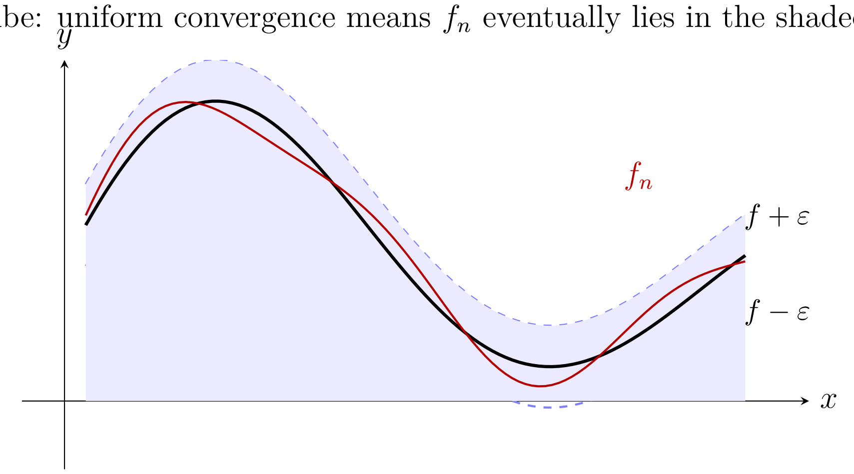
Definition 2.2 (Uniform convergence)**.** The sequence (f_n) *converges uniformly* to f on D if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ (independent of x) such that

$$n \geq N \implies \sup_{x\in D} |f_n(x) - f(x)| < \varepsilon.$$

Equivalently, $\|f_n - f\|_\infty \to 0$, where $\|g\|_\infty = \sup_{x\in D} |g(x)|$.

Remark 2.1 (Negation of uniform convergence)**.** (f_n) does *not* converge uniformly to f on D if there exists $\varepsilon_0 > 0$ and a sequence (x_n) in D such that $|f_n(x_n) - f(x_n)| \geq \varepsilon_0$ for infinitely many n.

The ε-tube: uniform convergence means f_n eventually lies in the shaded band



2.2 Uniform Cauchy Criterion

Theorem 2.1 (Uniform Cauchy criterion)**.** A sequence (f_n) of bounded functions on D converges uniformly on D if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \implies \sup_{x \in D} |f_m(x) - f_n(x)| < \varepsilon.$$

Proof. **(⇒).** If $f_n \to f$ uniformly, then for $\varepsilon > 0$ choose N so that $n \geq N$ implies $\|f_n - f\|_\infty < \varepsilon/2$. For $m, n \geq N$,

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(⇐). For each $x \in D$, the sequence $(f_n(x))$ is Cauchy in $\mathbb{R}$, hence converges; call the limit $f(x)$. We must show the convergence is uniform.

Given $\varepsilon > 0$, choose N from the Cauchy condition. For $m, n \geq N$ and any $x \in D$, $|f_m(x) - f_n(x)| < \varepsilon$. Fixing $n \geq N$ and letting $m \to \infty$, we obtain $|f(x) - f_n(x)| \leq \varepsilon$ for all $x \in D$. Since this bound is independent of x, $\|f - f_n\|_\infty \leq \varepsilon$, so $f_n \to f$ uniformly. □

2.3 A Fundamental Counterexample

Example 2.1 ($f_n(x) = x^n$ on $[0, 1]$)**.** Define $f_n \colon [0, 1] \to \mathbb{R}$ by $f_n(x) = x^n$. Each f_n is continuous.

Pointwise limit. For $x \in [0, 1)$, $x^n \to 0$; for $x = 1$, $x^n = 1$. Hence the pointwise limit is

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

This limit function is *discontinuous* at $x = 1$, even though every f_n is continuous.

Convergence is not uniform. Indeed, $\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} x^n = 1$ for every n (the supremum is not attained but equals 1). So $\|f_n - f\|_\infty = 1 \not\to 0$.

Alternatively, take $x_n = (1 - 1/n)$; then $f_n(x_n) = (1 - 1/n)^n \to 1/e \neq 0 = f(x_n)$.

This example shows that the pointwise limit of continuous functions need not be continuous. Uniform convergence is precisely the condition that prevents such pathologies.

2.4 Uniform Limit of Continuous Functions

Theorem 2.2 (Uniform limit of continuous functions is continuous)**.** Let (f_n) be a sequence of continuous functions $f_n \colon D \to \mathbb{R}$ that converges uniformly to $f \colon D \to \mathbb{R}$. Then f is continuous on D.

Proof. Fix $x_0 \in D$ and let $\varepsilon > 0$. We use the "$\varepsilon/3$-argument."

Step 1. Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in D} |f_N(x) - f(x)| < \frac{\varepsilon}{3}.$$

Step 2. Since f_N is continuous at x_0, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}.$$

Step 3. For $|x - x_0| < \delta$, the triangle inequality gives

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence f is continuous at x_0. Since x_0 was arbitrary, f is continuous on D. □

Remark 2.2 (Converse fails)**.** The converse is false: a sequence of continuous functions may converge pointwise to a continuous function without the convergence being uniform. For instance, $f_n(x) = x^n/n$ on $[0, 1]$ converges pointwise to $f \equiv 0$ (which is continuous), but $\|f_n\|_\infty = 1/n \to 0$—wait, that *is* uniform. A better example: $f_n(x) = nx(1 - x)^n$ on $[0, 1]$ converges pointwise to 0 (continuous), but $\sup f_n = n \cdot \frac{n}{(n+1)^{n+1}} \cdot (n + 1) \not\to 0$ in general—the point is that uniform convergence is a strictly stronger condition, and one must verify it explicitly.

Example 2.2 (Pointwise limit continuous but convergence not uniform)**.** Let $f_n(x) = \frac{nx}{1+n^2x^2}$ on $[0, 1]$. Then $f_n(x) \to 0$ pointwise for every $x \in [0, 1]$, and the limit $f \equiv 0$ is continuous. However, $f_n(1/n) = \frac{1}{2}$ for every n, so $\|f_n\|_\infty \geq 1/2$ and the convergence is not uniform.

2.5 Interchange of Limit and Integral

Theorem 2.3 (Uniform convergence and integration)**.** Let (f_n) be a sequence of Riemann-integrable functions on $[a,b]$ converging uniformly to f. Then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

In short, $\displaystyle\int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n$.

Proof. Step 1: f is integrable. Given $\varepsilon > 0$, choose N so that $\|f_N - f\|_\infty < \varepsilon/(3(b-a))$. Since f_N is integrable, there exists a partition P with $U(f_N, P) - L(f_N, P) < \varepsilon/3$.

On any subinterval $[x_{i-1}, x_i]$,

$$\begin{aligned}\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f &\le \sup(f - f_N) - \inf(f - f_N) + \sup f_N - \inf f_N \\ &\le 2\,\|f - f_N\|_\infty + (M_i^N - m_i^N),\end{aligned}$$

where M_i^N, m_i^N denote the sup and inf of f_N on the subinterval. Therefore

$$\begin{aligned}U(f,P) - L(f,P) &\le 2\,\|f - f_N\|_\infty\,(b-a) + U(f_N,P) - L(f_N,P) \\ &< 2 \cdot \frac{\varepsilon}{3(b-a)} \cdot (b-a) + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

By the integrability criterion, $f \in \mathcal{R}([a,b])$.

Step 2: convergence of integrals. For $n \ge N$,

$$|*| \int_a^b f_n\,\mathrm{d}x - \int_a^b f\,\mathrm{d}x = |*| \int_a^b (f_n - f)\,\mathrm{d}x \le \int_a^b |f_n - f|\,\mathrm{d}x \le \|f_n - f\|_\infty\,(b-a).$$

Since $\|f_n - f\|_\infty \to 0$, the right-hand side tends to zero. □

Example 2.3 (Failed interchange without uniform convergence)**.** Let $f_n\colon [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x \le 1/n, \\ 2n - n^2 x & \text{if } 1/n < x \le 2/n, \\ 0 & \text{if } 2/n < x \le 1. \end{cases}$$

Then $f_n(x) \to 0$ pointwise for every $x \in [0,1]$, but $\int_0^1 f_n\,\mathrm{d}x = 1$ for every n. The interchange fails because the convergence is not uniform (in fact $\sup f_n = n \to \infty$).

2.6 Interchange of Limit and Derivative

Theorem 2.4 (Uniform convergence and differentiation)**.** Let (f_n) be a sequence of C^1 functions on $[a,b]$. Suppose that:

(i) There exists $x_0 \in [a,b]$ such that $(f_n(x_0))$ converges.

(ii) The sequence of derivatives (f_n') converges uniformly on $[a,b]$ to some function

g.

Then (f_n) converges uniformly on $[a,b]$ to a C^1 function f, and $f' = g$. That is,

$$\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} f_n'.$$

Proof. Step 1: uniform convergence of (f_n). By the Fundamental Theorem of Calculus,

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,\mathrm{d}t.$$

Define $f(x) = \ell + \int_{x_0}^{x} g(t)\,\mathrm{d}t$, where $\ell = \lim_{n\to\infty} f_n(x_0)$.

For any $x \in [a,b]$,

$$\begin{aligned}|f_n(x) - f(x)| &\le |f_n(x_0) - \ell| + |*| \int_{x_0}^{x} \left(f_n'(t) - g(t)\right) \mathrm{d}t \\ &\le |f_n(x_0) - \ell| + \|f_n' - g\|_\infty \cdot (b-a).\end{aligned}$$

Both terms on the right tend to zero (the first by hypothesis (i), the second by hypothesis (ii)), independently of x. Hence $f_n \to f$ uniformly.

Step 2: f is C^1 and $f' = g$. Since g is the uniform limit of continuous functions f_n', it is continuous by Theorem 2.2. By the First Fundamental Theorem of Calculus, $f(x) = \ell + \displaystyle\int_{x_0}^{x} g(t)\,\mathrm{d}t$ is differentiable with $f'(x) = g(x)$. □

Remark 2.3 (Hypotheses are sharp). The condition that (f_n') converges *uniformly* cannot be weakened to pointwise. Mere pointwise convergence of (f_n) and (f_n') does not ensure $(\lim f_n)' = \lim f_n'$. For instance, $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ converges uniformly to 0 on $\mathbb{R}$, but $f_n'(x) = \sqrt{n}\cos(nx)$ does not converge at all.

2.7 Series of Functions

Definition 2.3 (Pointwise, uniform, and normal convergence of series). Let (u_n) be a sequence of functions $u_n \colon D \to \mathbb{R}$. The series $\sum_{n\ge 0} u_n$ is said to converge:

- **Pointwise** on D if for every $x \in D$, the numerical series $\sum u_n(x)$ converges.
- **Uniformly** on D if the sequence of partial sums $S_N = \sum_{n=0}^{N} u_n$ converges uniformly on D.
- **Normally** on D if $\sum_{n\ge 0} \|u_n\|_\infty < +\infty$, where $\|u_n\|_\infty = \sup_{x\in D} |u_n(x)|$.

Proposition 2.1 (Normal convergence implies uniform convergence). If $\sum u_n$ converges normally on D, then it converges uniformly (and hence pointwise) on D.

Proof. For $N < M$, $\|S_M - S_N\|_\infty = \left\|\sum_{n=N+1}^{M} u_n\right\|_\infty \le \sum_{n=N+1}^{M} \|u_n\|_\infty$. Since $\sum \|u_n\|_\infty$ converges, its tail tends to zero, so (S_N) is uniformly Cauchy, hence uniformly convergent. □

2.8 The Weierstrass M-Test

Theorem 2.5 (Weierstrass M-test)**.** Let (u_n) be a sequence of functions on D and suppose there exist constants $M_n \geq 0$ with $|u_n(x)| \leq M_n$ for all $x \in D$ and $\sum_{n\geq 0} M_n < +\infty$. Then $\sum u_n$ converges uniformly (and absolutely) on D.

Proof. We have $\|u_n\|_\infty \leq M_n$, so $\sum \|u_n\|_\infty \leq \sum M_n < +\infty$. Hence $\sum u_n$ converges normally, and by Proposition 2.1, it converges uniformly.

Absolute convergence at each x follows from $\sum |u_n(x)| \leq \sum M_n < +\infty$. □

Example 2.4 (Application of the M-test)**.** The series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly on $\mathbb{R}$. Indeed, $\left|\frac{\sin(nx)}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2} < +\infty$.

2.9 Interchange Theorems for Series

The interchange theorems for sequences translate directly into theorems for series via partial sums.

Corollary 2.1 (Term-by-term integration)**.** If $\sum u_n$ converges uniformly on $[a, b]$ and each u_n is Riemann integrable, then $\sum u_n$ is integrable and

$$\int_a^b \sum_{n=0}^{\infty} u_n(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_a^b u_n(x)\,\mathrm{d}x.$$

Proof. Apply Theorem 2.3 to the partial sums $S_N = \sum_{n=0}^{N} u_n$. □

Corollary 2.2 (Term-by-term differentiation)**.** Suppose each u_n is C^1 on $[a, b]$, $\sum u_n(x_0)$ converges for some $x_0 \in [a, b]$, and $\sum u_n'$ converges uniformly on $[a, b]$. Then $\sum u_n$ converges uniformly to a C^1 function S and

$$S'(x) = \sum_{n=0}^{\infty} u_n'(x).$$

Proof. Apply Theorem 2.4 to the partial sums. □

2.10 Power Series

Definition 2.4 (Power series)**.** A *power series* centered at $a \in \mathbb{R}$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n,$$

where $(c_n)_{n\geq 0}$ is a sequence of real coefficients.

Theorem 2.6 (Radius of convergence)**.** For every power series $\sum c_n(x-a)^n$ there exists a unique $R \in [0,+\infty]$ (the *radius of convergence*) such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$. Moreover,

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n} \qquad \text{(Cauchy–Hadamard formula)}.$$

Theorem 2.7 (Uniform convergence of power series on compact subsets)**.** Let $\sum c_n(x-a)^n$ have radius of convergence $R > 0$. Then for every $0 < r < R$, the series converges uniformly (in fact normally) on $[a-r, a+r]$.

Proof. For $|x-a| \le r < R$, $|c_n(x-a)^n| \le |c_n|\, r^n$. Since $r < R$, the series $\sum |c_n|\, r^n$ converges (by definition of the radius of convergence). The Weierstrass M-test with $M_n = |c_n|\, r^n$ gives uniform convergence on $[a-r, a+r]$. □

Theorem 2.8 (Term-by-term differentiation and integration of power series)**.** Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of convergence $R > 0$. Then:

(a) f is infinitely differentiable on $(a-R, a+R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n(x-a)^{n-1},$$

with the derived series having the same radius of convergence R.

(b) For any $x \in (a-R, a+R)$,

$$\int_a^x f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1},$$

with the integrated series having the same radius of convergence R.

Proof. **(a).** Let $g(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$. We first show that the derived series has radius of convergence R. Since $\limsup_{n\to\infty} |nc_n|^{1/n} = \limsup |c_n|^{1/n} \cdot \lim n^{1/n} = \limsup |c_n|^{1/n} \cdot 1 = 1/R$, the Cauchy–Hadamard formula gives the same radius.

Now fix $x_0 \in (a-R, a+R)$ and choose r with $|x_0 - a| < r < R$. On $[a-r, a+r]$, the series $\sum nc_n(x-a)^{n-1}$ converges uniformly (by the M-test with $M_n = n\,|c_n|\, r^{n-1}$, since $\sum M_n$ converges). Also, $\sum c_n(x_0-a)^n$ converges. By Corollary 2.2, f is differentiable and $f'(x) = g(x)$ on $(a-r, a+r)$. Since r can be taken arbitrarily close to R, this holds on all of $(a-R, a+R)$. Iterating gives infinite differentiability.

(b). The integrated series has the same radius by Cauchy–Hadamard (since $|c_n/(n+1)|^{1/n} \to |c_n|^{1/n}$ in the limsup). On any compact sub-interval of $(a-R, a+R)$ the original series converges uniformly, so term-by-term integration is justified by Corollary 2.1. □

2.11 Abel's Theorem

Theorem 2.9 (Abel's theorem)**.** Let $\sum_{n=0}^{\infty} c_n x^n$ have radius of convergence $R > 0$ (which may be $+\infty$). If the series $\sum_{n=0}^{\infty} c_n R^n$ converges (i.e. the power series converges at the right endpoint $x = R$), then

$$\lim_{x \to R^-} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n R^n.$$

That is, the function $f(x) = \sum c_n x^n$ is left-continuous at $x = R$. An analogous statement holds at $x = -R$.

Abel's theorem is a delicate result whose proof uses Abel summation (summation by parts). It has important applications, for instance in deriving $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ from the power series $\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ ($R = 1$, converges at $x = 1$ by the alternating series test).

2.12 The Weierstrass Approximation Theorem

Theorem 2.10 (Weierstrass approximation theorem)**.** Every continuous function $f : [a, b] \to \mathbb{R}$ can be uniformly approximated by polynomials. That is, for every $\varepsilon > 0$ there exists a polynomial p such that

$$\sup_{x \in [a,b]} |f(x) - p(x)| < \varepsilon.$$

Equivalently, the polynomials are dense in $(C([a, b]), \|\cdot\|_\infty)$.

Proof idea via Bernstein polynomials. Without loss of generality take $[a, b] = [0, 1]$. For $f \in C([0, 1])$ and $n \geq 1$, the n-th *Bernstein polynomial* of f is

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}.$$

This is a polynomial of degree at most n that samples f at the points k/n and takes a weighted average using the binomial distribution.

The key properties used in the proof are:

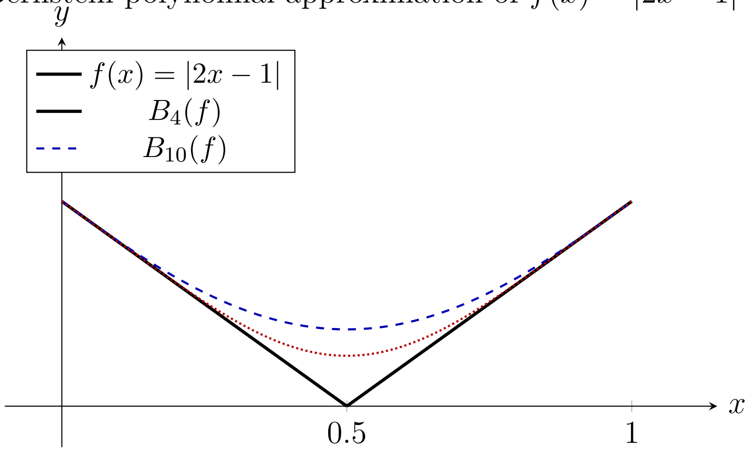
(i) $B_n(1)(x) = 1$ (the binomial theorem),

(ii) $B_n(t)(x) = x$,

(iii) $B_n(t^2)(x) = x^2 + \frac{x(1-x)}{n}$.

Using the uniform continuity of f on $[0, 1]$ and properties (i)–(iii) to control the variance, one shows that $\|B_n(f) - f\|_\infty \to 0$ as $n \to \infty$.

Bernstein polynomial approximation of $f(x) = |2x - 1|$



2.13 Counterexamples for Failed Interchanges

We collect several instructive counterexamples showing that the hypotheses of the interchange theorems cannot be relaxed.

Example 2.5 (Pointwise convergence does not preserve continuity)**.** As seen in Example 2.1, $f_n(x) = x^n$ on $[0, 1]$ converges pointwise to a discontinuous function.

Example 2.6 (Pointwise convergence and integration)**.** Define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = n \cdot \mathbf{1}_{(0,1/n]}(x)$. Then $f_n \to 0$ pointwise, but $\int_0^1 f_n \, dx = 1$ for all n.

Example 2.7 (Pointwise convergence and differentiation)**.** Let $f_n(x) = \frac{\sin(nx)}{n}$ on $\mathbb{R}$. Then $f_n \to 0$ uniformly, but $f_n'(x) = \cos(nx)$ does not converge at all (e.g. at $x = 0$, $f_n'(0) = 1$ for every n, while at $x = \pi$, $f_n'(\pi) = \cos(n\pi) = (-1)^n$). Note that (f_n') does *not* converge uniformly, so the hypotheses of Theorem 2.4 are not met.

Example 2.8 (Differentiation of a pointwise but non-uniformly convergent series)**.** The series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges pointwise on $\mathbb{R}$ (compare with $1/n^2$ for $x \neq 0$). Each term is C^1, but the series of derivatives $\sum \frac{1-nx^2}{n(1+nx^2)^2}$ diverges at $x = 0$. This shows that without uniform convergence of the derived series, term-by-term differentiation fails.

2.14 Exercises

Exercise 2.1 (Uniform convergence on sub-domains)**.** Show that $f_n(x) = x^n$ converges uniformly on $[0, a]$ for every $0 < a < 1$, but not uniformly on $[0, 1]$.

Exercise 2.2 (Sup-norm computation)**.** Let $f_n(x) = \frac{x}{1+nx^2}$ on $[0, +\infty)$. Compute $\|f_n\|_\infty$ and determine whether (f_n) converges uniformly on $[0, +\infty)$.

Exercise 2.3 (Uniform Cauchy criterion application)**.** Show that $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ converges uniformly on every interval $[-R, R]$, $R > 0$, by verifying the uniform Cauchy criterion.

Exercise 2.4 (Uniform limit preserves boundedness)**.** Prove that if each f_n is bounded on D and $f_n \to f$ uniformly on D, then f is bounded on D.

Exercise 2.5 (Weierstrass M-test applications)**.** Show that each series converges uniformly on the given domain:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^3}$ on $\mathbb{R}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $[-R, R]$ for any $R > 0$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n + x^2}$ on $\mathbb{R}$ (use Leibniz and then verify uniformity)

Exercise 2.6 (Continuity of a series)**.** Show that $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$ is continuous on $\mathbb{R}$.

Exercise 2.7 (Term-by-term integration)**.** Let $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$ for $|x| < 1$. By integrating term by term, derive the Leibniz formula $\displaystyle\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. Justify the interchange at $x = 1$ using Abel's theorem.

Exercise 2.8 (Interchange failure for derivatives)**.** Let $f_n(x) = \frac{1}{n}\sin(n^2 x)$. Show that $f_n \to 0$ uniformly on $\mathbb{R}$, but (f_n') does not converge at any x. Why does this not contradict Theorem 2.4?

Exercise 2.9 (Power series and radius)**.** Find the radius of convergence of each power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n$

Exercise 2.10 (Abel's theorem application)**.** Using the power series $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ ($|x| < 1$) and Abel's theorem, prove that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

Exercise 2.11 (Uniform but not normal convergence)**.** Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+x^2}$ converges uniformly on $[0, +\infty)$ but not normally. *Hint:* the sup-norm of the n-th term is $1/n$.

Exercise 2.12 (Dini's theorem)**.** Let (f_n) be a sequence of continuous functions on a compact set K converging pointwise to a continuous function f. If f_n is *monotone* (i.e. $f_1 \geq f_2 \geq \cdots$ or $f_1 \leq f_2 \leq \cdots$), prove that the convergence is uniform. *Hint:* consider $g_n = f_n - f$ (or $f - f_n$); use compactness to extract a finite subcover from the open cover $\{x : |g_n(x)| < \varepsilon\}$.

Chapter 8 — Summary

- **Pointwise convergence**: $f_n(x) \to f(x)$ for each x; the rate may depend on x.
- **Uniform convergence**: $\|f_n - f\|_\infty \to 0$; the rate is independent of x.
- The uniform limit of continuous functions is **continuous** ($\varepsilon/3$-argument).
- Under uniform convergence, one may **interchange limit and integral**.
- To interchange **limit and derivative**, one needs uniform convergence of the *derivatives* (plus pointwise convergence of the functions at one point).
- The **Weierstrass M-test** gives a practical criterion for uniform (in fact normal) convergence of series.
- **Power series** converge uniformly on compact subsets of their interval of convergence and can be differentiated and integrated term by term.
- **Abel's theorem** ensures continuity of a power series at an endpoint where it converges.
- The **Weierstrass approximation theorem**: every continuous function on $[a, b]$ is the uniform limit of polynomials.
- **Counterexamples** show that pointwise convergence alone does *not* justify interchanging limits with continuity, integration, or differentiation.

Chapter 3

Introduction to Multivariable Differential Calculus

Introduction

Throughout the preceding chapters we have studied functions of a single real variable. Yet the phenomena of physics, economics, and geometry are almost never governed by a single parameter. The temperature at a point in a room depends on three spatial coordinates and on time; the profit of a firm depends on quantities produced, prices, and advertising expenditures; the shape of a surface is described by a function $f(x, y)$.

This chapter extends differential calculus to functions $f : \mathbb{R}^n \to \mathbb{R}^m$. The central idea is that a *differentiable* function is one that admits a good *linear* approximation near each point. Making this precise requires us first to discuss the topology of $\mathbb{R}^n$.

3.1 Topology of $\mathbb{R}^n$

3.1.1 Norms on $\mathbb{R}^n$

Definition 3.1. Norm on $\mathbb{R}^n$norm-Rn A *norm* on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to [0, +\infty)$ satisfying, for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$:

(i) **Positive definiteness:** $\|x\| = 0 \iff x = 0$.

(ii) **Absolute homogeneity:** $\|\lambda x\| = |\lambda| \, \|x\|$.

(iii) **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$.

Example 3.1 (Common norms on $\mathbb{R}^n$)**.** Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

(a) **Euclidean norm**: $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.

(b) **1-norm**: $\|x\|_1 = \sum_{i=1}^n |x_i|$.

(c) **Sup-norm**: $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$.

Proposition 3.1 (Relationships between norms)**.** For every $x \in \mathbb{R}^n$,

$$\|x\|_\infty \;\le\; \|x\|_2 \;\le\; \|x\|_1 \;\le\; n\,\|x\|_\infty.$$

Proof. For the first inequality, $\|x\|_\infty = \max_i |x_i|$ and $\|x\|_2 = (\sum_i x_i^2)^{1/2} \ge (\max_i x_i^2)^{1/2} = \|x\|_\infty$. For the second, by Cauchy–Schwarz, $\|x\|_1 = \sum_i |x_i| \le \sqrt{n}\,(\sum_i x_i^2)^{1/2} = \sqrt{n}\,\|x\|_2$; the sharper bound $\|x\|_2 \le \|x\|_1$ follows from $(\sum_i |x_i|)^2 = \sum_i x_i^2 + 2\sum_{i<j} |x_i||x_j| \ge \sum_i x_i^2$. Finally, $\|x\|_1 = \sum_i |x_i| \le n \max_i |x_i| = n\|x\|_\infty$. □

Theorem 3.1 (Equivalence of norms in finite dimension)**.** All norms on $\mathbb{R}^n$ are equivalent: for any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^n$, there exist constants $0 < c \le C$ such that

$$c\,\|x\|_a \;\le\; \|x\|_b \;\le\; C\,\|x\|_a \qquad \text{for all } x \in \mathbb{R}^n.$$

Remark 3.1. The proof uses compactness of the unit sphere in $(\mathbb{R}^n, \|\cdot\|_2)$ and continuity of any norm with respect to the Euclidean norm. We refer to Rudin [1] for details. The conclusion is that in $\mathbb{R}^n$ the choice of norm does not affect notions of convergence, continuity, or openness.

3.1.2 Open sets, closed sets, and balls

Definition 3.2 (Open ball)**.** Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, the *open ball* of centre $a \in \mathbb{R}^n$ and radius $r > 0$ is

$$B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}.$$

Definition 3.3. Open and closed sets in $\mathbb{R}^n$open-closed-Rn A set $U \subseteq \mathbb{R}^n$ is *open* if for every $a \in U$ there exists $r > 0$ such that $B(a, r) \subseteq U$. A set F is *closed* if its complement $\mathbb{R}^n \setminus F$ is open.

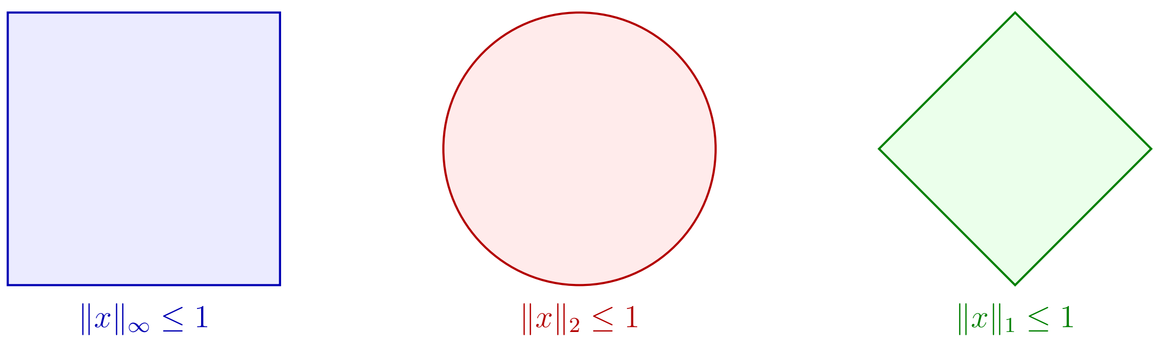


Figure 3.1: Unit balls in $\mathbb{R}^2$ for the sup-norm, Euclidean norm, and 1-norm.

3.1.3 Sequences in $\mathbb{R}^n$

Proposition 3.2 (Component-wise convergence)**.** Let $(x^{(k)})_{k\geq 1}$ be a sequence in $\mathbb{R}^n$ with $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})$, and let $a = (a_1, \dots, a_n)$. Then

$$x^{(k)} \to a \text{ in } \mathbb{R}^n \quad \Longleftrightarrow \quad x_i^{(k)} \to a_i \text{ in } \mathbb{R},\ \forall\, i = 1, \dots, n.$$

Proof. ($\Rightarrow$) For each i, $|x_i^{(k)} - a_i| \leq \|x^{(k)} - a\|_2 \to 0$.

($\Leftarrow$) $\|x^{(k)} - a\|_\infty = \max_i |x_i^{(k)} - a_i| \to 0$, and $\|x^{(k)} - a\|_2 \leq \sqrt{n}\, \|x^{(k)} - a\|_\infty \to 0$. □

3.2 Limits and Continuity in $\mathbb{R}^n$

Definition 3.4 (Limit of a function of several variables)**.** Let $U \subseteq \mathbb{R}^n$ be open, $a \in U$, and $f\colon U \setminus \{a\} \to \mathbb{R}^m$. We say $\lim_{x\to a} f(x) = \ell \in \mathbb{R}^m$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < \|x - a\| < \delta \implies \|f(x) - \ell\| < \varepsilon.$$

Definition 3.5 (Continuity)**.** A function $f\colon U \to \mathbb{R}^m$ is *continuous at* $a \in U$ if $\lim_{x\to a} f(x) = f(a)$. It is *continuous on* U if it is continuous at every point of U.

Example 3.2 (Iterated limits need not equal the double limit)**.** Define $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ by

$$f(x, y) = \frac{xy}{x^2 + y^2}.$$

For every fixed $y \neq 0$, $\lim_{x\to 0} f(x, y) = 0$; likewise $\lim_{y\to 0} f(x, y) = 0$ for fixed $x \neq 0$. Both iterated limits equal 0. However, along the line $y = x$, $f(x, x) = \frac{x^2}{2x^2} = \frac{1}{2}$, while along $y = -x$, $f(x, -x) = -\frac{1}{2}$. Therefore the double limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Theorem 3.2 (Extreme value theorem on compact sets)**.** Let $K \subseteq \mathbb{R}^n$ be compact (closed and bounded) and $f\colon K \to \mathbb{R}$ continuous. Then f attains its maximum and minimum on K: there exist $a, b \in K$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in K$.

Proof. Since K is compact and f is continuous, the image $f(K)$ is a compact subset of $\mathbb{R}$ (continuous image of a compact set is compact), hence closed and bounded. By completeness of $\mathbb{R}$, $\sup f(K)$ and $\inf f(K)$ exist and belong to $f(K)$ (since $f(K)$ is closed), giving points $a, b \in K$ with $f(a) = \inf f(K)$ and $f(b) = \sup f(K)$. □

3.3 Partial Derivatives

Definition 3.6 (Partial derivative)**.** Let $f\colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ with U open, and let $a = (a_1, \ldots, a_n) \in U$. The *partial derivative of f with respect to x_i at a* is

$$\frac{\partial f}{\partial x_i}(a) = \lim_{h\to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a)}{h},$$

provided this limit exists.

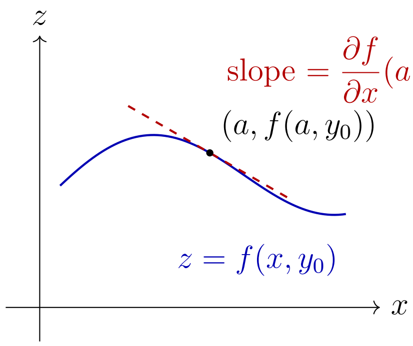


Figure 3.2: The partial derivative $\partial f/\partial x$ as the slope of the cross-section $y = y_0$.

Theorem 3.3 (Existence of partial derivatives does not imply continuity)**.** There exists a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that both $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist, yet f is not continuous at $(0,0)$.

Proof. Define

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0,0), \\ 0 & \text{if } (x, y) = (0,0). \end{cases}$$

Partial derivatives at the origin. $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$, since $f(h, 0) = 0$ for all $h \neq 0$. Similarly, $\frac{\partial f}{\partial y}(0,0) = 0$.

Discontinuity. Along the curve $y = x^2$,

$$f(x, x^2) = \frac{x^2 \cdot x^2}{x^4 + x^4} = \frac{x^4}{2x^4} = \frac{1}{2} \quad \text{for all } x \neq 0.$$

Therefore $\lim_{x\to 0} f(x, x^2) = \frac{1}{2} \neq 0 = f(0,0)$, so f is not continuous at the origin. □

3.4 Differentiability

Key Definition: Differentiability

Definition 3.7 (Differentiability)**.** Let $U \subseteq \mathbb{R}^n$ be open, $a \in U$, and $f\colon U \to \mathbb{R}^m$. We say f is *differentiable at* a if there exists a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(a+h) = f(a) + L(h) + \|h\|\,\varepsilon(h),$$

where $\varepsilon(h) \to 0$ as $h \to 0$. Equivalently,

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0.$$

The linear map L is unique; it is called the *differential* (or *total derivative*) of f at a and is denoted df_a or $Df(a)$.

Theorem 3.4 (Differentiable implies continuous)**.** If f is differentiable at a, then f is continuous at a.

Proof. Write $f(a+h) = f(a) + L(h) + \|h\|\,\varepsilon(h)$ with $\varepsilon(h) \to 0$. Since L is linear (hence continuous on $\mathbb{R}^n$), $L(h) \to 0$ as $h \to 0$, and $\|h\|\,\varepsilon(h) \to 0$ as $h \to 0$. Therefore $f(a+h) \to f(a)$. □

Theorem 3.5 (Differentiable implies partial derivatives exist)**.** If $f\colon U \to \mathbb{R}$ is differentiable at $a \in U$, then all partial derivatives $\frac{\partial f}{\partial x_i}(a)$ exist and

$$df_a(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\, h_i, \qquad h = (h_1, \ldots, h_n).$$

Proof. Let e_i be the i-th standard basis vector. Since f is differentiable,

$$f(a + te_i) = f(a) + t\,L(e_i) + |t|\,\varepsilon(te_i)$$

with $\varepsilon(te_i) \to 0$ as $t \to 0$. Hence

$$\frac{f(a+te_i) - f(a)}{t} = L(e_i) + \frac{|t|}{t}\,\varepsilon(te_i) \;\longrightarrow\; L(e_i)$$

as $t \to 0$. This shows $\frac{\partial f}{\partial x_i}(a) = L(e_i)$. Since L is linear and $h = \sum_i h_i e_i$, we get $L(h) = \sum_i h_i L(e_i) = \sum_i \frac{\partial f}{\partial x_i}(a)\, h_i$. □

Definition 3.8 (Jacobian matrix)**.** If $f = (f_1, \ldots, f_m)\colon U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, the *Jacobian matrix* of f at a is the $m \times n$ matrix

$$J_f(a) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(a) & \cdots & \dfrac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(a) & \cdots & \dfrac{\partial f_m}{\partial x_n}(a) \end{pmatrix}.$$

Then $df_a(h) = J_f(a)\, h$ for all $h \in \mathbb{R}^n$.

Theorem 3.6 (C^1 implies differentiable)**.** Let $f\colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ be such that all partial derivatives $\frac{\partial f}{\partial x_i}$ exist in a neighbourhood of a and are continuous at a. Then f is differentiable at a.

Proof for the case $n = 2$. Write $a = (a_1, a_2)$ and $h = (h_1, h_2)$ with $a + h \in U$. Then

$$f(a+h) - f(a) = \Big[f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2 + h_2)\Big] + \Big[f(a_1, a_2 + h_2) - f(a_1, a_2)\Big].$$

Apply the mean value theorem in one variable to each bracket. For the first, the function $t \mapsto f(t, a_2 + h_2)$ is differentiable on the segment $[a_1, a_1 + h_1]$, so there exists θ_1 between a_1 and $a_1 + h_1$ with

$$f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2 + h_2) = h_1 \frac{\partial f}{\partial x_1}(\theta_1, a_2 + h_2).$$

Similarly, there exists θ_2 between a_2 and $a_2 + h_2$ with

$$f(a_1, a_2 + h_2) - f(a_1, a_2) = h_2 \frac{\partial f}{\partial x_2}(a_1, \theta_2).$$

Therefore

$$f(a+h) - f(a) = h_1 \frac{\partial f}{\partial x_1}(a) + h_2 \frac{\partial f}{\partial x_2}(a) + R(h),$$

where

$$R(h) = h_1 \Big[\frac{\partial f}{\partial x_1}(\theta_1, a_2 + h_2) - \frac{\partial f}{\partial x_1}(a)\Big] + h_2 \Big[\frac{\partial f}{\partial x_2}(a_1, \theta_2) - \frac{\partial f}{\partial x_2}(a)\Big].$$

Since $(\theta_1, a_2 + h_2) \to a$ and $(a_1, \theta_2) \to a$ as $h \to 0$, the continuity of the partial derivatives at a gives

$$\frac{|R(h)|}{\|h\|} \leq \left|\frac{\partial f}{\partial x_1}(\theta_1, a_2 + h_2) - \frac{\partial f}{\partial x_1}(a)\right| + \left|\frac{\partial f}{\partial x_2}(a_1, \theta_2) - \frac{\partial f}{\partial x_2}(a)\right| \longrightarrow 0,$$

using $|h_i| \leq \|h\|$. Hence f is differentiable at a with $df_a(h) = \frac{\partial f}{\partial x_1}(a)h_1 + \frac{\partial f}{\partial x_2}(a)h_2$. □

3.5 The Chain Rule

Theorem 3.7 (Chain rule)**.** Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open. If $f\colon U \to V$ is differentiable at a and $g\colon V \to \mathbb{R}^p$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$d(g \circ f)_a = dg_{f(a)} \circ df_a.$$

In terms of Jacobian matrices, $J_{g\circ f}(a) = J_g(f(a)) \cdot J_f(a)$.

Proof. Write $b = f(a)$. By differentiability,

$$f(a+h) = f(a) + df_a(h) + \|h\|\,\varepsilon_1(h),$$
$$g(b+k) = g(b) + dg_b(k) + \|k\|\,\varepsilon_2(k),$$

where $\varepsilon_1(h) \to 0$ and $\varepsilon_2(k) \to 0$. Set $k = f(a+h) - f(a) = df_a(h) + \|h\|\varepsilon_1(h)$. Then $\|k\| \leq M\|h\|$ for some constant $M > 0$ (since df_a is a bounded linear map and $\|\varepsilon_1(h)\| \to 0$, so $\|k\| \leq (\|df_a\| + 1)\|h\|$ for h small).

Now

$$\begin{aligned} g(f(a+h)) &= g(b) + dg_b(k) + \|k\|\,\varepsilon_2(k) \\ &= g(b) + dg_b(df_a(h)) + dg_b(\|h\|\varepsilon_1(h)) + \|k\|\,\varepsilon_2(k). \end{aligned}$$

The remainder is

$$R(h) = \|h\|\,dg_b(\varepsilon_1(h)) + \|k\|\,\varepsilon_2(k).$$

Since dg_b is continuous and linear, $dg_b(\varepsilon_1(h)) \to 0$, so $\|h\|\,dg_b(\varepsilon_1(h)) = o(\|h\|)$. Also $\|k\|\,\|\varepsilon_2(k)\| \leq M\|h\|\,\|\varepsilon_2(k)\| = o(\|h\|)$ since $\varepsilon_2(k) \to 0$ as $k \to 0$ (and $k \to 0$ as $h \to 0$). Therefore $\|R(h)\|/\|h\| \to 0$, proving differentiability with $d(g \circ f)_a = dg_b \circ df_a$. □

3.6 Gradient and Geometric Interpretation

Definition 3.9 (Gradient)**.** If $f\colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at a, the *gradient* of f at a is the vector

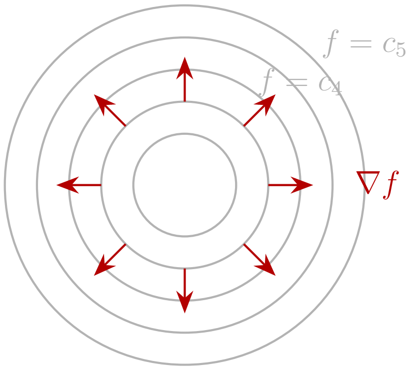
$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right) \in \mathbb{R}^n.$$

The differential can then be written $df_a(h) = \langle \nabla f(a), h\rangle$.

Proposition 3.3 (Direction of steepest ascent)**.** Among all unit vectors $v \in \mathbb{R}^n$ with $\|v\|_2 = 1$, the directional derivative $D_v f(a) = \langle \nabla f(a), v\rangle$ is maximised when $v = \nabla f(a)/\|\nabla f(a)\|_2$ (assuming $\nabla f(a) \neq 0$). The maximum value is $\|\nabla f(a)\|_2$.

Proof. By Cauchy–Schwarz, $|\langle \nabla f(a), v\rangle| \leq \|\nabla f(a)\|_2 \|v\|_2 = \|\nabla f(a)\|_2$, with equality when v is parallel to $\nabla f(a)$. □

Remark 3.2. The gradient $\nabla f(a)$ is perpendicular to the level set $\{x : f(x) = f(a)\}$ at the point a. This is because for any curve $\gamma(t)$ lying in the level set with $\gamma(0) = a$, differentiating $f(\gamma(t)) = f(a)$ at $t = 0$ gives $\langle \nabla f(a), \gamma'(0)\rangle = 0$.



Gradient vectors point radially outward,
perpendicular to level curves.

Figure 3.3: Gradient vectors on a contour plot of $f(x, y) = x^2 + y^2$.

3.7 Higher-Order Partial Derivatives and Schwarz's Theorem

Theorem 3.8 (Schwarz's theorem — symmetry of mixed partials)**.** Let $f\colon U \subseteq \mathbb{R}^2 \to \mathbb{R}$ with U open. Suppose $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$ all exist in U and that $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous at $(a, b) \in U$. Then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Proof. For h, k small enough, define

$$\Phi(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b).$$

Set $g(t) = f(t, b + k) - f(t, b)$. Then $\Phi(h, k) = g(a + h) - g(a)$. By the mean value theorem, there exists $\theta_1 \in (0, 1)$ such that

$$\Phi(h, k) = h\, g'(a + \theta_1 h) = h\Big[\frac{\partial f}{\partial x}(a + \theta_1 h, b + k) - \frac{\partial f}{\partial x}(a + \theta_1 h, b)\Big].$$

Applying the mean value theorem again (in the y-variable), there exists $\theta_2 \in (0, 1)$ such that

$$\Phi(h, k) = hk\, \frac{\partial^2 f}{\partial y \partial x}(a + \theta_1 h,\ b + \theta_2 k).$$

Similarly, writing $\Phi(h, k)$ using $\tilde{g}(s) = f(a + h, s) - f(a, s)$ and applying the mean value theorem twice in the other order yields

$$\Phi(h, k) = hk\, \frac{\partial^2 f}{\partial x \partial y}(a + \tilde{\theta}_1 h,\ b + \tilde{\theta}_2 k)$$

for some $\tilde{\theta}_1, \tilde{\theta}_2 \in (0, 1)$. Therefore

$$\frac{\partial^2 f}{\partial y \partial x}(a + \theta_1 h, b + \theta_2 k) = \frac{\partial^2 f}{\partial x \partial y}(a + \tilde{\theta}_1 h, b + \tilde{\theta}_2 k).$$

Letting $h, k \to 0$ and using continuity of both mixed partials at (a, b), we obtain

$$\frac{\partial^2 f}{\partial y \partial x}(a, b) = \frac{\partial^2 f}{\partial x \partial y}(a, b). \qquad \square$$

3.8 Taylor's Formula in Several Variables

Theorem 3.9 (Taylor's formula, order 1)**.** Let $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^1. Then for $a, a + h \in U$,

$$f(a + h) = f(a) + \langle \nabla f(a), h \rangle + o(\|h\|).$$

Theorem 3.10 (Taylor's formula, order 2)**.** Let $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^2. Then

$$f(a + h) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\, h_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\, h_i h_j + o(\|h\|^2).$$

Proof. Apply the single-variable Taylor formula to $\varphi(t) = f(a + th)$ at $t = 0$:

$$\varphi(1) = \varphi(0) + \varphi'(0) + \tfrac{1}{2}\varphi''(0) + o(1).$$

By the chain rule, $\varphi'(t) = \sum_i \frac{\partial f}{\partial x_i}(a+th)\, h_i$ and $\varphi''(t) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(a+th)\, h_i h_j$. Evaluating at $t = 0$ yields the stated formula. $\square$

3.9 Extrema of Functions of Several Variables

Definition 3.10 (Hessian matrix)**.** For $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ of class C^2, the *Hessian matrix* at $a \in U$ is the $n \times n$ symmetric matrix

$$H_f(a) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right)_{1 \le i,j \le n}.$$

Theorem 3.11 (Necessary condition for an extremum)**.** If f has a local extremum at an interior point a and f is differentiable at a, then $\nabla f(a) = 0$ (i.e. a is a *critical point*).

Proof. For each i, the function $t \mapsto f(a+te_i)$ has a local extremum at $t = 0$, so $\frac{\partial f}{\partial x_i}(a) = 0$ by the single-variable result. $\square$

Theorem 3.12 (Second-order criterion)**.** Let $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^2 and let $\nabla f(a) = 0$.

(i) If $H_f(a)$ is positive definite, then a is a strict local minimum.

(ii) If $H_f(a)$ is negative definite, then a is a strict local maximum.

(iii) If $H_f(a)$ is indefinite (has both positive and negative eigenvalues), then a is a *saddle point*.

Example 3.3 (Classification of critical points in $\mathbb{R}^2$)**.** For $f\colon \mathbb{R}^2 \to \mathbb{R}$ of class C^2 with $\nabla f(a) = 0$, set $A = \frac{\partial^2 f}{\partial x^2}(a)$, $B = \frac{\partial^2 f}{\partial x \partial y}(a)$, $C = \frac{\partial^2 f}{\partial y^2}(a)$. Let $\Delta = AC - B^2$.

- $\Delta > 0,\ A > 0$: local minimum.
- $\Delta > 0,\ A < 0$: local maximum.
- $\Delta < 0$: saddle point.
- $\Delta = 0$: test inconclusive.

Example 3.4. Consider $f(x, y) = x^2 - y^2$. Then $\nabla f = 0$ only at the origin. $A = 2$, $B = 0$, $C = -2$, $\Delta = 2 \cdot (-2) - 0 = -4 < 0$: saddle point.

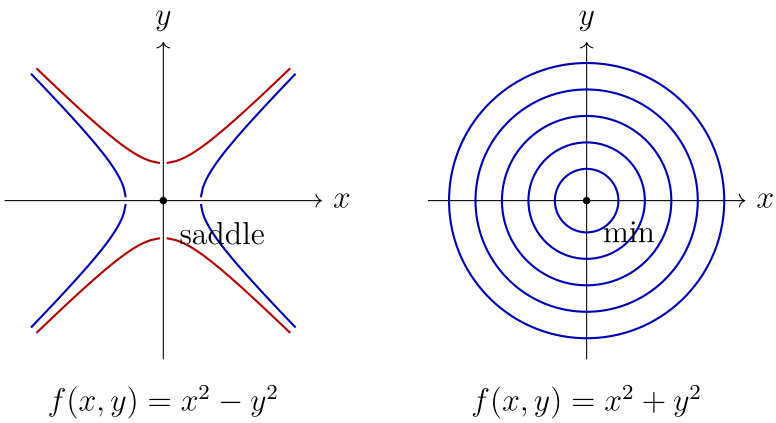


Figure 3.4: Contour plots: saddle point (left) versus local minimum (right).

3.10 Exercises

Exercise 3.1. Show that $\|x\|_2 \le \sqrt{n}\,\|x\|_\infty$ for all $x \in \mathbb{R}^n$.

Exercise 3.2. Let $f(x, y) = \frac{x^3 y}{x^6 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that f is not continuous at the origin, even though f is continuous along every line through the origin.

Exercise 3.3. Compute the Jacobian matrix of $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Show that $\det J_f = r$.

Exercise 3.4. Let $f(x, y) = e^x \cos y$. Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ everywhere.

Exercise 3.5. Let $f(x, y, z) = x^2 y + y z^2 + z x$. Compute ∇f and find all critical points.

Exercise 3.6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^4 + y^4 - 4xy + 1$. Find all critical points and classify them.

Exercise 3.7. Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable and define $h(r,\theta) = g(r\cos\theta, r\sin\theta)$. Express $\frac{\partial h}{\partial r}$ and $\frac{\partial h}{\partial \theta}$ in terms of the partial derivatives of g.

Exercise 3.8. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $f(tx) = t^k f(x)$ for all $t > 0$ (homogeneous of degree k), then $\langle \nabla f(x), x\rangle = kf(x)$ (Euler's identity).

Exercise 3.9. Let $f(x,y) = (x^2+y^2)e^{-(x^2+y^2)}$. Find and classify all critical points of f.

Exercise 3.10. Write the second-order Taylor expansion of $f(x,y) = \sin(x+2y)$ at $(0,0)$.

Exercise 3.11 ($\star\star$)**.** Show that if $f\colon U \to \mathbb{R}$ is differentiable at $a \in U$ and attains a local maximum at a subject to the constraint $g(x) = 0$ (where $g\colon U \to \mathbb{R}$ is C^1 with $\nabla g(a) \neq 0$), then there exists $\lambda \in \mathbb{R}$ such that $\nabla f(a) = \lambda \nabla g(a)$.

Chapter Summary

- All norms on $\mathbb{R}^n$ are equivalent; topology does not depend on the chosen norm.
- Existence of partial derivatives alone does not guarantee continuity or differentiability.
- Differentiability is defined via linear approximation: $f(a+h) = f(a)+L(h)+o(\|h\|)$.
- C^1 implies differentiable; the chain rule composes Jacobians.
- The gradient points in the direction of steepest ascent and is perpendicular to level sets.
- Critical points are classified by the Hessian matrix.

Chapter 4

Multiple Integrals and Change of Variables

Introduction

How does one compute the volume beneath a surface $z = f(x, y)$, the mass of a solid with variable density, or the centre of mass of a lamina? These problems demand integration of functions of several variables. In this chapter we define double and triple integrals via Riemann sums, establish Fubini's theorem (which reduces a multiple integral to iterated single-variable integrals), and present the change-of-variables formula — the multivariable analogue of integration by substitution.

4.1 Double Integrals over Rectangles

Definition 4.1. Riemann sum in $\mathbb{R}^2$riemann-sum-R2 Let $R = [a, b] \times [c, d]$ be a closed rectangle and $f\colon R \to \mathbb{R}$ bounded. A *partition* of R is $P = P_x \times P_y$ where $P_x\colon a = x_0 < x_1 < \cdots < x_p = b$ and $P_y\colon c = y_0 < y_1 < \cdots < y_q = d$. For each sub-rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, choose a sample point $(\xi_{ij}, \eta_{ij}) \in R_{ij}$. The *Riemann sum* is

$$S(f, P) = \sum_{i=1}^{p} \sum_{j=1}^{q} f(\xi_{ij}, \eta_{ij})\, \Delta x_i\, \Delta y_j,$$

where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$.

Definition 4.2 (Double integral)**.** The function f is *Riemann integrable* over R if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ with $|S(f, P) - I| < \varepsilon$ whenever $\|P\| < \delta$ (where $\|P\|$ is the mesh size), for every choice of sample points. We write $I = \iint_R f(x, y)\, dx\, dy$.

Theorem 4.1 (Integrability of continuous functions)**.** Every continuous function on a closed rectangle is Riemann integrable.

4.2 Fubini's Theorem

Theorem 4.2 (Fubini's theorem for continuous functions)**.** Let $R = [a,b] \times [c,d]$ and let $f\colon R \to \mathbb{R}$ be continuous. Then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b\!\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\!\left(\int_a^b f(x,y)\,dx\right)dy.$$

Proof. We prove the first equality; the second follows by symmetry. Define $F(x) = \int_c^d f(x,y)\,dy$. Since f is continuous on the compact set R, it is uniformly continuous: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - f(x',y')| < \varepsilon$ whenever $\|(x,y) - (x',y')\| < \delta$.

Let $P = \{x_0,\dots,x_p\} \times \{y_0,\dots,y_q\}$ be a partition of R with mesh $\|P\| < \delta$. For each sub-rectangle R_{ij}, the minimum and maximum of f on R_{ij} satisfy $M_{ij} - m_{ij} < \varepsilon$.

The lower and upper Darboux sums satisfy

$$L(f,P) \le \int_a^b F(x)\,dx \le U(f,P).$$

Indeed, for each i,

$$\sum_j m_{ij}\,\Delta y_j \le \int_c^d f(x,y)\,dy = F(x) \le \sum_j M_{ij}\,\Delta y_j \qquad \text{for all } x \in [x_{i-1},x_i],$$

so integrating over $[x_{i-1},x_i]$ and summing over i,

$$L(f,P) = \sum_{i,j} m_{ij}\Delta x_i\Delta y_j \le \int_a^b F(x)\,dx \le \sum_{i,j} M_{ij}\Delta x_i\Delta y_j = U(f,P).$$

Since f is integrable, $U(f,P) - L(f,P) < \varepsilon\,|R|$ (where $|R|$ is the area of R) for $\|P\|$ small enough. Because the double integral $\iint_R f$ also lies between $L(f,P)$ and $U(f,P)$, we conclude

$$\left|\iint_R f\,dx\,dy - \int_a^b F(x)\,dx\right| \le U(f,P) - L(f,P) < \varepsilon\,|R|.$$

Since ε is arbitrary, the two quantities are equal. □

4.3 Integration over Bounded Domains

Definition 4.3 (Type I and Type II domains)**.** A *Type I* domain has the form $D = \{(x,y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$ for continuous φ_1, φ_2. A *Type II* domain has the form $D = \{(x,y) : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$.

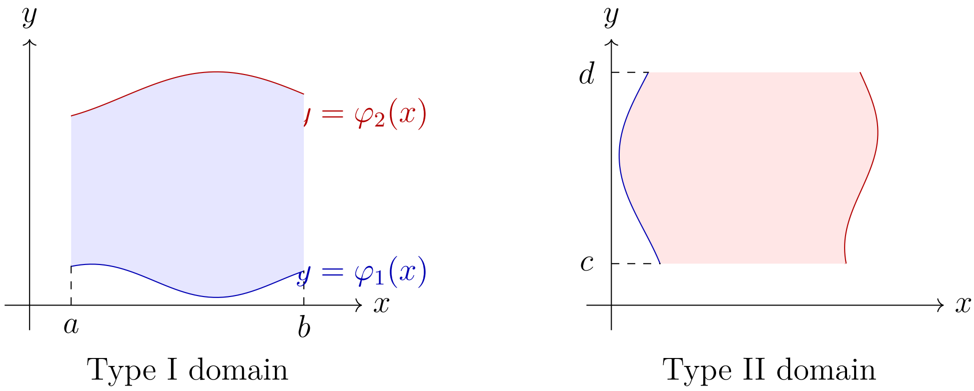


Figure 4.1: Type I (left) and Type II (right) integration domains.

For a Type I domain,

$$\iint_D f\,dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx.$$

Example 4.1 (Integration over a triangle)**.** Compute $\iint_D (x+y)\,dA$ where D is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.

The domain is of Type I: $0 \le x \le 1$, $0 \le y \le 1-x$. Thus

$$\begin{aligned}\iint_D (x+y)\,dA &= \int_0^1 \int_0^{1-x} (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{1-x} dx \\ &= \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2}\right) dx = \int_0^1 \frac{1-x^2}{2}\,dx \\ &= \frac{1}{2}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}.\end{aligned}$$

4.4 Change of Variables

Theorem 4.3 (Change of variables formula)**.** Let $\Phi\colon U \to V$ be a C^1 diffeomorphism between open sets $U, V \subseteq \mathbb{R}^n$, and let $f\colon V \to \mathbb{R}$ be continuous with compact support in V. Then

$$\int_V f(y)\,dy = \int_U f(\Phi(x))\,|\det J_\Phi(x)|\,dx.$$

Remark 4.1. The factor $|\det J_\Phi(x)|$ accounts for how the mapping Φ locally stretches or compresses volumes.

4.5 Polar Coordinates

The map $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$ has Jacobian matrix

$$J_\Phi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \qquad \det J_\Phi = r.$$

Therefore $dA = r\,dr\,d\theta$.

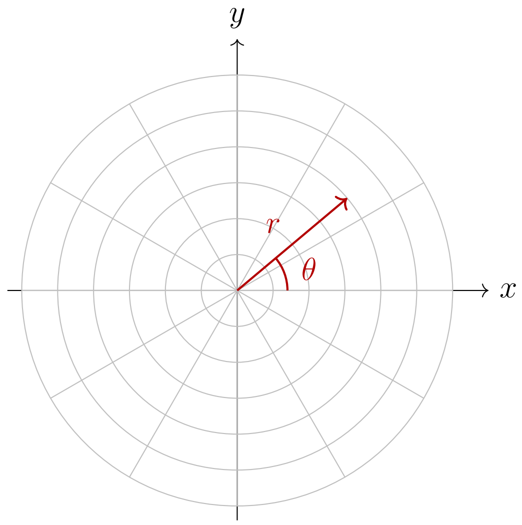


Figure 4.2: Polar coordinate grid: concentric circles $r = \text{const}$ and rays $\theta = \text{const}$.

Example 4.2 (Area of a disk)**.**

$$\text{Area}(D_R) = \iint_{D_R} dA = \int_0^{2\pi} \int_0^R r\,dr\,d\theta = 2\pi \cdot \frac{R^2}{2} = \pi R^2.$$

Theorem 4.4 (The Gaussian integral)**.**

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Proof. Let $I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$. Then

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy.$$

Switching to polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ with $dA = r\,dr\,d\theta$:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta = 2\pi \int_0^{\infty} r\,e^{-r^2}\,dr.$$

Substituting $u = r^2$, $du = 2r\,dr$:

$$\int_0^{\infty} r\,e^{-r^2}\,dr = \frac{1}{2}\int_0^{\infty} e^{-u}\,du = \frac{1}{2}.$$

Therefore $I^2 = 2\pi \cdot \frac{1}{2} = \pi$, and since $I > 0$, we conclude $I = \sqrt{\pi}$. □

4.6 Triple Integrals

Fubini's theorem extends naturally:

Theorem 4.5 (Fubini in $\mathbb{R}^3$)**.** If $f\colon [a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\to\mathbb{R}$ is continuous, then

$$\iiint f\,dx\,dy\,dz = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)\,dz\,dy\,dx,$$

and the order of integration may be permuted freely.

4.7 Cylindrical and Spherical Coordinates

4.7.1 Cylindrical coordinates

The map $(r,\theta,z)\mapsto(r\cos\theta, r\sin\theta, z)$ has Jacobian determinant r. Hence $dV = r\,dr\,d\theta\,dz$.

4.7.2 Spherical coordinates

Definition 4.4 (Spherical coordinates)**.** The map $\Phi(\rho,\theta,\varphi) = (\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)$, where $\rho\ge 0$, $\theta\in[0,2\pi)$, $\varphi\in[0,\pi]$.

The Jacobian matrix is

$$J_\Phi = \begin{pmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix}.$$

A direct computation gives $\det J_\Phi = -\rho^2\sin\varphi$, so

$$|\det J_\Phi| = \rho^2\sin\varphi, \qquad dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

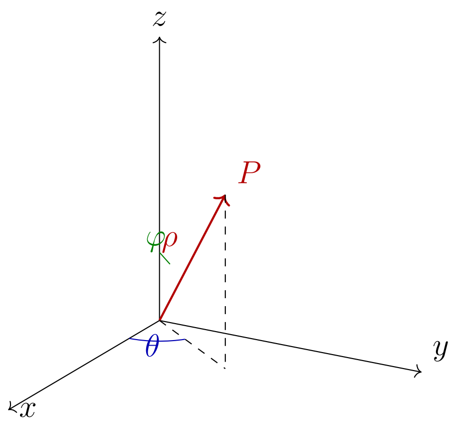


Figure 4.3: Spherical coordinates (ρ,θ,φ).

Example 4.3 (Volume of the ball B_R)**.**

$$\begin{aligned}\mathrm{Vol}(B_R) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\ &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\varphi\,d\varphi\right)\left(\int_0^{R}\rho^2\,d\rho\right) \\ &= 2\pi\cdot 2\cdot\frac{R^3}{3} = \frac{4\pi R^3}{3}.\end{aligned}$$

Example 4.4 (Moment of inertia of a solid ball)**.** The moment of inertia of a homogeneous ball of mass M and radius R about a diameter (say the z-axis) is

$$I_z = \frac{\rho_0}{1} \iiint_{B_R} (x^2 + y^2)\, dV,$$

where $\rho_0 = 3M/(4\pi R^3)$ is the density. In spherical coordinates, $x^2 + y^2 = \rho^2 \sin^2 \varphi$, so

$$\begin{aligned} I_z &= \rho_0 \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin^2 \varphi \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\ &= \rho_0 \cdot 2\pi \cdot \int_0^{\pi} \sin^3 \varphi \, d\varphi \cdot \int_0^{R} \rho^4 \, d\rho \\ &= \rho_0 \cdot 2\pi \cdot \frac{4}{3} \cdot \frac{R^5}{5} = \frac{3M}{4\pi R^3} \cdot \frac{8\pi R^5}{15} = \frac{2}{5} M R^2. \end{aligned}$$

4.8 Exercises

Exercise 4.1. Compute $\iint_R xy \, dA$ where $R = [0,1] \times [0,2]$.

Exercise 4.2. Compute $\iint_D e^{x+y} \, dA$ where $D = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}$.

Exercise 4.3. Switch the order of integration: $\int_0^1 \int_0^{\sqrt{x}} f(x, y) \, dy \, dx$.

Exercise 4.4. Use polar coordinates to compute $\iint_D \sqrt{x^2 + y^2} \, dA$ over the disk $D = \{x^2 + y^2 \le 4\}$.

Exercise 4.5. Compute $\iint_D \frac{1}{(1+x^2+y^2)^2} \, dA$ over $\mathbb{R}^2$.

Exercise 4.6. Compute $\iiint_E z \, dV$ where E is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Exercise 4.7. Use cylindrical coordinates to compute $\iiint_E dV$ where $E = \{x^2 + y^2 \le 1,\ 0 \le z \le 2 - x^2 - y^2\}$.

Exercise 4.8. Compute the volume enclosed by the sphere $\rho = 2$ and the cone $\varphi = \pi/3$ (using spherical coordinates).

Exercise 4.9. Find the centre of mass of the region bounded by $y = x^2$ and $y = x$ with uniform density.

Exercise 4.10. Show that $\int_0^\infty e^{-x^2} \cos(2bx) \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$ for all $b \in \mathbb{R}$. *Hint:* differentiate $I(b) = \int_0^\infty e^{-(x^2+2ibx)} \, dx$ under the integral sign.

Exercise 4.11 ($\star\star$)**.** Compute $\displaystyle\int_0^1 \int_0^1 \frac{1}{1 - xy} \, dx \, dy$ and deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Chapter Summary

- Double and triple integrals are defined via Riemann sums over partitions of rectangles/boxes.
- Fubini's theorem reduces multiple integrals to iterated single-variable integrals.
- The change-of-variables formula introduces the Jacobian determinant as a "volume scaling factor."
- Polar, cylindrical, and spherical coordinates simplify integration over regions with circular or spherical symmetry.
- The Gaussian integral $\int e^{-x^2} dx = \sqrt{\pi}$ is elegantly proved via polar coordinates.

Chapter 5

Curves and Line Integrals

Introduction

A particle moves through space along a trajectory; a force field does work on the particle as it travels. Computing this work requires integrating a vector field along a curve — a *line integral*. This chapter introduces parametric curves, arc length, and both scalar and vector line integrals. We then study conservative (gradient) fields, for which the line integral depends only on the endpoints, and conclude with Green's theorem, which relates a line integral around a closed curve to a double integral over the enclosed region.

5.1 Parametric Curves

Definition 5.1 (Parametric curve)**.** A *parametric curve* in $\mathbb{R}^n$ is a continuous map $\gamma\colon [a,b] \to \mathbb{R}^n$. The image $\gamma([a,b])$ is called the *trace* (or *support*) of the curve.

Definition 5.2 (Regular curve, piecewise C^1 curve)**.** A parametric curve $\gamma\colon [a,b] \to \mathbb{R}^n$ of class C^1 is *regular* if $\gamma'(t) \neq 0$ for all $t \in [a,b]$. It is *piecewise* C^1 if there exists a partition $a = t_0 < t_1 < \cdots < t_N = b$ such that $\gamma|_{[t_{k-1},t_k]}$ is C^1 for each k.

Definition 5.3 (Tangent vector and tangent line)**.** If γ is differentiable at t_0, the *tangent vector* is $\gamma'(t_0)$. The *tangent line* at $\gamma(t_0)$ is

$$\ell(s) = \gamma(t_0) + s\,\gamma'(t_0), \qquad s \in \mathbb{R}.$$

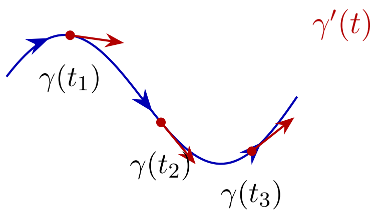


Figure 5.1: A parametric curve with tangent vectors at several points.

5.2 Examples of Curves

Example 5.1 (Circle)**.** $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. This is a regular C^∞ curve; $\gamma'(t) = (-\sin t, \cos t)$, $\|\gamma'(t)\| = 1$.

Example 5.2 (Ellipse)**.** $\gamma(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$, with $a, b > 0$. $\gamma'(t) = (-a\sin t, b\cos t) \neq 0$ since $a, b > 0$.

Example 5.3 (Cycloid)**.** $\gamma(t) = (t - \sin t, 1 - \cos t)$, $t \in [0, 2\pi]$. Here $\gamma'(t) = (1 - \cos t, \sin t)$, which vanishes at $t = 0$ and $t = 2\pi$ (cusps).

Example 5.4 (Helix)**.** $\gamma(t) = (\cos t, \sin t, t)$, $t \in [0, 4\pi]$. This is a regular curve in $\mathbb{R}^3$; $\gamma'(t) = (-\sin t, \cos t, 1)$, $\|\gamma'(t)\| = \sqrt{2}$.

5.3 Arc Length

Definition 5.4 (Arc length)**.** Let $\gamma\colon [a, b] \to \mathbb{R}^n$ be a continuous curve. For a partition $P\colon a = t_0 < t_1 < \cdots < t_N = b$, define the *polygonal length*

$$\ell(\gamma, P) = \sum_{k=1}^{N} \|\gamma(t_k) - \gamma(t_{k-1})\|.$$

The *arc length* (or *length*) of γ is

$$L(\gamma) = \sup_P \ell(\gamma, P).$$

If $L(\gamma) < \infty$, the curve is called *rectifiable.*

Theorem 5.1 (Arc length formula)**.** If $\gamma\colon [a, b] \to \mathbb{R}^n$ is C^1, then γ is rectifiable and

$$L(\gamma) = \int_a^b \|\gamma'(t)\|\, dt.$$

Proof. **Upper bound.** For any partition $P = \{t_0, \ldots, t_N\}$,

$$\|\gamma(t_k) - \gamma(t_{k-1})\| = \left\| \int_{t_{k-1}}^{t_k} \gamma'(t)\, dt \right\| \leq \int_{t_{k-1}}^{t_k} \|\gamma'(t)\|\, dt.$$

Summing over k: $\ell(\gamma, P) \leq \int_a^b \|\gamma'(t)\|\, dt$. Taking the supremum over P, $L(\gamma) \leq \int_a^b \|\gamma'(t)\|\, dt$.

Lower bound. Since γ' is continuous on $[a, b]$, it is uniformly continuous. Let $\varepsilon > 0$; choose $\delta > 0$ such that $|t - s| < \delta$ implies $\|\gamma'(t) - \gamma'(s)\| < \varepsilon$. Take a partition with $\|P\| < \delta$. For $t \in [t_{k-1}, t_k]$,

$$\gamma'(t) = \gamma'(t_{k-1}) + (\gamma'(t) - \gamma'(t_{k-1})),$$

so

$$\int_{t_{k-1}}^{t_k} \gamma'(t)\, dt = \gamma'(t_{k-1})\,(t_k - t_{k-1}) + R_k, \quad \|R_k\| \leq \varepsilon(t_k - t_{k-1}).$$

Therefore

$$\begin{aligned}\|\gamma(t_k)-\gamma(t_{k-1})\| &\geq \|\gamma'(t_{k-1})\|\,(t_k-t_{k-1})-\|R_k\|\\ &\geq \|\gamma'(t_{k-1})\|\,(t_k-t_{k-1})-\varepsilon(t_k-t_{k-1}).\end{aligned}$$

Summing,

$$\ell(\gamma,P)\geq\sum_k\|\gamma'(t_{k-1})\|\,(t_k-t_{k-1})-\varepsilon(b-a).$$

The sum on the right is a Riemann sum for $\int_a^b\|\gamma'(t)\|\,dt$, so for $\|P\|$ small enough it differs from the integral by less than ε. Thus $L(\gamma)\geq\int_a^b\|\gamma'(t)\|\,dt-2\varepsilon(b-a+1)$. Since ε is arbitrary, $L(\gamma)\geq\int_a^b\|\gamma'(t)\|\,dt$. □

Example 5.5 (Length of the helix)**.** For $\gamma(t)=(\cos t,\sin t,t)$ on $[0,2\pi]$, $\|\gamma'(t)\|=\sqrt{\sin^2 t+\cos^2 t+1}=\sqrt{2}$, so $L=\int_0^{2\pi}\sqrt{2}\,dt=2\pi\sqrt{2}$.

5.4 Arc Length Parameterisation

Definition 5.5 (Arc length parameter)**.** Given a regular C^1 curve $\gamma\colon[a,b]\to\mathbb{R}^n$, the *arc length function* is $s(t)=\int_a^t\|\gamma'(\tau)\|\,d\tau$. Since $s'(t)=\|\gamma'(t)\|>0$, the function s is strictly increasing with a C^1 inverse $t=t(s)$. The reparameterisation $\tilde{\gamma}(s)=\gamma(t(s))$ satisfies $\|\tilde{\gamma}'(s)\|=1$ for all s.

5.5 Scalar Line Integrals

Definition 5.6 (Scalar line integral)**.** Let $\gamma\colon[a,b]\to\mathbb{R}^n$ be piecewise C^1 and $f\colon\gamma([a,b])\to\mathbb{R}$ continuous. The *scalar line integral* of f along γ is

$$\int_\gamma f\,ds=\int_a^b f(\gamma(t))\,\|\gamma'(t)\|\,dt.$$

Example 5.6 (Mass of a wire)**.** A wire has the shape of the helix $\gamma(t)=(\cos t,\sin t,t)$, $t\in[0,2\pi]$, with linear density $\rho(x,y,z)=z$. Its mass is

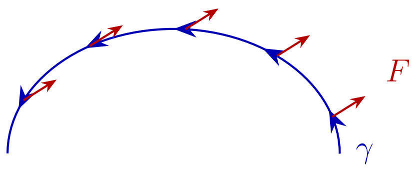
$$M=\int_\gamma\rho\,ds=\int_0^{2\pi}t\cdot\sqrt{2}\,dt=\sqrt{2}\cdot\frac{(2\pi)^2}{2}=2\pi^2\sqrt{2}.$$

5.6 Vector Line Integrals (Work)

Definition 5.7 (Vector line integral)**.** Let $F\colon U\subseteq\mathbb{R}^n\to\mathbb{R}^n$ be a continuous vector field and $\gamma\colon[a,b]\to U$ a piecewise C^1 curve. The *line integral of F along* γ is

$$\int_\gamma F\cdot d\gamma=\int_a^b\langle F(\gamma(t)),\gamma'(t)\rangle\,dt.$$

Remark 5.1. Physically, if F is a force field and γ is the trajectory of a particle, then $\int_\gamma F \cdot d\gamma$ is the *work* done by F on the particle.



$$\text{Work} = \int_\gamma F \cdot d\gamma$$

Figure 5.2: A force field F evaluated along a curve γ. The line integral gives the total work.

5.7 Gradient Fields and Potentials

Definition 5.8 (Conservative field)**.** A vector field $F\colon U \to \mathbb{R}^n$ is *conservative* (or a *gradient field*) if there exists a C^1 function $V: U \to \mathbb{R}$ such that $F = \nabla V$. The function V is called a *potential* for F.

Theorem 5.2 (Fundamental theorem for line integrals)**.** If $F = \nabla V$ on an open set U and $\gamma\colon [a,b] \to U$ is piecewise C^1, then

$$\int_\gamma F \cdot d\gamma = V(\gamma(b)) - V(\gamma(a)).$$

In particular, $\int_\gamma F \cdot d\gamma$ depends only on the endpoints, and $\oint_\gamma F \cdot d\gamma = 0$ for every closed curve.

Proof. By the chain rule,

$$\frac{d}{dt}\Big[V(\gamma(t))\Big] = \langle \nabla V(\gamma(t)), \gamma'(t)\rangle = \langle F(\gamma(t)), \gamma'(t)\rangle.$$

Integrating from a to b:

$$\int_a^b \langle F(\gamma(t)), \gamma'(t)\rangle\, dt = V(\gamma(b)) - V(\gamma(a)).$$ □

Proposition 5.1 (Necessary condition for a gradient field in $\mathbb{R}^2$)**.** Let $F = (P,Q)\colon U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 with U open. If $F = \nabla V$ for some C^2 function V, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{on } U.$$

Proof. $P = \frac{\partial V}{\partial x}$ and $Q = \frac{\partial V}{\partial y}$, so $\frac{\partial P}{\partial y} = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial Q}{\partial x}$ by Schwarz's theorem (Theorem 3.8). □

Remark 5.2. When U is *simply connected*, the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is also sufficient for F to be a gradient field. On domains that are not simply connected, this can fail; see the counterexample in Section 5.9.

5.8 Green's Theorem

Theorem 5.3 (Green's theorem)**.** Let $D \subseteq \mathbb{R}^2$ be a bounded domain whose boundary ∂D is a piecewise C^1 simple closed curve, oriented counterclockwise. Let $P, Q \colon \overline{D} \to \mathbb{R}$ be C^1. Then

$$\oint_{\partial D} (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Remark 5.3 (Proof idea)**.** For a Type I domain $D = \{(x, y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$, one shows separately that

$$\oint_{\partial D} P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA \qquad \text{and} \qquad \oint_{\partial D} Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA,$$

by direct computation using Fubini's theorem. For the first identity, the boundary integral over ∂D splits into a "bottom" piece $y = \varphi_1(x)$ (traversed left-to-right) and a "top" piece $y = \varphi_2(x)$ (right-to-left), and one verifies:

$$\oint_{\partial D} P\,dx = \int_a^b P(x, \varphi_1(x))\,dx - \int_a^b P(x, \varphi_2(x))\,dx = -\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}\,dy\,dx.$$

The argument for Q is analogous using a Type II decomposition. General domains are handled by decomposing into regions of these types.

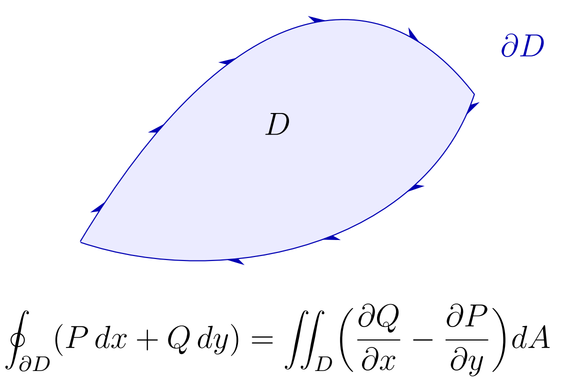


Figure 5.3: Green's theorem: the circulation around ∂D equals the double integral of the "curl" over D.

Corollary 5.1 (Area via line integrals)**.** Taking $P = -y/2$, $Q = x/2$, Green's theorem gives

$$\text{Area}(D) = \frac{1}{2}\oint_{\partial D} (x\,dy - y\,dx).$$

Example 5.7 (Area of an ellipse)**.** Parameterise ∂D by $\gamma(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$. Then

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^{2\pi} \Big(a\cos t \cdot b\cos t - b\sin t \cdot (-a\sin t)\Big) dt \\ &= \frac{ab}{2}\int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = \frac{ab}{2}\cdot 2\pi = \pi ab.\end{aligned}$$

5.9 A Closed Form That Is Not Exact

Example 5.8 (The angular form on $\mathbb{R}^2 \setminus \{0\}$)**.** Consider the vector field on $U = \mathbb{R}^2 \setminus \{(0,0)\}$:

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Setting $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$, one computes

$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2) + y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}.$$

So the necessary condition $\partial P/\partial y = \partial Q/\partial x$ is satisfied. Yet F is *not* a gradient field on U, because integrating around the unit circle $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$:

$$\oint_\gamma F\cdot d\gamma = \int_0^{2\pi} \Big((-\sin t)(-\sin t) + (\cos t)(\cos t)\Big) dt = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$$

A gradient field would give zero around any closed curve, by Theorem 5.2. The obstruction is topological: $U = \mathbb{R}^2 \setminus \{0\}$ is not simply connected.

5.10 Exercises

Exercise 5.1. Compute the arc length of the parabola $\gamma(t) = (t, t^2)$ for $t \in [0,1]$.

Exercise 5.2. Compute the arc length of one arch of the cycloid $\gamma(t) = (t - \sin t, 1 - \cos t)$ for $t \in [0, 2\pi]$.

Exercise 5.3. Find the arc length parameterisation of $\gamma(t) = (3\cos t, 3\sin t, 4t)$, $t \geq 0$.

Exercise 5.4. Compute $\int_\gamma (x^2+y^2)\, ds$ where γ is the circle of radius R centred at the origin.

Exercise 5.5. Compute $\int_\gamma F\cdot d\gamma$ where $F(x,y) = (y, x)$ and $\gamma(t) = (t, t^2)$, $t \in [0,1]$.

Exercise 5.6. Show that $F(x,y) = (2xy + 3, x^2 - 4y)$ is conservative. Find a potential V and use it to compute $\int_\gamma F\cdot d\gamma$ from $(0,0)$ to $(1,2)$.

Exercise 5.7. Use Green's theorem to evaluate $\oint_\gamma (y^2\,dx + 3xy\,dy)$ where γ is the boundary of the square $[0,1] \times [0,1]$ traversed counterclockwise.

Exercise 5.8. Use Green's theorem to compute the area enclosed by the astroid $\gamma(t) = (\cos^3 t, \sin^3 t)$, $t \in [0, 2\pi]$.

Exercise 5.9. Let $F(x,y,z) = (yz, xz, xy)$. Show that $F = \nabla V$ for $V(x,y,z) = xyz$ and compute $\int_\gamma F \cdot d\gamma$ along any path from $(1,0,0)$ to $(0,1,1)$.

Exercise 5.10. Let $F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ on $U = \mathbb{R}^2 \setminus \{0\}$. Compute $\int_\gamma F \cdot d\gamma$ where γ traverses the circle of radius 2 centred at the origin once counterclockwise. Does the result change if γ winds twice around the origin?

Exercise 5.11 ($\star\star$). Let $D \subseteq \mathbb{R}^2$ be a simply connected open set and $F = (P,Q) \in C^1(D;\mathbb{R}^2)$ with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in D. Prove that F is conservative, i.e. construct a potential $V(x,y) = \int_{(x_0,y_0)}^{(x,y)} F \cdot d\gamma$ and show it is well defined.

Chapter Summary

- A parametric curve $\gamma\colon [a,b] \to \mathbb{R}^n$ is regular if $\gamma'(t) \neq 0$; its arc length is $L = \int_a^b \|\gamma'(t)\|\,dt$.
- The scalar line integral $\int_\gamma f\,ds$ integrates a function with respect to arc length; the vector line integral $\int_\gamma F \cdot d\gamma$ computes work.
- For a gradient field $F = \nabla V$, the line integral depends only on the endpoints: $\int_\gamma F \cdot d\gamma = V(\text{end}) - V(\text{start})$.
- Green's theorem connects line integrals around a closed curve to double integrals over the enclosed region.
- A closed 1-form need not be exact on a non-simply-connected domain.

Concluding Remarks: Where to Go Next

The material in this course provides the foundation for several beautiful and deep areas of mathematics.

- **Measure theory and Lebesgue integration.** The Riemann integral, while powerful, has significant limitations (for instance, the limit of a pointwise convergent sequence of Riemann integrable functions need not be Riemann integrable). Lebesgue's theory provides a far more flexible framework, with powerful convergence theorems (monotone convergence, dominated convergence) that are indispensable in probability, functional analysis, and PDE theory.
- **Complex analysis.** Functions of a complex variable $f\colon \mathbb{C} \to \mathbb{C}$ that are differentiable (holomorphic) possess extraordinary properties: they are automatically C^∞, analytic, and satisfy Cauchy's integral formula. The residue theorem is a far-reaching generalisation of the techniques used to compute $\int e^{-x^2} dx$.

- **Differential geometry and manifolds.** Green's theorem is a special case of the general Stokes theorem on manifolds, which unifies Green's, Gauss's, and Stokes's classical theorems. The language of differential forms provides the natural setting for integration on curved spaces.
- **Partial differential equations.** Many of the tools developed here — multivariable derivatives, Green's theorem, change of variables — are the starting point for the study of the heat equation, wave equation, and Laplace's equation.

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