Real Analysis I

Course Notes

From Sequences to Derivatives

First Year — Mathematics

Academic Year 2025–2026

Contents

Preface

Welcome to Real Analysis.

If you are reading these notes, you are most likely beginning your first year of university mathematics. You may have enjoyed mathematics at school — perhaps you found satisfaction in solving equations, computing integrals, or sketching curves. The course you are about to undertake will be *very different* from what you have experienced before, and that is precisely what makes it exciting.

Why rigour? At school you were told that $\sqrt{2}$ is irrational, that continuous functions satisfy the Intermediate Value Theorem, and that every differentiable function is continuous. You likely accepted these facts on faith or saw informal justifications. In this course we will *prove* every single one of them, starting from clearly stated axioms. This is not pedantry; it is the only way to be certain that our conclusions are correct. History is full of examples where intuition alone led mathematicians astray — plausible-sounding statements that turn out to be false, and counter-intuitive results that turn out to be true. Rigorous proof is our shield against error.

What to expect. The first weeks may feel slow. We will spend a significant amount of time on logic, on the axioms of the real numbers, and on the precise definition of a limit — before we even begin to compute anything. This is intentional. Think of it as learning the grammar of a language before writing essays: the investment pays enormous dividends later. Once the foundations are solid, the major theorems of the course will unfold naturally and almost inevitably.

The journey. Here is a brief overview of the path ahead.

- **Chapter 1** sets up the language of mathematics: logic, quantifiers, proof techniques, sets, and functions. If you have never written a formal proof before, this chapter is your essential training ground.

- **Chapter 2** introduces the real number system and, most importantly, the *completeness axiom* — the single property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$. This axiom is the engine that powers nearly every deep result in the course.

- Later chapters (not included in this part) will develop the theory of sequences, series, continuous functions, and differentiation.

How to study. Mathematics is not a spectator sport. Reading these notes is necessary but far from sufficient. You must *write proofs yourself*, struggle with exercises, make mistakes, and correct them. When you read a proof, close the notes and try to reproduce the argument from memory. When you get stuck on an exercise, do not look at the solution immediately — give yourself at least thirty minutes of honest effort. The understanding that comes from wrestling with a problem is far deeper than the understanding that comes from reading someone else's solution.

A word of encouragement. If you find the first proofs difficult, that is completely normal. Every mathematician alive today once struggled with their first ε-δ argument. Persist, ask questions, and trust the process. By the end of the semester, you will look back at these first chapters and wonder why they ever seemed hard.

Good luck, and enjoy the journey.

Notation

The following symbols and conventions are used throughout these notes.

Symbol	Meaning
$\mathbb{N}$	The set of natural numbers $\{0, 1, 2, 3, \dots\}$
$\mathbb{N}^*$	The set of positive natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^*$	$\mathbb{R} \setminus \{0\}$
$\mathbb{R}_+$	$\{x \in \mathbb{R} : x \geq 0\}$
$\mathbb{C}$	The set of complex numbers
$\emptyset$	The empty set
ε	A strictly positive real number (usually "small")
$|x|$	Absolute value of x
$\|x\|$	Norm of x
$\forall$	"For all" (universal quantifier)
$\exists$	"There exists" (existential quantifier)
$\exists!$	"There exists a unique"
$\implies$	"Implies" (logical implication)
$\iff$	"If and only if" (logical equivalence)
$A \subset B$	A is a subset of B (possibly equal)
$A \subsetneq B$	A is a strict subset of B
$A \cup B$	Union of A and B
$A \cap B$	Intersection of A and B
$A \setminus B$	Set difference
A^c	Complement of A
$A \times B$	Cartesian product of A and B
$f : A \to B$	Function from A to B
$f \circ g$	Composition of f and g
$\sup A$	Supremum (least upper bound) of A
$\inf A$	Infimum (greatest lower bound) of A
$\max A$, $\min A$	Maximum, minimum of A (when they exist)
$\lfloor x \rfloor$	Floor (integer part) of x

Convention. Unless otherwise stated, the letters m, n, k, p denote natural numbers, the letters x, y, z, a, b, c denote real numbers, and $\varepsilon, \delta, \eta$ denote *strictly positive* real numbers.

Chapter 1

Logic, Sets, and Mathematical Reasoning

Before we can prove anything about real numbers, sequences, or functions, we need a precise language in which to express our statements and a toolkit of proof techniques to establish their truth. This chapter provides both. It is the single most important chapter in the course: every subsequent proof relies on the material developed here.

1.1 Propositions and Logical Connectives

> **Definition 1.1** (Proposition). A **proposition** (or **statement**) is a sentence that is either *true* (T) or *false* (F), but not both.

Example 1.1. The following are propositions:

(a) "$2 + 3 = 5$"   (true).

(b) "Every even number greater than 2 is the sum of two primes"   (this is Goldbach's conjecture — it is either true or false, even though we do not currently know which).

(c) "$\sqrt{2}$ is rational"   (false, as we shall prove in Section 1.3.3).

The sentence "Is 7 a prime number?" is *not* a proposition because it is a question, not a declarative statement. Similarly, "$x > 3$" is not a proposition by itself because its truth value depends on the unspecified variable x; it is a **predicate**.

> **Definition 1.2** (Logical connectives). Let P and Q be propositions. We define the following connectives:
>
> (i) **Negation** $\neg P$ ("not P"): true when P is false, and false when P is true.
>
> (ii) **Conjunction** $P \wedge Q$ ("P and Q"): true when both P and Q are true, false otherwise.
>
> (iii) **Disjunction** $P \vee Q$ ("P or Q"): true when at least one of P, Q is true, false only when both are false. (This is the *inclusive* "or".)
>
> (iv) **Implication** $P \implies Q$ ("if P then Q", or "P implies Q"): false *only* when P is true and Q is false; true in all other cases.

> (v) **Equivalence** $P \iff Q$ ("P if and only if Q"): true when P and Q have the same truth value, false otherwise.

Remark 1.1. The definition of implication often surprises students. In particular, the statement $P \implies Q$ is *true* whenever P is false, regardless of Q. This is called a **vacuously true** implication. For instance, "If $1 = 0$, then the moon is made of cheese" is a *true* implication. This convention is not arbitrary; it is the only definition that makes the logical calculus consistent and useful.

Truth tables. The following table summarises all the connectives.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Remark 1.2. Given the implication $P \implies Q$:

- The **converse** is $Q \implies P$. It is *not* logically equivalent to the original implication.

- The **contrapositive** is $\neg Q \implies \neg P$. It *is* logically equivalent to $P \implies Q$ (check the truth table!). This fact is the basis for proof by contrapositive.

1.2 Quantifiers

In mathematics we constantly make statements about all elements of a set or about the existence of elements with a particular property. The symbols $\forall$ and $\exists$ make such statements precise.

> **Definition 1.3** (Universal and existential quantifiers)**.** Let $P(x)$ be a predicate depending on a variable x ranging over a set E.
>
> (i) **Universal quantifier**: $\forall x \in E, \; P(x)$ means "for every element x of E, the property $P(x)$ holds."
>
> (ii) **Existential quantifier**: $\exists x \in E, \; P(x)$ means "there exists at least one element x of E such that $P(x)$ holds."

1.2.1 Negation of Quantified Statements

This is one of the most important skills you will develop in this course. The rules are:

$$\neg\Big(\forall x \in E, \; P(x)\Big) \iff \exists x \in E, \; \neg P(x), \tag{1.1}$$

$$\neg\Big(\exists x \in E, \; P(x)\Big) \iff \forall x \in E, \; \neg P(x). \tag{1.2}$$

In words: to negate a $\forall$, change it to $\exists$ and negate the predicate. To negate an $\exists$, change it to $\forall$ and negate the predicate. When there are *multiple* quantifiers, you proceed from left to right, flipping each one.

> **Strategy: Negating a multiply-quantified statement**
>
> Given a statement of the form $\forall x, \ \exists y, \ \forall z, \ P(x, y, z)$, its negation is obtained by flipping every quantifier and negating the innermost predicate: $\exists x, \ \forall y, \ \exists z, \ \neg P(x, y, z)$.

Example 1.2 (Negating the definition of convergence). Consider the statement: "The sequence (a_n) converges to ℓ", which is formally written as

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall n \geq N, \ |a_n - \ell| < \varepsilon. \tag{$\mathcal{S}$}$$

We negate this step by step.

Step 1. The outermost quantifier is "$\forall \varepsilon > 0$". Flip it to "$\exists \varepsilon > 0$" and negate everything that follows:

$$\neg(\mathcal{S}) \iff \exists \varepsilon > 0, \ \neg\Big(\exists N \in \mathbb{N}, \ \forall n \geq N, \ |a_n - \ell| < \varepsilon\Big).$$

Step 2. The next quantifier is "$\exists N \in \mathbb{N}$". Flip it to "$\forall N \in \mathbb{N}$" and negate the rest:

$$\neg(\mathcal{S}) \iff \exists \varepsilon > 0, \ \forall N \in \mathbb{N}, \ \neg\Big(\forall n \geq N, \ |a_n - \ell| < \varepsilon\Big).$$

Step 3. The next quantifier is "$\forall n \geq N$". Flip it to "$\exists n \geq N$" and negate the predicate:

$$\neg(\mathcal{S}) \iff \exists \varepsilon > 0, \ \forall N \in \mathbb{N}, \ \exists n \geq N, \ |a_n - \ell| \geq \varepsilon.$$

Interpretation. The sequence (a_n) does *not* converge to ℓ if and only if: *there exists* some $\varepsilon > 0$ such that, *no matter how large* we choose N, we can always *find* some $n \geq N$ for which $|a_n - \ell| \geq \varepsilon$. In other words, there are terms of the sequence that remain "far" from ℓ indefinitely.

Example 1.3. Consider the statement "The function f is bounded above on $[0, 1]$":

$$\exists M \in \mathbb{R}, \ \forall x \in [0, 1], \ f(x) \leq M.$$

Its negation is:

$$\forall M \in \mathbb{R}, \ \exists x \in [0, 1], \ f(x) > M.$$

This says: no matter how large a bound M you propose, I can find a point x in $[0, 1]$ where $f(x)$ exceeds M.

1.3 Proof Techniques

We now present the main methods of proof. For each method, we first explain the logical principle, then give worked examples.

1.3.1 Direct Proof

> **Strategy: Direct proof**
>
> To prove "$P \implies Q$", assume P is true and deduce, through a chain of logical steps, that Q is true.

Proposition 1.1 (Sum of two even integers)**.** The sum of two even integers is even.

Proof. Let a and b be even integers. By definition of "even", there exist integers k and m such that $a = 2k$ and $b = 2m$. Then

$$a + b = 2k + 2m = 2(k + m).$$

Since $k + m$ is an integer, $a + b$ is of the form $2 \cdot (\text{integer})$, hence $a + b$ is even. □

Proposition 1.2 (Product of rational and irrational)**.** Let $q \in \mathbb{Q}$ with $q \neq 0$, and let $\alpha \notin \mathbb{Q}$. Then $q\alpha \notin \mathbb{Q}$.

Proof. We prove the contrapositive in the next subsection; here we give a direct argument. Suppose for the sake of deriving the conclusion directly. We have $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$, $a \neq 0$. Suppose, aiming for a contradiction with the hypothesis that α is irrational, that $q\alpha$ were rational. Then $q\alpha = \frac{p}{r}$ for some $p, r \in \mathbb{Z}$, $r \neq 0$. But then $\alpha = \frac{q\alpha}{q} = \frac{p}{r} \cdot \frac{b}{a} = \frac{pb}{ra} \in \mathbb{Q}$, contradicting the assumption that α is irrational. Therefore $q\alpha \notin \mathbb{Q}$. □

1.3.2 Proof by Contrapositive

Recall that $P \implies Q$ is logically equivalent to $\neg Q \implies \neg P$. Sometimes it is easier to prove the contrapositive.

Strategy: Proof by contrapositive

To prove "$P \implies Q$", assume $\neg Q$ is true and deduce $\neg P$.

Proposition 1.3. Let n be an integer. If n^2 is even, then n is even.

Proof. We prove the contrapositive: "If n is odd, then n^2 is odd."

Assume n is odd. Then there exists an integer k such that $n = 2k + 1$. Therefore

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k$ is an integer, n^2 is of the form $2 \cdot (\text{integer}) + 1$, so n^2 is odd. This completes the proof of the contrapositive, and hence the original statement is proved. □

Proposition 1.4. Let $n \in \mathbb{Z}$. If n^2 is not divisible by 3, then n is not divisible by 3.

Proof. We prove the contrapositive: "If n is divisible by 3, then n^2 is divisible by 3."

Assume $3 \mid n$. Then $n = 3k$ for some $k \in \mathbb{Z}$. Hence $n^2 = 9k^2 = 3(3k^2)$, so $3 \mid n^2$. □

1.3.3 Proof by Contradiction

> **Strategy: Proof by contradiction**
>
> To prove a statement P, assume $\neg P$ and derive a logical contradiction (i.e., show that $\neg P$ implies something that is both true and false). Since mathematics cannot contain contradictions, our assumption $\neg P$ must have been wrong, so P is true.

> **Theorem 1.1** ($\sqrt{2}$ is irrational). There is no rational number whose square equals 2. In other words, $\sqrt{2} \notin \mathbb{Q}$.

Proof. Suppose, for the sake of contradiction, that $\sqrt{2} \in \mathbb{Q}$. Then we can write

$$\sqrt{2} = \frac{p}{q},$$

where $p, q \in \mathbb{N}^*$ and the fraction $\frac{p}{q}$ is in lowest terms (i.e., $\gcd(p, q) = 1$).

Squaring both sides gives $2 = \frac{p^2}{q^2}$, hence

$$p^2 = 2q^2. \tag{1.3}$$

This tells us that p^2 is even. By Proposition 1.3 (or its contrapositive, which we proved above), p itself must be even. So we can write $p = 2k$ for some $k \in \mathbb{N}^*$.

Substituting into (1.3):

$$(2k)^2 = 2q^2 \implies 4k^2 = 2q^2 \implies q^2 = 2k^2.$$

By the same reasoning, q^2 is even, so q is even.

But now both p and q are even, which contradicts our assumption that $\gcd(p, q) = 1$ (since they share the common factor 2).

This contradiction shows that the initial assumption "$\sqrt{2} \in \mathbb{Q}$" is false. Therefore $\sqrt{2} \notin \mathbb{Q}$. $\qquad\square$

> **Proposition 1.5** (Infinitude of primes). There are infinitely many prime numbers.

Proof. Suppose, for the sake of contradiction, that there are only finitely many primes, say $p_1, p_2, \ldots, p_k$. Consider the number

$$N = p_1 \cdot p_2 \cdots p_k + 1.$$

Since $N > 1$, either N is itself prime or N has a prime factor. In either case, there exists a prime p that divides N. Since $p_1, \ldots, p_k$ are all the primes, we must have $p = p_i$ for some i.

But p_i divides the product $p_1 p_2 \cdots p_k$, and p_i divides N, so p_i divides

$$N - p_1 p_2 \cdots p_k = 1.$$

No prime divides 1, so we have a contradiction. Therefore the set of primes is infinite. $\quad\square$

1.3.4 Mathematical Induction

> **Theorem 1.2** (Principle of Mathematical Induction)**.** Let $P(n)$ be a statement depending on a natural number $n \geq n_0$. Suppose:
>
> (i) **Base case:** $P(n_0)$ is true.
>
> (ii) **Inductive step:** For every $n \geq n_0$, $P(n) \implies P(n+1)$.
>
> Then $P(n)$ is true for every $n \geq n_0$.

Remark 1.3. Think of induction like an infinite row of dominoes. The base case knocks over the first domino. The inductive step guarantees that each falling domino knocks over the next. Together, they ensure that *every* domino falls.

Example 1.4 (Sum of the first n natural numbers). We prove that for every $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{1.4}$$

Base case $(n = 1)$. The left-hand side is 1. The right-hand side is $\frac{1 \cdot 2}{2} = 1$. So $P(1)$ holds.

Inductive step. Let $n \geq 1$ and assume $P(n)$ is true, i.e., $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. We must show $P(n+1)$:

$$
\begin{aligned}
1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the inductive hypothesis)} \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2},
\end{aligned}
$$

which is exactly the formula with n replaced by $n+1$. So $P(n+1)$ holds.

By the Principle of Mathematical Induction, (1.4) holds for all $n \geq 1$.

Example 1.5 (Bernoulli's inequality). Let $x \geq -1$. We prove by induction that for every $n \geq 1$,

$$(1 + x)^n \geq 1 + nx. \tag{1.5}$$

Base case $(n = 1)$. $(1+x)^1 = 1 + x = 1 + 1 \cdot x$. The inequality holds with equality.

Inductive step. Assume $(1+x)^n \geq 1 + nx$ for some $n \geq 1$. Then

$$
\begin{aligned}
(1 + x)^{n+1} &= (1 + x)^n \cdot (1 + x) \\
&\geq (1 + nx)(1 + x) \quad \text{(inductive hypothesis; note } 1 + x \geq 0\text{)} \\
&= 1 + nx + x + nx^2 \\
&= 1 + (n+1)x + nx^2 \\
&\geq 1 + (n+1)x \quad \text{(since } nx^2 \geq 0\text{)}.
\end{aligned}
$$

This is exactly the desired inequality with n replaced by $n+1$. By induction, (1.5) holds for all $n \geq 1$.

> **Theorem 1.3** (Principle of Strong Induction)**.** Let $P(n)$ be a statement depending on a natural number $n \geq n_0$. Suppose:

(i) $P(n_0)$ is true.

(ii) For every $n \geq n_0$: if $P(k)$ is true for *all $n_0 \leq k \leq n$*, then $P(n+1)$ is true.

Then $P(n)$ is true for every $n \geq n_0$.

Example 1.6 (Every integer ≥ 2 has a prime factor). We prove by strong induction that every integer $n \geq 2$ has at least one prime factor.

Base case $(n = 2)$.   2 is itself prime, so it is its own prime factor.

Inductive step.   Let $n \geq 2$ and assume that every integer k with $2 \leq k \leq n$ has a prime factor. Consider $n + 1$. There are two cases:

- If $n + 1$ is prime, then $n + 1$ is its own prime factor.

- If $n + 1$ is not prime, then $n + 1 = ab$ where $2 \leq a, b \leq n$. By the strong inductive hypothesis, a has a prime factor p. Since $p \mid a$ and $a \mid (n + 1)$, we have $p \mid (n + 1)$.

In both cases, $n+1$ has a prime factor. By strong induction, the result holds for all $n \geq 2$.

1.4   Sets and Set Operations

Definition 1.4 (Set, element, subset)**.** A **set** is a collection of distinct objects, called its **elements**. We write $x \in A$ to mean "x is an element of A", and $x \notin A$ for its negation.

A set A is a **subset** of B, written $A \subset B$, if every element of A is also an element of B:
$$A \subset B \iff \forall x, (x \in A \implies x \in B).$$
Two sets are **equal** if and only if $A \subset B$ and $B \subset A$.

Definition 1.5 (Set operations)**.** Let A and B be subsets of a universal set Ω.

(i) **Union**: $A \cup B = \{ x \in \Omega \mid x \in A \text{ or } x \in B \}$.

(ii) **Intersection**: $A \cap B = \{ x \in \Omega \mid x \in A \text{ and } x \in B \}$.

(iii) **Set difference**: $A \setminus B = \{ x \in \Omega \mid x \in A \text{ and } x \notin B \}$.

(iv) **Complement**: $A^c = \Omega \setminus A = \{ x \in \Omega \mid x \notin A \}$.

(v) **Cartesian product**: $A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$.

$A \cup B$ $A \cap B$ A^c
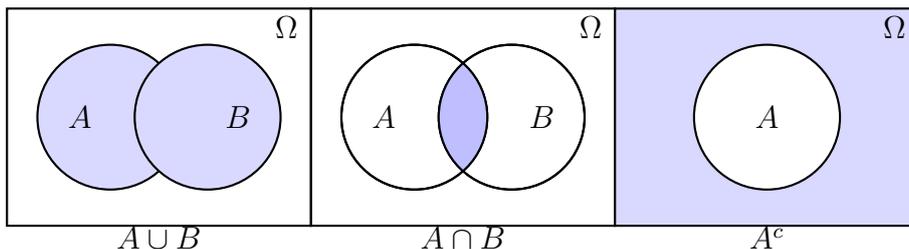
> **Theorem 1.4** (De Morgan's laws)**.** Let A and B be subsets of a universal set Ω. Then
>
> (i) $(A \cup B)^c = A^c \cap B^c$,
>
> (ii) $(A \cap B)^c = A^c \cup B^c$.

Proof. We prove (i); the proof of (ii) is similar and left as an exercise.

We show the two sets are equal by showing each is a subset of the other.

($\subset$) Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$. This means $x \notin A$ *and* $x \notin B$ (because if x were in either A or B, then x would be in $A \cup B$). So $x \in A^c$ and $x \in B^c$, hence $x \in A^c \cap B^c$.

($\supset$) Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, i.e., $x \notin A$ and $x \notin B$. Therefore $x \notin A \cup B$, so $x \in (A \cup B)^c$.

Since each set is a subset of the other, they are equal. $\qquad\square$

1.5 Functions

> **Definition 1.6** (Function)**.** A **function** $f : A \to B$ is a rule that assigns to each element $a \in A$ a *unique* element $f(a) \in B$. The set A is called the **domain** and B the **codomain**.

> **Definition 1.7** (Injection, surjection, bijection)**.** Let $f : A \to B$ be a function.
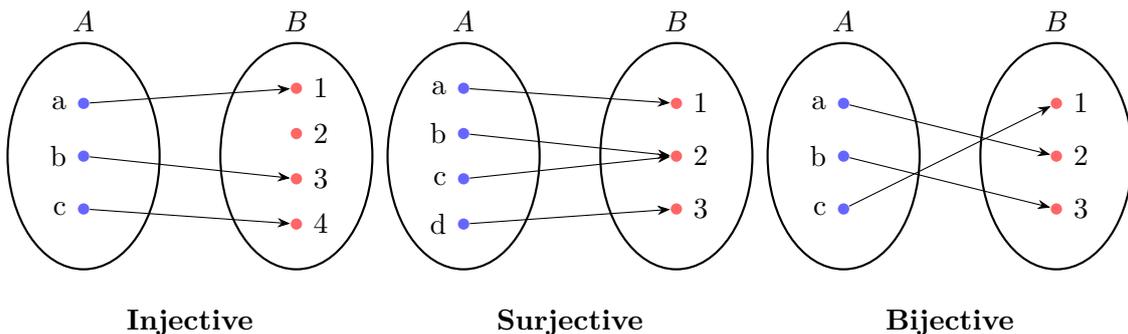>
> (i) f is **injective** (one-to-one) if
> $$\forall a_1, a_2 \in A, \quad f(a_1) = f(a_2) \implies a_1 = a_2.$$
>
> (ii) f is **surjective** (onto) if
> $$\forall b \in B, \ \exists a \in A, \quad f(a) = b.$$
>
> (iii) f is **bijective** if it is both injective and surjective.

Injective   Surjective   Bijective

1.6 Equivalence Relations and Order Relations

> **Definition 1.8** (Equivalence relation)**.** A relation $\sim$ on a set E is an **equivalence relation** if it is:
>
> (i) **Reflexive**: $\forall x \in E,\ x \sim x$.
>
> (ii) **Symmetric**: $\forall x, y \in E,\ x \sim y \implies y \sim x$.
>
> (iii) **Transitive**: $\forall x, y, z \in E,\ (x \sim y \text{ and } y \sim z) \implies x \sim z$.
>
> The **equivalence class** of x is $[x] = \{\, y \in E \mid y \sim x \,\}$.

Example 1.7. Fix $n \in \mathbb{N}^*$. Define $a \sim b$ on $\mathbb{Z}$ by $n \mid (a - b)$ (congruence modulo n). This is an equivalence relation:

- Reflexive: $n \mid (a - a) = 0$.

- Symmetric: if $n \mid (a - b)$, then $n \mid (-(a - b)) = (b - a)$.

- Transitive: if $n \mid (a - b)$ and $n \mid (b - c)$, then $n \mid ((a - b) + (b - c)) = (a - c)$.

> **Definition 1.9** (Order relation)**.** A relation $\leq$ on a set E is a **(partial) order** if it is:
>
> (i) **Reflexive**: $\forall x \in E,\ x \leq x$.
>
> (ii) **Antisymmetric**: $\forall x, y \in E,\ (x \leq y \text{ and } y \leq x) \implies x = y$.
>
> (iii) **Transitive**: $\forall x, y, z \in E,\ (x \leq y \text{ and } y \leq z) \implies x \leq z$.
>
> The order is **total** if additionally $\forall x, y \in E$, either $x \leq y$ or $y \leq x$.

Example 1.8. The usual ordering $\leq$ on $\mathbb{R}$ is a total order. The inclusion relation $\subset$ on subsets of a given set is a partial order that is *not* total in general (e.g., $\{1\} \not\subset \{2\}$ and $\{2\} \not\subset \{1\}$).

1.7 Common Errors in Logic

> **Common Errors — Watch Out!**
>
> 1. **Confusing a statement with its converse.** "$P \implies Q$" is *not* the same as "$Q \implies P$". Example: "If it rains, the ground is wet" does not mean "If the ground is wet, it rained" (someone might have used a hose).
>
> 2. **Incorrect negation of quantifiers.** The negation of "$\forall x,\ P(x)$" is *not* "$\forall x,\ \neg P(x)$". It is "$\exists x,\ \neg P(x)$". You only need *one* counterexample to disprove a universal statement.
>
> 3. **Using a specific example as a proof.** Checking that a formula works for $n = 1, 2, 3$ does *not* prove it works for all n. You need induction or a general argument.

> **4. Assuming what you want to prove.** In a direct proof of "$A \implies B$", you may assume A but you must *derive* B. You may not write B and then "verify" it.
>
> **5. Confusing "or" with "exclusive or".** In mathematics, "P or Q" means "at least one of P, Q is true" and allows both to be true.

1.8 Exercises

Exercise 1.1 (Truth tables). [⋆] Construct truth tables for each of the following compound propositions:

(a) $P \implies (Q \vee R)$,

(b) $(P \wedge Q) \implies P$,

(c) $(P \implies Q) \iff (\neg Q \implies \neg P)$.

For (c), verify that the result is always true (a *tautology*).

Exercise 1.2 (Negations). [⋆] Write the negation of each of the following statements:

(a) $\forall x \in \mathbb{R},\ x^2 \geq 0$.

(b) $\exists n \in \mathbb{N},\ n^2 = n$.

(c) $\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathbb{R},\ |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$.

(d) $\forall x \in \mathbb{R},\ (x > 0 \implies \exists n \in \mathbb{N},\ \frac{1}{n} < x)$.

Exercise 1.3 (Direct proof). [⋆]Prove that the product of two odd integers is odd.

Exercise 1.4 (Contrapositive). [⋆⋆]Let $n \in \mathbb{Z}$. Prove that if n^2 is divisible by 5, then n is divisible by 5. *(Hint: consider the possible remainders of n modulo 5.)*

Exercise 1.5 (Contradiction). [⋆⋆]Prove that $\sqrt{3}$ is irrational. *(Model your proof on the proof that $\sqrt{2}$ is irrational.)*

Exercise 1.6 (Induction: sum of squares). [⋆]Prove by induction that for every $n \geq 1$:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise 1.7 (Induction: divisibility). [⋆⋆]Prove that for every $n \geq 1$, 6 divides $n^3 - n$.

Exercise 1.8 (Induction: geometric sum). [⋆]Let $q \neq 1$. Prove by induction that for every $n \geq 0$:

$$\sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}.$$

Exercise 1.9 (Sets: De Morgan). [⋆⋆]Prove the second De Morgan law: $(A \cap B)^c = A^c \cup B^c$.

Exercise 1.10 (Functions). [⋆⋆]Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 3$. Prove that f is bijective. Find f^{-1}.

Exercise 1.11 (Composition and injectivity). [⋆⋆⋆]Let $f : A \to B$ and $g : B \to C$. Prove:

(a) If $g \circ f$ is injective, then f is injective.

(b) If $g \circ f$ is surjective, then g is surjective.

(c) Give an example showing that $g \circ f$ can be bijective even if neither f nor g is bijective.

Exercise 1.12 (Equivalence relation). [★★]Define a relation on $\mathbb{Z} \times \mathbb{Z}^*$ by $(a, b) \sim (c, d) \iff ad = bc$. Prove that $\sim$ is an equivalence relation. *(This is the construction of $\mathbb{Q}$.)*

Chapter Summary

> ### Chapter Summary
>
> - A **proposition** is a statement that is either true or false.
>
> - The logical connectives are $\neg, \wedge, \vee, \implies, \iff$.
>
> - The **contrapositive** of $P \implies Q$ is $\neg Q \implies \neg P$, and it is logically equivalent.
>
> - To **negate quantifiers**: flip $\forall \leftrightarrow \exists$ and negate the innermost predicate.
>
> - **Proof methods**: direct proof, contrapositive, contradiction, induction (ordinary and strong).
>
> - **Set operations**: $\cup, \cap, \setminus$, complement, Cartesian product. **De Morgan's laws** relate unions, intersections, and complements.
>
> - A function is **injective** (one-to-one), **surjective** (onto), or **bijective** (both).
>
> - An **equivalence relation** is reflexive, symmetric, and transitive. An **order relation** is reflexive, antisymmetric, and transitive.

Chapter 2

The Real Numbers — Axioms and Completeness

The real numbers are the natural setting for analysis. In this chapter we state the axioms that define $\mathbb{R}$, and we single out the one axiom that makes $\mathbb{R}$ special: the *completeness axiom*. From this single principle, we will derive the Archimedean property, the density of $\mathbb{Q}$ in $\mathbb{R}$, and the nested intervals theorem — tools that we will use over and over again in subsequent chapters.

2.1 Why $\mathbb{Q}$ Is Not Enough

We proved in Theorem 1.1 that $\sqrt{2} \notin \mathbb{Q}$. This means that the equation $x^2 = 2$ has no solution in $\mathbb{Q}$. But the problem runs deeper: the rational number line has "holes" in it.

Consider the set
$$A = \left\{ r \in \mathbb{Q} \mid r > 0 \text{ and } r^2 < 2 \right\}.$$

This set is nonempty ($1 \in A$) and is bounded above in $\mathbb{Q}$ (for instance, 2 is an upper bound, since $r \in A$ implies $r < 2$). However, A has *no least upper bound in* $\mathbb{Q}$. Intuitively, the least upper bound "should be" $\sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$.

This is not just an isolated curiosity. Many fundamental constructions in analysis — limits of sequences, the Intermediate Value Theorem, the existence of integrals — require the number system to have "no holes." The rational numbers fail this requirement. The real numbers $\mathbb{R}$ are designed precisely to fill every such hole.

2.2 The Ordered Field Axioms

The real numbers $\mathbb{R}$ satisfy three groups of axioms.

Definition 2.1 (Field axioms)**.** $(\mathbb{R}, +, \cdot)$ is a **field**, meaning it satisfies the following for all $a, b, c \in \mathbb{R}$:

Addition axioms:

(A1) **Associativity:** $(a + b) + c = a + (b + c)$.

(A2) **Commutativity:** $a + b = b + a$.

(A3) **Identity:** There exists $0 \in \mathbb{R}$ such that $a + 0 = a$.

(A4) **Inverses:** For each a, there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0$.

Multiplication axioms:

(M1) **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(M2) **Commutativity:** $a \cdot b = b \cdot a$.

(M3) **Identity:** There exists $1 \in \mathbb{R}$, $1 \neq 0$, such that $a \cdot 1 = a$.

(M4) **Inverses:** For each $a \neq 0$, there exists $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$.

Distributivity:

(D) $a \cdot (b + c) = a \cdot b + a \cdot c$.

Definition 2.2 (Order axioms)**.** $\mathbb{R}$ is equipped with a total order $\leq$ satisfying for all $a, b, c \in \mathbb{R}$:

(O1) **Compatibility with addition:** $a \leq b \implies a + c \leq b + c$.

(O2) **Compatibility with multiplication:** $a \leq b$ and $0 \leq c \implies ac \leq bc$.

A field equipped with such an order is called an **ordered field**.

Remark 2.1. $\mathbb{Q}$ also satisfies all of the above axioms. What distinguishes $\mathbb{R}$ from $\mathbb{Q}$ is the *completeness axiom*, which we state in Section 2.5.

2.3 Absolute Value

Definition 2.3 (Absolute value)**.** For $x \in \mathbb{R}$, the **absolute value** of x is

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Proposition 2.1 (Properties of absolute value)**.** For all $x, y \in \mathbb{R}$:

(i) $|x| \geq 0$, and $|x| = 0 \iff x = 0$.

(ii) $|-x| = |x|$.

(iii) $|xy| = |x| \, |y|$.

(iv) $-|x| \leq x \leq |x|$.

(v) $|x| \leq a \iff -a \leq x \leq a$ (for $a \geq 0$).

Proof. We prove each property by case analysis on the sign of x (and y where relevant).
(i) If $x \geq 0$, then $|x| = x \geq 0$. If $x < 0$, then $|x| = -x > 0$. In either case $|x| \geq 0$. Moreover, $|x| = 0$ implies either $x = 0$ (if $x \geq 0$) or $-x = 0$ (if $x < 0$), both giving $x = 0$.

Conversely, $|0| = 0$.

(ii) If $x \geq 0$, then $-x \leq 0$, so $|-x| = -(-x) = x = |x|$. If $x < 0$, then $-x > 0$, so $|-x| = -x = |x|$.

(iii) Consider the four sign combinations. For instance, if $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x| \, |y|$. The other cases are similar.

(iv) If $x \geq 0$, then $|x| = x$, and $-|x| = -x \leq 0 \leq x$. If $x < 0$, then $|x| = -x$, and $x < 0 \leq -x = |x|$, while $-|x| = x$. In both cases, $-|x| \leq x \leq |x|$.

(v) ($\Rightarrow$) If $|x| \leq a$, then by (iv), $-a \leq -|x| \leq x \leq |x| \leq a$. ($\Leftarrow$) If $-a \leq x \leq a$, then $x \leq a$ and $-x \leq a$. Since $|x|$ equals either x or $-x$, we get $|x| \leq a$. $\qquad\square$

Theorem 2.1 (Triangle inequality)**.** For all $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y| \, .$$

Proof. By Proposition 2.1(iv), we have

$$-|x| \leq x \leq |x| \qquad \text{and} \qquad -|y| \leq y \leq |y| \, .$$

Adding these two inequalities:

$$-(|x| + |y|) \leq x + y \leq |x| + |y| \, .$$

By the characterisation in Proposition 2.1(v) (with $a = |x| + |y| \geq 0$), this is equivalent to $|x + y| \leq |x| + |y|$. $\qquad\square$

Corollary 2.1 (Reverse triangle inequality)**.** For all $x, y \in \mathbb{R}$,

$$\Big| |x| - |y| \Big| \leq |x - y| \, .$$

Proof. By the triangle inequality applied to $x = (x - y) + y$:

$$|x| = |(x - y) + y| \leq |x - y| + |y| \, ,$$

so $|x| - |y| \leq |x - y|$. Swapping the roles of x and y: $|y| - |x| \leq |y - x| = |x - y|$. Since both $|x| - |y|$ and $|y| - |x|$ are $\leq |x - y|$, we conclude $\big| |x| - |y| \big| \leq |x - y|$. $\qquad\square$

2.4   Bounded Sets, Supremum, and Infimum

Definition 2.4 (Bounded sets)**.** Let $A \subset \mathbb{R}$ be a nonempty set.

   (i) A is **bounded above** if there exists $M \in \mathbb{R}$ such that $\forall a \in A$, $a \leq M$. Such an M is called an **upper bound** of A.

   (ii) A is **bounded below** if there exists $m \in \mathbb{R}$ such that $\forall a \in A$, $a \geq m$. Such an m is called a **lower bound** of A.

   (iii) A is **bounded** if it is bounded both above and below.

> **Definition 2.5** (Supremum and infimum). Let $A \subset \mathbb{R}$ be nonempty.
>
> (i) The **supremum** (or **least upper bound**) of A, denoted $\sup A$, is a number $s \in \mathbb{R}$ such that:
>
>   (a) s is an upper bound of A: $\forall a \in A$, $a \leq s$.
>
>   (b) s is the *least* such bound: if M is any upper bound of A, then $s \leq M$.
>
> (ii) The **infimum** (or **greatest lower bound**) of A, denoted $\inf A$, is defined analogously: the greatest number t that is a lower bound of A.

Remark 2.2. If a supremum exists, it is unique. Indeed, if s_1 and s_2 are both suprema of A, then s_1 is an upper bound so $s_2 \leq s_1$ (since s_2 is the *least* upper bound), and s_2 is an upper bound so $s_1 \leq s_2$. Hence $s_1 = s_2$.

Remark 2.3 (Sup vs. Max). The **maximum** of A is an element $a_0 \in A$ such that $a_0 \geq a$ for all $a \in A$. If $\max A$ exists, then $\sup A = \max A$. But $\sup A$ may exist even when $\max A$ does not. For instance, $A = (0, 1)$ has $\sup A = 1$, but $1 \notin A$, so A has no maximum.

$$A = [a, b)$$

$$\inf A = a \qquad \sup A = b$$

lower bounds $\qquad\qquad$ upper bounds
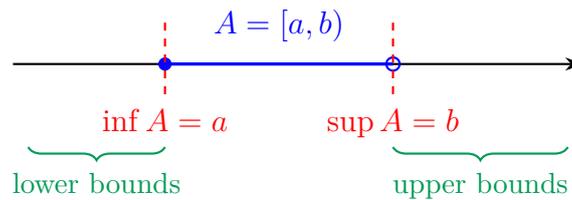
> **Theorem 2.2** (ε-characterisation of the supremum). Let $A \subset \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$. Then $s = \sup A$ if and only if:
>
> (i) $\forall a \in A$, $a \leq s$ $\quad$ (s is an upper bound), and
>
> (ii) $\forall \varepsilon > 0$, $\exists a \in A$, $a > s - \varepsilon$ $\quad$ (no number smaller than s is an upper bound).

Proof. ($\Rightarrow$) Suppose $s = \sup A$. Condition (i) holds because the supremum is an upper bound. For (ii), let $\varepsilon > 0$. Then $s - \varepsilon < s$. Since s is the *least* upper bound, $s - \varepsilon$ is *not* an upper bound of A. Therefore, there exists $a \in A$ such that $a > s - \varepsilon$.

($\Leftarrow$) Suppose s satisfies (i) and (ii). By (i), s is an upper bound of A, so $\sup A \leq s$ (since $\sup A$ is the least upper bound). We now show $s \leq \sup A$. Suppose for contradiction that $s > \sup A$. Set $\varepsilon = s - \sup A > 0$. By (ii), there exists $a \in A$ with $a > s - \varepsilon = \sup A$. But this contradicts the fact that $\sup A$ is an upper bound of A. Therefore $s \leq \sup A$, and combined with $\sup A \leq s$, we get $s = \sup A$. $\qquad\square$

Remark 2.4. There is an analogous characterisation for the infimum: $t = \inf A$ if and only if t is a lower bound of A and for every $\varepsilon > 0$ there exists $a \in A$ with $a < t + \varepsilon$.

2.5 The Completeness Axiom

We now state the axiom that makes $\mathbb{R}$ fundamentally different from $\mathbb{Q}$.

> **Definition 2.6** (Completeness axiom (Least Upper Bound Property)). Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Remark 2.5. Equivalently (by considering $-A = \{-a : a \in A\}$), every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$.

Why is this axiom so important? Without completeness, we cannot guarantee that limits exist. For instance, consider the sequence of rational numbers

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \ldots$$

which gives better and better decimal approximations to $\sqrt{2}$. Intuitively this sequence "converges," but its limit is $\sqrt{2}$, which is not in $\mathbb{Q}$. In $\mathbb{Q}$, this sequence has no limit. The completeness axiom ensures that such pathologies do not occur in $\mathbb{R}$: every "hole" is filled.

$\mathbb{Q}$ **lacks the completeness property.** We saw in Section 2.1 that the set $A = \{r \in \mathbb{Q} \mid r > 0,\ r^2 < 2\}$ is nonempty and bounded above in $\mathbb{Q}$, but has no least upper bound in $\mathbb{Q}$. This shows that $\mathbb{Q}$ does not satisfy the completeness axiom. The real numbers $\mathbb{R}$ are, in a precise sense, the "completion" of $\mathbb{Q}$: we adjoin all the missing limits.

2.6 Consequences of Completeness

2.6.1 The Archimedean Property

> **Theorem 2.3** (Archimedean property). For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.
> Equivalently: for every $\varepsilon > 0$ and every $M > 0$, there exists $n \in \mathbb{N}^*$ such that $n\varepsilon > M$ (i.e., the multiples of any positive number are unbounded).

Proof. We prove the first formulation. Suppose for contradiction that there exists $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$. Then $\mathbb{N}$ is bounded above by x. Since $\mathbb{N}$ is nonempty and bounded above, the completeness axiom guarantees that $s = \sup \mathbb{N}$ exists in $\mathbb{R}$.

Now consider $s - 1$. Since $s - 1 < s$ and s is the *least* upper bound of $\mathbb{N}$, the number $s - 1$ is not an upper bound of $\mathbb{N}$. Therefore there exists $n_0 \in \mathbb{N}$ with $n_0 > s - 1$, i.e., $n_0 + 1 > s$.

But $n_0 + 1 \in \mathbb{N}$ (since $\mathbb{N}$ is closed under the successor operation), and $n_0 + 1 > s$ contradicts the fact that s is an upper bound of $\mathbb{N}$.

This contradiction shows that $\mathbb{N}$ is not bounded above, proving the result. $\square$

Remark 2.6. An immediate consequence: for every $\varepsilon > 0$, there exists $n \in \mathbb{N}^*$ such that $\frac{1}{n} < \varepsilon$. Indeed, by the Archimedean property applied to $x = \frac{1}{\varepsilon}$, there exists $n > \frac{1}{\varepsilon}$, hence $\frac{1}{n} < \varepsilon$. This fact is used constantly in analysis.

2.6.2 Density of $\mathbb{Q}$ in $\mathbb{R}$

> **Theorem 2.4** (Density of $\mathbb{Q}$ in $\mathbb{R}$). Between any two distinct real numbers, there exists a rational number. Formally: if $a, b \in \mathbb{R}$ with $a < b$, then there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof. Since $a < b$, we have $b - a > 0$. By the Archimedean property, there exists $n \in \mathbb{N}^*$ such that

$$n(b - a) > 1, \quad \text{i.e.,} \quad \frac{1}{n} < b - a. \tag{2.1}$$

This means the intervals of length $\frac{1}{n}$ are narrow enough to "fit between" a and b.

Now we need to find an integer m such that $a < \frac{m}{n} < b$, i.e., $na < m < nb$. We must show that there is an integer in the open interval (na, nb).

Again by the Archimedean property, there exists a positive integer M_1 with $M_1 > na$ and a positive integer M_2 with $M_2 > -na$. So the set of integers greater than na is nonempty. Let

$$m = \min \left\{ k \in \mathbb{Z} \mid k > na \right\}.$$

(This minimum exists because the set of integers greater than na is nonempty and bounded below by na, and every nonempty set of integers bounded below has a minimum.)

By definition of m, we have $m > na$, i.e., $\frac{m}{n} > a$. Also, $m - 1 \leq na$ (since m is the *smallest* integer exceeding na), so $m \leq na + 1$. Therefore

$$m \leq na + 1 < na + n(b - a) = nb,$$

where we used (2.1) in the strict inequality. Hence $\frac{m}{n} < b$.

Setting $q = \frac{m}{n} \in \mathbb{Q}$, we have $a < q < b$. $\square$

> **Corollary 2.2** (Density of irrationals)**.** Between any two distinct real numbers, there exists an irrational number.

Proof. Let $a < b$. Then $a - \sqrt{2} < b - \sqrt{2}$. By the density of $\mathbb{Q}$ (Theorem 2.4), there exists $q \in \mathbb{Q}$ with $a - \sqrt{2} < q < b - \sqrt{2}$. Set $\alpha = q + \sqrt{2}$. Then $a < \alpha < b$. We claim α is irrational: if α were rational, then $\alpha - q = \sqrt{2}$ would be rational (as the difference of two rationals), contradicting Theorem 1.1. Therefore $\alpha \notin \mathbb{Q}$. $\square$

2.6.3 The Floor Function

> **Proposition 2.2** (Existence and uniqueness of the floor)**.** For every $x \in \mathbb{R}$, there exists a unique integer $n \in \mathbb{Z}$ such that $n \leq x < n + 1$. This integer is called the **floor** of x and is denoted $\lfloor x \rfloor$.

Proof. **Existence.** By the Archimedean property, the set $S = \left\{ k \in \mathbb{Z} \mid k > x \right\}$ is nonempty. Similarly, the set $\left\{ k \in \mathbb{Z} \mid k \leq x \right\}$ is nonempty (take k to be an integer less than or equal to x, which exists by applying the Archimedean property to $-x$).

Let $m = \min S$ (the smallest integer strictly greater than x; this exists because S is a nonempty set of integers bounded below). Set $n = m - 1$. Then $n = m - 1 \leq x$ (since m is the smallest integer $> x$, we have $m - 1 \leq x$) and $n + 1 = m > x$. So $n \leq x < n + 1$.
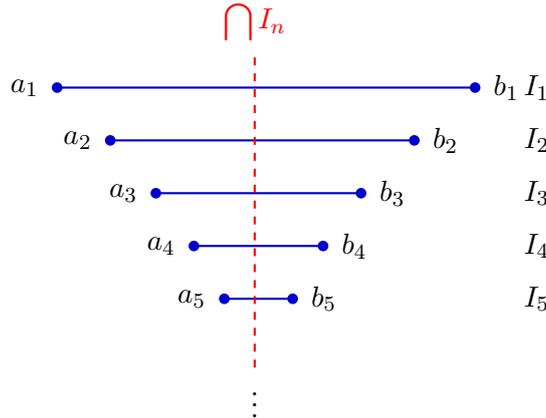
Uniqueness. Suppose n and n' both satisfy $n \leq x < n + 1$ and $n' \leq x < n' + 1$. Then $n \leq x < n' + 1$ gives $n < n' + 1$, so $n \leq n'$. Similarly $n' \leq n$. Hence $n = n'$. $\square$

2.6.4 The Nested Intervals Theorem

> **Theorem 2.5** (Nested intervals theorem). Let $(I_n)_{n \geq 1}$ be a sequence of closed bounded intervals $I_n = [a_n, b_n]$ such that $I_{n+1} \subset I_n$ for all n (i.e., the intervals are nested). Then
> $$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$
> Moreover, if $\lim_{n \to \infty}(b_n - a_n) = 0$, then the intersection contains exactly one point.

Proof. Since $I_{n+1} \subset I_n$, we have $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all n. Moreover, $a_n \leq b_n$ for all n. In fact, $a_m \leq b_n$ for *all* $m, n \geq 1$ (indeed, if $m \leq n$ then $a_m \leq a_n \leq b_n$, and if $m > n$ then $a_m \leq b_m \leq b_n$).

Step 1: The set of left endpoints is bounded above. The set $A = \{a_n : n \geq 1\}$ is nonempty and bounded above by every b_n (by the observation above). By the completeness axiom, $s = \sup A$ exists.

Step 2: s belongs to every interval. For each n, b_n is an upper bound of A, so $s \leq b_n$ (since s is the *least* upper bound). Also, $a_n \in A$ and $s = \sup A \geq a_n$. Therefore $a_n \leq s \leq b_n$, which means $s \in [a_n, b_n] = I_n$ for every n. Hence $s \in \bigcap_{n=1}^{\infty} I_n$, and this intersection is nonempty.

Step 3: Uniqueness when lengths tend to zero. Suppose $\lim_{n \to \infty}(b_n - a_n) = 0$ and let $x, y \in \bigcap I_n$. Then for every n, $a_n \leq x \leq b_n$ and $a_n \leq y \leq b_n$, so $|x - y| \leq b_n - a_n$. Since $b_n - a_n \to 0$, we get $|x - y| = 0$, i.e., $x = y$. $\qquad\square$

2.7 Historical Remarks

The rigorous construction of the real numbers was one of the great achievements of nineteenth-century mathematics. Two independent approaches emerged around 1872:

- **Dedekind cuts** (Richard Dedekind, 1872): A real number is defined as a partition of $\mathbb{Q}$ into two nonempty sets L and R such that every element of L is less than every element of R, and L has no greatest element. The "cut" between L and R represents the real number. Irrational numbers correspond to cuts where R has no least element either.

- **Cauchy sequences** (Georg Cantor, 1872): A real number is defined as an equivalence class of Cauchy sequences of rational numbers, where two sequences are equivalent if their difference converges to zero. This approach makes the completeness of $\mathbb{R}$ immediate: every Cauchy sequence of real numbers converges (this will be proved in a later chapter).

Both constructions yield the same object: the unique (up to isomorphism) complete ordered field $\mathbb{R}$. In this course, we take the axiomatic approach: we *postulate* that $\mathbb{R}$ exists and satisfies the field axioms, the order axioms, and the completeness axiom, and we derive everything else from these.

2.8 Common Errors

> **Common Errors — Watch Out!**
>
> 1. **Confusing** $\sup A$ **with** $\max A$. The supremum of a set *need not belong to the set*. For example, $\sup(0,1) = 1$, but $1 \notin (0,1)$, so $(0,1)$ has no maximum. Always check whether the supremum is attained before calling it a maximum.
>
> 2. **Claiming** $\sup A \in A$ **without justification.** This is a frequent error. If you need the supremum to be an element of the set, you must *prove* it, or state that A has a maximum.
>
> 3. **Forgetting that** $\sup$ **applies to** *nonempty* **bounded sets.** The completeness axiom requires A to be nonempty and bounded above. The supremum of the empty set is not defined.
>
> 4. **Applying the Archimedean property to non-positive numbers.** The statement "for every $\varepsilon > 0$ there exists n with $\frac{1}{n} < \varepsilon$" requires $\varepsilon > 0$. For $\varepsilon \le 0$ the statement is meaningless.
>
> 5. **Assuming** $\mathbb{R}$ **is "just** $\mathbb{Q}$ **with more digits."** The decimal expansion viewpoint is useful for intuition but dangerous for proofs. The completeness axiom is the correct foundation.

2.9 Exercises

Exercise 2.1 (Supremum practice). [⋆]Find $\sup A$ and $\inf A$ for each of the following sets, and determine whether the supremum (respectively, infimum) is a maximum (respectively, minimum).

(a) $A = \left\{ \frac{1}{n} : n \in \mathbb{N}^* \right\}$.

(b) $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$.

(c) $A = \left\{ (-1)^n \left(1 + \frac{1}{n} \right) : n \in \mathbb{N}^* \right\}$.

Exercise 2.2 (Supremum of a union). [⋆⋆]Let $A, B \subset \mathbb{R}$ be nonempty and bounded above. Prove that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

Exercise 2.3 (ε-characterisation). [★★]Let $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}^*\right\}$. Use the ε-characterisation (Theorem 2.2) to prove that $\sup A = 1$.

Exercise 2.4 (Supremum of a sum). [★★★]Let $A, B \subset \mathbb{R}$ be nonempty and bounded above. Define $A + B = \{a + b : a \in A, \ b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$.

Exercise 2.5 (Archimedean consequences). [★]Using the Archimedean property, prove:

(a) For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n \le x < n + 1$.

(b) $\inf \left\{\frac{1}{n} : n \in \mathbb{N}^*\right\} = 0$.

Exercise 2.6 (Density). [★★]

(a) Prove that between any two distinct real numbers, there exist infinitely many rational numbers.

(b) Prove that between any two distinct real numbers, there exist infinitely many irrational numbers.

Exercise 2.7 (Completeness is essential). [★★★]Let $A = \{r \in \mathbb{Q} \mid r > 0, \ r^2 < 2\}$.

(a) Prove that A is nonempty and bounded above in $\mathbb{Q}$.

(b) Prove that A has no maximum in $\mathbb{Q}$. *(Hint: given $r \in A$, show that $r' = r + \frac{2 - r^2}{r + 2}$ satisfies $r' \in A$ and $r' > r$.)*

(c) Explain why A has no least upper bound in $\mathbb{Q}$.

Exercise 2.8 (Nested intervals). [★★]Let $I_n = \left[1 - \frac{1}{n}, \ 2 + \frac{1}{n}\right]$ for $n \ge 1$. Determine $\bigcap_{n=1}^{\infty} I_n$.

Exercise 2.9 (Nested intervals — singleton). [★★]Let $I_n = \left[\frac{n}{n+1}, \ \frac{n+2}{n+1}\right]$ for $n \ge 1$.

(a) Show that the intervals are nested.

(b) Compute $b_n - a_n$ and its limit.

(c) Determine $\bigcap_{n=1}^{\infty} I_n$.

Exercise 2.10 (Triangle inequality applications). [★]Let $a, b, c \in \mathbb{R}$. Prove:

(a) $|a - c| \le |a - b| + |b - c|$.

(b) $|a + b + c| \le |a| + |b| + |c|$.

Exercise 2.11 (Absolute value equations). [★]Solve and prove your answer:

(a) $|2x - 3| = 5$.

(b) $|x - 1| + |x + 1| \le 4$.

Exercise 2.12 (Floor function properties). [★★]Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Prove:

(a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

(b) $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ -1 & \text{if } x \notin \mathbb{Z}. \end{cases}$

Chapter Summary

Chapter Summary

- $\mathbb{Q}$ is an ordered field but is **incomplete**: bounded sets may lack a supremum in $\mathbb{Q}$.

- $\mathbb{R}$ is the unique **complete ordered field**: every nonempty subset bounded above has a supremum in $\mathbb{R}$ (completeness axiom).

- The **absolute value** $|x|$ satisfies the triangle inequality: $|x + y| \leq |x| + |y|$.

- The **supremum** $\sup A$ is the least upper bound. The ε-characterisation: $s = \sup A$ iff s is an upper bound and $\forall \varepsilon > 0, \exists a \in A, a > s - \varepsilon$.

- **Archimedean property**: $\mathbb{N}$ is unbounded above. Equivalently, for all $\varepsilon > 0$, $\exists n \in \mathbb{N}^*, \frac{1}{n} < \varepsilon$.

- **Density of** $\mathbb{Q}$: between any two reals lies a rational (and an irrational).

- The **floor function** $\lfloor x \rfloor$ is the unique integer with $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

- **Nested intervals theorem**: a decreasing sequence of closed bounded intervals has nonempty intersection; if the lengths tend to zero, the intersection is a singleton.

Chapter 3

Sequences of Real Numbers

Motivation: what does "approaching" mean?

Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ The terms get *closer and closer* to 0, yet no term actually equals 0. But what does "closer and closer" mean *precisely*? How close is close enough?

   The answer—one of the great achievements of 19th-century mathematics—is the ε– N definition of convergence. It replaces the vague intuition of "approaching" with a rigorous, quantified statement that can be proved or disproved. Mastering this definition is the single most important step in a first course in analysis.

3.1 Sequences: first definitions

> **Definition 3.1** (Sequence of real numbers)**.** A **sequence of real numbers** is a function
> $$u \colon \mathbb{N} \longrightarrow \mathbb{R}, \qquad n \longmapsto u_n.$$
> We write $(u_n)_{n \in \mathbb{N}}$, or simply (u_n), and call u_n the n-**th term** (or **general term**) of the sequence.

Remark 3.1. Sometimes the index starts at $n = 1$ or another integer; the theory is the same. We may also write (a_n), (x_n), etc.

Example 3.1.

1. $u_n = \dfrac{1}{n+1}$: the terms are $1, \frac{1}{2}, \frac{1}{3}, \ldots$

2. $v_n = (-1)^n$: the terms alternate between 1 and -1.

3. $w_n = \dfrac{n}{n+1}$: the terms are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

4. $a_n = 2^n$: the terms grow without bound.

3.2 Convergence of a sequence

The following definition is the cornerstone of the entire course. Read it slowly, several times.

> **Definition (Convergence of a sequence)**
>
> Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. We say that (u_n) **converges** to $\ell \in \mathbb{R}$ if
>
> $$\boxed{\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n \geq N, \quad |u_n - \ell| < \varepsilon.}$$
>
> We then write $\lim_{n \to \infty} u_n = \ell$, or $u_n \xrightarrow[n \to \infty]{} \ell$.
>
> **Parsing each quantifier:**
>
> 1. "$\forall \varepsilon > 0$": for **every** tolerance ε, no matter how small—you do not choose ε; it is given to you by an adversary.
>
> 2. "$\exists N \in \mathbb{N}$": there **exists** a rank N (which **may depend on** ε)—your job is to find (or prove the existence of) such an N.
>
> 3. "$\forall n \geq N$": for **all** indices n at least as large as N—that is, from rank N onward, with **no exceptions**.
>
> 4. "$|u_n - \ell| < \varepsilon$": the distance from u_n to ℓ is strictly less than ε.
>
> In plain language: *no matter how narrow a band $(\ell - \varepsilon, \ell + \varepsilon)$ you draw around the limit, all but finitely many terms of the sequence lie inside that band.*

Remark 3.2. The order of quantifiers is critical. "$\forall \varepsilon > 0, \exists N$" means N may depend on ε. If the order were reversed, N would have to work for *every* ε simultaneously—an impossibly strong demand.

> **Definition 3.2** (Divergent sequence)**.** A sequence that does not converge is called **divergent**.

Visualization: the ε-band
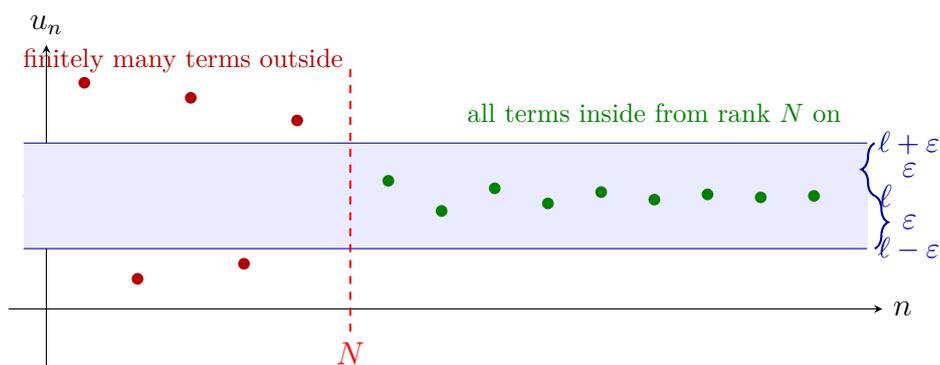
3.3 First proofs of convergence

3.3.1 Strategy: scratch work then formal proof

Every ε–N proof proceeds in two stages:

1. **Scratch work (rough computation).** Assume $|u_n - \ell| < \varepsilon$ and work backward to find what condition on n guarantees this. This is *not* part of the final proof—it is how you *discover* the right N.

2. **Formal proof.** Start with "Let $\varepsilon > 0$", choose N (the value you found in the scratch work), and verify that $n \geq N \implies |u_n - \ell| < \varepsilon$.

3.3.2 Worked example: $\lim_{n \to \infty} \frac{1}{n} = 0$

Example 3.2 (Scratch work). We want $\left| \frac{1}{n} - 0 \right| < \varepsilon$, i.e. $\frac{1}{n} < \varepsilon$. This is equivalent to $n > \frac{1}{\varepsilon}$. So any integer $N > \frac{1}{\varepsilon}$ will work. By the Archimedean property, such an N exists.

> **Proposition 3.1.** $\lim_{n \to \infty} \dfrac{1}{n} = 0.$

Proof. Let $\varepsilon > 0$. By the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \dfrac{1}{\varepsilon}$, i.e. $\dfrac{1}{N} < \varepsilon$.

Let $n \geq N$. Then $n \geq N > \dfrac{1}{\varepsilon}$, so

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

This shows that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \implies \left| \frac{1}{n} - 0 \right| < \varepsilon$. Therefore $\lim_{n \to \infty} \frac{1}{n} = 0$. $\qquad \square$

3.3.3 Worked example: $\lim_{n \to \infty} \frac{n}{n+1} = 1$

Example 3.3 (Scratch work). We compute

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{n - (n+1)}{n+1} \right| = \frac{1}{n+1}.$$

We need $\dfrac{1}{n+1} < \varepsilon$, i.e. $n + 1 > \dfrac{1}{\varepsilon}$, i.e. $n > \dfrac{1}{\varepsilon} - 1$.

So it suffices to take N any integer with $N > \dfrac{1}{\varepsilon} - 1$ (or, more simply, $N > \dfrac{1}{\varepsilon}$, since that is larger).

> **Proposition 3.2.** $\lim_{n \to \infty} \dfrac{n}{n+1} = 1.$

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > \dfrac{1}{\varepsilon}$ (Archimedean property). For every $n \geq N$,

$$\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence $\dfrac{n}{n+1} \to 1$ as $n \to \infty$. $\qquad \square$

3.3.4 Worked example: $((-1)^n)$ diverges

Proposition 3.3. The sequence $((-1)^n)_{n \in \mathbb{N}}$ does not converge.

Proof. Suppose for contradiction that $((-1)^n)$ converges to some $\ell \in \mathbb{R}$. Then, by the definition of convergence, applied with $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N, \quad |(-1)^n - \ell| < \tfrac{1}{2}.$$

In particular, taking $n = 2N$ (which is even and $\geq N$) and $n = 2N+1$ (which is odd and $\geq N$):

$$|1 - \ell| < \tfrac{1}{2} \quad \text{and} \quad |-1 - \ell| < \tfrac{1}{2}.$$

By the triangle inequality,

$$2 = |1 - (-1)| = |(1 - \ell) + (\ell - (-1))| \leq |1 - \ell| + |-1 - \ell| < \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

This gives $2 < 1$, a contradiction. Therefore $((-1)^n)$ diverges. $\qquad\square$

3.4 Uniqueness of the limit

Theorem 3.1 (Uniqueness of limits). If a sequence (u_n) converges, then its limit is unique. That is, if $u_n \to \ell$ and $u_n \to \ell'$, then $\ell = \ell'$.

Proof. Suppose $u_n \to \ell$ and $u_n \to \ell'$. Let $\varepsilon > 0$.

- Since $u_n \to \ell$, there exists $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|u_n - \ell| < \frac{\varepsilon}{2}$.

- Since $u_n \to \ell'$, there exists $N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$, $|u_n - \ell'| < \frac{\varepsilon}{2}$.

Set $N = \max(N_1, N_2)$. For every $n \geq N$, the triangle inequality gives

$$|\ell - \ell'| = |(\ell - u_n) + (u_n - \ell')| \leq |u_n - \ell| + |u_n - \ell'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $|\ell - \ell'| < \varepsilon$ for every $\varepsilon > 0$.

Since $|\ell - \ell'|$ is a fixed non-negative real number that is smaller than every positive real number, it must equal 0. Hence $\ell = \ell'$. $\qquad\square$

3.5 Bounded sequences

Definition 3.3 (Bounded sequence). A sequence (u_n) is **bounded** if there exists $M \geq 0$ such that
$$\forall n \in \mathbb{N}, \quad |u_n| \leq M.$$

Theorem 3.2. Every convergent sequence is bounded.

Proof. Let (u_n) be a sequence converging to $\ell \in \mathbb{R}$. Apply the definition of convergence with $\varepsilon = 1$: there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N, \quad |u_n - \ell| < 1,$$

which gives $|u_n| \leq |u_n - \ell| + |\ell| < 1 + |\ell|$ for all $n \geq N$.

The finitely many terms $u_0, u_1, \ldots, u_{N-1}$ form a finite set, so

$$M = \max\big(|u_0|, |u_1|, \ldots, |u_{N-1}|, \ 1 + |\ell|\big)$$

is well-defined. For every $n \in \mathbb{N}$, we have $|u_n| \leq M$, so (u_n) is bounded. $\qquad \square$

Remark 3.3. The converse is false: $((-1)^n)$ is bounded but not convergent.

3.6 Squeeze theorem

> **Theorem 3.3** (Squeeze theorem / Sandwich theorem)**.** Let (u_n), (v_n), (w_n) be sequences such that
> $$\forall n \in \mathbb{N}, \quad u_n \leq v_n \leq w_n,$$
> and suppose $u_n \to \ell$ and $w_n \to \ell$ for the same $\ell \in \mathbb{R}$. Then $v_n \to \ell$.

Proof. Let $\varepsilon > 0$.

- Since $u_n \to \ell$, there exists N_1 such that $\forall n \geq N_1$, $|u_n - \ell| < \varepsilon$, i.e. $\ell - \varepsilon < u_n < \ell + \varepsilon$.

- Since $w_n \to \ell$, there exists N_2 such that $\forall n \geq N_2$, $|w_n - \ell| < \varepsilon$, i.e. $\ell - \varepsilon < w_n < \ell + \varepsilon$.

Let $N = \max(N_1, N_2)$. For every $n \geq N$,

$$\ell - \varepsilon < u_n \leq v_n \leq w_n < \ell + \varepsilon,$$

so $|v_n - \ell| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $v_n \to \ell$. $\qquad \square$

Example 3.4. Let $u_n = \dfrac{\sin n}{n}$. Since $-1 \leq \sin n \leq 1$ for all n,

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Both $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$, so by the squeeze theorem, $\dfrac{\sin n}{n} \to 0$.

3.7 Algebra of limits

> **Theorem 3.4** (Algebra of limits)**.** Let (u_n) and (v_n) be convergent sequences with $u_n \to \ell$ and $v_n \to \ell'$. Then:
>
> 1. $(u_n + v_n) \to \ell + \ell'$.
>
> 2. For every $\lambda \in \mathbb{R}$, $(\lambda\, u_n) \to \lambda\, \ell$.
>
> 3. $(u_n \cdot v_n) \to \ell \cdot \ell'$.
>
> 4. If $\ell' \neq 0$, then $(u_n/v_n) \to \ell/\ell'$ (and $v_n \neq 0$ for all n large enough).

Proof of 1 (sum). Let $\varepsilon > 0$.

- Since $u_n \to \ell$, there exists N_1 with $\forall n \geq N_1$, $|u_n - \ell| < \frac{\varepsilon}{2}$.

- Since $v_n \to \ell'$, there exists N_2 with $\forall n \geq N_2$, $|v_n - \ell'| < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$. For $n \geq N$,

$$|(u_n + v_n) - (\ell + \ell')| = |(u_n - \ell) + (v_n - \ell')| \leq |u_n - \ell| + |v_n - \ell'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $(u_n + v_n) \to \ell + \ell'$. $\hfill\square$

Proof of 3 (product). We use the identity

$$u_n v_n - \ell\ell' = (u_n - \ell)\, v_n + \ell\, (v_n - \ell').$$

Since (v_n) converges, it is bounded (Theorem 3.2): there exists $M > 0$ with $|v_n| \leq M$ for all n.

Let $\varepsilon > 0$.

- There exists N_1 with $\forall n \geq N_1$, $|u_n - \ell| < \dfrac{\varepsilon}{2M}$.

- There exists N_2 with $\forall n \geq N_2$, $|v_n - \ell'| < \dfrac{\varepsilon}{2(|\ell| + 1)}$.

(We write $|\ell| + 1$ instead of $|\ell|$ to avoid division by zero when $\ell = 0$.)

Let $N = \max(N_1, N_2)$. For $n \geq N$,

$$\begin{aligned}
|u_n v_n - \ell\ell'| &\leq |u_n - \ell|\,|v_n| + |\ell|\,|v_n - \ell'| \\
&< \frac{\varepsilon}{2M} \cdot M + |\ell| \cdot \frac{\varepsilon}{2(|\ell| + 1)} \\
&= \frac{\varepsilon}{2} + \frac{|\ell|}{|\ell| + 1} \cdot \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $(u_n v_n) \to \ell\ell'$. $\hfill\square$

3.8   Monotone sequences and the Monotone Convergence Theorem

Definition 3.4 (Monotone sequence)**.** A sequence (u_n) is

- **increasing** if $u_n \leq u_{n+1}$ for all $n \in \mathbb{N}$;

- **strictly increasing** if $u_n < u_{n+1}$ for all $n \in \mathbb{N}$;

- **decreasing** if $u_n \geq u_{n+1}$ for all $n \in \mathbb{N}$;

- **strictly decreasing** if $u_n > u_{n+1}$ for all $n \in \mathbb{N}$.

A sequence that is either increasing or decreasing is called **monotone**.

> **Theorem 3.5** (Monotone Convergence Theorem)**.** Every bounded monotone sequence of real numbers converges.
>
> More precisely:
>
> 1. If (u_n) is increasing and bounded above, then $\lim\limits_{n\to\infty} u_n = \sup\limits_{n\in\mathbb{N}} u_n$.
>
> 2. If (u_n) is decreasing and bounded below, then $\lim\limits_{n\to\infty} u_n = \inf\limits_{n\in\mathbb{N}} u_n$.

Proof. We prove (1); the proof of (2) is analogous.

Let (u_n) be increasing and bounded above. The set $A = \{u_n : n \in \mathbb{N}\}$ is non-empty and bounded above, so by the completeness axiom of $\mathbb{R}$, $\ell = \sup A$ exists in $\mathbb{R}$.

We claim that $u_n \to \ell$. Let $\varepsilon > 0$. Since ℓ is the *least* upper bound of A and $\ell - \varepsilon < \ell$, the number $\ell - \varepsilon$ is *not* an upper bound of A. Hence there exists $N \in \mathbb{N}$ with

$$u_N > \ell - \varepsilon.$$

Since (u_n) is increasing and ℓ is an upper bound, for every $n \geq N$ we have

$$\ell - \varepsilon < u_N \leq u_n \leq \ell < \ell + \varepsilon.$$

Therefore $|u_n - \ell| < \varepsilon$ for all $n \geq N$. This proves $u_n \to \ell = \sup A$. $\qquad\square$

Remark 3.4. This theorem relies on the completeness of $\mathbb{R}$. It fails in $\mathbb{Q}$: the sequence $1, 1.4, 1.41, 1.414, \ldots$ of truncations of $\sqrt{2}$ is increasing and bounded in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$.

3.9 Subsequences and the Bolzano–Weierstrass theorem

> **Definition 3.5** (Subsequence)**.** Let $(u_n)_{n\in\mathbb{N}}$ be a sequence and let $\varphi \colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. The sequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ is called a **subsequence** of (u_n).

Remark 3.5. Common notation: $(u_{n_k})_{k\in\mathbb{N}}$ where (n_k) is a strictly increasing sequence of natural numbers. For instance, (u_{2k}) is the subsequence of even-indexed terms.

> **Lemma 3.1.** If $\varphi \colon \mathbb{N} \to \mathbb{N}$ is strictly increasing, then $\varphi(n) \geq n$ for all $n \in \mathbb{N}$.

Proof. By induction. $\varphi(0) \geq 0$ since $\varphi(0) \in \mathbb{N}$. If $\varphi(k) \geq k$, then $\varphi(k+1) > \varphi(k) \geq k$, so $\varphi(k+1) \geq k+1$. $\qquad\square$

> **Proposition 3.4.** If $u_n \to \ell$, then every subsequence of (u_n) also converges to ℓ.

Proof. Let $\varepsilon > 0$. There exists N with $|u_n - \ell| < \varepsilon$ for $n \geq N$. For $k \geq N$, Lemma 3.1 gives $\varphi(k) \geq k \geq N$, so $\left|u_{\varphi(k)} - \ell\right| < \varepsilon$. $\qquad\square$

> **Definition 3.6** (Limit point / cluster value)**.** A real number ℓ is a **limit point** (or **cluster value**, French: *valeur d'adhérence*) of a sequence (u_n) if some subsequence of (u_n) converges to ℓ.

Example 3.5. The sequence $u_n = (-1)^n$ has exactly two limit points: 1 (via the subsequence u_{2k}) and -1 (via u_{2k+1}).

> **Theorem 3.6** (Bolzano–Weierstrass)**.** Every bounded sequence of real numbers has a convergent subsequence (equivalently, at least one limit point).

Proof. Let (u_n) be a bounded sequence: there exist $a, b \in \mathbb{R}$ with $a \leq u_n \leq b$ for all n. We construct a convergent subsequence by *interval bisection.*

Set $[a_0, b_0] = [a, b]$. At each stage, consider the midpoint $m = \frac{a_k + b_k}{2}$ and the two halves $[a_k, m]$ and $[m, b_k]$. At least one of these halves contains *infinitely many* terms of (u_n) (since the union of two finite sets is finite, and $[a_k, b_k]$ contains infinitely many terms). Choose such a half and call it $[a_{k+1}, b_{k+1}]$.

This gives a nested sequence of closed intervals $[a_0, b_0] \supset [a_1, b_1] \supset \cdots$ with $b_k - a_k = \dfrac{b-a}{2^k} \to 0$.

Now extract a subsequence: pick n_0 such that $u_{n_0} \in [a_0, b_0]$. Having chosen $n_0 < n_1 < \cdots < n_k$ with $u_{n_j} \in [a_j, b_j]$, the interval $[a_{k+1}, b_{k+1}]$ contains infinitely many terms, so we can find $n_{k+1} > n_k$ with $u_{n_{k+1}} \in [a_{k+1}, b_{k+1}]$.

By the nested interval property, $\displaystyle\bigcap_{k=0}^{\infty} [a_k, b_k] = \{\ell\}$ for some $\ell \in \mathbb{R}$.

Since $u_{n_k} \in [a_k, b_k]$ and $b_k - a_k \to 0$, we have

$$|u_{n_k} - \ell| \leq b_k - a_k = \frac{b-a}{2^k} \to 0.$$

Hence $u_{n_k} \to \ell$, and (u_{n_k}) is the desired convergent subsequence. $\qquad\square$

3.10 Cauchy sequences

> **Definition 3.7** (Cauchy sequence)**.** A sequence (u_n) is a **Cauchy sequence** if
>
> $$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall p, q \geq N, \quad |u_p - u_q| < \varepsilon.$$
>
> In words: the terms become arbitrarily close to *each other*—without needing to know the limit.

> **Theorem 3.7.** Every convergent sequence is a Cauchy sequence.

Proof. Let $u_n \to \ell$ and let $\varepsilon > 0$. There exists N with $|u_n - \ell| < \frac{\varepsilon}{2}$ for all $n \geq N$. For $p, q \geq N$,

$$|u_p - u_q| \leq |u_p - \ell| + |\ell - u_q| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\qquad\square$

> **Theorem 3.8** (Cauchy criterion for convergence in $\mathbb{R}$)**.** A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof. ($\Rightarrow$) Theorem 3.7.

($\Leftarrow$) Let (u_n) be a Cauchy sequence. We proceed in three steps.

Step 1: (u_n) is bounded. Take $\varepsilon = 1$. There exists N with $|u_p - u_q| < 1$ for all $p, q \geq N$. In particular, $|u_n| \leq |u_n - u_N| + |u_N| < 1 + |u_N|$ for $n \geq N$. Setting $M = \max(|u_0|, \ldots, |u_{N-1}|, 1 + |u_N|)$, we get $|u_n| \leq M$ for all n.

Step 2: (u_n) has a convergent subsequence. Since (u_n) is bounded, the Bolzano–Weierstrass theorem (Theorem 3.6) gives a subsequence (u_{n_k}) converging to some $\ell \in \mathbb{R}$.

Step 3: (u_n) converges to ℓ. Let $\varepsilon > 0$.

- Since (u_n) is Cauchy, there exists N_1 with $|u_p - u_q| < \frac{\varepsilon}{2}$ for $p, q \geq N_1$.

- Since $u_{n_k} \to \ell$, there exists K with $|u_{n_K} - \ell| < \frac{\varepsilon}{2}$ and $n_K \geq N_1$.

For $n \geq N_1$,
$$|u_n - \ell| \leq |u_n - u_{n_K}| + |u_{n_K} - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $u_n \to \ell$. $\qquad\square$

3.11 Recursive sequences

Many sequences are defined by a **recurrence relation** $u_{n+1} = f(u_n)$ for some function $f \colon \mathbb{R} \to \mathbb{R}$ and an initial value u_0.

General method

1. **Existence and well-definedness.** The sequence is defined by induction—always valid.

2. **Conjecture monotonicity and bounds.** Compute a few terms; guess that (u_n) is, say, increasing and bounded above.

3. **Prove monotonicity** (typically by induction): show $u_{n+1} - u_n \geq 0$ or $u_{n+1}/u_n \geq 1$.

4. **Prove the bound** (again by induction).

5. **Conclude convergence** by the Monotone Convergence Theorem (3.5).

6. **Identify the limit.** If $u_n \to \ell$ and f is continuous at ℓ, then passing to the limit in $u_{n+1} = f(u_n)$ gives $\ell = f(\ell)$. Solve this equation for ℓ, then use the bounds/monotonicity to select the correct root.

Example 3.6. Let $u_0 = 1$ and $u_{n+1} = \dfrac{1}{2}\left(u_n + \dfrac{2}{u_n}\right)$.

Claim: (u_n) converges to $\sqrt{2}$.

Step 1: $u_n > 0$ for all n. By induction: $u_0 = 1 > 0$, and if $u_n > 0$ then u_{n+1} is the average of two positive numbers.

Step 2: $u_n \geq \sqrt{2}$ **for all** $n \geq 1$. By AM–GM: $u_{n+1} = \frac{1}{2}(u_n + \frac{2}{u_n}) \geq \sqrt{u_n \cdot \frac{2}{u_n}} = \sqrt{2}$.

Step 3: $(u_n)_{n \geq 1}$ **is decreasing.** $u_{n+1} - u_n = \frac{1}{2}\left(\frac{2}{u_n} - u_n\right) = \frac{2 - u_n^2}{2u_n} \leq 0$ since $u_n^2 \geq 2$ by Step 2.

Step 4: Convergence. $(u_n)_{n \geq 1}$ is decreasing and bounded below by $\sqrt{2}$, hence converges by the MCT.

Step 5: Identify the limit. Let $\ell = \lim u_n$. Then $\ell = \frac{1}{2}(\ell + 2/\ell)$, giving $2\ell = \ell + 2/\ell$, so $\ell = 2/\ell$, hence $\ell^2 = 2$. Since $\ell \geq \sqrt{2} > 0$, we get $\ell = \sqrt{2}$.

Example 3.7. Let $u_0 = 0$ and $u_{n+1} = \dfrac{u_n + 3}{4}$.

Claim: (u_n) converges to 1.

Step 1: $0 \leq u_n \leq 1$ **for all** n. Induction: $u_0 = 0 \in [0,1]$. If $0 \leq u_n \leq 1$, then $u_{n+1} = \frac{u_n + 3}{4} \in [\frac{3}{4}, 1] \subset [0,1]$.

Step 2: (u_n) **is increasing.** $u_{n+1} - u_n = \frac{u_n + 3}{4} - u_n = \frac{3 - 3u_n}{4} = \frac{3(1 - u_n)}{4} \geq 0$ since $u_n \leq 1$.

Step 3: Convergence and limit. By MCT, (u_n) converges. If $\ell = \lim u_n$, then $\ell = \frac{\ell + 3}{4}$, giving $4\ell = \ell + 3$, so $\ell = 1$.

3.12 Adjacent sequences

> **Definition 3.8** (Adjacent sequences)**.** Two sequences (u_n) and (v_n) are called **adjacent** if:
>
> 1. (u_n) is increasing,
>
> 2. (v_n) is decreasing,
>
> 3. $\lim_{n \to \infty}(v_n - u_n) = 0$.

> **Theorem 3.9.** If (u_n) and (v_n) are adjacent, then both converge to a common limit ℓ satisfying $u_n \leq \ell \leq v_n$ for all n.

Proof. Since $v_n - u_n \to 0$ and $v_n - u_n$ is eventually non-negative (indeed, one can check $u_n \leq v_n$ for all n: the sequence $(v_n - u_n)$ is decreasing, and its limit is 0, so $v_n - u_n \geq 0$).

Hence $u_n \leq v_n$ for all n, so (u_n) is increasing and bounded above by v_0, and (v_n) is decreasing and bounded below by u_0. By the MCT, both converge: $u_n \to \ell_1$, $v_n \to \ell_2$. Then $\ell_2 - \ell_1 = \lim(v_n - u_n) = 0$, so $\ell_1 = \ell_2 = \ell$. Finally, $u_n \leq \ell$ (since (u_n) is increasing with limit ℓ) and $v_n \geq \ell$ (since (v_n) is decreasing with limit ℓ). $\qquad\square$

3.13 Additional visualizations

Monotone bounded   **Cauchy sequence**
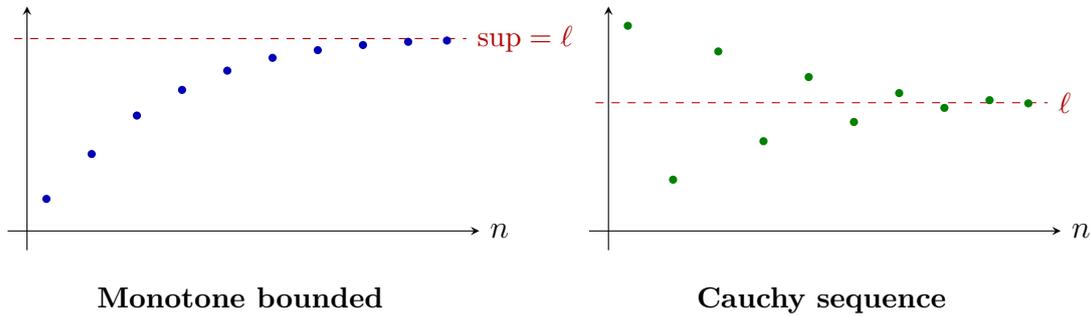
3.14 Common errors

Common Errors — Sequences

1. **Confusing bounded and convergent.** Bounded $\not\Longrightarrow$ convergent (e.g. $(-1)^n$). Convergent $\Longrightarrow$ bounded, but not conversely.

2. **Choosing ε in a proof.** In an ε–N proof, you do *not* pick a specific value of ε. You write "Let $\varepsilon > 0$" and find N that works for *that particular* (but arbitrary) ε.

3. N **must not depend on n.** The rank N depends on ε but not on n. Writing "take $N = n + \ldots$" is invalid.

4. **Assuming the limit exists.** To prove convergence, you cannot start by assuming $\lim u_n = \ell$ and then compute ℓ (circular reasoning). Use monotonicity + boundedness or the Cauchy criterion first.

5. **Divergent $\neq$ tends to $+\infty$.** The sequence $(-1)^n$ diverges but does *not* tend to $\pm\infty$.

6. **Forgetting that a Cauchy sequence argument requires $\mathbb{R}$.** The implication Cauchy $\Longrightarrow$ convergent uses the completeness of $\mathbb{R}$. It is false in $\mathbb{Q}$.

3.15 Exercises

Exercise 3.1 (★). Prove from the ε–N definition that $\displaystyle\lim_{n\to\infty} \frac{2n+1}{n+3} = 2$.

Exercise 3.2 (★). Prove that $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$.

Exercise 3.3 (★). Let $u_n = \dfrac{3n + (-1)^n}{n}$. Prove that $u_n \to 3$.

Exercise 3.4 (★). Using the squeeze theorem, find $\displaystyle\lim_{n\to\infty} \frac{\cos(n^2)}{n+1}$.

Exercise 3.5 (★★). Let $u_n = \left(1 + \dfrac{1}{n}\right)^n$. Show that (u_n) is increasing and bounded above by 3. *(Hint: use the binomial theorem and compare term by term.)*

Exercise 3.6 (★★). Let $u_0 = 2$ and $u_{n+1} = \dfrac{1}{2}\left(u_n + \dfrac{3}{u_n}\right)$. Show that $(u_n)_{n \geq 1}$ is decreasing, bounded below by $\sqrt{3}$, and find its limit.

Exercise 3.7 (★★). Let $u_n = \sum_{k=0}^{n} \dfrac{1}{k!}$. Show that (u_n) is increasing and bounded above. *(Hint: compare $k!$ with 2^{k-1} for $k \geq 1$.)*

Exercise 3.8 (★★). Let (u_n) be a Cauchy sequence and let (u_{n_k}) be a subsequence converging to ℓ. Prove directly (without using Theorem 3.8) that $u_n \to \ell$.

Exercise 3.9 (★★). Let (u_n) and (v_n) be adjacent sequences with $u_0 = 0$, $v_0 = 1$, $u_{n+1} = \dfrac{u_n + v_n}{2}$, $v_{n+1} = \dfrac{u_{n+1} + v_n}{2}$. Show that (u_n) and (v_n) are adjacent and find their common limit.

Exercise 3.10 (★★). Prove that the sequence $u_n = \sum_{k=1}^{n} \dfrac{1}{k} - \ln n$ is decreasing and bounded below by 0. *(This limit is the Euler–Mascheroni constant $\gamma \approx 0.5772$.)*

Exercise 3.11 (★★★). Let $u_0 \in (0, 1)$ and $u_{n+1} = u_n(1 - u_n)$. Show that $u_n \to 0$. Then show that $n\, u_n \to 1$.

Exercise 3.12 (★★★). *(Stolz–Cesàro lemma.)* Let (b_n) be a strictly increasing sequence with $b_n \to +\infty$, and let (a_n) be a sequence such that $\dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} \to \ell \in \mathbb{R}$. Prove that $\dfrac{a_n}{b_n} \to \ell$.

Exercise 3.13 (★★★). Prove that every sequence in $\mathbb{R}$ has a monotone subsequence. *(Hint: consider "peak" indices n where $u_n \geq u_m$ for all $m \geq n$.)*

Chapter summary

> **Chapter 3 — Summary**
>
> - A sequence converges to ℓ if: $\forall \varepsilon > 0,\ \exists N,\ \forall n \geq N,\ |u_n - \ell| < \varepsilon$.
>
> - The limit, when it exists, is unique.
>
> - Convergent $\implies$ bounded (converse is false).
>
> - Squeeze theorem: $u_n \leq v_n \leq w_n$ with $u_n, w_n \to \ell \implies v_n \to \ell$.
>
> - Algebra of limits: sums, products, quotients of convergent sequences behave as expected.
>
> - Monotone convergence theorem: increasing + bounded above $\implies$ converges to sup.
>
> - Bolzano–Weierstrass: bounded $\implies$ has a convergent subsequence.
>
> - Cauchy criterion: (u_n) converges $\iff$ (u_n) is Cauchy (in $\mathbb{R}$).
>
> - Recursive sequences: prove monotonicity + bounds, then pass to the limit.
>
> - Adjacent sequences converge to a common limit.

Chapter 4

Numerical Series

Motivation: adding infinitely many numbers

Can we make sense of an infinite sum such as $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$? What about $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$?

Intuitively, the first sum "should" equal 2, while the second grows without bound. To make these ideas rigorous, we reduce infinite sums to the theory of sequences developed in Chapter 3: the infinite sum is the *limit of partial sums*.

4.1   Definitions

> **Definition 4.1** (Series, partial sums). Let $(a_n)_{n \geq 0}$ be a sequence of real numbers. The **series** $\sum a_n$ (or $\sum_{n=0}^{\infty} a_n$) is the sequence of **partial sums**
>
> $$S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n.$$
>
> The series **converges** if the sequence (S_n) converges, and in that case we write
>
> $$\sum_{n=0}^{\infty} a_n = \lim_{n \to \infty} S_n.$$
>
> If (S_n) diverges, the series **diverges**. The number a_n is called the **general term** of the series.

> **Definition 4.2** (Remainder). If $\sum a_n$ converges with sum S, the n**-th remainder** is
>
> $$R_n = S - S_n = \sum_{k=n+1}^{\infty} a_k.$$
>
> Note that $R_n \to 0$ as $n \to \infty$.

4.2 Necessary condition for convergence

> **Theorem 4.1** (Necessary condition). If the series $\sum a_n$ converges, then $a_n \to 0$.

Proof. If $\sum a_n$ converges, then (S_n) converges to some $S \in \mathbb{R}$. Since $a_n = S_n - S_{n-1}$ for $n \geq 1$, and both $S_n \to S$ and $S_{n-1} \to S$, the algebra of limits gives

$$a_n = S_n - S_{n-1} \to S - S = 0. \qquad \square$$

> **WARNING: the converse is false!**
>
> The condition $a_n \to 0$ is **necessary but not sufficient**. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ has $\frac{1}{n} \to 0$, yet it **diverges**.

> **Proposition 4.1** (Divergence of the harmonic series). The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. We show that the partial sums are unbounded. Group the terms:

$$
\begin{aligned}
S_1 &= 1, \\
S_2 &= 1 + \tfrac{1}{2}, \\
S_4 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) \geq 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{2}, \\
S_8 &= S_4 + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \geq S_4 + 4 \cdot \tfrac{1}{8} = S_4 + \tfrac{1}{2}.
\end{aligned}
$$

In general, $S_{2^k} \geq 1 + \frac{k}{2}$, which tends to $+\infty$. Hence (S_n) is unbounded, so the harmonic series diverges. $\qquad \square$

4.3 Geometric series

> **Theorem 4.2** (Geometric series). Let $a \in \mathbb{R}$ with $a \neq 0$ and $r \in \mathbb{R}$. The geometric series $\sum_{n=0}^{\infty} a\, r^n$ converges if and only if $|r| < 1$, and in that case
>
> $$\sum_{n=0}^{\infty} a\, r^n = \frac{a}{1 - r}.$$

Proof. The partial sum is

$$S_n = a \sum_{k=0}^{n} r^k = a \cdot \frac{1 - r^{n+1}}{1 - r} \quad \text{for } r \neq 1.$$

(This identity is proved by multiplying both sides by $1 - r$.)

- If $|r| < 1$, then $r^{n+1} \to 0$ (one can show $|r|^n \to 0$ using, e.g., $|r|^n \leq \frac{1}{(1+\delta)^n} \to 0$ where $|r| = \frac{1}{1+\delta}$ with $\delta > 0$). Hence $S_n \to \frac{a}{1-r}$.

- If $|r| \geq 1$, then $|a\,r^n| = |a|\,|r|^n \not\to 0$, so the necessary condition (Theorem 4.1) fails, and the series diverges.

- If $r = 1$, then $S_n = a(n+1) \to \pm\infty$. □

Example 4.1.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2.$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} = \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}.$

4.4 Telescoping series

Proposition 4.2. Let (b_n) be a sequence. The series $\displaystyle\sum_{n=0}^{\infty}(b_n - b_{n+1})$ converges if and only if (b_n) converges, and in that case

$$\sum_{n=0}^{\infty}(b_n - b_{n+1}) = b_0 - \lim_{n\to\infty} b_n.$$

Proof. The partial sum telescopes: $S_n = \sum_{k=0}^{n}(b_k - b_{k+1}) = b_0 - b_{n+1}$. Hence S_n converges iff b_{n+1} converges, and the formula follows. □

Example 4.2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \lim_{n\to\infty}\frac{1}{n+1} = 1.$
(Use partial fractions: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.)

Example 4.3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right).$
The partial sum is $\frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) \to \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$

4.5 Series with non-negative terms

For a series $\sum a_n$ with $a_n \geq 0$, the partial sums (S_n) form an increasing sequence. Therefore, by the MCT, $\sum a_n$ **converges if and only if (S_n) is bounded above**.

4.5.1 Comparison test

Theorem 4.3 (Comparison test). Let $\sum a_n$ and $\sum b_n$ be series with $0 \leq a_n \leq b_n$ for all n (or at least for all n large enough). Then:

1. If $\sum b_n$ converges, then $\sum a_n$ converges.

2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof. Let $S_n = \sum_{k=0}^{n} a_k$ and $T_n = \sum_{k=0}^{n} b_k$. Since $a_k \leq b_k$, we have $S_n \leq T_n$ for all n.

(1) If $\sum b_n$ converges, then (T_n) is bounded above by $T = \sum_{k=0}^{\infty} b_k$. Hence $S_n \leq T_n \leq T$ for all n, so (S_n) is increasing and bounded above, hence converges by the MCT.

(2) is the contrapositive of (1). $\qquad\square$

4.5.2 Ratio test (d'Alembert)

> **Theorem 4.4** (d'Alembert's ratio test)**.** Let $\sum a_n$ be a series with $a_n > 0$ for all n. Suppose
> $$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$
> exists (possibly $+\infty$). Then:
>
> 1. If $L < 1$, the series converges.
>
> 2. If $L > 1$ (including $L = +\infty$), the series diverges.
>
> 3. If $L = 1$, the test is inconclusive.

Proof. Case $L < 1$. Choose r with $L < r < 1$. Since $\frac{a_{n+1}}{a_n} \to L < r$, there exists N such that $\frac{a_{n+1}}{a_n} \leq r$ for all $n \geq N$. Then, by induction,

$$a_{N+k} \leq a_N \cdot r^k \quad \text{for all } k \geq 0.$$

Since $\sum_{k=0}^{\infty} a_N r^k = \frac{a_N}{1-r} < \infty$ (geometric series with ratio $r < 1$), the comparison test gives the convergence of $\sum_{n=N}^{\infty} a_n$, hence of $\sum_{n=0}^{\infty} a_n$.

Case $L > 1$. There exists N with $\frac{a_{n+1}}{a_n} > 1$ for $n \geq N$, so $(a_n)_{n \geq N}$ is increasing and $a_n \geq a_N > 0$ for $n \geq N$. Hence $a_n \not\to 0$, so the series diverges by the necessary condition. $\qquad\square$

Remark 4.1. When $L = 1$: $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges, yet both have ratio limit 1. So the test tells us nothing.

4.5.3 Root test (Cauchy)

> **Theorem 4.5** (Cauchy's root test)**.** Let $\sum a_n$ be a series with $a_n \geq 0$. Suppose
> $$L = \lim_{n \to \infty} \sqrt[n]{a_n}$$
> exists (possibly $+\infty$). Then:
>
> 1. If $L < 1$, the series converges.
>
> 2. If $L > 1$, the series diverges.
>
> 3. If $L = 1$, the test is inconclusive.

Proof. Case $L < 1$. Choose r with $L < r < 1$. There exists N with $\sqrt[n]{a_n} \leq r$ for all $n \geq N$, i.e. $a_n \leq r^n$. Since $\sum r^n$ converges (geometric with ratio $r < 1$), the comparison test gives convergence.

Case $L > 1$. There exists N with $\sqrt[n]{a_n} > 1$ for $n \geq N$, i.e. $a_n > 1$ for $n \geq N$. Hence $a_n \not\to 0$, and the series diverges. $\qquad\square$

4.5.4 Integral test

> **Theorem 4.6** (Integral test). Let $f \colon [1, +\infty) \to [0, +\infty)$ be a continuous, decreasing function. Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

The proof uses the comparison between the sum and upper/lower Riemann sums; we omit it here and refer to the chapter on integration.

Example 4.4 (*p*-series / Riemann series). For $\alpha \in \mathbb{R}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ converges if and only if $\alpha > 1$.

Apply the integral test with $f(x) = x^{-\alpha}$:

$$\int_1^N \frac{dx}{x^{\alpha}} = \begin{cases} \dfrac{N^{1-\alpha} - 1}{1 - \alpha} & \text{if } \alpha \neq 1, \\ \ln N & \text{if } \alpha = 1. \end{cases}$$

As $N \to \infty$: if $\alpha > 1$, the integral converges (to $\frac{1}{\alpha-1}$); if $\alpha \leq 1$, it diverges.

4.6 Absolute and conditional convergence

> **Definition 4.3.** A series $\sum a_n$ is **absolutely convergent** if the series $\sum |a_n|$ converges. It is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

> **Theorem 4.7.** If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

Proof. We use the Cauchy criterion for series. Let $\varepsilon > 0$. Since $\sum |a_n|$ converges, the sequence of its partial sums is Cauchy: there exists N such that for all $q > p \geq N$,

$$\sum_{k=p+1}^{q} |a_k| < \varepsilon.$$

Then, by the triangle inequality,

$$\left| \sum_{k=p+1}^{q} a_k \right| \leq \sum_{k=p+1}^{q} |a_k| < \varepsilon.$$

Hence the partial sums of $\sum a_n$ are Cauchy, so $\sum a_n$ converges (by the Cauchy criterion for sequences in $\mathbb{R}$). $\qquad\square$

Remark 4.2. The converse is false: the alternating harmonic series is the standard counterexample (see below).

4.7 Alternating series: the Leibniz criterion

> **Definition 4.4.** An **alternating series** has the form $\sum_{n=0}^{\infty}(-1)^n b_n$ where $b_n \geq 0$ for all n.

> **Theorem 4.8** (Leibniz criterion / Alternating series test)**.** Let $(b_n)_{n\geq 0}$ be a sequence such that:
>
> 1. (b_n) is decreasing: $b_{n+1} \leq b_n$ for all n,
>
> 2. $b_n \to 0$.
>
> Then the series $\sum_{n=0}^{\infty}(-1)^n b_n$ converges. Moreover, if S denotes its sum, then
>
> $$|S - S_n| \leq b_{n+1} \quad \text{and} \quad S_{2n} \leq S \leq S_{2n+1} \text{ (or vice versa)}.$$

Proof. Consider the even and odd partial sums separately.

Even partial sums.
$$S_{2n+2} = S_{2n} + b_{2n+1} - b_{2n+2}.$$

Since $b_{2n+1} \geq b_{2n+2}$ (decreasing), we have $S_{2n+2} \geq S_{2n}$. Thus (S_{2n}) is increasing.

Odd partial sums.
$$S_{2n+1} = S_{2n-1} - b_{2n} + b_{2n+1}.$$

Since $b_{2n} \geq b_{2n+1}$, we have $S_{2n+1} \leq S_{2n-1}$. Thus (S_{2n+1}) is decreasing.

Comparison. $S_{2n+1} = S_{2n} + (-1)^{2n+1}b_{2n+1} = S_{2n} - b_{2n+1} \leq S_{2n}$? Wait — actually $S_{2n+1} = S_{2n} + (-1)^{2n+1}b_{2n+1}$. Since the sign convention is $(-1)^n b_n$, we have:

$$S_{2n+1} - S_{2n} = (-1)^{2n+1}b_{2n+1} = -b_{2n+1} \leq 0,$$

so $S_{2n} \geq S_{2n+1}$. But also

$$S_{2n} - S_{2n-1} = (-1)^{2n}b_{2n} = b_{2n} \geq 0,$$

so $S_{2n} \geq S_{2n-1}$.

Combining, for all n:

$$S_0 \geq S_1, \qquad S_{2n} \leq S_{2n+2}, \qquad S_{2n+1} \geq S_{2n+3}, \qquad S_{2n+1} \leq S_{2n}.$$

Actually, let us be more careful. We have:

- (S_{2n}) is increasing (shown above) and bounded above by $S_1 + b_1 = S_0 = b_0$. More precisely, $S_{2n} \leq S_0 = b_0$ since $S_{2n} = b_0 - (b_1 - b_2) - (b_3 - b_4) - \cdots - (b_{2n-1} - b_{2n}) \leq b_0$.

- (S_{2n+1}) is decreasing and bounded below by 0 since $S_{2n+1} = (b_0 - b_1) + (b_2 - b_3) + \cdots + (b_{2n} - b_{2n+1}) \geq 0$.

- $S_{2n} \geq S_{2n+1}$ for all n (since $S_{2n} - S_{2n+1} = b_{2n+1} \geq 0$).

Wait, let us recheck the sign: $S_{2n+1} = S_{2n} + (-1)^{2n+1}b_{2n+1} = S_{2n} - b_{2n+1}$, so indeed $S_{2n} - S_{2n+1} = b_{2n+1} \geq 0$.

By the monotone convergence theorem:

- (S_{2n}) converges to some ℓ_{even},

- (S_{2n+1}) converges to some ℓ_{odd}.

Since $S_{2n} - S_{2n+1} = b_{2n+1} \to 0$, we have $\ell_{\text{even}} = \ell_{\text{odd}} =: S$.

Since (S_{2n}) and (S_{2n+1}) are the two subsequences that together exhaust all terms, $S_n \to S$.

Error bound. Since $S_{2n} \leq S \leq S_{2n+1}$ (the even subsequence increases to S, the odd one decreases to S—actually, let us verify: (S_{2n}) increases to S, so $S_{2n} \leq S$; (S_{2n+1}) decreases to S, so $S_{2n+1} \geq S$. Hence $S_{2n} \leq S \leq S_{2n+1}$.)

For any n:
$$|S - S_n| \leq |S_{n+1} - S_n| = b_{n+1}. \qquad \square$$

Example 4.5 (Alternating harmonic series). The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges by the Leibniz criterion (with $b_n = \frac{1}{n}$, which is decreasing and tends to 0). Its sum is $\ln 2$.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so the series is **conditionally convergent**, not absolutely convergent.

4.8 Power series

> **Definition 4.5** (Power series). A **power series** centered at 0 is a series of the form
>
> $$\sum_{n=0}^{\infty} a_n x^n,$$
>
> where $(a_n)_{n \geq 0}$ is a sequence of real numbers (the **coefficients**) and x is a real variable.
>
> More generally, a power series centered at $c \in \mathbb{R}$ is $\sum_{n=0}^{\infty} a_n (x - c)^n$.

> **Theorem 4.9** (Radius of convergence). For every power series $\sum a_n x^n$, there exists a unique $R \in [0, +\infty]$ (called the **radius of convergence**) such that:
>
> 1. The series converges absolutely for $|x| < R$.
>
> 2. The series diverges for $|x| > R$.
>
> 3. At $|x| = R$, anything can happen (convergence, divergence, or conditional convergence).
>
> The interval $(-R, R)$ is called the **interval of convergence** (behavior at the endpoints $\pm R$ must be checked separately).

(endpoints require separate analysis)
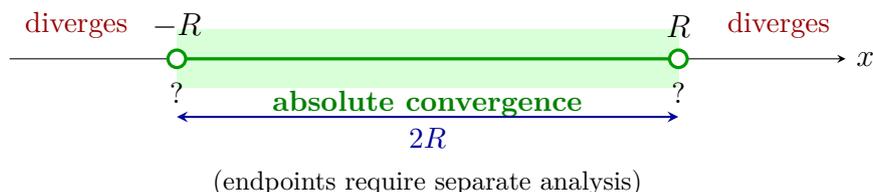
Theorem 4.10 (Hadamard formula)**.** The radius of convergence of $\sum a_n x^n$ is

$$R = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{|a_n|}},$$

with the conventions $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$.

Remark 4.3. In practice, one often computes R via the ratio test: $R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$ when this limit exists.

4.8.1   Classical power series

Example 4.6 (Exponential)*.*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad R = +\infty.$$

Ratio: $\frac{a_{n+1}}{a_n} \cdot |x| = \frac{|x|}{n+1} \to 0 < 1$ for all $x \in \mathbb{R}$.

Example 4.7 (Sine and cosine)*.*

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n}}{(2n)!}, \quad R = +\infty.$$

Example 4.8 (Logarithm)*.*

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \, x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad R = 1.$$

Converges for $x \in (-1, 1]$ (converges at $x = 1$ by Leibniz; diverges at $x = -1$).

Example 4.9 (Binomial series)*.* For $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \text{where} \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad R = 1.$$

4.9   Cauchy product

Theorem 4.11 (Cauchy product / Mertens' theorem)**.** If $\sum a_n$ and $\sum b_n$ are absolutely convergent series with sums A and B respectively, then the **Cauchy product**

$$c_n = \sum_{k=0}^{n} a_k \, b_{n-k}$$

defines an absolutely convergent series $\sum c_n$ with $\sum_{n=0}^{\infty} c_n = A \cdot B$.

Example 4.10 (Product $e^x \cdot e^y = e^{x+y}$). Let $A = \sum \frac{x^n}{n!}$ and $B = \sum \frac{y^n}{n!}$. The Cauchy product has general term

$$c_n = \sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \frac{(x+y)^n}{n!}.$$

Hence $e^x \cdot e^y = \sum \frac{(x+y)^n}{n!} = e^{x+y}$.

4.10 Visualizations
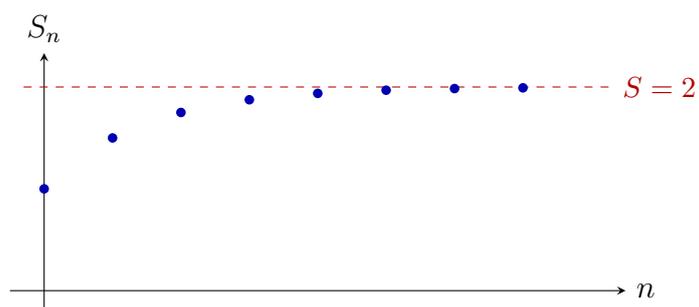
Partial sums of $\sum \frac{1}{2^n}$
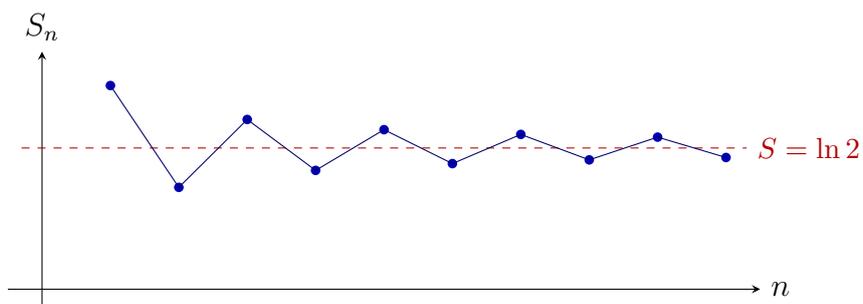
Partial sums of $\sum (-1)^{n+1}/n$

4.11 Common errors

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Common Errors — Series

1. **"$a_n \to 0$ so the series converges."** NO! This is a necessary condition, not sufficient. The harmonic series is the classic counterexample.

2. **Confusing the series $\sum a_n$ with the sequence (a_n).** A series is the *sequence of partial sums*, not the sequence of terms. When we say "the series converges," we mean (S_n) converges.

3. **Applying the ratio/root test when $L = 1$.** When $L = 1$, the test gives **no conclusion**. You must use a different method.

4. **Forgetting to check endpoints of power series.** The radius of convergence tells you about the open interval $(-R, R)$. At $x = \pm R$, you must analyze convergence separately.

5. **Conditionally convergent $\neq$ divergent.** A conditionally convergent series *does* converge; it is just not absolutely convergent. (Riemann's rearrangement theorem shows its terms can be rearranged to give any sum.)

6. **Comparison test requires non-negative terms (or absolute values).** Make sure $a_n \geq 0$ and $b_n \geq 0$ before applying the comparison test.

</div>

4.12 Exercises

Exercise 4.1 (★). Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ converges.

Exercise 4.2 (★). Compute $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4^n}$.

Exercise 4.3 (★). Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$. *(Hint: partial fractions and telescoping.)*

Exercise 4.4 (★). Using the ratio test, determine convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$.

Exercise 4.5 (★). Using the root test, determine convergence of $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$.

Exercise 4.6 (★★). Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. *(Hint: compare with $\sum \frac{1}{n(n-1)}$ for $n \geq$ 2.)*

Exercise 4.7 (★★). Study the convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

Exercise 4.8 (★★). Show that $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. *(Hint: relate to the power series of* arctan x *evaluated at $x = 1$.)*

Exercise 4.9 (★★). Find the radius of convergence and determine convergence at the endpoints for $\sum_{n=1}^{\infty} \dfrac{x^n}{n \cdot 2^n}$.

Exercise 4.10 (★★). Prove that if $\sum a_n$ converges absolutely and (b_n) is bounded, then $\sum a_n b_n$ converges absolutely.

Exercise 4.11 (★★★). (*Abel's summation.*) Let (a_n) be a decreasing sequence with $a_n \to 0$, and let (b_n) be a sequence whose partial sums $B_n = \sum_{k=0}^{n} b_k$ are bounded. Prove that $\sum a_n b_n$ converges. (*Hint: Abel summation by parts:* $\sum_{k=p}^{q} a_k b_k = a_q B_q - a_p B_{p-1} - \sum_{k=p}^{q-1}(a_{k+1} - a_k)B_k$.)

Exercise 4.12 (★★★). Find the radius of convergence of $\sum_{n=0}^{\infty} \dfrac{(2n)!}{(n!)^2} x^n$. (*Hint: use the ratio test and Stirling's approximation, or compute* a_{n+1}/a_n *directly.*)

Exercise 4.13 (★★★). Prove the **condensation test**: if (a_n) is a decreasing sequence with $a_n \geq 0$, then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. Use this to give another proof that $\sum 1/n^\alpha$ converges iff $\alpha > 1$.

Chapter summary

Chapter 4 — Summary

- A series $\sum a_n$ converges iff its partial sums (S_n) converge.

- Necessary condition: $a_n \to 0$ (not sufficient—harmonic series!).

- Geometric series $\sum r^n$: converges iff $|r| < 1$, sum $= \frac{1}{1-r}$.

- Telescoping series: $\sum(b_n - b_{n+1}) = b_0 - \lim b_n$.

- For series with non-negative terms:

    - Comparison: $0 \leq a_n \leq b_n$ and $\sum b_n$ conv. $\implies$ $\sum a_n$ conv.
    - Ratio test: $\frac{a_{n+1}}{a_n} \to L$; $L < 1 \implies$ conv., $L > 1 \implies$ div.
    - Root test: $\sqrt[n]{a_n} \to L$; $L < 1 \implies$ conv., $L > 1 \implies$ div.
    - Integral test: compare with $\int f$.

- Absolutely convergent $\implies$ convergent (converse false).

- Leibniz: alternating series with $b_n \searrow 0$ converges; error $\leq b_{n+1}$.

- Power series $\sum a_n x^n$: radius R (Hadamard), converges absolutely on $(-R, R)$.

- Cauchy product: if $\sum a_n, \sum b_n$ abs. conv., then $(\sum a_n)(\sum b_n) = \sum c_n$ with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Chapter 5

Limits and Continuity of Functions

THIS CHAPTER develops the theory of limits and continuity for real-valued functions of a real variable. These ideas sit at the very heart of analysis: differentiation, integration, and the study of series all depend on a precise understanding of what it means for a function to "approach a value" or to "have no jumps." We give every ε–δ argument in complete detail, including the scratch work that motivates the choice of δ.

5.1 Limits of Functions

5.1.1 The ε–δ Definition

Let us fix the setting. We consider a function $f\colon D \to \mathbb{R}$, where $D \subset \mathbb{R}$, and a point $a \in \mathbb{R}$ that is a *limit point* (accumulation point) of D: every open interval containing a meets $D \setminus \{a\}$.

Definition 5.1 (Limit of a function)**.** Let $f\colon D \to \mathbb{R}$ and let a be a limit point of D. We say that

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x \in D, \quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Reading the definition

In words: no matter how small the tolerance $\varepsilon > 0$ is, we can find a "response" $\delta > 0$ so that every x in the domain within distance δ of a (but $x \neq a$) has its image $f(x)$ within distance ε of L.

Key points:
1. The value $f(a)$, if it exists, plays *no role*; we require $0 < |x - a|$.
2. δ is allowed to depend on ε (and on a and f).
3. The quantifier order matters: $\forall\, \varepsilon\ \exists\, \delta\ \forall\, x$.
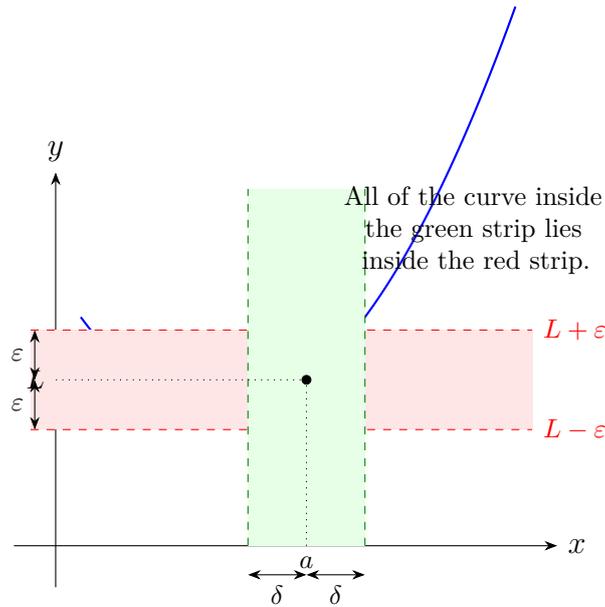
Figure 5.1: The ε–δ definition of $\lim_{x \to a} f(x) = L$.

5.1.2 Sequential Characterization (Heine's Criterion)

The following theorem connects the limit of a function with sequential limits, which are often easier to work with.

> **Theorem 5.1** (Heine's criterion)**.** Let $f : D \to \mathbb{R}$ and let a be a limit point of D. Then
> $$\lim_{x \to a} f(x) = L$$
> if and only if for every sequence (x_n) in $D \setminus \{a\}$ with $x_n \to a$ we have $f(x_n) \to L$.

Proof. We prove both directions in full detail.

($\Rightarrow$) Assume $\lim_{x \to a} f(x) = L$. Let (x_n) be any sequence in $D \setminus \{a\}$ with $x_n \to a$. We must show $f(x_n) \to L$.

Fix $\varepsilon > 0$. By the limit hypothesis, there exists $\delta > 0$ such that
$$\forall x \in D, \quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Since $x_n \to a$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - a| < \delta$. Because $x_n \neq a$ for every n (they belong to $D \setminus \{a\}$), we have $0 < |x_n - a| < \delta$ for $n \geq N$, and therefore $|f(x_n) - L| < \varepsilon$ for $n \geq N$. This proves $f(x_n) \to L$.

($\Leftarrow$) We prove the contrapositive. Assume that $\lim_{x \to a} f(x) \neq L$; we construct a sequence (x_n) in $D \setminus \{a\}$ with $x_n \to a$ but $f(x_n) \not\to L$.

The negation of the ε–δ definition gives:
$$\exists \varepsilon_0 > 0, \ \forall \delta > 0, \ \exists x \in D, \quad 0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \varepsilon_0.$$

Apply this with $\delta = \frac{1}{n}$ for each $n \geq 1$: we obtain $x_n \in D$ with $0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$. Then $x_n \in D \setminus \{a\}$, $x_n \to a$ (by the squeeze theorem, since $|x_n - a| < 1/n \to 0$), yet $f(x_n) \not\to L$ because $|f(x_n) - L| \geq \varepsilon_0$ for every n. $\qquad\square$

Remark 5.1. Heine's criterion is especially useful for *disproving* the existence of a limit: find two sequences converging to a whose images converge to different values.

5.1.3 Worked Example: $\lim_{x\to 2}(3x-1) = 5$

Example 5.1. **Claim.** $\lim_{x\to 2}(3x-1) = 5$.

Scratch work (not part of the proof). We need $|f(x) - L| < \varepsilon$, i.e. $|(3x-1) - 5| < \varepsilon$, i.e. $|3x - 6| < \varepsilon$, i.e. $3|x - 2| < \varepsilon$, i.e. $|x - 2| < \varepsilon/3$. So $\delta = \varepsilon/3$ should work.

Formal proof. Let $\varepsilon > 0$. Set $\delta = \varepsilon/3 > 0$. Then for every $x \in \mathbb{R}$ with $0 < |x - 2| < \delta$,

$$|(3x - 1) - 5| = |3x - 6| = 3|x - 2| < 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

By definition, $\lim_{x\to 2}(3x-1) = 5$. □

5.1.4 Worked Example: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Example 5.2. **Claim.** $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Geometric argument. For $0 < x < \pi/2$, consider the unit circle. The area of the inscribed triangle, the circular sector, and the circumscribed triangle give the chain of inequalities

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2}.$$

Dividing by $\frac{\sin x}{2} > 0$:

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}.$$

Taking reciprocals (all terms positive):

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

Since $\cos x \to 1$ as $x \to 0^+$, the squeeze theorem gives $\frac{\sin x}{x} \to 1$ as $x \to 0^+$.

For $x < 0$, set $u = -x > 0$; then $\frac{\sin x}{x} = \frac{\sin(-u)}{-u} = \frac{\sin u}{u} \to 1$.

Therefore $\lim_{x\to 0} \frac{\sin x}{x} = 1$. □

5.2 One-Sided Limits, Infinite Limits, Limits at Infinity

> **Definition 5.2** (One-sided limits).
>
> - **Right-hand limit:** $\lim_{x\to a^+} f(x) = L$ means
>
> $$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall x \in D, \quad a < x < a + \delta \implies |f(x) - L| < \varepsilon.$$
>
> - **Left-hand limit:** $\lim_{x\to a^-} f(x) = L$ means
>
> $$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall x \in D, \quad a - \delta < x < a \implies |f(x) - L| < \varepsilon.$$

Proposition 5.1. $\lim_{x \to a} f(x) = L$ if and only if both one-sided limits exist and equal L.

Proof. The forward direction is immediate (restrict the quantifier). For the converse, given $\varepsilon > 0$, let δ_1 work for the right-hand limit and δ_2 for the left-hand limit; take $\delta = \min(\delta_1, \delta_2)$. $\square$

Definition 5.3 (Infinite limits and limits at infinity)**.**

- $\lim_{x \to a} f(x) = +\infty$ means

$$\forall M > 0, \ \exists \delta > 0, \ \forall x \in D, \quad 0 < |x - a| < \delta \implies f(x) > M.$$

- $\lim_{x \to +\infty} f(x) = L$ means

$$\forall \varepsilon > 0, \ \exists A > 0, \ \forall x \in D, \quad x > A \implies |f(x) - L| < \varepsilon.$$

- $\lim_{x \to +\infty} f(x) = +\infty$ means

$$\forall M > 0, \ \exists A > 0, \ \forall x \in D, \quad x > A \implies f(x) > M.$$

Analogous definitions hold for $-\infty$.

Example 5.3. $\displaystyle \lim_{x \to 0^+} \frac{1}{x} = +\infty.$

Proof. Let $M > 0$. Set $\delta = 1/M$. If $0 < x < \delta$, then $1/x > 1/\delta = M$. $\square$

5.3 Continuity at a Point

Definition 5.4 (Continuity at a point)**.** Let $f \colon D \to \mathbb{R}$ and $a \in D$. We say f is **continuous at** a if

$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall x \in D, \quad |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Equivalently, if a is a limit point of D: $\lim_{x \to a} f(x) = f(a)$. If a is an isolated point of D, then f is automatically continuous at a.

Limit vs. continuity

Comparing Definition 5.1 and Definition 5.4:
- For the *limit*, we require $0 < |x - a|$ (we exclude a itself).
- For *continuity*, we do *not* exclude a, and the target is $f(a)$.
- Continuity at a *requires* $a \in D$ (so $f(a)$ exists).

> **Proposition 5.2** (Sequential characterization of continuity)**.** Let $f\colon D \to \mathbb{R}$ and $a \in D$. Then f is continuous at a if and only if for every sequence (x_n) in D with $x_n \to a$, we have $f(x_n) \to f(a)$.

Proof. The proof is essentially the same as for Theorem 5.1, but now we allow $x_n = a$ and compare with $f(a)$ instead of L. We omit the repetition and invite the reader to write the details as an exercise. $\qquad\square$

5.3.1   Examples of Continuous Functions

Example 5.4 (Polynomials are continuous). Every polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.

Proof sketch.   The constant function $x \mapsto c$ is continuous (take any δ), and $x \mapsto x$ is continuous (take $\delta = \varepsilon$). Continuity is preserved by sums and products (limit laws), so every polynomial is continuous.

Example 5.5 ($|x|$ is continuous). The function $f(x) = |x|$ is continuous on $\mathbb{R}$.

ε–δ **proof.**   Fix $a \in \mathbb{R}$ and $\varepsilon > 0$. Set $\delta = \varepsilon$. For $|x - a| < \delta$:

$$\big||x| - |a|\big| \;\le\; |x - a| < \delta = \varepsilon,$$

where the first inequality is the *reverse triangle inequality*. Hence f is continuous at a. $\square$

Example 5.6 (Dirichlet function — nowhere continuous). Define $\mathbf{1}_{\mathbb{Q}}\colon \mathbb{R} \to \mathbb{R}$ by

$$\mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This function is **nowhere continuous**.

Proof.   Fix any $a \in \mathbb{R}$. Take $\varepsilon_0 = \frac{1}{2}$. By the density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$, every interval $(a - \delta, a + \delta)$ contains both a rational r and an irrational s. Then $|\mathbf{1}_{\mathbb{Q}}(r) - \mathbf{1}_{\mathbb{Q}}(s)| = 1 \ge \varepsilon_0$. So either $|\mathbf{1}_{\mathbb{Q}}(r) - \mathbf{1}_{\mathbb{Q}}(a)|$ or $|\mathbf{1}_{\mathbb{Q}}(s) - \mathbf{1}_{\mathbb{Q}}(a)|$ is $\ge \frac{1}{2} = \varepsilon_0$. Hence f is not continuous at a. $\qquad\square$

Alternatively, using the sequential characterization: if a is rational, pick an irrational sequence $s_n \to a$; then $f(s_n) = 0 \not\to 1 = f(a)$. Similarly if a is irrational.

5.4   Algebra of Continuous Functions

> **Proposition 5.3.** If f, g are continuous at a, then so are $f + g$, $f \cdot g$, λf $(\lambda \in \mathbb{R})$, and f/g (provided $g(a) \neq 0$). The composition $g \circ f$ is continuous at a whenever f is continuous at a and g is continuous at $f(a)$.

Proof. Each statement reduces to the corresponding limit law via $\lim_{x \to a} f(x) = f(a)$. For the composition: let $\varepsilon > 0$. By continuity of g at $f(a)$, find $\eta > 0$ with $|y - f(a)| < \eta \Rightarrow |g(y) - g(f(a))| < \varepsilon$. By continuity of f at a, find $\delta > 0$ with $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \eta$. Then $|x - a| < \delta \Rightarrow |g(f(x)) - g(f(a))| < \varepsilon$. $\qquad\square$

5.5   The Intermediate Value Theorem

> **Theorem 5.2** (Intermediate Value Theorem (IVT)). Let $f \colon [a,b] \to \mathbb{R}$ be continuous with $f(a) < f(b)$ (the case $f(a) > f(b)$ is analogous). Then for every value γ with $f(a) < \gamma < f(b)$, there exists $c \in (a,b)$ with $f(c) = \gamma$.

Proof. Define

$$S = \{x \in [a,b] : f(x) < \gamma\}.$$

S is nonempty ($a \in S$) and bounded above (by b), so $c := \sup S$ exists. We show $f(c) = \gamma$.

Step 1: $f(c) \leq \gamma$. Since $c = \sup S$, for each $n \geq 1$ there exists $x_n \in S$ with $c - \frac{1}{n} < x_n \leq c$. Then $x_n \to c$ and $f(x_n) < \gamma$ for every n. By continuity at c, $f(c) = \lim f(x_n) \leq \gamma$.

Step 2: $f(c) \geq \gamma$. If $c = b$, then $f(c) = f(b) > \gamma$ and we are done. If $c < b$, then for every n large enough that $c + \frac{1}{n} \leq b$, the point $c + \frac{1}{n} \notin S$ (it exceeds the supremum), so $f(c + \frac{1}{n}) \geq \gamma$. Letting $n \to \infty$, by continuity, $f(c) \geq \gamma$.

Steps 1 and 2 together give $f(c) = \gamma$. Moreover $c \neq a$ (because $f(a) < \gamma$ and continuity would force nearby values $< \gamma$, giving $c > a$) and similarly $c \neq b$, so $c \in (a,b)$. $\square$
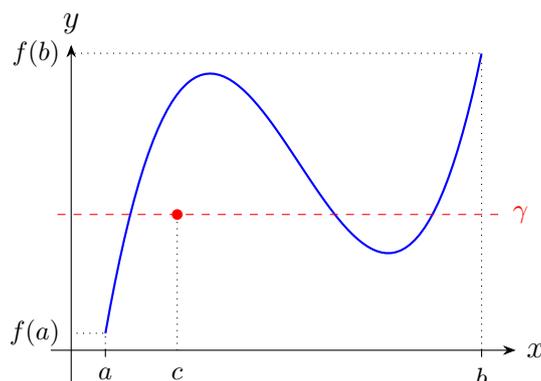
Figure 5.2: The Intermediate Value Theorem: the graph must cross the horizontal line $y = \gamma$.

> **Corollary 5.1** (Existence of nth roots). For every $y > 0$ and every $n \in \mathbb{N}_{\geq 1}$, there exists a unique $x > 0$ with $x^n = y$.

Proof. The function $f(x) = x^n$ is continuous on $[0, \infty)$ with $f(0) = 0$ and $f(x) \to +\infty$. Choosing b large enough so that $b^n > y$, the IVT on $[0,b]$ gives c with $c^n = y$. Uniqueness follows because f is strictly increasing on $[0, \infty)$. $\square$

> **Corollary 5.2** (Odd-degree polynomials have a real root). Every polynomial of odd degree with real coefficients has at least one real root.

Proof. If $p(x) = x^{2k+1} + $ lower-order terms, then $p(x) \to +\infty$ as $x \to +\infty$ and $p(x) \to -\infty$ as $x \to -\infty$. So there exist $a < b$ with $p(a) < 0 < p(b)$. Apply the IVT. $\square$

5.6 The Extreme Value Theorem

> **Theorem 5.3** (Extreme Value Theorem). Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is bounded and attains its maximum and minimum: there exist $x_m, x_M \in [a,b]$ with
> $$f(x_m) \leq f(x) \leq f(x_M) \quad \text{for all } x \in [a,b].$$

Proof. We prove that f attains its supremum (the infimum is analogous).

Step 1: f is bounded above. Suppose not. Then for each $n \geq 1$, there exists $x_n \in [a,b]$ with $f(x_n) > n$. By the Bolzano–Weierstrass theorem, (x_n) has a convergent subsequence $x_{n_k} \to c \in [a,b]$. By continuity, $f(x_{n_k}) \to f(c)$, contradicting $f(x_{n_k}) > n_k \to \infty$.

Step 2: The supremum is attained. Let $M = \sup_{x \in [a,b]} f(x) < \infty$ (Step 1). By definition of supremum, for each $n \geq 1$ there exists $x_n \in [a,b]$ with $M - \frac{1}{n} < f(x_n) \leq M$. Again by Bolzano–Weierstrass, extract a convergent subsequence $x_{n_k} \to x_M \in [a,b]$. By continuity, $f(x_{n_k}) \to f(x_M)$. Since $M - \frac{1}{n_k} < f(x_{n_k}) \leq M$ and both sides converge to M, the squeeze theorem gives $f(x_M) = M$. $\square$

5.7 Uniform Continuity

> **Definition 5.5** (Uniform continuity). A function $f\colon D \to \mathbb{R}$ is **uniformly continuous** on D if
> $$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x, y \in D, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

> **Uniform vs. pointwise continuity**
>
> - **Pointwise:** $\forall\, a \in D,\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x \in D,\ \dots$ (δ may depend on a).
> - **Uniform:** $\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x, y \in D,\ \dots$ (*one δ works for *all* pairs*).
>
> Uniform $\Rightarrow$ pointwise (set $y = a$). The converse is false.
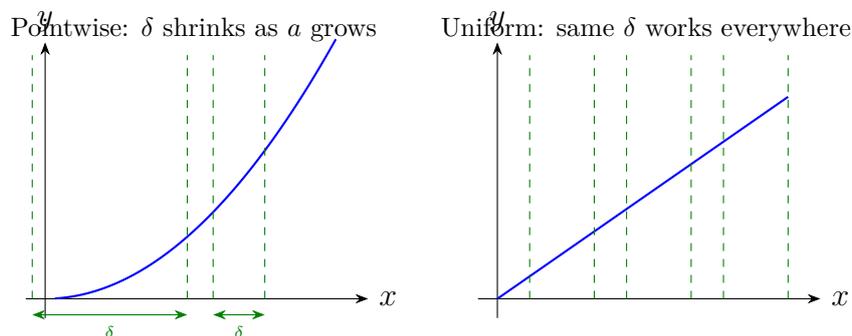
Figure 5.3: Pointwise continuity (left) vs. uniform continuity (right).

Example 5.7 (x^2 is not uniformly continuous on $\mathbb{R}$). Take $\varepsilon_0 = 1$. For any $\delta > 0$, choose $x = 1/\delta$ and $y = x + \delta/2$. Then $|x - y| = \delta/2 < \delta$ but

$$|x^2 - y^2| = |x - y| \cdot |x + y| = \frac{\delta}{2}\left(\frac{2}{\delta} + \frac{\delta}{2}\right) = 1 + \frac{\delta^2}{4} > 1 = \varepsilon_0.$$

So $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

> **Theorem 5.4** (Heine's theorem)**.** If $f\colon [a,b] \to \mathbb{R}$ is continuous, then f is uniformly continuous on $[a,b]$.

Proof. Suppose for contradiction that f is *not* uniformly continuous. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exist $x, y \in [a,b]$ with $|x-y| < \delta$ but $|f(x)-f(y)| \geq \varepsilon_0$.

Applying this with $\delta = 1/n$, we get sequences $(x_n), (y_n)$ in $[a,b]$ with $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

By Bolzano–Weierstrass, (x_n) has a subsequence $x_{n_k} \to c \in [a,b]$. Since $|y_{n_k} - x_{n_k}| < 1/n_k \to 0$, we also have $y_{n_k} \to c$. By continuity at c:

$$f(x_{n_k}) \to f(c) \quad \text{and} \quad f(y_{n_k}) \to f(c).$$

Hence $|f(x_{n_k}) - f(y_{n_k})| \to 0$, contradicting $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0 > 0$. $\qquad\square$

> **Definition 5.6** (Lipschitz continuity)**.** $f\colon D \to \mathbb{R}$ is **Lipschitz** (with constant K) if
> $$\forall\, x, y \in D, \quad |f(x) - f(y)| \leq K|x - y|.$$

> **Proposition 5.4.** Lipschitz $\Longrightarrow$ uniformly continuous.

Proof. Given $\varepsilon > 0$, take $\delta = \varepsilon/K$. Then $|x - y| < \delta$ implies $|f(x) - f(y)| \leq K|x - y| < K\delta = \varepsilon$. $\qquad\square$

Example 5.8. $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but *not* Lipschitz on $[0, \infty)$ (the derivative $1/(2\sqrt{x}) \to \infty$ as $x \to 0^+$). This shows the converse of Proposition 5.4 is false.

5.8   Common Errors

> **Frequent mistakes in Chapter 5**
>
> 1. **Forgetting $0 < |x - a|$ in the limit definition.** The limit is about behaviour *near* a, not *at* a.
> 2. **Choosing δ that depends on x.** In ε–δ proofs, δ is chosen *before* x is specified.
> 3. **Confusing pointwise and uniform continuity.** Check the quantifier order!
> 4. **Applying the IVT to a function that is not continuous.** The step function $\lfloor x \rfloor$ has range $\mathbb{Z}$ despite $f(0) = 0$, $f(3/2) = 1$; it skips non-integer values because it is discontinuous.
> 5. **Applying the Extreme Value Theorem on an open interval.** $f(x) = 1/x$ on $(0, 1)$ is continuous but unbounded.
> 6. **Writing "f is continuous so f is uniformly continuous" without specifying a closed bounded interval.** Heine's theorem requires $[a, b]$.

5.9   Exercises

Exercise 5.1. Prove from the ε–δ definition that $\lim_{x\to 3}(x^2) = 9$.
  Hint: factor $|x^2 - 9| = |x - 3| \cdot |x + 3|$ and bound $|x + 3|$ by restricting $\delta \leq 1$.

Exercise 5.2. Prove that $\lim_{x\to 1} \dfrac{1}{x} = 1$ using ε–δ.

Exercise 5.3. Use Heine's criterion to show that $\lim_{x\to 0} \sin(1/x)$ does not exist.

Exercise 5.4. Let $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is continuous at 0.

Exercise 5.5. Prove that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$ using ε–δ.
  Hint: $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}$.

Exercise 5.6. Show that if f is continuous on $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$, then there exists $m > 0$ with $f(x) \geq m$ for all x.

Exercise 5.7. Let $f \colon \mathbb{R} \to \mathbb{R}$ be continuous and suppose $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. Prove that f is bounded and attains its supremum or infimum.

Exercise 5.8. Show that $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.
  Hint: $|\sin x - \sin y| = 2|\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})| \leq 2 \cdot |\frac{x-y}{2}| = |x - y|$.

Exercise 5.9. Prove that if f is uniformly continuous on $\mathbb{R}$ and (x_n) is Cauchy, then $(f(x_n))$ is Cauchy. Show by example this can fail if f is only (pointwise) continuous.

Exercise 5.10. **(Fixed-point theorem).** Let $f \colon [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point, i.e., there exists $c \in [0, 1]$ with $f(c) = c$.
  Hint: consider $g(x) = f(x) - x$ and apply the IVT.

Exercise 5.11. Let $f \colon [a, b] \to \mathbb{R}$ be continuous and injective. Show that f is strictly monotone.

Exercise 5.12. **(Thomae's function).** Define $t \colon [0, 1] \to \mathbb{R}$ by $t(p/q) = 1/q$ (in lowest terms, $q \geq 1$) for rationals, and $t(x) = 0$ for irrationals. Prove that t is continuous at every irrational and discontinuous at every rational.

Chapter Summary

Key results of Chapter 5

- Limit of a function: ε–δ definition and sequential characterization (Heine).
- Continuity: ε–δ formulation; algebraic operations preserve continuity.
- **Intermediate Value Theorem** (IVT): continuous image of an interval is an interval.
- **Extreme Value Theorem**: continuous on $[a, b]$ $\Rightarrow$ bounded and attains bounds.
- **Heine's theorem**: continuous on $[a, b]$ $\Rightarrow$ uniformly continuous.
- Lipschitz $\Rightarrow$ uniformly continuous $\Rightarrow$ continuous (none reversed in general).

Chapter 6

Differentiation

D IFFERENTIATION makes precise the intuitive notion of "instantaneous rate of change." In this chapter we develop the theory rigorously from the limit definition, prove all the standard rules, and establish the powerful Mean Value Theorem and its consequences. We finish with Taylor's formula, which gives polynomial approximations of arbitrary order.

6.1 The Derivative

Definition of the derivative

Let $f\colon I \to \mathbb{R}$ where I is an open interval, and let $a \in I$. The **derivative of f at a** is
$$f'(a) \;=\; \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \;=\; \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$
provided this limit exists (and is finite). If $f'(a)$ exists, we say f is **differentiable at a**.

Theorem 6.1 (Differentiable $\Rightarrow$ continuous)**.** If f is differentiable at a, then f is continuous at a.

Proof. Write, for $x \neq a$,

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a).$$

As $x \to a$, the first factor tends to $f'(a)$ (finite) and the second to 0. By the product rule for limits,
$$\lim_{x\to a} \big(f(x) - f(a) \big) = f'(a) \cdot 0 = 0,$$

so $f(x) \to f(a)$, i.e. f is continuous at a. $\qquad\square$

Remark 6.1. The converse is false. The function $f(x) = |x|$ is continuous at 0 but not differentiable there, as we show next.

6.1.1 Counterexamples

Example 6.1 (|x| at 0). We have

$$\frac{|h| - |0|}{h} = \frac{|h|}{h} = \begin{cases} 1 & h > 0, \\ -1 & h < 0. \end{cases}$$

The right-hand limit is 1, the left-hand limit is -1; they differ, so $f'(0)$ does not exist.

y

$y = |x|$

slope $= -1$ slope $= +1$

x

Corner: not differentiable
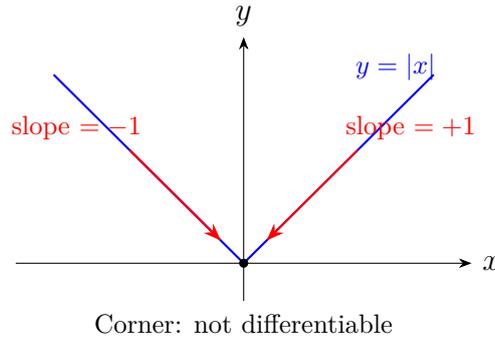
Figure 6.1: $|x|$ has a corner at the origin.

Example 6.2 ($x^2 \sin(1/x)$). Define $g(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$. Then

$$\frac{g(h) - g(0)}{h} = \frac{h^2 \sin(1/h)}{h} = h \sin(1/h) \to 0 \quad (h \to 0),$$

since $|h \sin(1/h)| \leq |h| \to 0$. So $g'(0) = 0$. However,

$$g'(x) = 2x \sin(1/x) - \cos(1/x) \quad (x \neq 0),$$

and $\lim_{x \to 0} g'(x)$ does not exist (the $\cos(1/x)$ term oscillates). So g is differentiable everywhere, but g' is not continuous at 0.

6.2 Differentiation Rules

Theorem 6.2 (Sum rule). If f and g are differentiable at a, then $f + g$ is differentiable at a and $(f + g)'(a) = f'(a) + g'(a)$.

Proof.

$$\frac{(f+g)(a+h) - (f+g)(a)}{h} = \frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h} \to f'(a) + g'(a). \quad \square$$

Theorem 6.3 (Product rule / Leibniz rule). If f and g are differentiable at a, then

$$(fg)'(a) = f'(a)\, g(a) + f(a)\, g'(a).$$

Proof. We use the standard trick of adding and subtracting $f(a + h)g(a)$:

$$\frac{f(a+h)g(a+h) - f(a)g(a)}{h} = \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}$$

$$= f(a+h) \cdot \frac{g(a+h) - g(a)}{h} + g(a) \cdot \frac{f(a+h) - f(a)}{h}.$$

As $h \to 0$: $f(a+h) \to f(a)$ (by Theorem 6.1), $\frac{g(a+h)-g(a)}{h} \to g'(a)$, and $\frac{f(a+h)-f(a)}{h} \to f'(a)$. Hence

$$(fg)'(a) = f(a)\, g'(a) + g(a)\, f'(a). \qquad \square$$

Theorem 6.4 (Quotient rule). If f and g are differentiable at a and $g(a) \neq 0$, then

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\, g(a) - f(a)\, g'(a)}{g(a)^2}.$$

Proof. We first show that $(1/g)'(a) = -g'(a)/g(a)^2$. Write

$$\frac{1/g(a+h) - 1/g(a)}{h} = \frac{g(a) - g(a+h)}{h\, g(a+h)\, g(a)} = \frac{-1}{g(a+h)\, g(a)} \cdot \frac{g(a+h) - g(a)}{h}.$$

As $h \to 0$, $g(a+h) \to g(a) \neq 0$ (continuity) and the difference quotient $\to g'(a)$. Hence $(1/g)'(a) = -g'(a)/g(a)^2$.

The full quotient rule follows from $f/g = f \cdot (1/g)$ and the product rule:

$$(f/g)' = f' \cdot (1/g) + f \cdot (1/g)' = \frac{f'}{g} - \frac{f\, g'}{g^2} = \frac{f'g - fg'}{g^2}. \qquad \square$$

Theorem 6.5 (Chain rule). Let f be differentiable at a and g differentiable at $f(a)$. Then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'\big(f(a)\big) \cdot f'(a).$$

Proof. Define an auxiliary function

$$\varphi(y) = \begin{cases} \dfrac{g(y) - g(b)}{y - b} & y \neq b, \\[2mm] g'(b) & y = b, \end{cases}$$

where $b = f(a)$. Then φ is continuous at b (because g is differentiable at b) and

$$g(y) - g(b) = \varphi(y)\, (y - b) \quad \text{for all } y.$$

Setting $y = f(x)$:

$$g(f(x)) - g(f(a)) = \varphi(f(x)) \big(f(x) - f(a)\big).$$

Dividing by $x - a$ ($x \neq a$):

$$\frac{g(f(x)) - g(f(a))}{x - a} = \varphi(f(x)) \cdot \frac{f(x) - f(a)}{x - a}.$$

As $x \to a$: $f(x) \to f(a) = b$ (continuity of f), so $\varphi(f(x)) \to \varphi(b) = g'(b)$, and $\frac{f(x)-f(a)}{x-a} \to f'(a)$. Hence $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$. $\qquad \square$

Theorem 6.6 (Derivative of an inverse function). Let $f \colon I \to J$ be continuous, strictly monotone, and differentiable at $a \in I$ with $f'(a) \neq 0$. Then f^{-1} is differen-

tiable at $b = f(a)$ and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

Proof. Let $b = f(a)$ and set $g = f^{-1}$. For $k \neq 0$ with $b + k \in J$, let $h = g(b+k) - g(b) = g(b+k) - a$. Then $h \neq 0$ (because f is injective) and $f(a+h) = b+k$, so $k = f(a+h) - f(a)$. Thus

$$\frac{g(b+k) - g(b)}{k} = \frac{h}{f(a+h) - f(a)} = \frac{1}{\dfrac{f(a+h) - f(a)}{h}}.$$

As $k \to 0$, we have $b + k \to b$, so $g(b+k) \to g(b) = a$ (continuity of g), hence $h \to 0$. The denominator tends to $f'(a) \neq 0$. Therefore $g'(b) = 1/f'(a)$. $\qquad\square$

6.3   Local Extrema and the Mean Value Theorem

Theorem 6.7 (Fermat's theorem)**.** Let $f \colon (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. If f has a local extremum at c, then $f'(c) = 0$.

Proof. Suppose f has a local maximum at c (the minimum case is analogous). There exists $r > 0$ with $(c - r, c + r) \subset (a, b)$ and $f(x) \leq f(c)$ for $|x - c| < r$.

- For $0 < h < r$: $f(c+h) - f(c) \leq 0$, so $\frac{f(c+h) - f(c)}{h} \leq 0$. Taking $h \to 0^+$: $f'(c) \leq 0$.

- For $-r < h < 0$: $f(c+h) - f(c) \leq 0$ and $h < 0$, so $\frac{f(c+h) - f(c)}{h} \geq 0$. Taking $h \to 0^-$: $f'(c) \geq 0$.

Together, $f'(c) = 0$. $\qquad\square$

Theorem 6.8 (Rolle's theorem)**.** Let $f \colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), with $f(a) = f(b)$. Then there exists $c \in (a, b)$ with $f'(c) = 0$.

Proof. By the Extreme Value Theorem (Theorem 5.3), f attains its maximum M and minimum m on $[a, b]$.
    Case 1: $m = M$. Then f is constant, so $f'(c) = 0$ for every $c \in (a, b)$.
    Case 2: $m < M$. Since $f(a) = f(b)$, the maximum or the minimum (or both) must be attained at some interior point $c \in (a, b)$. By Fermat's theorem, $f'(c) = 0$. $\qquad\square$

Theorem 6.9 (Mean Value Theorem (MVT))**.** Let $f \colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define the auxiliary function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g is continuous on $[a,b]$, differentiable on (a,b), and

$$g(a) = f(a), \quad g(b) = f(b) - \big(f(b) - f(a)\big) = f(a).$$

So $g(a) = g(b)$. By Rolle's theorem, there exists $c \in (a,b)$ with $g'(c) = 0$, i.e.

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

giving $f'(c) = \frac{f(b) - f(a)}{b - a}$. □

> **Corollary 6.1** (Zero derivative $\Rightarrow$ constant). If f is differentiable on an open interval I and $f'(x) = 0$ for all $x \in I$, then f is constant on I.

Proof. For any $x_1 < x_2$ in I, the MVT gives $c \in (x_1, x_2)$ with $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$. □

> **Corollary 6.2** (Positive derivative $\Rightarrow$ increasing). If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b).

Proof. For $x_1 < x_2$: $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$ since $f'(c) > 0$ and $x_2 - x_1 > 0$. □

6.4 L'Hôpital's Rule

> **Theorem 6.10** (L'Hôpital's rule, 0/0 case). Let $f, g \colon (a,b) \to \mathbb{R}$ be differentiable with $g'(x) \neq 0$ on (a,b). Suppose
>
> $$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x) \quad \text{and} \quad \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}.$$
>
> Then $\displaystyle\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Proof. Extend f and g continuously to $[a,b)$ by setting $f(a) = g(a) = 0$. Fix $x \in (a,b)$. By the Cauchy (generalized) Mean Value Theorem (apply Rolle to $h(t) = f(t)\big(g(x) - g(a)\big) - g(t)\big(f(x) - f(a)\big)$), there exists $c_x \in (a,x)$ with

$$f'(c_x)\big(g(x) - g(a)\big) = g'(c_x)\big(f(x) - f(a)\big).$$

Since $f(a) = g(a) = 0$:

$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)},$$

provided $g(x) \neq 0$ (which holds for x close to a, since $g'(c_x) \neq 0$ and $g(a) = 0$ would force $g(x) \neq 0$ for small $x - a$ by the MVT). As $x \to a^+$, we have $c_x \to a^+$ (because $a < c_x < x$), so $\frac{f'(c_x)}{g'(c_x)} \to L$. □

Example 6.3. $\displaystyle\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$ (applying L'Hôpital twice; each application is of the 0/0 form).

6.5 Taylor's Formula

6.5.1 Taylor Polynomial and Lagrange Remainder

> **Definition 6.1** (Taylor polynomial)**.** Let f be n times differentiable at a. The **Taylor polynomial of order n of f at a** is
>
> $$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

> **Theorem 6.11** (Taylor–Lagrange)**.** Let $f\colon [a,x] \to \mathbb{R}$ (or $[x,a]$) be such that $f^{(n)}$ is continuous on the closed interval and $f^{(n+1)}$ exists on the open interval. Then there exists c strictly between a and x with
>
> $$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$
>
> The last term is the **Lagrange remainder** $R_n(x)$.

Proof. Define $R_n(x) = f(x) - T_n(x)$. Note that $R_n(a) = R_n'(a) = \cdots = R_n^{(n)}(a) = 0$ (by construction of the Taylor polynomial) and $R_n^{(n+1)} = f^{(n+1)}$.

We seek c with $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$. Define $g(t) = (x-t)^{n+1}$; note $g(a) = (x-a)^{n+1}$, $g(x) = 0$.

Apply the Cauchy MVT to R_n and g on $[a,x]$:

$$\frac{R_n(x) - R_n(a)}{g(x) - g(a)} = \frac{R_n'(c_1)}{g'(c_1)}$$

for some c_1 between a and x. Since $R_n(a) = 0$ and $g(x) = 0$:

$$\frac{R_n(x)}{-(x-a)^{n+1}} = \frac{R_n'(c_1)}{-(n+1)(x-c_1)^n}.$$

Apply the Cauchy MVT again to R_n' and $(x-t)^n$ on $[a,c_1]$ (using $R_n'(a) = 0$), and iterate this process n times. After $n+1$ applications, we arrive at a point $c = c_{n+1}$ between a and x with

$$R_n(x) = \frac{R_n^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \qquad \square$$

6.5.2 Taylor–Young Formula (Landau Notation)

> **Definition 6.2** (Landau notation)**.** We write $f(x) = o(g(x))$ as $x \to a$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ (assuming $g(x) \neq 0$ near a, a excluded). In particular, $f(x) = o(1)$ means $f(x) \to 0$.

Theorem 6.12 (Taylor–Young)**.** If f is n times differentiable at a, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o\left((x-a)^n\right) \quad \text{as } x \to a.$$

Proof. We prove by induction on n.

Base case $n = 0$: $f(x) = f(a) + o(1)$ says $f(x) \to f(a)$, which is continuity at a. Differentiability at a implies continuity, so this holds.

Base case $n = 1$: $f(x) = f(a) + f'(a)(x-a) + o(x-a)$ is exactly the definition of differentiability.

Inductive step: Assume the result for $n-1$ applied to f' (which is $(n-1)$-times differentiable at a):

$$f'(x) = \sum_{k=0}^{n-1} \frac{f^{(k+1)}(a)}{k!}(x-a)^k + o\left((x-a)^{n-1}\right).$$

Define $R_n(x) = f(x) - T_n(x)$, so $R_n(a) = 0$ and $R_n'(x) = f'(x) - T_n'(x)$. One checks that $T_n'(x) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{(k-1)!}(x-a)^{k-1} = \sum_{j=0}^{n-1} \frac{f^{(j+1)}(a)}{j!}(x-a)^j$. By the inductive hypothesis, $R_n'(x) = o((x-a)^{n-1})$.

We need $R_n(x) = o((x-a)^n)$. By the Mean Value Theorem applied to R_n on $[a, x]$: $R_n(x) = R_n(x) - R_n(a) = R_n'(\xi)(x-a)$ for some ξ between a and x. Then

$$\frac{R_n(x)}{(x-a)^n} = \frac{R_n'(\xi)}{(x-a)^{n-1}}.$$

For any $\varepsilon > 0$, by the inductive hypothesis there exists $\delta > 0$ with $|R_n'(t)| \le \varepsilon|t-a|^{n-1}$ for $|t-a| < \delta$. Since $|\xi-a| \le |x-a|$, for $|x-a| < \delta$ we get $|R_n'(\xi)| \le \varepsilon|\xi-a|^{n-1} \le \varepsilon|x-a|^{n-1}$, hence $\left|\frac{R_n(x)}{(x-a)^n}\right| \le \varepsilon$. Thus $R_n(x) = o((x-a)^n)$. $\qquad\square$

6.5.3 Standard Taylor Expansions

Table of standard Taylor expansions at $a = 0$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \tag{6.1}$$

$$\sin x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \tag{6.2}$$

$$\cos x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + o(x^{2n+1}) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \tag{6.3}$$

$$\ln(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k+1} x^k}{k} + o(x^n) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{6.4}$$

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + o(x^n), \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \tag{6.5}$$
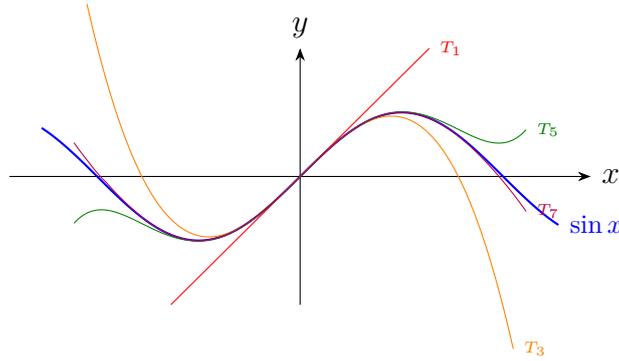
Figure 6.2: Taylor polynomials T_1, T_3, T_5, T_7 of $\sin x$ at 0.

6.5.4 Operations on Taylor Expansions and Applications

Taylor expansions can be combined using the following rules:

- **Sum/Difference:** Expand each function, add/subtract term by term, discard terms beyond order n.

- **Product:** Expand each factor, multiply formally, keep only terms up to order n.

- **Composition:** If $f(x) = o(1)$, substitute the expansion of f into the expansion of g and truncate.

- **Quotient:** Use long division or write $f/g = f \cdot (1/g)$ and expand $1/g$ via the geometric series or successive identification.

Example 6.4 (Application to a limit). Compute $\displaystyle\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

Solution. $e^x = 1 + x + \frac{x^2}{2} + o(x^2)$, so $e^x - 1 - x = \frac{x^2}{2} + o(x^2)$. Hence

$$\frac{e^x - 1 - x}{x^2} = \frac{\frac{x^2}{2} + o(x^2)}{x^2} = \frac{1}{2} + o(1) \to \frac{1}{2}. \quad \square$$

Example 6.5 (Application to a limit). Compute $\displaystyle\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$.

Solution. $\sin x = x - \frac{x^3}{6} + o(x^3), \; \cos x = 1 - \frac{x^2}{2} + o(x^2)$, so

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{6} + o(x^3)}{1 - \frac{x^2}{2} + o(x^2)}.$$

Using long division (or multiplying numerator by $1 + \frac{x^2}{2} + o(x^2)$):

$$\tan x = x + \frac{x^3}{3} + o(x^3).$$

Therefore

$$\frac{\tan x - \sin x}{x^3} = \frac{\left(x + \frac{x^3}{3} + o(x^3)\right) - \left(x - \frac{x^3}{6} + o(x^3)\right)}{x^3} = \frac{\frac{x^3}{2} + o(x^3)}{x^3} = \frac{1}{2} + o(1) \to \frac{1}{2}. \quad \square$$

Example 6.6 (Asymptotic expansion). Find the asymptotic expansion of $\sqrt{1+x} - \sqrt{1-x}$ as $x \to 0$, up to order 3.

Solution. Using (6.5) with $\alpha = 1/2$:

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + o(x^3),$$

$$(1-x)^{1/2} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + o(x^3).$$

Subtracting:

$$\sqrt{1+x} - \sqrt{1-x} = x + \tfrac{1}{8}x^3 + o(x^3). \quad \square$$

6.6 Common Errors

Frequent mistakes in Chapter 6

1. **"f continuous at $a \Rightarrow f$ differentiable at a."** False: $|x|$ is continuous but not differentiable at 0.
2. **Using L'Hôpital when the hypotheses are not satisfied.** Check that both numerator and denominator tend to 0 (or ∞) and that the derivative of the denominator is nonzero.
3. **Forgetting the chain rule.** $\frac{d}{dx}\sin(x^2) \neq \cos(x^2)$; the correct answer is $2x\cos(x^2)$.
4. **Confusing Taylor–Lagrange and Taylor–Young.** Lagrange gives an exact remainder with an unknown c; Young gives a qualitative $o((x-a)^n)$ remainder.
5. **Truncating Taylor expansions at the wrong order.** When computing a limit $f(x)/x^n$, expand f to at least order n to get the leading constant.
6. **MVT: forgetting the hypotheses.** The function must be continuous on the *closed* interval and differentiable on the *open* interval.

6.7 Exercises

Exercise 6.1. Using the definition, compute $f'(x)$ for $f(x) = x^3$.

Exercise 6.2. Let $f(x) = x^{1/3}$. Show that f is not differentiable at 0.

Exercise 6.3. Prove the **generalized product rule** (Leibniz formula):

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Hint: induction on n.

Exercise 6.4. Let $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$ Show f is differentiable on $\mathbb{R}$ but f' is discontinuous at 0.

Exercise 6.5. Use the MVT to prove: $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Exercise 6.6. Show that $e^x > 1 + x$ for all $x \neq 0$.
   Hint: study $g(x) = e^x - 1 - x$.

Exercise 6.7. Compute the following limits using Taylor expansions:

1. $\lim\limits_{x \to 0} \dfrac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

2. $\lim\limits_{x \to 0} \dfrac{\ln(1 + x) - x + \frac{x^2}{2}}{x^3}$

3. $\lim\limits_{x \to 0} \dfrac{e^{\sin x} - 1 - x}{x^2}$

Exercise 6.8. Find the Taylor expansion of $\frac{1}{1-x}$ at $a = 0$ to order n and identify the Lagrange remainder explicitly.

Exercise 6.9. Prove that if $f''(x) \geq 0$ on an interval I, then f is convex, i.e., $f\big(\lambda x + (1 - \lambda)y\big) \leq \lambda f(x) + (1 - \lambda)f(y)$ for all $x, y \in I$, $\lambda \in [0, 1]$.
    Hint: use the MVT on both subintervals.

Exercise 6.10. **(Darboux's theorem).** Let f be differentiable on $[a, b]$. Show that f' has the intermediate value property (even though f' may not be continuous).
    Hint: consider $g(x) = f(x) - \lambda x$ and apply Fermat's theorem.

Exercise 6.11. Compute the Taylor expansion of $\arctan x$ at $a = 0$ to order $2n + 1$.
    Hint: integrate the geometric series $\frac{1}{1+t^2} = \sum_{k=0}^{n}(-1)^k t^{2k} + o(t^{2n})$.

Exercise 6.12. Let $f \colon \mathbb{R} \to \mathbb{R}$ be twice differentiable with $f(0) = 0$, $f(1) = 1$, $f'(0) = f'(1) = 0$. Prove there exists $c \in (0, 1)$ with $|f''(c)| \geq 4$.
    Hint: apply the MVT twice, or use Taylor–Lagrange at $a = 0$ and $a = 1$ evaluated at $x = 1/2$.

Chapter Summary

> **Key results of Chapter 6**
>
> - Derivative: defined as a limit; differentiable $\Rightarrow$ continuous (not conversely).
> - Algebraic rules: sum, product (Leibniz), quotient, chain rule, inverse function — all proved.
> - **Fermat's theorem**: local extremum $\Rightarrow f'(c) = 0$.
> - **Rolle's theorem**: $f(a) = f(b) \Rightarrow \exists c, \; f'(c) = 0$.
> - **Mean Value Theorem**: $f'(c) = \frac{f(b) - f(a)}{b - a}$.
> - Corollaries: $f' = 0 \Rightarrow$ constant; $f' > 0 \Rightarrow$ increasing.
> - **L'Hôpital's rule** for indeterminate forms $0/0$.
> - **Taylor–Lagrange** (exact remainder) and **Taylor–Young** (asymptotic remainder).
> - Standard expansions: $e^x, \sin x, \cos x, \ln(1 + x), (1 + x)^\alpha$.
> - Taylor expansions as a tool for computing limits and asymptotic analysis.

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