Complex Analysis

Lecture Notes

Licence L3 — 2025–2026

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"The shortest path between two truths in the real domain passes through the complex domain."
— Jacques Hadamard

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Preface

This course is an introduction to the theory of functions of a single complex variable, aimed at third-year undergraduate students in mathematics. The subject is one of the great triumphs of nineteenth-century analysis, and its influence permeates nearly every branch of modern mathematics, from number theory and algebraic geometry to theoretical physics and engineering.

Why is complex differentiability so much richer than real differentiability?

In real analysis, a function $f : \mathbb{R} \to \mathbb{R}$ can be differentiable once without being differentiable twice; it can be infinitely differentiable without being analytic (i.e., without equalling its Taylor series). The landscape of real-differentiable functions is vast and somewhat wild.

In stark contrast, if a function $f : \mathbb{C} \to \mathbb{C}$ is complex-differentiable (holomorphic) on an open set, then it is automatically:

- **infinitely differentiable** — all higher-order derivatives exist;
- **analytic** — it equals its Taylor series in a neighbourhood of every point;
- **determined by its values on tiny sets** — the identity principle tells us that if two holomorphic functions agree on a set with an accumulation point, they agree everywhere on their connected domain;
- **conformal** — wherever $f'(z) \neq 0$, the mapping preserves angles between curves.

The source of this rigidity is the *Cauchy–Riemann equations*: the requirement that the limit defining $f'(z_0)$ be the same regardless of the *direction* from which z approaches z_0 in the plane. In $\mathbb{R}$, there are only two directions (left and right); in $\mathbb{C} \simeq \mathbb{R}^2$, there are infinitely many. This single constraint binds the real and imaginary parts of f into a system of partial differential equations that forces extraordinary regularity.

The crowning achievement is **Cauchy's integral formula**: the value of a holomorphic function at a point is completely determined by its values on any surrounding closed curve. From this single formula flow nearly all the major theorems of the subject: Liouville's theorem, the maximum modulus principle, the residue theorem, and the open mapping theorem.

We have organised the material into the following chapters:

1. **Complex Numbers** — algebraic and geometric review, topology of the complex plane.
2. **Holomorphic Functions** — complex derivative, Cauchy–Riemann equations, conformality, harmonic functions.

3. **Complex Integration** — line integrals, Cauchy's theorem and integral formula.
4. **Power Series and Analytic Functions** — Taylor and Laurent series, isolated singularities.
5. **The Residue Theorem and Applications** — computation of real integrals, argument principle, Rouché's theorem.

Each chapter contains complete proofs, worked examples, and exercises graded by difficulty: one star (★) for routine practice, two stars (★★) for problems requiring synthesis, and three stars (★★★) for challenging or theoretical problems.

Prerequisites. We assume familiarity with: linear algebra over $\mathbb{R}$; real analysis in one and several variables (limits, continuity, differentiation, Riemann integration); and basic point-set topology (open and closed sets, compactness, connectedness).

The Author, 2025

Notation

Symbol	Meaning
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	Natural numbers, integers, rationals, reals, complex numbers
$\mathrm{Re}(z),\ \mathrm{Im}(z)$	Real and imaginary parts of $z \in \mathbb{C}$
$\lvert z\rvert$	Modulus of z: $\lvert z\rvert = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$
$\overline{z}$	Complex conjugate of z: $\overline{x+iy} = x - iy$
$\mathrm{Arg}(z)$	Principal argument of z, taken in $(-\pi, \pi]$
$\arg(z)$	Multi-valued argument of z
$\mathrm{Log}(z)$	Principal branch of the logarithm: $\mathrm{Log}(z) = \ln\lvert z\rvert + i\,\mathrm{Arg}(z)$
$D(a, r)$	Open disk of centre a and radius r: $\{z \in \mathbb{C} : \lvert z - a\rvert < r\}$
$\overline{D}(a, r)$	Closed disk: $\{z \in \mathbb{C} : \lvert z - a\rvert \leq r\}$
$C(a, r)$	Circle of centre a and radius r: $\{z \in \mathbb{C} : \lvert z - a\rvert = r\}$
$\mathbb{D}$	Unit disk $D(0, 1)$
$\mathbb{T}$	Unit circle $C(0, 1)$
$\mathbb{C}^*$	$\mathbb{C} \setminus \{0\}$
$\hat{\mathbb{C}}$	Extended complex plane (Riemann sphere): $\mathbb{C} \cup \{\infty\}$
$\mathscr{O}(\Omega)$	Set of holomorphic functions on the open set $\Omega \subseteq \mathbb{C}$
$f'(z_0)$	Complex derivative of f at z_0
$\partial f/\partial z,\ \partial f/\partial \overline{z}$	Wirtinger derivatives
$\mathrm{d}z,\ \mathrm{d}\overline{z}$	Complex differentials
$\oint_\gamma f(z)\,\mathrm{d}z$	Complex line integral of f along the curve γ
$\mathrm{ind}(\gamma, a)$	Winding number (index) of γ about a
$\mathrm{Res}(f, a)$	Residue of f at a
Δ	Laplacian: $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$

Contents

Chapter 1

Complex Numbers — Review and Plane Geometry

1.1 Algebraic Construction of $\mathbb{C}$

Definition 1.1 (The field of complex numbers)**.** The *field of complex numbers* is the set $\mathbb{C} = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ equipped with the operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2),$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

We write $z = x + iy$ where $i = (0, 1)$ satisfies $i^2 = -1$. The real number $x = \operatorname{Re}(z)$ is the *real part* and $y = \operatorname{Im}(z)$ is the *imaginary part* of z.

Proposition 1.2 ($\mathbb{C}$ is a field)**.** *$(\mathbb{C}, +, \cdot)$ is a commutative field. The additive identity is $0 = (0, 0)$, and the multiplicative identity is $1 = (1, 0)$. For $z = x + iy \neq 0$, the multiplicative inverse is*

$$z^{-1} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}.$$

Proof. Commutativity and associativity of addition are inherited from $\mathbb{R}^2$. For multiplication, commutativity follows from the symmetry of the formula. Associativity is verified by direct computation: for $z_j = x_j + iy_j$,

$$\begin{aligned}(z_1z_2)z_3 &= \Big((x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)\Big)(x_3 + iy_3)\\ &= (x_1x_2x_3 - y_1y_2x_3 - x_1y_2y_3 - y_1x_2y_3)\\ &\quad + i(x_1x_2y_3 - y_1y_2y_3 + x_1y_2x_3 + y_1x_2x_3),\end{aligned}$$

which is symmetric under re-bracketing as $z_1(z_2z_3)$. Distributivity is similarly verified. For the inverse, if $z = x + iy \neq 0$ then $|z|^2 = x^2 + y^2 > 0$ and

$$z \cdot \frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1. \qquad \square$$

Remark 1.3 (The field $\mathbb{C}$ is not ordered). There is no total order $\leq$ on $\mathbb{C}$ that is compatible with the field operations. Indeed, if such an order existed, then either $i > 0$ or $i < 0$. In the first case, $i^2 = -1 > 0$, a contradiction. In the second case, $(-i) > 0$, so $(-i)^2 = -1 > 0$, the same contradiction.

Definition 1.4 (Conjugate)**.** For $z = x + iy \in \mathbb{C}$, the *complex conjugate* is $\overline{z} = x - iy$. The conjugation map $z \mapsto \overline{z}$ is the unique non-trivial field automorphism of $\mathbb{C}$ that fixes $\mathbb{R}$ pointwise.

Proposition 1.5 (Properties of conjugation)**.** *For all $z, w \in \mathbb{C}$:*

(i) $\overline{z + w} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\,\overline{w}$.

(ii) $\overline{\overline{z}} = z$.

(iii) $z \in \mathbb{R}$ *if and only if* $\overline{z} = z$*; and* z *is purely imaginary if and only if* $\overline{z} = -z$.

(iv) $\operatorname{Re}(z) = \dfrac{z + \overline{z}}{2}$ *and* $\operatorname{Im}(z) = \dfrac{z - \overline{z}}{2i}$.

Proof. All statements follow by writing $z = x + iy$, $w = u + iv$ and computing directly. For instance, $\overline{zw} = \overline{(xu - yv) + i(xv + yu)} = (xu - yv) - i(xv + yu) = (x - iy)(u - iv) = \overline{z}\,\overline{w}$. ☐

1.2 Modulus, Argument, and Exponential Form

Definition 1.6 (Modulus)**.** The *modulus* of $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}.$$

Geometrically, $|z|$ is the Euclidean distance from the origin to the point (x, y) in $\mathbb{R}^2$.

Proposition 1.7 (Properties of the modulus)**.** *For all $z, w \in \mathbb{C}$:*

(i) $|z| \geq 0$, *with equality if and only if* $z = 0$.

(ii) $|zw| = |z|\,|w|$.

(iii) $|z + w| \leq |z| + |w|$ (triangle inequality).

(iv) $\big||z| - |w|\big| \leq |z - w|$ (reverse triangle inequality).

(v) $|\operatorname{Re}(z)| \leq |z|$ *and* $|\operatorname{Im}(z)| \leq |z|$.

Proof. (i) and (v) are immediate.

(ii) $|zw|^2 = (zw)\overline{(zw)} = (zw)(\overline{z}\,\overline{w}) = (z\overline{z})(w\overline{w}) = |z|^2\,|w|^2$.

(iii) We compute:

$$\begin{aligned}|z+w|^2 &= (z+w)\overline{(z+w)} = |z|^2 + z\overline{w} + \overline{z}w + |w|^2 \\ &= |z|^2 + 2\,\mathrm{Re}(z\overline{w}) + |w|^2 \\ &\le |z|^2 + 2\,|z\overline{w}| + |w|^2 = |z|^2 + 2\,|z|\,|w| + |w|^2 \\ &= (|z|+|w|)^2.\end{aligned}$$

(iv) From (iii), $|z| = |(z-w)+w| \le |z-w| + |w|$, so $|z| - |w| \le |z-w|$. Exchanging the roles of z and w gives $|w| - |z| \le |z-w|$, whence the result. □

Definition 1.8 (Argument)**.** For $z \in \mathbb{C}^*$, an *argument* of z is any real number θ such that

$$z = |z|\,(\cos\theta + i\sin\theta).$$

The set of all arguments of z is $\arg(z) = \{\theta_0 + 2k\pi : k \in \mathbb{Z}\}$ for any particular argument θ_0. The *principal argument* $\mathrm{Arg}(z) \in (-\pi, \pi]$ is the unique argument in this interval:

$$\mathrm{Arg}(z) = \begin{cases} \arctan(y/x) & \text{if } x > 0, \\ \arctan(y/x) + \pi & \text{if } x < 0,\ y \ge 0, \\ \arctan(y/x) - \pi & \text{if } x < 0,\ y < 0, \\ \pi/2 & \text{if } x = 0,\ y > 0, \\ -\pi/2 & \text{if } x = 0,\ y < 0. \end{cases}$$

Theorem 1.9 (Euler's formula)**.** *For every $\theta \in \mathbb{R}$,*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This can be taken as a definition (from the power series of e^z) or proved by showing that $f(\theta) = e^{-i\theta}(\cos\theta + i\sin\theta)$ satisfies $f'(\theta) = 0$ and $f(0) = 1$.

Proof. Consider the function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(\theta) = e^{-i\theta}(\cos\theta + i\sin\theta)$. Differentiating:

$$\begin{aligned}f'(\theta) &= -ie^{-i\theta}(\cos\theta + i\sin\theta) + e^{-i\theta}(-\sin\theta + i\cos\theta) \\ &= e^{-i\theta}\Big[-i\cos\theta + \sin\theta - \sin\theta + i\cos\theta\Big] = 0.\end{aligned}$$

Since f is constant and $f(0) = 1\cdot(1+0) = 1$, we have $f(\theta) = 1$ for all θ, giving $e^{i\theta} = \cos\theta + i\sin\theta$. □

Remark 1.10. Using Euler's formula, every $z \in \mathbb{C}^*$ can be written in *exponential form*:

$$z = r\,e^{i\theta}, \qquad r = |z|\,, \quad \theta \in \arg(z).$$

Multiplication in exponential form is especially transparent: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2\, e^{i(\theta_1+\theta_2)}.$$

Thus multiplication *multiplies moduli and adds arguments.*

Example 1.11 (Computing powers)**.** Let $z = 1+i$. Then $|z| = \sqrt{2}$ and $\operatorname{Arg}(z) = \pi/4$, so $z = \sqrt{2}\, e^{i\pi/4}$. Therefore

$$z^{10} = (\sqrt{2})^{10}\, e^{i\cdot 10\pi/4} = 2^5\, e^{i\cdot 5\pi/2} = 32\, e^{i\pi/2} = 32i.$$

Example 1.12 (De Moivre's formula)**.** For $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

This follows immediately from $(e^{i\theta})^n = e^{in\theta}$. Taking $n = 3$ and expanding the left side gives the triple-angle formulas:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta,$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

1.3 Roots of Unity

Theorem 1.13 (n-th roots of a complex number)**.** *Let $w = \rho\, e^{i\phi} \in \mathbb{C}^*$ and $n \geq 1$ be an integer. The equation $z^n = w$ has exactly n distinct solutions in $\mathbb{C}$, given by*

$$z_k = \rho^{1/n} \exp\left(i\,\frac{\phi + 2k\pi}{n}\right), \qquad k = 0, 1, \ldots, n-1.$$

Proof. Write $z = r\, e^{i\theta}$. Then $z^n = w$ becomes $r^n e^{in\theta} = \rho\, e^{i\phi}$. Equating moduli: $r^n = \rho$, so $r = \rho^{1/n}$ (the unique positive real n-th root). Equating arguments: $n\theta = \phi + 2k\pi$ for some $k \in \mathbb{Z}$, giving $\theta = (\phi + 2k\pi)/n$. For $k = 0, 1, \ldots, n-1$, these give n distinct values in $[0, 2\pi)$; for other values of k, the angle differs by a multiple of 2π and so the same complex number is obtained. □

Definition 1.14 (Roots of unity)**.** The *n-th roots of unity* are the n solutions of $z^n = 1$:

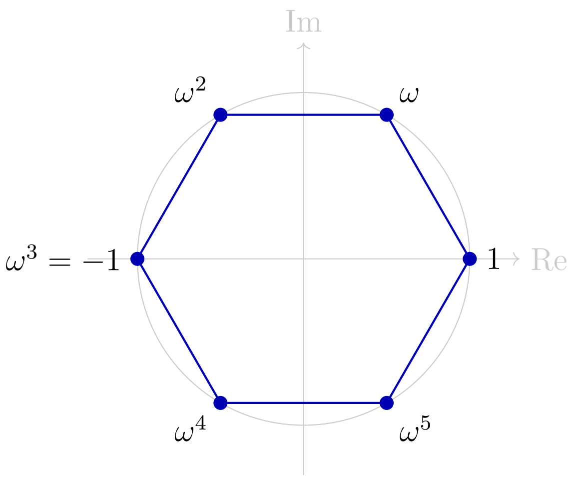
$$\omega_k = e^{2ik\pi/n}, \qquad k = 0, 1, \ldots, n-1.$$

Setting $\omega = \omega_1 = e^{2i\pi/n}$, the roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. They form a cyclic group of order n under multiplication, and they are the vertices of a regular n-gon inscribed in the unit circle.

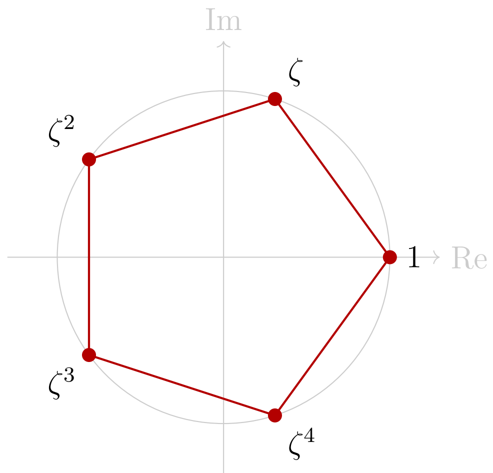
Proposition 1.15 (Sum of roots of unity)**.** *For every integer $n \geq 2$,*

$$\sum_{k=0}^{n-1} \omega^k = 0 \qquad \textit{and} \qquad \prod_{k=0}^{n-1} \omega^k = (-1)^{n+1}.$$

Proof. For the sum, since $\omega \neq 1$ we use the geometric series formula: $\sum_{k=0}^{n-1} \omega^k = (1 - \omega^n)/(1-\omega) = 0$. For the product, $\prod_{k=0}^{n-1} \omega^k = \omega^{0+1+\cdots+(n-1)} = \omega^{n(n-1)/2} = e^{i\pi(n-1)} = (-1)^{n-1} = (-1)^{n+1}$. □



The 6th roots of unity ($\omega = e^{i\pi/3}$)



The 5th roots of unity ($\zeta = e^{2i\pi/5}$)

Example 1.16 (Cube roots of $-8i$)**.** We wish to solve $z^3 = -8i$. First, write $-8i = 8\,e^{-i\pi/2}$ (since $|-8i| = 8$ and $\operatorname{Arg}(-8i) = -\pi/2$). Then

$$z_k = 8^{1/3}\exp\left(i\,\frac{-\pi/2 + 2k\pi}{3}\right) = 2\,\exp\left(i\,\frac{(4k-1)\pi}{6}\right), \quad k = 0, 1, 2.$$

Explicitly:

$$\begin{aligned} z_0 &= 2e^{-i\pi/6} = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \sqrt{3} - i, \\ z_1 &= 2e^{i\pi/2} = 2i, \\ z_2 &= 2e^{i7\pi/6} = 2\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -\sqrt{3} - i. \end{aligned}$$

1.4 Inversion and Möbius Transformations

Definition 1.17 (Complex inversion)**.** The *inversion* map is $z \mapsto 1/z$, defined on $\mathbb{C}^*$. In exponential form, if $z = re^{i\theta}$ then $1/z = (1/r)\, e^{-i\theta}$: inversion inverts the modulus and negates the argument.

Proposition 1.18 (Inversion maps circles/lines to circles/lines)**.** *The inversion $z \mapsto 1/z$ maps:*

(i) *a circle not passing through the origin to a circle not passing through the origin;*

(ii) *a circle passing through the origin to a line not passing through the origin;*

(iii) *a line not passing through the origin to a circle passing through the origin;*

(iv) *a line passing through the origin to itself* (as a set).

Proof. A general circle or line in $\mathbb{C}$ can be written

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R},$$

which in complex notation (using $x = (z + \overline{z})/2$, $y = (z - \overline{z})/(2i)$, $x^2 + y^2 = |z|^2$) becomes

$$\alpha\, |z|^2 + \mathrm{Re}((\beta - i\gamma)z) + \delta = 0.$$

Under the substitution $z \mapsto w = 1/z$, i.e., $z = 1/w$, and using $|1/w|^2 = 1/\,|w|^2$:

$$\alpha/\,|w|^2 + \mathrm{Re}((\beta - i\gamma)/w) + \delta = 0.$$

Multiplying through by $|w|^2$:

$$\alpha + \mathrm{Re}((\beta - i\gamma)\overline{w}) + \delta\, |w|^2 = 0.$$

This is again a circle/line equation with parameters $(\delta, \beta, \gamma, \alpha)$. If $\alpha \neq 0$ and $\delta \neq 0$, both equations describe circles not through the origin. If $\alpha = 0$ (the original is a line) then the image has δ in the role of the leading coefficient, giving a circle through the origin, etc. □

Definition 1.19 (Möbius transformation)**.** A *Möbius transformation* (or *fractional linear transformation*) is a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

When $c \neq 0$, we set $f(-d/c) = \infty$ and $f(\infty) = a/c$. When $c = 0$, we set $f(\infty) = \infty$. The condition $ad - bc \neq 0$ ensures that f is not constant.

Proposition 1.20 (Basic properties of Möbius transformations)**.**

(i) *Every Möbius transformation is a bijection* $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$*, with inverse*

$$f^{-1}(w) = \frac{dw - b}{-cw + a}.$$

(ii) *The composition of two Möbius transformations is a Möbius transformation, with the associated matrix being the product of the individual matrices:*

$$f \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad g \circ f \leftrightarrow \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(iii) *Every Möbius transformation is a composition of translations* $z \mapsto z + b$*, dilations/rotations* $z \mapsto az$ $(a \neq 0)$*, and the inversion* $z \mapsto 1/z$*.*

(iv) *Möbius transformations map circles and lines in* $\hat{\mathbb{C}}$ *to circles and lines.*

Proof. (i) Direct verification: $f(f^{-1}(w)) = w$.

(ii) If $f(z) = (az + b)/(cz + d)$ and $g(w) = (a'w + b')/(c'w + d')$, then

$$g(f(z)) = \frac{a' \frac{az+b}{cz+d} + b'}{c' \frac{az+b}{cz+d} + d'} = \frac{(a'a + b'c)z + (a'b + b'd)}{(c'a + d'c)z + (c'b + d'd)},$$

which corresponds to the matrix product.

(iii) If $c = 0$, then $f(z) = (a/d)z + b/d$ is an affine map (composition of dilation and translation). If $c \neq 0$, write

$$f(z) = \frac{a}{c} - \frac{ad - bc}{c^2} \cdot \frac{1}{z + d/c},$$

which is a translation, then inversion, then dilation, then translation.

(iv) This follows from (iii), since translations and rotations obviously preserve circles/lines, and inversion does so by Proposition 1.18. ☐

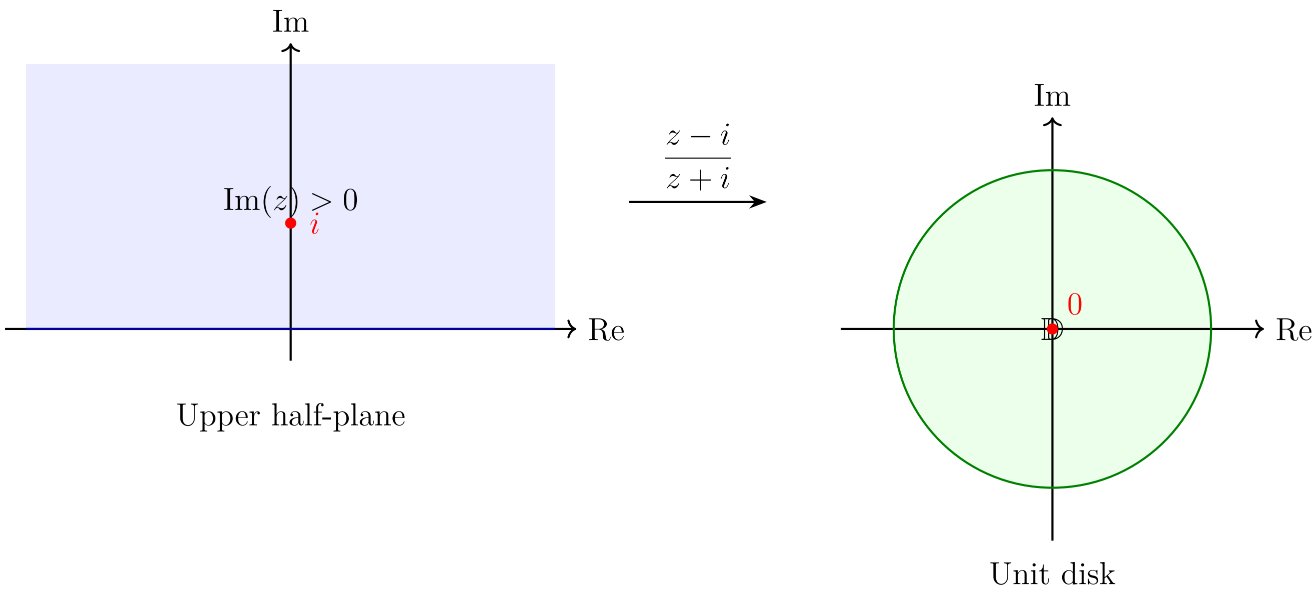
Example 1.21 (Cayley transform)**.** The Möbius transformation

$$w = \frac{z - i}{z + i}$$

maps the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ biholomorphically onto the unit disk $\mathbb{D}$. Indeed, for $z = x + iy$ with $y > 0$:

$$|w|^2 = \frac{|z - i|^2}{|z + i|^2} = \frac{x^2 + (y - 1)^2}{x^2 + (y + 1)^2} = \frac{x^2 + y^2 - 2y + 1}{x^2 + y^2 + 2y + 1} < 1,$$

since $-2y < 2y$ when $y > 0$. The real axis maps to $|w| = 1$, i.e., the unit circle. The inverse (Cayley transform) is $z = i(1 + w)/(1 - w)$.



1.5 Topology of $\mathbb{C}$

Since $\mathbb{C}$ is identified with $\mathbb{R}^2$, it inherits the Euclidean metric $d(z, w) = |z - w|$. We recall the essential topological notions.

Definition 1.22 (Open and closed sets)**.** Let $A \subseteq \mathbb{C}$.

- A is *open* if for every $z \in A$, there exists $r > 0$ such that $D(z, r) \subseteq A$.
- A is *closed* if $\mathbb{C} \setminus A$ is open, equivalently if A contains all its limit points.
- The *interior* of A is $\mathring{A} = \bigcup\{U \subseteq A : U \text{ open}\}$, the largest open set contained in A.
- The *closure* of A is $\overline{A} = \bigcap\{F \supseteq A : F \text{ closed}\}$, the smallest closed set containing A.
- The *boundary* of A is $\partial A = \overline{A} \setminus \mathring{A}$.

Example 1.23 (Open disk and closed disk)**.** The open disk $D(a, r) = \{z : |z - a| < r\}$ is open; the closed disk $\overline{D}(a, r) = \{z : |z - a| \leq r\}$ is closed; the circle $C(a, r) = \{z : |z - a| = r\}$ is the boundary of both.

Definition 1.24 (Connected and path-connected sets)**.**

- $A \subseteq \mathbb{C}$ is *connected* if it cannot be written as a disjoint union of two non-empty open subsets of A.
- A is *path-connected* if for every $z_0, z_1 \in A$, there exists a continuous path $\gamma : [0, 1] \to A$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$.

A non-empty connected open set in $\mathbb{C}$ is called a *domain*.

Proposition 1.25. *In $\mathbb{C}$ (more generally in $\mathbb{R}^n$), a non-empty open set is connected if and only if it is path-connected.*

Proof. Path-connected clearly implies connected (standard argument). For the converse, let $\Omega \subseteq \mathbb{C}$ be open and connected. Fix $z_0 \in \Omega$ and let $U = \{z \in \Omega : z \text{ can be joined to } z_0 \text{ by a path in } \Omega\}$. Then U is non-empty ($z_0 \in U$). U is open: if $z \in U$ then since Ω is open, there is a disk $D(z, r) \subseteq \Omega$; any point in this disk can be joined to z by a line segment, hence to z_0 by concatenation. Similarly, $\Omega \setminus U$ is open: if $w \in \Omega \setminus U$ then a disk $D(w, r) \subseteq \Omega$ consists entirely of points not joinable to z_0 (otherwise w would be joinable). Since Ω is connected and $U \neq \emptyset$, we must have $U = \Omega$. □

Definition 1.26 (Simply connected domain)**.** A domain $\Omega \subseteq \mathbb{C}$ is *simply connected* if every closed curve in Ω can be continuously deformed (within Ω) to a point. Intuitively, Ω has "no holes." Formally, for every continuous closed curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = \gamma(1)$, there exists a continuous map $H : [0, 1] \times [0, 1] \to \Omega$ with $H(t, 0) = \gamma(t)$, $H(t, 1) = \gamma(0)$ for all t, and $H(0, s) = H(1, s)$ for all s.

Example 1.27.

(i) $\mathbb{C}$, any open disk $D(a, r)$, any open half-plane, any convex open set: all simply connected.

(ii) $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$: connected but *not* simply connected (the unit circle cannot be contracted to a point within $\mathbb{C}^*$).

(iii) An annulus $\{z : r < |z| < R\}$: connected but not simply connected.

Definition 1.28 (Compact sets)**.** A set $K \subseteq \mathbb{C}$ is *compact* if every open cover of K has a finite subcover. By the Heine–Borel theorem, $K \subseteq \mathbb{C}$ is compact if and only if K is closed and bounded.

1.5.1 The Extended Complex Plane and the Riemann Sphere

Definition 1.29 (Riemann sphere)**.** The *extended complex plane* (or *Riemann sphere*) is $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where ∞ is a single additional point called the *point at infinity*. We define a topology on $\hat{\mathbb{C}}$ by declaring the open sets to be:

- all open subsets of $\mathbb{C}$, and
- sets of the form $\{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$ for $R > 0$ (neighbourhoods of ∞).

The space $\hat{\mathbb{C}}$ is homeomorphic to the 2-sphere $S^2 \subset \mathbb{R}^3$ via stereographic projection.

Proposition 1.30 (Stereographic projection)**.** *Consider the unit sphere $S^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$ and identify $\mathbb{C}$ with the equatorial plane*

$\{Z = 0\}$. *The stereographic projection from the north pole* $N = (0, 0, 1)$ *is the map* $\pi : S^2 \setminus \{N\} \to \mathbb{C}$ *given by*

$$\pi(X, Y, Z) = \frac{X + iY}{1 - Z}.$$

Its inverse is

$$\pi^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{|z|^2 + 1}, \frac{2\operatorname{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

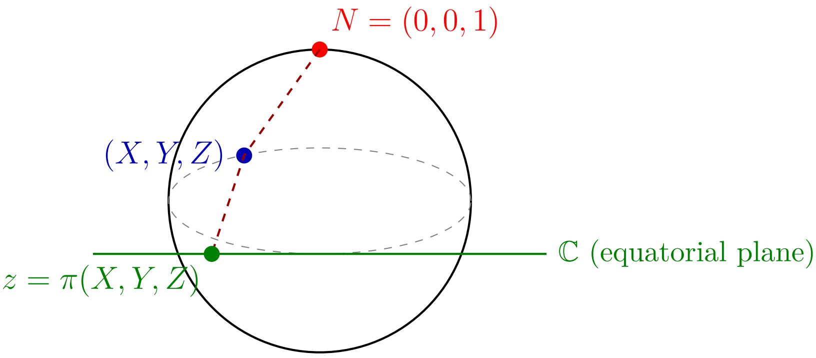
Setting $\pi(N) = \infty$ *extends* π *to a homeomorphism* $S^2 \to \hat{\mathbb{C}}$.

Proof. A point $(X, Y, Z) \in S^2 \setminus \{N\}$ and $N = (0, 0, 1)$ determine a line in $\mathbb{R}^3$:

$$\ell(t) = (1 - t)N + t(X, Y, Z) = (tX,\ tY,\ 1 - t + tZ).$$

This intersects the plane $Z = 0$ when $1 - t + tZ = 0$, i.e., $t = 1/(1 - Z)$. The intersection point is $(X/(1 - Z),\ Y/(1 - Z),\ 0)$, corresponding to $z = \frac{X+iY}{1-Z}$.

For the inverse, set $z = x + iy$ and solve: from $x = X/(1 - Z)$, $y = Y/(1 - Z)$ and $X^2 + Y^2 + Z^2 = 1$, we get $x^2 + y^2 = (1 - Z^2)/(1 - Z)^2 = (1 + Z)/(1 - Z)$. Then $|z|^2 (1 - Z) = 1 + Z$, giving $Z = (|z|^2 - 1)/(|z|^2 + 1)$ and $1 - Z = 2/(|z|^2 + 1)$, whence $X = 2x/(|z|^2 + 1)$ and $Y = 2y/(|z|^2 + 1)$. □



Remark 1.31 (Chordal metric). The *chordal metric* on $\hat{\mathbb{C}}$ is defined by

$$\chi(z, w) = \frac{2\,|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, \qquad \chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

This is the Euclidean distance between the stereographic images on S^2. The chordal metric makes $\hat{\mathbb{C}}$ into a compact metric space.

1.6 Exercises for Chapter 1

Exercise 1.1 (★). Let $z = 3 + 4i$ and $w = 1 - 2i$. Compute $z + w$, zw, z/w, $|z|$, $|w|$, $\overline{z}w$, and $\operatorname{Arg}(z)$.

Exercise 1.2 (★). Show that for all $z, w \in \mathbb{C}$,

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2) \qquad \text{(parallelogram law)}.$$

Give a geometric interpretation.

Exercise 1.3 (★). Find all $z \in \mathbb{C}$ such that $z^4 = -16$. Express the solutions in both exponential and Cartesian form, and plot them.

Exercise 1.4 (★). Let $\omega = e^{2i\pi/7}$. Compute $\sum_{k=0}^{6} \omega^{3k}$.

Exercise 1.5 (★★). Prove that for any $n \geq 1$ and $\theta \in \mathbb{R}$ with $\theta \neq 2k\pi/n$ for any $k \in \mathbb{Z}$:

$$\sum_{k=0}^{n-1} \cos(k\theta) = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{(n-1)\theta}{2}\right).$$

Hint: Compute $\mathrm{Re}\left(\sum_{k=0}^{n-1} e^{ik\theta}\right)$.

Exercise 1.6 (★★). Show that the Möbius transformation $f(z) = (z-1)/(z+1)$ maps the right half-plane $\{\mathrm{Re}(z) > 0\}$ onto the unit disk $\mathbb{D}$. What is the image of the imaginary axis?

Exercise 1.7 (★★). Let $a \in \mathbb{D}$. Show that the *Blaschke factor*

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

maps $\mathbb{D}$ bijectively onto $\mathbb{D}$ and $\mathbb{T}$ onto $\mathbb{T}$. Compute φ_a^{-1}.

Exercise 1.8 (★★). Let $f(z) = (az+b)/(cz+d)$ with $ad - bc \neq 0$. Show that f has exactly 1 or 2 fixed points in $\hat{\mathbb{C}}$ (i.e., solutions of $f(z) = z$). When does it have exactly one?

Exercise 1.9 (★★★). Show that the group of Möbius transformations preserving the unit disk $\mathbb{D}$ is

$$\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z} : \theta \in \mathbb{R},\ a \in \mathbb{D} \right\}.$$

Hint: If f is such a Möbius transformation, let $a = f^{-1}(0)$ and consider $\varphi_a \circ f^{-1}$.

Exercise 1.10 (★★★). (Cross-ratio.) For four distinct points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, define the *cross-ratio*

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

(a) Show that the cross-ratio is invariant under Möbius transformations: $(f(z_1), f(z_2); f(z_3), f(z_4)) = (z_1, z_2; z_3, z_4)$.

(b) Show that four distinct points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ lie on a circle or line if and only if $(z_1, z_2; z_3, z_4) \in \mathbb{R}$.

(c) Given three distinct points $z_2, z_3, z_4 \in \hat{\mathbb{C}}$, show that $z \mapsto (z, z_2; z_3, z_4)$ is the unique Möbius transformation sending $z_2 \mapsto 1$, $z_3 \mapsto 0$, $z_4 \mapsto \infty$.

Chapter 2

Holomorphic Functions — Cauchy–Riemann Conditions

In this chapter we introduce the central concept of the course: *holomorphic* (complex-differentiable) functions. We shall see that a single requirement — that the complex derivative exist — imposes far stronger constraints than real differentiability. The key link is the system of *Cauchy–Riemann equations*, which connect the partial derivatives of the real and imaginary parts.

2.1 The Complex Derivative

Definition 2.1 (Holomorphic function)**.** Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. We say that f is *complex-differentiable* (or *holomorphic*) at a point $z_0 \in \Omega$ if the limit

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in $\mathbb{C}$. Here $h \in \mathbb{C}^*$ and $h \to 0$ means $|h| \to 0$ with h approaching 0 from *any direction* in the plane. The number $f'(z_0)$ is called the *complex derivative* of f at z_0. We say that f is *holomorphic on* Ω if it is holomorphic at every point of Ω. We write $f \in \mathscr{O}(\Omega)$.

Remark 2.2 (The crucial difference from $\mathbb{R}$). In the real case, the variable h in $f'(x_0) = \lim_{h \to 0}(f(x_0 + h) - f(x_0))/h$ can only approach 0 from the left or from the right — two directions. In the complex case, h can approach 0 from *any direction* in $\mathbb{R}^2$: along the real axis, the imaginary axis, spirals, or any other path. The requirement that the same limit be obtained for *all* directions is extremely restrictive and is the source of the remarkable rigidity of holomorphic functions.

Proposition 2.3 (Holomorphic implies continuous)**.** *If f is holomorphic at z_0, then f is continuous at z_0.*

Proof. If $f'(z_0)$ exists, then for $h \neq 0$:

$$f(z_0+h) - f(z_0) = \frac{f(z_0+h)-f(z_0)}{h} \cdot h \to f'(z_0) \cdot 0 = 0 \quad \text{as } h \to 0. \qquad \square$$

Proposition 2.4 (Algebraic rules for the complex derivative)**.** *Let $f, g \in \mathscr{O}(\Omega)$ and $\alpha \in \mathbb{C}$. Then:*

(i) $(\alpha f + g)'(z) = \alpha f'(z) + g'(z)$.

(ii) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ (product rule).

(iii) *If $g(z) \neq 0$, then* $(f/g)'(z) = \big(f'(z)g(z) - f(z)g'(z)\big)/g(z)^2$ (quotient rule).

(iv) *If $f : \Omega \to \Omega'$ and $g : \Omega' \to \mathbb{C}$ are holomorphic, then* $(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$ (chain rule).

Proof. The proofs are identical to the real case. We give the chain rule as an example. Let $w_0 = f(z_0)$ and define

$$\varphi(w) = \begin{cases} \dfrac{g(w)-g(w_0)}{w-w_0} & \text{if } w \neq w_0, \\ g'(w_0) & \text{if } w = w_0. \end{cases}$$

Then φ is continuous at w_0 (since g is holomorphic at w_0) and $g(w)-g(w_0) = \varphi(w)(w-w_0)$ for all w. Setting $w = f(z)$:

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \varphi(f(z)) \cdot \frac{f(z)-f(z_0)}{z-z_0}.$$

As $z \to z_0$, $f(z) \to f(z_0) = w_0$ (by continuity of f), so $\varphi(f(z)) \to \varphi(w_0) = g'(w_0)$, and the second factor tends to $f'(z_0)$. $\square$

Example 2.5 (Polynomials and rational functions)**.**

(i) **Constant functions:** $f(z) = c$ has $f'(z) = 0$ everywhere.

(ii) **The identity:** $f(z) = z$ has $f'(z) = 1$.

(iii) **Powers:** $f(z) = z^n$ ($n \in \mathbb{N}$) has $f'(z) = nz^{n-1}$ (by induction using the product rule).

(iv) **Polynomials:** $p(z) = \sum_{k=0}^{n} a_k z^k$ is holomorphic on $\mathbb{C}$ with $p'(z) = \sum_{k=1}^{n} k a_k z^{k-1}$.

(v) **Rational functions:** $f(z) = p(z)/q(z)$ is holomorphic on $\mathbb{C} \setminus \{z : q(z) = 0\}$.

Example 2.6 (The exponential function)**.** Define $e^z = e^x(\cos y + i \sin y)$ for $z = x + iy$. Then

$$\frac{e^{z+h} - e^z}{h} = e^z \cdot \frac{e^h - 1}{h}.$$

We claim $\lim_{h\to 0}(e^h - 1)/h = 1$. Indeed, writing $h = s + it$ with $|h|$ small:

$$e^h - 1 = e^s \cos t - 1 + ie^s \sin t = (e^s - 1)\cos t + (\cos t - 1) + ie^s \sin t.$$

Using $e^s = 1 + s + O(s^2)$, $\cos t = 1 - t^2/2 + O(t^4)$, $\sin t = t + O(t^3)$:

$$e^h - 1 = s + it + O(|h|^2) = h + O(|h|^2),$$

so $(e^h - 1)/h \to 1$. Therefore $(e^z)' = e^z$ for all $z \in \mathbb{C}$.

Example 2.7 (Conjugation is not holomorphic)**.** The function $f(z) = \overline{z}$ is *not* holomorphic anywhere. Indeed,

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{h}}{h}.$$

If $h = t \in \mathbb{R}^*$, this equals $t/t = 1$. If $h = it$ with $t \in \mathbb{R}^*$, this equals $-it/(it) = -1$. Since the limit along the real axis gives 1 and the limit along the imaginary axis gives -1, the limit does not exist.

Example 2.8 (The modulus squared)**.** The function $f(z) = |z|^2 = z\overline{z}$ is holomorphic *only at the origin*. Indeed,

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h)\overline{(z+h)} - z\overline{z}}{h} = \overline{z} + z\frac{\overline{h}}{h} + \overline{h}.$$

For this to have a limit as $h \to 0$, the term $z \cdot \overline{h}/h$ must have a limit. By the same argument as above, $\overline{h}/h$ has no limit unless $z = 0$. When $z = 0$, the expression becomes $\overline{h}$, which tends to 0. So $f'(0) = 0$ and f is not holomorphic at any $z \neq 0$.

2.2 The Cauchy–Riemann Equations

The following theorem provides an analytic criterion for holomorphy in terms of the real and imaginary parts of f.

Write $f = u + iv$ where $u, v : \Omega \to \mathbb{R}$ and identify $z = x + iy$.

Theorem 2.9 (Cauchy–Riemann equations — necessary condition)**.** *If $f = u + iv$ is holomorphic at $z_0 = x_0 + iy_0$, then the partial derivatives u_x, u_y, v_x, v_y all exist at (x_0, y_0) and satisfy the* Cauchy–Riemann equations*:*

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.}$$

Moreover, the complex derivative is given by

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Proof. Since $f'(z_0)$ exists, the limit $\lim_{h\to 0}(f(z_0+h)-f(z_0))/h$ is the same for all directions of approach.

Approach along the real axis: Set $h = t \in \mathbb{R}$, $t \to 0$:

$$\begin{aligned} f'(z_0) &= \lim_{t\to 0} \frac{f(x_0+t+iy_0)-f(x_0+iy_0)}{t} \\ &= \lim_{t\to 0} \frac{u(x_0+t,y_0)-u(x_0,y_0)}{t} + i\lim_{t\to 0}\frac{v(x_0+t,y_0)-v(x_0,y_0)}{t} \\ &= u_x(x_0,y_0)+iv_x(x_0,y_0). \end{aligned}$$

Approach along the imaginary axis: Set $h = it$, $t \in \mathbb{R}$, $t \to 0$:

$$\begin{aligned} f'(z_0) &= \lim_{t\to 0} \frac{f(x_0+i(y_0+t))-f(x_0+iy_0)}{it} \\ &= \frac{1}{i}\left[u_y(x_0,y_0)+iv_y(x_0,y_0)\right] \\ &= -iu_y(x_0,y_0)+v_y(x_0,y_0). \end{aligned}$$

Equating the two expressions for $f'(z_0)$:

$$u_x + iv_x = v_y - iu_y.$$

Equating real and imaginary parts: $u_x = v_y$ and $v_x = -u_y$. ☐

Remark 2.10 (The Cauchy–Riemann equations are not sufficient without regularity). The Cauchy–Riemann equations at a point (x_0, y_0) are necessary but not sufficient for holomorphy. Consider

$$f(z) = \begin{cases} \exp(-1/z^4) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

One can verify that $u_x = v_y = u_y = -v_x = 0$ at the origin, yet f is not even continuous at 0 (the limit depends on the direction of approach). The additional requirement is that the partial derivatives be *continuous*.

Theorem 2.11 (Cauchy–Riemann equations — sufficient condition)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and $f = u + iv : \Omega \to \mathbb{C}$. Suppose that the partial derivatives u_x, u_y, v_x, v_y exist in Ω and are* continuous *at $z_0 = x_0 + iy_0$, and that the Cauchy–Riemann equations hold at (x_0, y_0):*

$$u_x(x_0,y_0) = v_y(x_0,y_0), \qquad u_y(x_0,y_0) = -v_x(x_0,y_0).$$

Then f is holomorphic at z_0, with $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$.

Proof. Since u has continuous partial derivatives at (x_0, y_0), it is (real) differentiable there:

$$u(x_0+s, y_0+t) = u(x_0,y_0) + u_x s + u_y t + \varepsilon_1(s,t)\sqrt{s^2+t^2},$$

where $\varepsilon_1(s,t) \to 0$ as $(s,t) \to (0,0)$. Similarly for v:

$$v(x_0+s, y_0+t) = v(x_0,y_0) + v_x s + v_y t + \varepsilon_2(s,t)\sqrt{s^2+t^2}.$$

Setting $h = s + it$ so that $|h| = \sqrt{s^2 + t^2}$:

$$f(z_0 + h) - f(z_0) = (u_x s + u_y t + \varepsilon_1 |h|) + i(v_x s + v_y t + \varepsilon_2 |h|).$$

Using the Cauchy–Riemann equations ($v_y = u_x$ and $v_x = -u_y$):

$$\begin{aligned} f(z_0 + h) - f(z_0) &= u_x s - v_x t + i(v_x s + u_x t) + (\varepsilon_1 + i\varepsilon_2) |h| \\ &= (u_x + iv_x)(s + it) + (\varepsilon_1 + i\varepsilon_2) |h| \\ &= (u_x + iv_x)\, h + (\varepsilon_1 + i\varepsilon_2) |h| . \end{aligned}$$

Dividing by h:

$$\frac{f(z_0 + h) - f(z_0)}{h} = u_x + iv_x + (\varepsilon_1 + i\varepsilon_2) \frac{|h|}{h}.$$

Since $||h| /h| = 1$ and $\varepsilon_1, \varepsilon_2 \to 0$, the last term vanishes as $h \to 0$. Therefore $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. □

Corollary 2.12. *If $f = u + iv : \Omega \to \mathbb{C}$ is of class C^1 (i.e., u_x, u_y, v_x, v_y exist and are continuous on Ω), then f is holomorphic on Ω if and only if the Cauchy–Riemann equations hold at every point of Ω.*

Example 2.13 (Verification for e^z)**.** For $f(z) = e^z = e^x \cos y + ie^x \sin y$, we have $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Then:

$$u_x = e^x \cos y = v_y, \qquad u_y = -e^x \sin y = -v_x.$$

The Cauchy–Riemann equations hold everywhere, with continuous partial derivatives. Therefore f is holomorphic on $\mathbb{C}$, with $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z$.

Example 2.14 (Verification for $\overline{z}$ — failure of Cauchy–Riemann)**.** For $f(z) = \overline{z} = x - iy$, we have $u = x$, $v = -y$. Then $u_x = 1$ but $v_y = -1$, so $u_x \neq v_y$. The Cauchy–Riemann equations fail everywhere, confirming that $\overline{z}$ is nowhere holomorphic.

Definition 2.15 (Wirtinger derivatives)**.** The *Wirtinger operators* are defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Proposition 2.16 (Holomorphy via Wirtinger derivatives)**.** *A C^1 function $f : \Omega \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial f}{\partial \overline{z}} = 0 \quad \text{on } \Omega.$$

In this case, $f'(z) = \dfrac{\partial f}{\partial z}$.

Proof. Write $f = u + iv$. Then

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) \\ &= \frac{1}{2}\Big[(u_x - v_y) + i(v_x + u_y)\Big].\end{aligned}$$

This vanishes if and only if $u_x = v_y$ and $u_y = -v_x$, which are the Cauchy–Riemann equations. □

Remark 2.17. The condition $\partial f/\partial \bar{z} = 0$ has an elegant interpretation: a holomorphic function is one that "depends on z but not on $\bar{z}$." This perspective is fundamental in several complex variables and in complex differential geometry.

2.3 Geometric Interpretation: Conformality

One of the most striking properties of holomorphic functions is that they preserve angles. This makes them fundamental in applications ranging from fluid dynamics to cartography.

Definition 2.18 (Angle between curves)**.** Let $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C}$ be two smooth curves with $\gamma_1(0) = \gamma_2(0) = z_0$ and $\gamma_1'(0) \neq 0$, $\gamma_2'(0) \neq 0$. The *angle* between γ_1 and γ_2 at z_0 is

$$\angle(\gamma_1, \gamma_2) = \arg\left(\frac{\gamma_2'(0)}{\gamma_1'(0)}\right) \pmod{2\pi}.$$

This is the signed angle from the tangent of γ_1 to the tangent of γ_2 at z_0.

Theorem 2.19 (Holomorphic maps are conformal)**.** *Let $f : \Omega \to \mathbb{C}$ be holomorphic with $f'(z_0) \neq 0$. Then f is* conformal *at z_0: it preserves angles between curves and preserves orientation. More precisely, if γ_1, γ_2 are smooth curves through z_0, then the image curves $f \circ \gamma_1$ and $f \circ \gamma_2$ meet at $f(z_0)$ with the same angle:*

$$\angle(f \circ \gamma_1,\ f \circ \gamma_2) = \angle(\gamma_1, \gamma_2).$$

Proof. By the chain rule, $(f \circ \gamma_j)'(0) = f'(z_0) \cdot \gamma_j'(0)$ for $j = 1, 2$. Therefore

$$\frac{(f \circ \gamma_2)'(0)}{(f \circ \gamma_1)'(0)} = \frac{f'(z_0)\gamma_2'(0)}{f'(z_0)\gamma_1'(0)} = \frac{\gamma_2'(0)}{\gamma_1'(0)}.$$

Taking arguments: $\angle(f \circ \gamma_1, f \circ \gamma_2) = \arg(\gamma_2'(0)/\gamma_1'(0)) = \angle(\gamma_1, \gamma_2)$. □

Remark 2.20. The condition $f'(z_0) \neq 0$ is essential. At a zero of f' of order k (i.e., $f(z) = f(z_0) + c(z - z_0)^{k+1} + \cdots$ with $c \neq 0$), angles are multiplied by $k + 1$. For instance, $f(z) = z^2$ doubles angles at the origin.

Proposition 2.21 (Conformal ⇔ holomorphic or anti-holomorphic)**.** *Let $f : \Omega \to \mathbb{C}$ be a C^1 map with non-vanishing Jacobian. Then f is angle-preserving (conformal)* *at every point if and only if f is holomorphic with $f' \neq 0$ everywhere. If instead f reverses angles (anti-conformal), then $\overline{f}$ is holomorphic.*

Proof. The real Jacobian matrix of $f = u + iv$ is

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

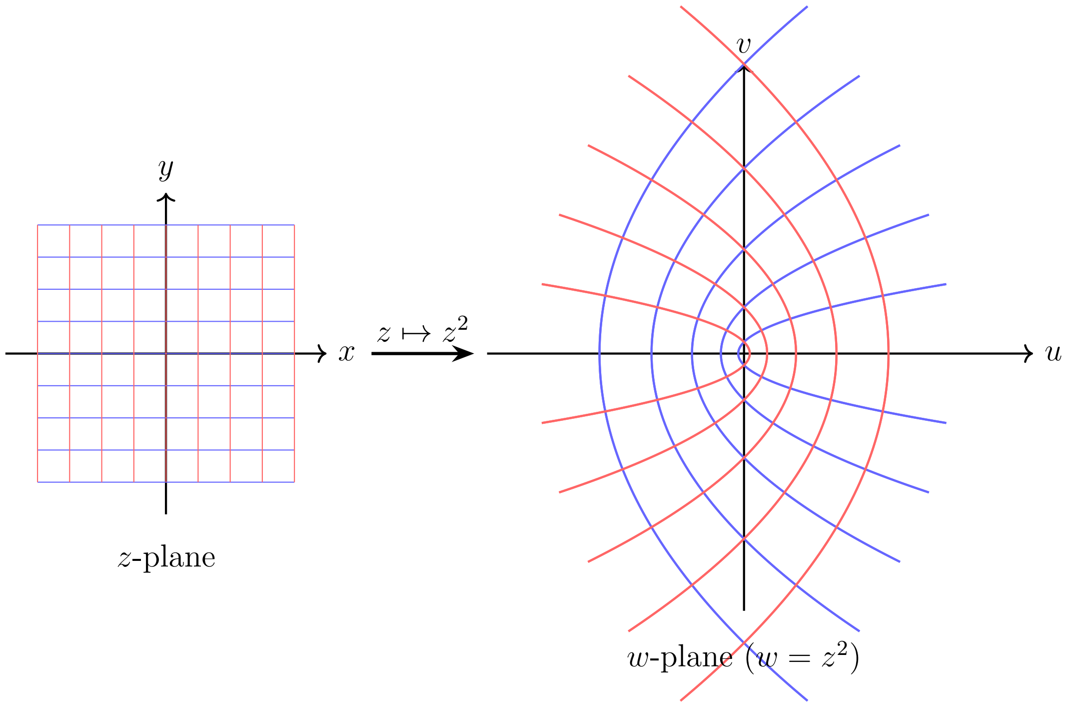
Conformality means J_f is a positive scalar multiple of a rotation matrix. A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a positive scalar times a rotation if and only if $a = d$ and $c = -b$ (with $a^2 + b^2 > 0$). This gives $u_x = v_y$ and $u_y = -v_x$, which are the Cauchy–Riemann equations. Anti-conformality similarly gives $u_x = -v_y$ and $u_y = v_x$, the Cauchy–Riemann equations for $\overline{f}$. □

Example 2.22 (Conformal image of a grid under $z \mapsto z^2$)**.** The map $f(z) = z^2$ is conformal away from $z = 0$ (where $f'(0) = 0$). Let us see what happens to the standard grid of horizontal and vertical lines.

A horizontal line $z = t + ib$ ($t \in \mathbb{R}$) maps to $f(z) = (t + ib)^2 = t^2 - b^2 + 2ibt$, i.e., the parametric curve $x = t^2 - b^2$, $y = 2bt$. Eliminating t: $t = y/(2b)$, so $x = y^2/(4b^2) - b^2$, a parabola opening to the right.

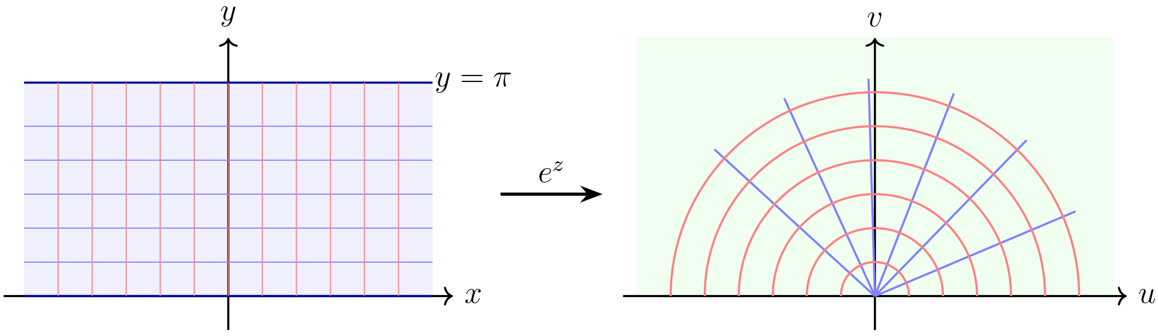
A vertical line $z = a + it$ maps to $(a + it)^2 = a^2 - t^2 + 2iat$, giving $x = a^2 - t^2$, $y = 2at$, so $x = a^2 - y^2/(4a^2)$, a parabola opening to the left.

These two families of parabolas intersect at right angles, illustrating the conformal property.



Example 2.23 (Conformal image under $z \mapsto e^z$)**.** The map $f(z) = e^z$ sends the horizontal strip $\{x + iy : x \in \mathbb{R},\ 0 < y < \pi\}$ conformally onto the upper half-plane.

Indeed, $e^{x+iy} = e^x e^{iy}$ has modulus $e^x \in (0, \infty)$ and argument $y \in (0, \pi)$, which sweeps out all of $\{\text{Im}(w) > 0\}$.
Horizontal lines $y = c$ map to rays from the origin at angle c, and vertical lines $x = a$ map to circles of radius e^a. These two families are orthogonal in the w-plane.



z-plane (strip $0 < y < \pi$)

w-plane (upper half-plane)

2.4 Harmonic Functions and Their Connection to Holomorphy

Definition 2.24 (Harmonic function)**.** A C^2 function $\varphi : \Omega \to \mathbb{R}$ (where $\Omega \subseteq \mathbb{R}^2$ is open) is *harmonic* if it satisfies the *Laplace equation*:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0.$$

Theorem 2.25 (Real and imaginary parts of holomorphic functions are harmonic)**.** *If $f = u + iv$ is holomorphic on a domain Ω, then u and v are harmonic on Ω.*

Proof. We shall use the fact (proved later via Cauchy's integral formula) that a holomorphic function is infinitely differentiable. Granted this, the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ can be differentiated:

$$u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy}.$$

Since v is C^2, Schwarz's theorem gives $v_{xy} = v_{yx}$, hence

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Similarly, differentiating the Cauchy–Riemann equations the other way:

$$v_{xx} = -u_{yx}, \qquad v_{yy} = u_{xy}.$$

So $\Delta v = v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$ □

Definition 2.26 (Harmonic conjugate)**.** Let $u : \Omega \to \mathbb{R}$ be harmonic on a domain Ω. A harmonic function $v : \Omega \to \mathbb{R}$ is called a *harmonic conjugate* of u if $f = u + iv$ is holomorphic on Ω, i.e., if u and v satisfy the Cauchy–Riemann equations.

Theorem 2.27 (Existence of harmonic conjugate on simply connected domains)**.** *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $u : \Omega \to \mathbb{R}$ a harmonic function. Then u has a harmonic conjugate v on Ω, unique up to an additive constant. Consequently, $f = u + iv$ is holomorphic on Ω.*

Proof. We need to find v satisfying $v_x = -u_y$ and $v_y = u_x$. Fix a point $(x_0, y_0) \in \Omega$ and define

$$v(x, y) = \int_\gamma (-u_y \, dx + u_x \, dy) \, ,$$

where γ is any piecewise-smooth path in Ω from (x_0, y_0) to (x, y). For this to be well-defined (independent of the path), we need the 1-form $\omega = -u_y \, dx + u_x \, dy$ to be closed, which means

$$\frac{\partial}{\partial y}(-u_y) = \frac{\partial}{\partial x}(u_x) \quad \Longleftrightarrow \quad -u_{yy} = u_{xx} \quad \Longleftrightarrow \quad \Delta u = 0.$$

This holds since u is harmonic. On a simply connected domain, a closed 1-form is exact (Poincaré lemma), so the integral is path-independent and defines a well-defined function v.

By the fundamental theorem of calculus for line integrals, $v_x = -u_y$ and $v_y = u_x$, which are the Cauchy–Riemann equations. Uniqueness up to a constant follows because if v_1, v_2 are two harmonic conjugates, then $(v_1 - v_2)_x = (v_1 - v_2)_y = 0$, so $v_1 - v_2$ is constant on the connected set Ω. ☐

Remark 2.28. Simple connectivity is essential. On $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the function $u(x, y) = \frac{1}{2} \ln(x^2 + y^2)$ is harmonic, but its conjugate $v = \arctan(y/x)$ is multi-valued (the argument function). There is no single-valued harmonic conjugate on all of $\mathbb{C}^*$.

Example 2.29 (Finding a harmonic conjugate)**.** Let $u(x, y) = x^2 - y^2$ (which is harmonic since $u_{xx} + u_{yy} = 2 + (-2) = 0$). We seek v with $v_x = -u_y = 2y$ and $v_y = u_x = 2x$. From the first equation: $v(x, y) = 2xy + g(y)$. Substituting into the second: $2x + g'(y) = 2x$, so $g'(y) = 0$, i.e., $g(y) = C$. Therefore $v(x, y) = 2xy + C$, and

$$f(z) = (x^2 - y^2) + i(2xy + C) = z^2 + iC.$$

Example 2.30 (Harmonic conjugate of $\ln |z|$)**.** On $\Omega = \mathbb{C} \setminus (-\infty, 0]$ (the slit plane), $u(x, y) = \frac{1}{2} \ln(x^2 + y^2) = \ln |z|$ is harmonic. The harmonic conjugate is $v = \operatorname{Arg}(z)$ (the principal argument), giving $f = u + iv = \operatorname{Log}(z)$, the principal branch of the logarithm.

Proposition 2.31 (Laplacian in terms of Wirtinger derivatives)**.** *For a C^2 function*

$\varphi : \Omega \to \mathbb{R}$,

$$\Delta\varphi = 4\,\frac{\partial^2\varphi}{\partial z\,\partial\overline{z}}.$$

Proof. Using $\partial/\partial z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial/\partial\overline{z} = \frac{1}{2}(\partial_x + i\partial_y)$:

$$4\frac{\partial^2\varphi}{\partial z\,\partial\overline{z}} = 4\cdot\frac{1}{2}(\partial_x - i\partial_y)\cdot\frac{1}{2}(\partial_x + i\partial_y)\varphi = (\partial_x - i\partial_y)(\partial_x + i\partial_y)\varphi = (\partial_x^2 + \partial_y^2)\varphi = \Delta\varphi.$$

(The cross terms cancel: $i\partial_x\partial_y - i\partial_y\partial_x = 0$ by Schwarz.) □

Proposition 2.32 (Mean value property for harmonic functions). *If φ is harmonic on a domain Ω containing $\overline{D}(a,r)$, then*

$$\varphi(a) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(a + re^{i\theta})\,d\theta.$$

That is, the value at the centre equals the average on any circle centred there.

Proof. This will be proved as a consequence of the Cauchy integral formula in Chapter 3. For now we state it to illustrate the power of harmonicity. □

Proposition 2.33 (Maximum principle for harmonic functions). *Let φ be harmonic on a domain Ω. If φ attains a maximum or minimum value at an interior point of Ω, then φ is constant.*

Proof. Suppose $\varphi(a) = \max_{z\in\Omega}\varphi(z)$ for some $a \in \Omega$. By the mean value property, for any small $r > 0$:

$$\varphi(a) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(a + re^{i\theta})\,d\theta.$$

Since $\varphi(a + re^{i\theta}) \leq \varphi(a)$ for all θ and the average equals $\varphi(a)$, we must have $\varphi(a + re^{i\theta}) = \varphi(a)$ for all θ (a non-negative continuous function with zero integral is identically zero). Thus φ is constant on $D(a,r)$. The set $\{z \in \Omega : \varphi(z) = \varphi(a)\}$ is both open (by the argument above) and closed (by continuity), hence equal to Ω by connectedness. □

2.5 Entire Functions and First Glimpses Beyond

Definition 2.34 (Entire function). A function $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic on all of $\mathbb{C}$ is called an *entire function*. Examples include polynomials, e^z, $\sin z$, $\cos z$.

Proposition 2.35 (Trigonometric functions as entire functions). *The functions*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

are entire, with $(\sin z)' = \cos z$ and $(\cos z)' = -\sin z$. Note that unlike their real counterparts, $\sin z$ and $\cos z$ are unbounded *on $\mathbb{C}$: for instance, $\sin(iy) = i\sinh(y) \to \infty$ as $y \to \infty$.*

Example 2.36 (The hyperbolic functions)**.** The hyperbolic functions are defined by

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}.$$

They are entire and satisfy $\sin(iz) = i \sinh z$ and $\cos(iz) = \cosh z$.

Remark 2.37 (Preview: Liouville's theorem). We shall prove in Chapter 3 that every *bounded* entire function is constant (Liouville's theorem). This is a dramatic consequence of the rigidity of holomorphic functions, with no analogue in real analysis. As a corollary, one obtains the *fundamental theorem of algebra*: every non-constant polynomial has a root in $\mathbb{C}$.

2.6 Exercises for Chapter 2

Exercise 2.1 (★)**.** Determine which of the following functions are holomorphic, and compute the derivative where it exists:

(a) $f(z) = z^3 - 2z + i$

(b) $f(z) = |z|^2$

(c) $f(z) = \mathrm{Re}(z)$

(d) $f(z) = e^{z^2}$

(e) $f(z) = z\overline{z} + z^2$

Exercise 2.2 (★)**.** Verify the Cauchy–Riemann equations for $f(z) = \sin z$ by writing $f = u + iv$ explicitly.

Exercise 2.3 (★)**.** Find the harmonic conjugate of $u(x, y) = e^x \cos y$ and express the resulting holomorphic function in terms of z.

Exercise 2.4 (★)**.** Show that $u(x, y) = x^3 - 3xy^2$ is harmonic and find a holomorphic function f with $\mathrm{Re}(f) = u$.

Exercise 2.5 (★★)**.** Let $f : \Omega \to \mathbb{C}$ be holomorphic and suppose $\mathrm{Re}(f)$ is constant on Ω (a connected open set). Show that f is constant.

Exercise 2.6 (★★)**.** Let $f : \Omega \to \mathbb{C}$ be holomorphic. Show that if $|f|$ is constant on Ω (connected), then f is constant.

Hint: If $|f|^2 = u^2 + v^2 = c$ is constant, differentiate and use the Cauchy–Riemann equations.

Exercise 2.7 (★★)**.** Let $f = u + iv$ be holomorphic on a domain Ω. Show that the level curves $\{u = \text{const}\}$ and $\{v = \text{const}\}$ are orthogonal at every point where $f'(z) \neq 0$.

Hint: The gradient of u is $\nabla u = (u_x, u_y)$ and that of v is $\nabla v = (v_x, v_y)$. Compute $\nabla u \cdot \nabla v$ using the Cauchy–Riemann equations.

Exercise 2.8 (★★). Show that $f(z) = \sqrt{|xy|}$ (where $z = x + iy$) satisfies the Cauchy–Riemann equations at the origin but is *not* holomorphic there.

Hint: Compute the partial derivatives at the origin and check whether f is differentiable (in the real sense) at the origin.

Exercise 2.9 (★★). Describe the image of the following sets under $f(z) = z^2$:

(a) The first quadrant $\{x > 0, y > 0\}$.

(b) The half-strip $\{0 < x < 1, y > 0\}$.

(c) The sector $\{0 < \arg z < \pi/4\}$.

In each case, verify conformality by describing what happens to the boundary curves.

Exercise 2.10 (★★). Let $f(z) = z + 1/z$ (defined on $\mathbb{C}^*$).

(a) Show that f maps the unit circle $|z| = 1$ onto the interval $[-2, 2]$.

(b) Show that f maps $\mathbb{C} \setminus \overline{\mathbb{D}}$ (the exterior of the closed unit disk) conformally onto $\mathbb{C} \setminus [-2, 2]$.

(c) This is the *Joukowski map*, fundamental in aerodynamics. Describe the images of circles $|z| = r$ for $r > 1$.

Exercise 2.11 (★★★). (Harmonic functions and the Dirichlet problem.) Let u be a continuous function on $\overline{\mathbb{D}}$ that is harmonic on $\mathbb{D}$. Suppose $u = 0$ on $\mathbb{T} = \partial\mathbb{D}$. Prove that $u \equiv 0$ on $\mathbb{D}$.

Hint: Apply the maximum principle to both u and $-u$.

Exercise 2.12 (★★★). (Looman–Menchov theorem preview.) Show that if $f = u + iv$ is continuous on a domain Ω, the partial derivatives u_x, u_y, v_x, v_y exist everywhere in Ω, and the Cauchy–Riemann equations hold at every point, then f is holomorphic. (This is significantly harder than Theorem 2.11 because we do not assume continuity of the partial derivatives. You may use without proof the fact that a continuous function satisfying the Cauchy–Riemann equations is holomorphic if it has only countably many points where it is not differentiable in the real sense.)

Exercise 2.13 (★★★). Let Ω be a domain in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a holomorphic function. Show that if f maps Ω into a straight line, then f is constant.

Hint: After composing with an affine transformation, reduce to the case $\mathrm{Im}(f) = 0$.

Exercise 2.14 (★★★). Let Ω be a domain and $f : \Omega \to \mathbb{C}$ holomorphic with $f' \neq 0$ on Ω. Show that f is an open mapping (images of open sets are open) by using the inverse function theorem for $\mathbb{R}^2$ and the fact that the Jacobian determinant of f is $|f'(z)|^2 > 0$.

Chapter 3

Elementary Functions

Introduction

In real analysis, the elementary functions — exponential, logarithm, trigonometric and hyperbolic functions — are old friends. In the complex setting they acquire a richer and sometimes surprising personality. The exponential function e^z becomes periodic in the imaginary direction; the logarithm splits into infinitely many *branches*; and the trigonometric functions lose their boundedness. This chapter is devoted to a careful study of these functions, their mapping properties, and the subtleties that arise from multi-valuedness.

3.1 The Complex Exponential Function

3.1.1 Definition and first properties

Definition 3.1 (Complex exponential)**.** For $z = x + iy \in \mathbb{C}$ (with $x, y \in \mathbb{R}$), the **complex exponential function** is defined by

$$\exp(z) \;=\; e^z \;=\; e^x(\cos y + i \sin y).$$

Equivalently, using the power series that converges for every $z \in \mathbb{C}$,

$$\exp(z) \;=\; \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Proposition 3.2 (Basic properties of exp)**.** *The complex exponential satisfies the following properties.*

(i) ***Homomorphism.*** *For all $z, w \in \mathbb{C}$, $e^{z+w} = e^z \cdot e^w$.*

(ii) ***Non-vanishing.*** *$e^z \neq 0$ for every $z \in \mathbb{C}$, and $(e^z)^{-1} = e^{-z}$.*

(iii) ***Modulus and argument.*** *$|e^z| = e^{\operatorname{Re} z}$ and $\arg(e^z) = \operatorname{Im} z + 2\pi\mathbb{Z}$.*

(iv) ***Periodicity.*** *$e^{z+2\pi i} = e^z$ for all $z \in \mathbb{C}$; moreover $2\pi i$ is a* fundamental period*: if $e^z = e^w$ then $z - w \in 2\pi i\mathbb{Z}$.*

(v) ***Holomorphicity.*** exp *is entire and* $\dfrac{\mathrm{d}}{\mathrm{d}z}e^z = e^z$.

Proof. **(i)** Using the power series, the Cauchy product gives

$$e^z \cdot e^w = \left(\sum_{n\geq 0}\frac{z^n}{n!}\right)\left(\sum_{m\geq 0}\frac{w^m}{m!}\right) = \sum_{k\geq 0}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}z^j w^{k-j} = \sum_{k\geq 0}\frac{(z+w)^k}{k!} = e^{z+w}.$$

(ii) From (i), $e^z \cdot e^{-z} = e^0 = 1$, so $e^z \neq 0$.

(iii) Write $z = x+iy$. Then $|e^z| = |e^x(\cos y + i\sin y)| = e^x\sqrt{\cos^2 y + \sin^2 y} = e^x = e^{\operatorname{Re} z}$. The argument statement follows from $\cos y + i\sin y = e^{iy}$.

(iv) $e^{z+2\pi i} = e^z e^{2\pi i} = e^z \cdot 1 = e^z$. If $e^z = e^w$, then $e^{z-w} = 1$, so $e^{x'-x''}(\cos(y'-y'') + i\sin(y'-y'')) = 1$ where $z - w = (x'-x'') + i(y'-y'')$. Taking moduli, $e^{x'-x''} = 1$ so $x' = x''$. Then $\cos(y'-y'') = 1$ and $\sin(y'-y'') = 0$, giving $y' - y'' \in 2\pi\mathbb{Z}$.

(v) The power series $\sum z^n/n!$ has infinite radius of convergence, so exp is entire. Term-by-term differentiation gives $\exp'(z) = \sum_{n\geq 1} z^{n-1}/(n-1)! = \exp(z)$. ☐

Remark 3.3 (Surjectivity). The map $\exp\colon \mathbb{C} \to \mathbb{C}$ is *not* surjective onto $\mathbb{C}$ (the value 0 is never attained), but it *is* surjective onto $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Indeed, given $w = \rho e^{i\theta} \neq 0$, we can choose $z = \ln\rho + i\theta$ and verify that $e^z = w$.

3.1.2 Mapping properties of exp

The exponential maps horizontal lines to rays from the origin and vertical lines to circles centered at the origin. More precisely:

Proposition 3.4 (Image of lines and strips)**.**

(i) *The horizontal line* $\{\, x + iy_0 : x \in \mathbb{R} \,\}$ *is mapped onto the ray* $\{\, re^{iy_0} : r > 0 \,\}$.

(ii) *The vertical line* $\{\, x_0 + iy : y \in \mathbb{R} \,\}$ *is mapped onto the circle* $\{\, e^{x_0}e^{iy} : y \in \mathbb{R} \,\} = \{w \in \mathbb{C} : |w| = e^{x_0}\}$.

(iii) *The horizontal strip* $S_\alpha = \{\, x + iy : x \in \mathbb{R},\ \alpha < y < \alpha + 2\pi \,\}$ *is mapped* bijectively *onto* $\mathbb{C}^*$.

Proof. Parts (i) and (ii) are immediate from $e^{x+iy} = e^x e^{iy}$. For (iii), surjectivity follows from Remark 3.3: given $w \in \mathbb{C}^*$, write $w = |w|\,e^{i\theta}$ with θ chosen uniquely in $(\alpha, \alpha + 2\pi)$, and set $z = \ln|w| + i\theta \in S_\alpha$. Injectivity follows from Proposition 3.2(iv): if $e^{z_1} = e^{z_2}$ with $z_1, z_2 \in S_\alpha$, then $z_1 - z_2 \in 2\pi i\mathbb{Z}$ and $|\mathrm{Im}(z_1 - z_2)| < 2\pi$, so $z_1 = z_2$. ☐

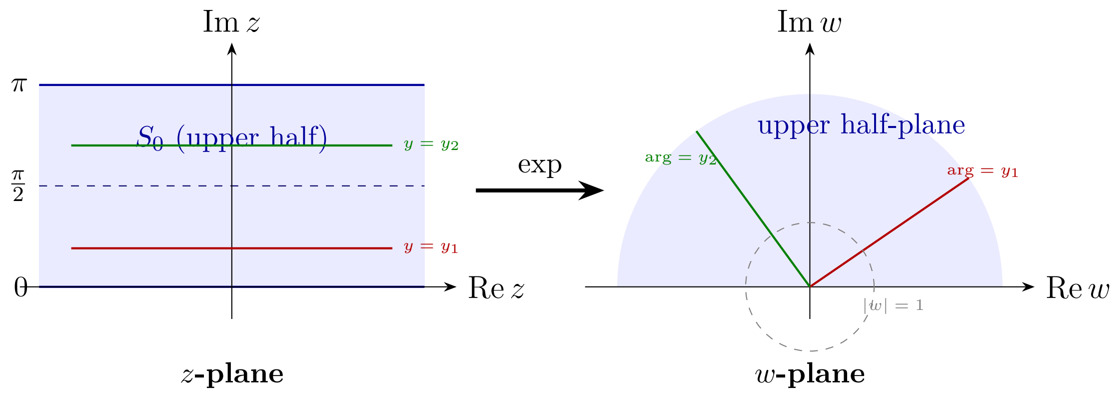


Figure 3.1: The exponential maps the strip $\{0 < \operatorname{Im} z < \pi\}$ bijectively onto the upper half of $\mathbb{C}^*$. Horizontal lines become rays; vertical lines (not shown) become circles.

Example 3.5 (Image of a rectangle)**.** Consider the rectangle $R = \{x + iy : 0 \leq x \leq 1,\ 0 \leq y \leq \pi/2\}$. Under exp:

- The bottom edge ($y = 0$, $0 \leq x \leq 1$) maps to the segment $[1, e]$ on the positive real axis.
- The top edge ($y = \pi/2$, $0 \leq x \leq 1$) maps to the segment $[i, ei]$ on the positive imaginary axis.
- The left edge ($x = 0$, $0 \leq y \leq \pi/2$) maps to the arc $\{e^{iy} : 0 \leq y \leq \pi/2\}$, a quarter of the unit circle.
- The right edge ($x = 1$, $0 \leq y \leq \pi/2$) maps to the arc $\{ee^{iy} : 0 \leq y \leq \pi/2\}$, a quarter of the circle of radius e.

The image is the sector $\{w \in \mathbb{C} : 1 \leq |w| \leq e,\ 0 \leq \arg w \leq \pi/2\}$.

3.2 The Complex Logarithm

3.2.1 Motivation and multi-valuedness

We seek a function log that "inverts" the exponential: $e^{\log w} = w$ for $w \in \mathbb{C}^*$. If $w = \rho e^{i\theta}$ with $\rho > 0$ and $\theta \in \mathbb{R}$, then $z = \ln \rho + i\theta$ satisfies $e^z = w$. But so does $z + 2k\pi i$ for every $k \in \mathbb{Z}$. The complex logarithm is therefore intrinsically *multi-valued*.

Definition 3.6 (Multi-valued logarithm)**.** For $w \in \mathbb{C}^*$, the **multi-valued logarithm** is the set

$$\log w \;=\; \Big\{ \ln|w| + i \arg w + 2k\pi i : k \in \mathbb{Z} \Big\} \;=\; \ln|w| + i \operatorname{Arg} w + 2\pi i \mathbb{Z},$$

where $\operatorname{Arg} w \in (-\pi, \pi]$ denotes the principal value of the argument. Each element of this set is called a **value** (or **determination**) of $\log w$.

3.2.2 The principal branch

Definition 3.7 (Principal logarithm)**.** The **principal logarithm** (or **principal branch of the logarithm**) is the function

$$\operatorname{Log}\colon \mathbb{C}\setminus(-\infty,0] \longrightarrow \mathbb{C}, \qquad \operatorname{Log} w \;=\; \ln|w| + i\operatorname{Arg} w,$$

where $\operatorname{Arg} w$ is the unique argument of w in $(-\pi,\pi)$. The set $(-\infty,0]=\{x\in\mathbb{R} : x\le 0\}$ is called the **branch cut**.

Remark 3.8. The branch cut $(-\infty,0]$ is a somewhat arbitrary choice. One could instead remove any ray from the origin to infinity and define a corresponding branch. The important point is that one must remove *some* curve from 0 to ∞ in order to obtain a single-valued, continuous logarithm. The reason is topological: the argument function cannot be defined continuously on all of $\mathbb{C}^*$, because $\mathbb{C}^*$ has a "hole" at the origin.

Theorem 3.9 (Properties of Log)**.**

(i) Log *is holomorphic on* $\mathbb{C}\setminus(-\infty,0]$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}z}\operatorname{Log} z \;=\; \frac{1}{z}.$$

(ii) $e^{\operatorname{Log} z}=z$ *for all* $z\in\mathbb{C}\setminus(-\infty,0]$.

(iii) $\operatorname{Log}(e^z)=z$ *if and only if* $\operatorname{Im} z\in(-\pi,\pi)$.

(iv) *The identity* $\operatorname{Log}(z_1z_2)=\operatorname{Log} z_1+\operatorname{Log} z_2$ *holds* modulo $2\pi i$*: there exists* $k\in\{-1,0,1\}$ *such that* $\operatorname{Log}(z_1z_2)=\operatorname{Log} z_1+\operatorname{Log} z_2+2k\pi i$.

Proof. **(i)** Write $z=re^{i\theta}$ with $r>0$ and $\theta\in(-\pi,\pi)$. In terms of $u=\operatorname{Re} z$ and $v=\operatorname{Im} z$ we have $r=\sqrt{u^2+v^2}$ and $\theta=\arctan(v/u)$ (with appropriate adjustments for $u\le 0$, but we have excluded the negative real axis). Then

$$\operatorname{Log} z=\tfrac{1}{2}\ln(u^2+v^2)+i\arctan(v/u).$$

Both components are smooth, and a direct computation verifies the Cauchy–Riemann equations. For the derivative, let $g=\operatorname{Log}$ and note $e^{g(z)}=z$, so differentiating: $g'(z)\cdot e^{g(z)}=1$, giving $g'(z)=1/e^{g(z)}=1/z$.

(ii) Let $z=re^{i\theta}$ with $\theta\in(-\pi,\pi)$. Then $e^{\operatorname{Log} z}=e^{\ln r+i\theta}=re^{i\theta}=z$.

(iii) $\operatorname{Log}(e^z)=\ln|e^z|+i\operatorname{Arg}(e^z)=\operatorname{Re} z+i\operatorname{Arg}(e^z)$. This equals $z=\operatorname{Re} z+i\operatorname{Im} z$ if and only if $\operatorname{Arg}(e^z)=\operatorname{Im} z$, which holds precisely when $\operatorname{Im} z\in(-\pi,\pi)$.

(iv) Both sides of $\operatorname{Log}(z_1z_2)=\operatorname{Log} z_1+\operatorname{Log} z_2$ are values of $\log(z_1z_2)$, so they differ by $2k\pi i$ for some $k\in\mathbb{Z}$. Since arguments lie in $(-\pi,\pi)$, the difference of the two sides has imaginary part in $(-3\pi,3\pi)$, forcing $k\in\{-1,0,1\}$. ☐

3.2.3 General branches of the logarithm

Definition 3.10 (Branch of the logarithm)**.** Let $\Omega \subseteq \mathbb{C}^*$ be a domain (connected open set). A **branch of the logarithm** on Ω is a continuous function $L\colon \Omega \to \mathbb{C}$ satisfying $e^{L(z)} = z$ for all $z \in \Omega$. Any such branch is necessarily holomorphic with $L'(z) = 1/z$, and any two branches on Ω differ by a constant $2k\pi i$ with $k \in \mathbb{Z}$.

Proposition 3.11 (Existence of branches)**.** *A branch of the logarithm exists on a domain $\Omega \subseteq \mathbb{C}^*$ if and only if the function $1/z$ has a primitive on Ω. In particular, a branch exists whenever Ω is* simply connected *(i.e., every closed curve in Ω can be continuously deformed to a point within Ω).*

Proof. If L is a branch of log on Ω, then $L'(z) = 1/z$, so L is a primitive of $1/z$. Conversely, if F is a primitive of $1/z$ on Ω, fix $z_0 \in \Omega$ and choose $c \in \mathbb{C}$ with $e^{F(z_0)+c} = z_0$ (this is possible since $z_0 \neq 0$). Define $L(z) = F(z)+c$. Then $\frac{\mathrm{d}}{\mathrm{d}z}\left[z\, e^{-L(z)}\right] = e^{-L(z)} - z \cdot \frac{1}{z} \cdot e^{-L(z)} = 0$, so $z\, e^{-L(z)}$ is constant on Ω, equal to $z_0\, e^{-L(z_0)} = 1$. Hence $e^{L(z)} = z$.

The existence of primitives on simply connected domains will follow from Cauchy's theorem (Chapter 4). □

3.2.4 Intuition of Riemann surfaces

Remark 3.12 (The logarithmic Riemann surface). Rather than choosing a branch, one can "unwind" all the values of $\log z$ into a single geometric object: the **Riemann surface** of the logarithm. Imagine infinitely many copies of $\mathbb{C}^*$, labelled by $k \in \mathbb{Z}$, each slit along the negative real axis. The upper lip of the k-th sheet is glued to the lower lip of the $(k+1)$-th sheet. The result is a *helicoid*-like surface on which the logarithm becomes a well-defined, single-valued holomorphic function.

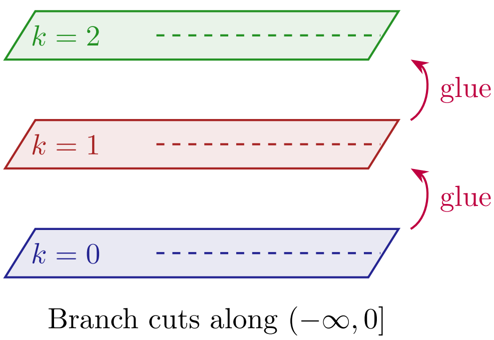


Figure 3.2: Schematic of the Riemann surface of $\log z$: infinitely many sheets, each carrying one branch $\ln|z| + i(\operatorname{Arg} z + 2k\pi)$, glued along the branch cut.

3.3 Complex Powers

Definition 3.13 (Complex power)**.** Let $\alpha \in \mathbb{C}$. For $z \in \mathbb{C} \setminus (-\infty, 0]$, the **principal value** of z^α is

$$z^\alpha = \exp\big(\alpha \operatorname{Log} z\big) = e^{\alpha \operatorname{Log} z}.$$

More generally, using any branch L of the logarithm on a domain $\Omega \subseteq \mathbb{C}^*$, one defines $z^\alpha = e^{\alpha L(z)}$.

Proposition 3.14 (Properties of complex powers)**.** *Let $\alpha, \beta \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. Using the principal branch:*

(i) *z^α is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ with $\frac{\mathrm{d}}{\mathrm{d}z} z^\alpha = \alpha\, z^{\alpha-1}$.*

(ii) *$z^\alpha \cdot z^\beta = z^{\alpha+\beta}$ (principal values).*

(iii) *$(z^\alpha)^\beta = z^{\alpha\beta}$ holds for principal values only up to a factor $e^{2k\pi i\beta}$ with $k \in \mathbb{Z}$.*

(iv) *If $\alpha = n \in \mathbb{Z}$, then z^n coincides with the usual integer power, and no branch cut is needed.*

(v) *If $\alpha = 1/n$ with $n \in \mathbb{Z}_{\geq 1}$, then $z^{1/n}$ gives the principal n-th root.*

Proof. **(i)** Since $\operatorname{Log} z$ and exp are holomorphic, so is their composition. By the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}z} e^{\alpha \operatorname{Log} z} = e^{\alpha \operatorname{Log} z} \cdot \frac{\alpha}{z} = \alpha \frac{z^\alpha}{z} = \alpha\, z^{\alpha-1}.$$

(ii) $z^\alpha \cdot z^\beta = e^{\alpha \operatorname{Log} z} \cdot e^{\beta \operatorname{Log} z} = e^{(\alpha+\beta) \operatorname{Log} z} = z^{\alpha+\beta}$.

(iii) $(z^\alpha)^\beta = e^{\beta \operatorname{Log}(z^\alpha)}$, and $\operatorname{Log}(z^\alpha)$ may differ from $\alpha \operatorname{Log} z$ by $2k\pi i$.

(iv) When $\alpha = n \in \mathbb{Z}$, $e^{n \operatorname{Log} z} = e^{n \ln|z| + in \operatorname{Arg} z} = |z|^n e^{in \operatorname{Arg} z} = z^n$ by Euler's formula. Since all branches of log give $e^{n \cdot (L(z))} = z^n$ (the $2k\pi i n$ disappears under exp), no branch cut is needed.

(v) Follows from the definition with $\alpha = 1/n$. □

Example 3.15 (Powers of i)**.** Since $\operatorname{Log} i = \ln 1 + i\,\frac{\pi}{2} = \frac{i\pi}{2}$, we have:

- $i^i = e^{i \operatorname{Log} i} = e^{i \cdot i\pi/2} = e^{-\pi/2} \approx 0.2079$. This is a *real* number!
- $i^{1/2} = e^{\frac{1}{2} \operatorname{Log} i} = e^{i\pi/4} = \frac{\sqrt{2}}{2}(1+i)$.
- More generally, $i^\alpha = e^{i\alpha\pi/2}$ for any $\alpha \in \mathbb{C}$ (principal value).

Using other branches $\log i = i\pi/2 + 2k\pi i$, we get the other values: $i^i = e^{-\pi/2 - 2k\pi}$ for $k \in \mathbb{Z}$.

Example 3.16 (The function $z^{1/2}$)**.** The principal square root $z^{1/2} = e^{\frac{1}{2} \operatorname{Log} z}$ satisfies:

- For $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi)$: $z^{1/2} = \sqrt{r}\, e^{i\theta/2}$.

- The image of $z^{1/2}$ is the right half-plane $\{\operatorname{Re} w > 0\}$ together with the positive imaginary axis.
- On the negative real axis, $z^{1/2}$ is discontinuous: approaching $x < 0$ from above gives $\sqrt{|x|}\, i$, from below gives $-\sqrt{|x|}\, i$.

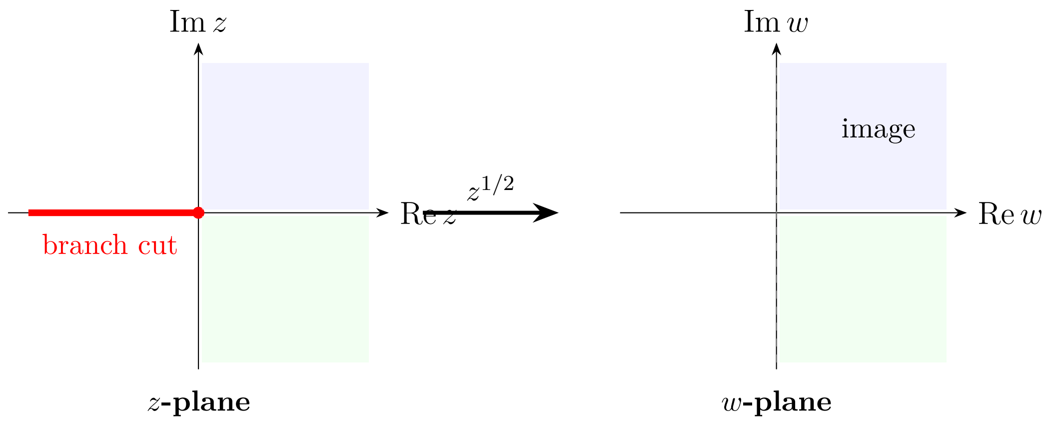


Figure 3.3: The principal square root maps $\mathbb{C} \setminus (-\infty, 0]$ bijectively onto the open right half-plane $\{\operatorname{Re} w > 0\}$.

3.4 Trigonometric and Hyperbolic Functions

3.4.1 Definitions via the exponential

Definition 3.17 (Complex trigonometric functions)**.** For $z \in \mathbb{C}$, the **complex cosine** and **complex sine** are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The other trigonometric functions are defined as usual:

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}, \qquad \sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z},$$

wherever the denominators are nonzero.

Definition 3.18 (Complex hyperbolic functions)**.** For $z \in \mathbb{C}$, the **hyperbolic cosine** and **hyperbolic sine** are defined by

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

Similarly, $\tanh z = \sinh z / \cosh z$, etc.

3.4.2 Properties

Proposition 3.19 (Properties of complex trigonometric functions)**.**

(i) ***Entirety.*** cos *and* sin *are entire functions with derivatives* $\cos' z = -\sin z$ *and* $\sin' z = \cos z$*.*

(ii) ***Euler identity.*** $e^{iz} = \cos z + i \sin z$ *for all* $z \in \mathbb{C}$*.*

(iii) ***Pythagorean identity.*** $\cos^2 z + \sin^2 z = 1$*.*

(iv) ***Periodicity.*** cos *and* sin *have period* 2π*:* $\cos(z + 2\pi) = \cos z$*,* $\sin(z + 2\pi) = \sin z$*.*

(v) ***Zeros.*** $\sin z = 0 \iff z = n\pi \ (n \in \mathbb{Z})$*;* $\cos z = 0 \iff z = (n + \frac{1}{2})\pi \ (n \in \mathbb{Z})$*.*

(vi) ***Unboundedness.*** *Unlike the real case,* cos *and* sin *are unbounded on* $\mathbb{C}$*. For instance,* $|\sin(iy)| = |\sinh y| \to \infty$ *as* $|y| \to \infty$*.*

(vii) ***Addition formulas.*** *All real addition formulas extend to* $\mathbb{C}$*:*

$$\cos(z + w) = \cos z \cos w - \sin z \sin w,$$
$$\sin(z + w) = \sin z \cos w + \cos z \sin w.$$

Proof. All these follow from the definitions and properties of exp.

(i) is immediate since e^{iz} and e^{-iz} are entire.

(ii) $\cos z + i \sin z = \frac{e^{iz}+e^{-iz}}{2} + i \cdot \frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} + \frac{e^{iz}-e^{-iz}}{2} = e^{iz}$.

(iii) $\cos^2 z + \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$. Expanding: $= \frac{e^{2iz}+2+e^{-2iz}}{4} - \frac{e^{2iz}-2+e^{-2iz}}{4} = \frac{4}{4} = 1$.

(v) $\sin z = 0 \iff e^{iz} = e^{-iz} \iff e^{2iz} = 1 \iff 2iz \in 2\pi i\mathbb{Z} \iff z \in \pi\mathbb{Z}$. The argument for cos is similar.

(vi) $\sin(iy) = \frac{e^{-y}-e^{y}}{2i} = i\frac{e^{y}-e^{-y}}{2} = i \sinh y$, so $|\sin(iy)| = |\sinh y| \to \infty$. □

Proposition 3.20 (Relations between trigonometric and hyperbolic functions)**.** *For all* $z \in \mathbb{C}$*:*

$$\cos(iz) = \cosh z, \qquad \sin(iz) = i \sinh z, \qquad \cosh(iz) = \cos z, \qquad \sinh(iz) = i \sin z.$$

Proof. Direct computation: $\cos(iz) = \frac{e^{i(iz)}+e^{-i(iz)}}{2} = \frac{e^{-z}+e^{z}}{2} = \cosh z$. The other identities are proved similarly. □

Proposition 3.21 (Decomposition of sin and cos into real and imaginary parts)**.** *Writing* $z = x + iy$ *with* $x, y \in \mathbb{R}$*:*

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$
$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

In particular,

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y.$$

Proof. By the addition formula, $\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy) = \sin x \cosh y + i \cos x \sinh y$. Then $|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x(1+\sinh^2 y) + (1 - \sin^2 x)\sinh^2 y = \sin^2 x + \sinh^2 y$. The computation for $\cos z$ is analogous. □

3.4.3 Mapping properties of sin and cos

Example 3.22 (Image of a vertical strip under sin)**.** Consider $\sin z$ restricted to the strip $S = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$. The function sin maps S conformally onto $\mathbb{C}$ minus the two rays $(-\infty, -1]$ and $[1, +\infty)$ on the real axis. This is because:

- On the boundary $\operatorname{Re} z = \pm\pi/2$, $\sin z$ takes values on the real axis with $|\sin z| \geq 1$ (since $\sin(\pm\pi/2 + iy) = \pm\cosh y$).
- On the real segment $(-\pi/2, \pi/2)$, sin maps onto $(-1, 1)$.
- The imaginary axis is mapped onto itself (since $\sin(iy) = i \sinh y$).

3.5 Inverse Trigonometric Functions

Proposition 3.23 (Complex arcsine)**.** *The equation* $\sin w = z$ *can be solved for* w *in terms of* z*:*

$$\arcsin z = -i \operatorname{Log}\Big(iz + (1-z^2)^{1/2}\Big),$$

where $(1-z^2)^{1/2}$ *denotes a branch of the square root. This defines a multi-valued function; the principal branch is obtained by using the principal* Log *and the principal square root.*

Proof. Let w satisfy $\sin w = z$, i.e., $\frac{e^{iw} - e^{-iw}}{2i} = z$. Set $\zeta = e^{iw}$. Then $\zeta - \zeta^{-1} = 2iz$, so $\zeta^2 - 2iz\zeta - 1 = 0$. By the quadratic formula,

$$\zeta = iz + \sqrt{1-z^2},$$

(choosing a square root). Then $iw = \log\zeta$, so $w = -i\log\zeta$. □

Remark 3.24. Similarly, $\arccos z = -i \operatorname{Log}\Big(z + i(1-z^2)^{1/2}\Big)$ and $\arctan z = \frac{1}{2i}\operatorname{Log}\Big(\frac{1+iz}{1-iz}\Big)$.

3.6 Worked Examples

Example 3.25 (Computing Log for various values)**.**

(a) $\operatorname{Log}(-1) = \ln 1 + i\pi = i\pi$. (Since $\operatorname{Arg}(-1) = \pi\ldots$ but wait: we defined Arg to lie in $(-\pi, \pi)$. This means -1 lies on the branch cut, so $\operatorname{Log}(-1)$ is *not defined* under our convention. If we adopt the convention $\operatorname{Arg} \in (-\pi, \pi]$, then $\operatorname{Log}(-1) = i\pi$.)

(b) $\operatorname{Log}(-i) = \ln 1 + i(-\pi/2) = -i\pi/2$.

(c) $\operatorname{Log}(1+i) = \ln\sqrt{2} + i\pi/4 = \frac{1}{2}\ln 2 + i\pi/4$.

(d) $\operatorname{Log}(e) = \ln e + i \cdot 0 = 1$.

(e) $\operatorname{Log}(-e^2) = \ln(e^2) + i\pi = 2 + i\pi$ (if we allow $\operatorname{Arg}(-e^2) = \pi$).

Example 3.26 (Computing complex powers)**.**

(a) $(-1)^i = e^{i\operatorname{Log}(-1)}$. Using $\operatorname{Log}(-1) = i\pi$: $(-1)^i = e^{i \cdot i\pi} = e^{-\pi} \approx 0.0432$.

(b) $2^i = e^{i\operatorname{Log} 2} = e^{i\ln 2} = \cos(\ln 2) + i\sin(\ln 2) \approx 0.7692 + 0.6390\, i$.

(c) $(1+i)^{1+i}$: We have $\operatorname{Log}(1+i) = \frac{1}{2}\ln 2 + i\frac{\pi}{4}$. Then

$$\begin{aligned}(1+i)^{1+i} &= e^{(1+i)(\frac{1}{2}\ln 2 + i\frac{\pi}{4})} = e^{\frac{1}{2}\ln 2 - \frac{\pi}{4} + i(\frac{1}{2}\ln 2 + \frac{\pi}{4})} \\ &= e^{\frac{1}{2}\ln 2 - \frac{\pi}{4}}\Big[\cos\Big(\tfrac{1}{2}\ln 2 + \tfrac{\pi}{4}\Big) + i\sin\Big(\tfrac{1}{2}\ln 2 + \tfrac{\pi}{4}\Big)\Big].\end{aligned}$$

Numerically, $|(1+i)^{1+i}| = e^{(\ln 2)/2 - \pi/4} \approx 0.6440$.

Example 3.27 (Solving $e^z = -2$)**.** We need z such that $e^z = -2$. Since $-2 = 2e^{i\pi}$, the general solution is $z = \ln 2 + i(\pi + 2k\pi) = \ln 2 + i(2k+1)\pi$ for $k \in \mathbb{Z}$.

Example 3.28 (Solving $\cos z = 3$)**.** This equation has *no real solution*, since $|\cos x| \le 1$ for $x \in \mathbb{R}$. But it has complex solutions: $\cos z = 3$ means $\frac{e^{iz} + e^{-iz}}{2} = 3$, so $e^{2iz} - 6e^{iz} + 1 = 0$. Setting $\zeta = e^{iz}$: $\zeta = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$. Both values are positive reals, so $iz = \ln(3 \pm 2\sqrt{2}) + 2k\pi i$, hence $z = 2k\pi - i\ln(3 \pm 2\sqrt{2})$ for $k \in \mathbb{Z}$.
Note: $\ln(3 - 2\sqrt{2}) = -\ln(3 + 2\sqrt{2})$ since $(3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 1$. So the solutions are $z = 2k\pi \pm i\ln(3 + 2\sqrt{2})$ for $k \in \mathbb{Z}$.

3.7 Exercises

Exercise 3.1 (Exponential equations)**.** Solve the following equations for $z \in \mathbb{C}$.

(a) $e^z = 1 + i$.

(b) $e^z = -e^3$.

(c) $e^{2z} - 3e^z + 2 = 0$.

(d) $e^z + e^{-z} = 1$.

Exercise 3.2 (Logarithm computations)**.** Compute all values of $\log z$ and the principal value $\operatorname{Log} z$ for:

(a) $z = -3$.

(b) $z = 1 - i$.

(c) $z = -1 - i\sqrt{3}$.

(d) $z = e^{2+i\pi/3}$.

Exercise 3.3 (Complex powers)**.** Compute the principal value and all values of:

(a) $(-1)^{1/3}$.

(b) $(1 - i)^{2i}$.

(c) 3^{3i}.

(d) $i^{1/i}$.

Exercise 3.4 (Trigonometric equations)**.** Find all $z \in \mathbb{C}$ satisfying:

(a) $\sin z = 2$.

(b) $\cos z = i$.

(c) $\tan z = 2i$.

Exercise 3.5 (Mapping properties)**.**

(a) Describe the image of the strip $\{z : 0 < \operatorname{Im} z < \pi/4\}$ under exp.

(b) Describe the image of the strip $\{z : -\pi < \operatorname{Im} z < 0\}$ under exp.

(c) Show that $w = e^z$ maps the rectangle $\{a < \operatorname{Re} z < b,\ \alpha < \operatorname{Im} z < \beta\}$ (with $\beta - \alpha < 2\pi$) onto the region $\{e^a < |w| < e^b,\ \alpha < \arg w < \beta\}$.

Exercise 3.6 (Identities)**.** Prove the following for all $z \in \mathbb{C}$:

(a) $|\cos z|^2 + |\sin z|^2 = \cosh(2y) + \cos(2x)$ does *not* hold. Find the correct identity.

(b) $\cos(2z) = \cos^2 z - \sin^2 z$.

(c) $\sin(z + \pi/2) = \cos z$.

(d) $\cosh^2 z - \sinh^2 z = 1$.

Exercise 3.7 (Branches of the logarithm)**.** Let $\Omega = \mathbb{C} \setminus [0, +\infty)$ (the plane with the non-negative real axis removed).

(a) Show that a branch of log exists on Ω with $\operatorname{Im}(\log z) \in (0, 2\pi)$.

(b) Compute this branch at $z = -1$, $z = i$, $z = -i$.

(c) Is this branch continuous on Ω? Holomorphic?

Exercise 3.8 (Continuity across the branch cut)**.** Show that $\operatorname{Log} z$ is discontinuous along $(-\infty, 0)$: for $x < 0$,

$$\lim_{\varepsilon \to 0^+} \operatorname{Log}(x + i\varepsilon) - \lim_{\varepsilon \to 0^+} \operatorname{Log}(x - i\varepsilon) = 2\pi i.$$

This "jump" of $2\pi i$ is characteristic of the branch cut.

Exercise 3.9 (The function $z^{1/3}$)**.** Using the principal branch of Log:

(a) Show that $z^{1/3} = |z|^{1/3} e^{i \operatorname{Arg}(z)/3}$ on $\mathbb{C} \setminus (-\infty, 0]$.

(b) Determine the image of $\mathbb{C} \setminus (-\infty, 0]$ under $z \mapsto z^{1/3}$.

(c) Find all three cube roots of -8 and identify which one is the principal value.

Exercise 3.10 (The Joukowski function)**.** The **Joukowski function** $J(z) = \frac{1}{2}(z + 1/z)$ appears in aerodynamics.

(a) Show that J maps circles $|z| = r > 1$ to ellipses, and circles $|z| = 1$ to the segment $[-1, 1]$.

(b) Show that J is conformal on $\{|z| > 1\}$.

(c) Express $\cos z$ in terms of J and exp.

Chapter 4

Complex Integration — Cauchy's Theorem

Introduction

Integration in the complex plane is a far richer theory than its real counterpart. The integral $\int_\gamma f(z)\,\mathrm{d}z$ of a complex-valued function along a curve γ depends, in general, on the *path* taken — not just on the endpoints. The spectacular discovery of Cauchy is that for *holomorphic* functions, this path-dependence disappears under topological conditions on the domain: the integral around any closed curve that can be continuously shrunk to a point is *zero*. This is Cauchy's theorem, the cornerstone of complex analysis, from which an enormous number of consequences will flow in the chapters ahead.

4.1 Curves and Contours

Definition 4.1 (Parametrized curve)**.** A **parametrized curve** (or **path**) in $\mathbb{C}$ is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$ where $[a,b] \subseteq \mathbb{R}$ is a closed interval. We call $\gamma(a)$ the **initial point** and $\gamma(b)$ the **terminal point**. The **trace** (or **image**) of γ is the set $\gamma^* = \gamma([a,b]) \subseteq \mathbb{C}$.

Definition 4.2 (Smooth and piecewise smooth curves)**.** A parametrized curve $\gamma\colon [a,b] \to \mathbb{C}$ is **smooth** if γ is continuously differentiable on $[a,b]$ (with one-sided derivatives at the endpoints) and $\gamma'(t) \neq 0$ for all $t \in [a,b]$.
A curve is **piecewise smooth** (or a **contour**) if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_{k-1},t_k]}$ is smooth for each $k = 1,\ldots,n$.

Definition 4.3 (Closed curve and simple curve)**.** A curve $\gamma\colon [a,b] \to \mathbb{C}$ is **closed** if $\gamma(a) = \gamma(b)$. It is **simple** (or a **Jordan curve**) if γ is injective on $[a,b)$, i.e., γ does not cross itself except possibly at the endpoints. A simple closed curve is called a **Jordan curve**.

Definition 4.4 (Opposite curve)**.** Given a curve $\gamma\colon [a,b] \to \mathbb{C}$, the **opposite curve** (or **reverse**) is $\gamma^-\colon [a,b] \to \mathbb{C}$ defined by $\gamma^-(t) = \gamma(a+b-t)$. It traverses the same trace in the opposite direction.

Definition 4.5 (Concatenation)**.** If $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [b,c] \to \mathbb{C}$ satisfy $\gamma_1(b) = \gamma_2(b)$, their **concatenation** is the curve $\gamma_1 + \gamma_2\colon [a,c] \to \mathbb{C}$ defined by

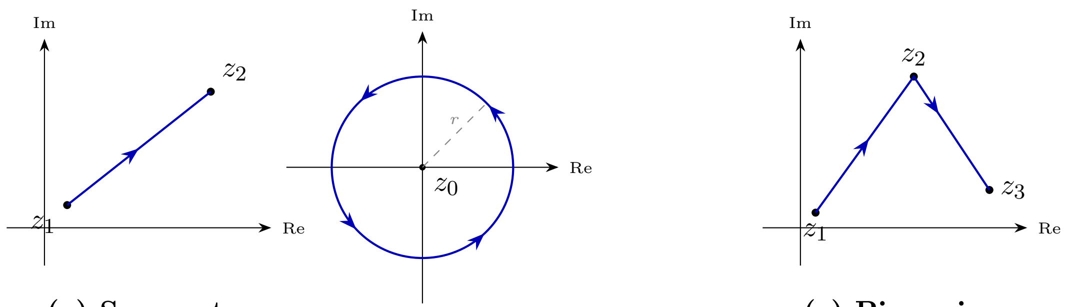
$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a,b], \\ \gamma_2(t) & \text{if } t \in [b,c]. \end{cases}$$

Example 4.6 (Standard parametrizations)**.**

(a) **Line segment.** The segment from z_1 to z_2: $\gamma(t) = (1-t)z_1 + tz_2$, $t \in [0,1]$. Then $\gamma'(t) = z_2 - z_1$.

(b) **Circle.** The circle of center z_0 and radius $r > 0$, traversed counterclockwise: $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then $\gamma'(t) = ire^{it}$.

(c) **Circular arc.** The arc from angle α to β: $\gamma(t) = z_0 + re^{it}$, $t \in [\alpha, \beta]$.



(a) Segment **(b) Circle** **(c) Piecewise**

Figure 4.1: Standard types of contours: (a) a line segment, (b) a circle traversed counterclockwise, (c) a piecewise linear path.

4.2 The Complex Line Integral

Definition 4.7 (Complex line integral)**.** Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth curve, and let $f\colon \gamma^* \to \mathbb{C}$ be continuous. The **complex line integral** of f along γ is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f\big(\gamma(t)\big)\,\gamma'(t)\,\mathrm{d}t.$$

Remark 4.8. The right-hand side is a standard Riemann integral of a complex-valued function of a real variable: if $g\colon [a,b] \to \mathbb{C}$ is continuous, then $\int_a^b g(t)\,\mathrm{d}t = \int_a^b \operatorname{Re} g(t)\,\mathrm{d}t +$

$i\int_a^b \operatorname{Im} g(t)\,\mathrm{d}t$.

Proposition 4.9 (Elementary properties). *Let f, g be continuous on the traces of the indicated curves, and let $\alpha \in \mathbb{C}$.*

(i) ***Linearity.*** $\displaystyle\int_\gamma (\alpha f + g)\,\mathrm{d}z = \alpha \int_\gamma f\,\mathrm{d}z + \int_\gamma g\,\mathrm{d}z.$

(ii) ***Reversal.*** $\displaystyle\int_{\gamma^-} f\,\mathrm{d}z = -\int_\gamma f\,\mathrm{d}z.$

(iii) ***Concatenation.*** *If $\gamma = \gamma_1 + \gamma_2$, then* $\displaystyle\int_\gamma f\,\mathrm{d}z = \int_{\gamma_1} f\,\mathrm{d}z + \int_{\gamma_2} f\,\mathrm{d}z.$

(iv) ***Reparametrization invariance.*** *The integral depends only on the oriented curve, not on the particular parametrization.*

Proof. All parts follow from the corresponding properties of the real Riemann integral.

(ii) Let $\gamma\colon [a,b] \to \mathbb{C}$. Setting $s = a + b - t$:

$$\int_{\gamma^-} f\,\mathrm{d}z = \int_a^b f(\gamma(a+b-t)) \cdot (-\gamma'(a+b-t))\,\mathrm{d}t = -\int_a^b f(\gamma(s))\gamma'(s)\,\mathrm{d}s = -\int_\gamma f\,\mathrm{d}z.$$

The other parts are left as exercises. □

Theorem 4.10 (ML-inequality). *If γ is a piecewise smooth curve of length $L(\gamma)$ and $|f(z)| \le M$ for all $z \in \gamma^*$, then*

$$|[|\Big]\int_\gamma f(z)\,\mathrm{d}z \;\le\; M \cdot L(\gamma),$$

*where $L(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t$ is the **arc length** of γ.*

Proof. Using the standard estimate for integrals of complex-valued functions:

$$|[|\Big]\int_\gamma f(z)\,\mathrm{d}z = |[|\Big]\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t \le M\int_a^b |\gamma'(t)|\,\mathrm{d}t = ML(\gamma).$$

□

4.2.1 First computations

Example 4.11 (Integral of z^n around a circle). Let $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, and let $n \in \mathbb{Z}$. We compute $\int_\gamma (z - z_0)^n\,\mathrm{d}z$.
We have $z - z_0 = re^{it}$ and $\mathrm{d}z = ire^{it}\,\mathrm{d}t$, so

$$\int_\gamma (z - z_0)^n\,\mathrm{d}z = \int_0^{2\pi} r^n e^{int} \cdot ire^{it}\,\mathrm{d}t = ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,\mathrm{d}t.$$

Case $n \neq -1$:

$$\int_0^{2\pi} e^{i(n+1)t}\,\mathrm{d}t = \left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_0^{2\pi} = \frac{e^{2\pi i(n+1)}-1}{i(n+1)} = 0,$$

since $e^{2\pi i(n+1)} = 1$.
Case $n = -1$:

$$\int_\gamma \frac{\mathrm{d}z}{z-z_0} = ir^0 \int_0^{2\pi} e^0\,\mathrm{d}t = i\cdot 2\pi = 2\pi i.$$

Therefore:

$$\boxed{\int_\gamma (z-z_0)^n\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n=-1,\\ 0 & \text{if } n\in\mathbb{Z},\ n\neq -1.\end{cases}} \tag{4.1}$$

This result is fundamental and will be used repeatedly.

Example 4.12 (Integral of $\overline{z}$ around the unit circle)**.** Let $\gamma(t) = e^{it}$, $t\in[0,2\pi]$. Note that $\overline{z}$ is *not* holomorphic. We compute:

$$\int_\gamma \overline{z}\,\mathrm{d}z = \int_0^{2\pi} \overline{e^{it}}\cdot ie^{it}\,\mathrm{d}t = \int_0^{2\pi} e^{-it}\cdot ie^{it}\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i.$$

Example 4.13 (Integral along different paths)**.** Compute $\int_\gamma z\,\mathrm{d}z$ where γ goes from 0 to $1+i$.
Path 1: the straight line $\gamma_1(t) = (1+i)t$, $t\in[0,1]$. $\gamma_1'(t) = 1+i$, so

$$\int_{\gamma_1} z\,\mathrm{d}z = \int_0^1 (1+i)t\cdot(1+i)\,\mathrm{d}t = (1+i)^2\int_0^1 t\,\mathrm{d}t = 2i\cdot\tfrac{1}{2} = i.$$

Path 2: first along the real axis from 0 to 1, then vertically from 1 to $1+i$. Let $\gamma_2 = \alpha+\beta$ with $\alpha(t) = t$, $t\in[0,1]$, and $\beta(t) = 1+it$, $t\in[0,1]$.

$$\begin{aligned}\int_\alpha z\,\mathrm{d}z &= \int_0^1 t\cdot 1\,\mathrm{d}t = \tfrac{1}{2},\\ \int_\beta z\,\mathrm{d}z &= \int_0^1 (1+it)\cdot i\,\mathrm{d}t = i\int_0^1 \mathrm{d}t - \int_0^1 t\,\mathrm{d}t = i-\tfrac{1}{2}.\end{aligned}$$

So $\int_{\gamma_2} z\,\mathrm{d}z = \frac{1}{2}+i-\frac{1}{2} = i$.
Both paths give the same answer. This is no accident: $f(z) = z$ is entire and has a primitive $F(z) = z^2/2$.

4.3 Primitives and Path Independence

Definition 4.14 (Primitive)**.** Let $\Omega\subseteq\mathbb{C}$ be open and $f\colon\Omega\to\mathbb{C}$ continuous. A **primitive** (or **antiderivative**) of f on Ω is a holomorphic function $F\colon\Omega\to\mathbb{C}$ such that $F'(z) = f(z)$ for all $z\in\Omega$.

Theorem 4.15 (Fundamental theorem of contour integration). *Let $\Omega \subseteq \mathbb{C}$ be open, $f\colon \Omega \to \mathbb{C}$ continuous, and suppose f has a primitive F on Ω. Then for every piecewise smooth curve $\gamma\colon [a,b] \to \Omega$,*

$$\int_\gamma f(z)\,\mathrm{d}z \;=\; F(\gamma(b)) - F(\gamma(a)). \tag{4.2}$$

In particular, if γ is closed then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

Proof. By the chain rule (for functions of a real variable composed with holomorphic functions),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[F(\gamma(t))\Big] = F'(\gamma(t))\cdot\gamma'(t) = f(\gamma(t))\cdot\gamma'(t).$$

Hence

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\Big[F(\gamma(t))\Big]\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a)).$$ □

Corollary 4.16. *If f has a primitive on a domain Ω, then the integral $\int_\gamma f\,\mathrm{d}z$ depends only on the endpoints of γ, not on the particular path within Ω.*

Example 4.17 (Using the fundamental theorem).

(a) $\displaystyle\int_\gamma z^3\,\mathrm{d}z = \left[\frac{z^4}{4}\right]_0^{1+i} = \frac{(1+i)^4}{4} = \frac{-4}{4} = -1$ for any path from 0 to $1+i$, since $z^4/4$ is a primitive.

(b) $\displaystyle\int_\gamma e^z\,\mathrm{d}z = \Big[e^z\Big]_0^{i\pi} = e^{i\pi} - e^0 = -1-1 = -2$ for any path from 0 to $i\pi$.

(c) $\displaystyle\int_\gamma \frac{\mathrm{d}z}{z}$ from 1 to -1 **depends on the path**: through the upper half-plane gives $i\pi$, through the lower gives $-i\pi$. There is no primitive of $1/z$ on all of $\mathbb{C}^*$.

Theorem 4.18 (Characterization of existence of primitives). *Let $\Omega \subseteq \mathbb{C}$ be a domain and $f\colon \Omega \to \mathbb{C}$ continuous. The following are equivalent:*

(i) *f has a primitive on Ω.*

(ii) *$\int_\gamma f\,\mathrm{d}z = 0$ for every closed piecewise smooth curve γ in Ω.*

(iii) *$\int_\gamma f\,\mathrm{d}z$ depends only on the endpoints for every piecewise smooth curve γ in Ω.*

Proof. (i)⇒(ii): Theorem 4.15.

(ii)⇒(iii): If γ_1 and γ_2 have the same endpoints, then $\gamma_1 + \gamma_2^-$ is closed, so $0 = \int_{\gamma_1+\gamma_2^-} f\,\mathrm{d}z = \int_{\gamma_1} f\,\mathrm{d}z - \int_{\gamma_2} f\,\mathrm{d}z$.

(iii)⇒(i): Fix $z_0 \in \Omega$ and define $F(z) = \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta$ where γ_z is any piecewise smooth path from z_0 to z in Ω. By (iii), F is well-defined. For h small enough,

$$F(z+h) - F(z) = \int_{[z,z+h]} f(\zeta)\,\mathrm{d}\zeta = f(z)\cdot h + \int_{[z,z+h]} \Big(f(\zeta) - f(z)\Big)\,\mathrm{d}\zeta.$$

By continuity of f, the last integral is $o(|h|)$, so $F'(z) = \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$. □

4.4 Cauchy's Theorem — Statement and Proof

Cauchy's theorem asserts that the integral of a holomorphic function around a closed contour is zero, under appropriate topological conditions on the domain or the contour. We present several versions, starting with the simplest.

4.4.1 Simply connected domains

Definition 4.19 (Simply connected domain). A domain $\Omega \subseteq \mathbb{C}$ is **simply connected** if every closed curve in Ω can be continuously deformed (within Ω) to a point. Equivalently, Ω has "no holes": $\mathbb{C} \setminus \Omega$ is connected in $\mathbb{C} \cup \{\infty\}$ (the Riemann sphere).

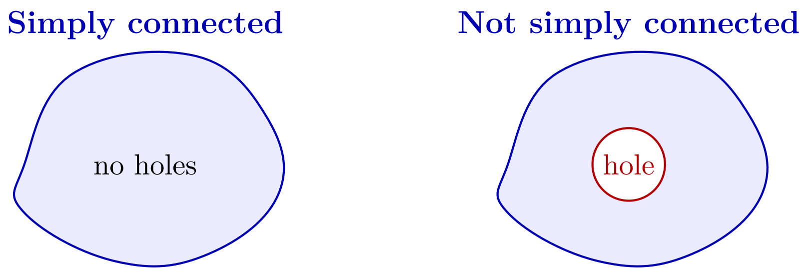


Figure 4.2: A simply connected domain (left) versus a multiply connected domain with a hole (right).

Example 4.20 (Simply connected domains). The following are simply connected:

- $\mathbb{C}$ itself, and any open disk $D(z_0, R)$.
- Any convex open set (e.g., a half-plane, a strip).
- Any star-shaped domain: a domain Ω such that there exists $z_0 \in \Omega$ with $[z_0, z] \subseteq \Omega$ for all $z \in \Omega$.
- $\mathbb{C} \setminus (-\infty, 0]$ (the slit plane).

The following are *not* simply connected:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (a loop around the origin cannot be shrunk).
- An annulus $\{z : r < |z| < R\}$.

4.4.2 Cauchy's theorem for triangles (Goursat's lemma)

We begin with a version that requires only holomorphicity — no assumption on the continuity of the derivative.

Theorem 4.21 (Goursat's lemma)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{C}$ holomorphic. If Δ is a closed triangle (together with its interior) contained in Ω, and $\partial\Delta$ denotes its boundary traversed counterclockwise, then*

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z \;=\; 0.$$

Proof. We use a bisection argument. Let $\Delta^{(0)} = \Delta$ and let $I_0 = \int_{\partial\Delta^{(0)}} f(z)\,\mathrm{d}z$. Join the midpoints of the three sides of $\Delta^{(0)}$ to form four congruent triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. Orienting each sub-triangle counterclockwise, the integrals over the internal edges cancel in pairs, so

$$I_0 = \sum_{j=1}^{4} \int_{\partial\Delta_j} f(z)\,\mathrm{d}z.$$

By the triangle inequality, at least one of the four sub-triangles, call it $\Delta^{(1)}$, satisfies

$$\|[|\Big]\int_{\partial\Delta^{(1)}} f(z)\,\mathrm{d}z \;\geq\; \frac{|I_0|}{4}.$$

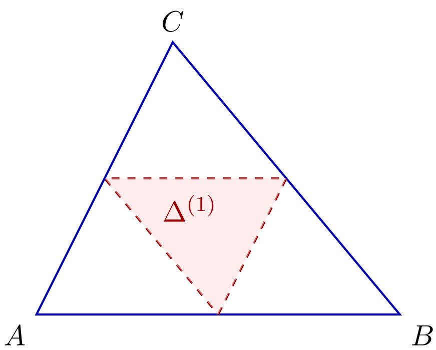


Figure 4.3: Bisection in Goursat's proof: the triangle $\Delta^{(0)}$ is subdivided into four; we select the sub-triangle with the largest integral.

Repeating inductively, we obtain a nested sequence of triangles $\Delta^{(0)} \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \cdots$ with

$$\|[|\Big]\int_{\partial\Delta^{(n)}} f(z)\,\mathrm{d}z \geq \frac{|I_0|}{4^n}, \quad L(\partial\Delta^{(n)}) = \frac{L}{2^n}, \quad \operatorname{diam}(\Delta^{(n)}) = \frac{d}{2^n},$$

where $L = L(\partial\Delta^{(0)})$ and $d = \operatorname{diam}(\Delta^{(0)})$.

By the nested compact sets theorem, $\bigcap_n \Delta^{(n)} = \{z_*\}$ for some $z_* \in \Omega$. Since f is holomorphic at z_*,

$$f(z) = f(z_*) + f'(z_*)(z - z_*) + \varphi(z)(z - z_*),$$

where $\varphi(z) \to 0$ as $z \to z_*$. The function $g(z) = f(z_*) + f'(z_*)(z - z_*)$ is a polynomial and thus has a primitive, so $\int_{\partial\Delta^{(n)}} g(z)\,\mathrm{d}z = 0$. Therefore

$$\int_{\partial\Delta^{(n)}} f(z)\,\mathrm{d}z = \int_{\partial\Delta^{(n)}} \varphi(z)(z - z_*)\,\mathrm{d}z.$$

For any $\varepsilon > 0$, there exists N such that for $n \geq N$, $|\varphi(z)| < \varepsilon$ on $\Delta^{(n)}$. Then by the ML-inequality:

$$\|[|\Big]\int_{\partial\Delta^{(n)}} f(z)\,\mathrm{d}z \leq \varepsilon \cdot \frac{d}{2^n} \cdot \frac{L}{2^n} = \frac{\varepsilon\, dL}{4^n}.$$

Combined with the lower bound:

$$\frac{|I_0|}{4^n} \le \frac{\varepsilon\, dL}{4^n},$$

so $|I_0| \le \varepsilon\, dL$. Since $\varepsilon > 0$ is arbitrary, $I_0 = 0$. □

4.4.3 Cauchy's theorem for convex and star-shaped domains

Theorem 4.22 (Cauchy's theorem for star-shaped domains)**.** *Let $\Omega \subseteq \mathbb{C}$ be a star-shaped domain, and let $f\colon \Omega \to \mathbb{C}$ be holomorphic. Then:*

(i) *f has a primitive F on Ω.*

(ii) *$\displaystyle\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed piecewise smooth curve γ in Ω.*

Proof. Let z_0 be a star center of Ω, i.e., $[z_0, z] \subseteq \Omega$ for all $z \in \Omega$. Define

$$F(z) = \int_{[z_0,z]} f(\zeta)\,\mathrm{d}\zeta = \int_0^1 f\Big(z_0 + t(z - z_0)\Big)(z - z_0)\,\mathrm{d}t.$$

We show $F'(z) = f(z)$. For h small, consider the triangle with vertices $z_0, z, z + h$. By Goursat's lemma (Theorem 4.21),

$$\int_{[z_0,z]} f\,\mathrm{d}\zeta + \int_{[z,z+h]} f\,\mathrm{d}\zeta + \int_{[z+h,z_0]} f\,\mathrm{d}\zeta = 0.$$

Hence $F(z + h) - F(z) = \int_{[z,z+h]} f(\zeta)\,\mathrm{d}\zeta$. Since $\int_{[z,z+h]} \mathrm{d}\zeta = h$, we can write

$$F(z + h) - F(z) - f(z)h = \int_{[z,z+h]} \Big(f(\zeta) - f(z)\Big)\,\mathrm{d}\zeta.$$

By continuity of f and the ML-inequality, $|F(z + h) - F(z) - f(z)h| \le \sup_{\zeta\in[z,z+h]} |f(\zeta) - f(z)| \cdot |h| = o(|h|)$. Thus $F'(z) = f(z)$. Part (ii) follows from Theorem 4.15. □

4.4.4 Cauchy's theorem via Green's theorem

When f' is continuous (not just existent), one can give a short proof using Green's theorem from multivariable calculus. Although this assumption is slightly stronger, the argument is instructive.

Theorem 4.23 (Green's theorem — recalled)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary $\partial\Omega$ oriented counterclockwise. If $P, Q\colon \overline{\Omega} \to \mathbb{R}$ are C^1, then*

$$\oint_{\partial\Omega} (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

Theorem 4.24 (Cauchy's theorem via Green's theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piecewise smooth boundary $\partial\Omega$ oriented positively, and let f be holomorphic* *on an open set containing $\overline{\Omega}$, with f' continuous. Then*

$$\oint_{\partial\Omega} f(z)\,\mathrm{d}z \;=\; 0.$$

Proof. Write $f = u + iv$ and $z = x + iy$, so $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$. Then

$$f(z)\,\mathrm{d}z = (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) = (u\,\mathrm{d}x - v\,\mathrm{d}y) + i(v\,\mathrm{d}x + u\,\mathrm{d}y).$$

By Green's theorem applied to each real part:

$$\mathrm{Re}\oint_{\partial\Omega} f\,\mathrm{d}z = \oint_{\partial\Omega}(u\,\mathrm{d}x - v\,\mathrm{d}y) = \iint_{\Omega}\Big(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\mathrm{d}x\,\mathrm{d}y,$$
$$\mathrm{Im}\oint_{\partial\Omega} f\,\mathrm{d}z = \oint_{\partial\Omega}(v\,\mathrm{d}x + u\,\mathrm{d}y) = \iint_{\Omega}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big)\mathrm{d}x\,\mathrm{d}y.$$

By the Cauchy–Riemann equations ($u_x = v_y$ and $u_y = -v_x$), both integrands vanish identically. □

4.4.5 The homotopy version

The most general version of Cauchy's theorem involves the topological notion of homotopy.

Definition 4.25 (Homotopy of curves)**.** Let $\Omega \subseteq \mathbb{C}$ be open, and let $\gamma_0, \gamma_1 \colon [a,b] \to \Omega$ be two closed curves (with the same base point, or free homotopy). We say γ_0 and γ_1 are **homotopic** in Ω if there exists a continuous function $H \colon [a,b] \times [0,1] \to \Omega$ such that

$$H(t,0) = \gamma_0(t), \quad H(t,1) = \gamma_1(t), \quad H(a,s) = H(b,s) \quad \text{for all } t \in [a,b],\ s \in [0,1].$$

The map H is called a **homotopy** between γ_0 and γ_1. A closed curve homotopic to a constant (a point) is called **null-homotopic**.

Theorem 4.26 (Cauchy's theorem — homotopy version)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and $f \colon \Omega \to \mathbb{C}$ holomorphic. If γ_0 and γ_1 are homotopic closed piecewise smooth curves in Ω, then*

$$\int_{\gamma_0} f(z)\,\mathrm{d}z \;=\; \int_{\gamma_1} f(z)\,\mathrm{d}z.$$

In particular, if γ is null-homotopic in Ω, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

Proof sketch. The continuous homotopy H maps the rectangle $[a,b] \times [0,1]$ into Ω. By compactness, the image $H([a,b] \times [0,1])$ has a positive distance to $\mathbb{C} \setminus \Omega$. We can therefore approximate H by a "piecewise linear" homotopy through a fine grid. Each elementary deformation (moving the curve across one small rectangle in the grid) changes the integral by the integral around a small rectangle contained in Ω. By Goursat's lemma (applied to triangulate each rectangle), each such integral is zero. Hence the total integral is unchanged by the homotopy.

A complete proof can be found in standard references; it relies on a careful approximation argument and the compactness of $[a,b] \times [0,1]$. □

Corollary 4.27 (Cauchy's theorem for simply connected domains). *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $f\colon \Omega \to \mathbb{C}$ holomorphic. Then:*

(i) $\oint_\gamma f(z)\,\mathrm{d}z = 0$ *for every closed piecewise smooth curve γ in Ω.*

(ii) *f has a primitive on Ω.*

Proof. **(i)** In a simply connected domain, every closed curve is null-homotopic. By Theorem 4.26, the integral is zero.

(ii) By Theorem 4.18, (i) implies the existence of a primitive. □

4.5 The Winding Number

Definition 4.28 (Winding number). Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed piecewise smooth curve and let $w \notin \gamma^*$. The **winding number** (or **index**) of γ with respect to w is

$$\operatorname{ind}(\gamma, w) \;=\; n(\gamma, w) \;=\; \frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}z}{z - w}.$$

Theorem 4.29 (Properties of the winding number).

(i) $\operatorname{ind}(\gamma, w) \in \mathbb{Z}$ *for every $w \notin \gamma^*$.*

(ii) *The function $w \mapsto \operatorname{ind}(\gamma, w)$ is continuous on $\mathbb{C} \setminus \gamma^*$, hence locally constant (since it is integer-valued and continuous).*

(iii) $\operatorname{ind}(\gamma, w) = 0$ *for all w in the unbounded component of $\mathbb{C} \setminus \gamma^*$.*

(iv) *If $\gamma(t) = z_0 + re^{int}$, $t \in [0, 2\pi]$ (the circle of center z_0 and radius r, traversed n times), then $\operatorname{ind}(\gamma, z_0) = n$.*

Proof. **(i)** Define $\varphi(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - w}\,\mathrm{d}s$ for $t \in [a,b]$. Then $\varphi'(t) = \frac{\gamma'(t)}{\gamma(t) - w}$ (at points of smoothness). Consider

$$h(t) \;=\; (\gamma(t) - w)\, e^{-\varphi(t)}.$$

Then $h'(t) = \gamma'(t)e^{-\varphi(t)} - (\gamma(t) - w)\varphi'(t)e^{-\varphi(t)} = \gamma'(t)e^{-\varphi(t)} - \gamma'(t)e^{-\varphi(t)} = 0$, so h is constant: $h(t) = h(a) = (\gamma(a) - w)e^0 = \gamma(a) - w$. Therefore $(\gamma(t) - w)e^{-\varphi(t)} = \gamma(a) - w$ for all t. At $t = b$, using $\gamma(b) = \gamma(a)$: $(\gamma(a) - w)e^{-\varphi(b)} = \gamma(a) - w$, so $e^{-\varphi(b)} = 1$ (since $\gamma(a) \neq w$). This means $\varphi(b) \in 2\pi i\mathbb{Z}$, i.e., $\frac{1}{2\pi i}\varphi(b) = \operatorname{ind}(\gamma, w) \in \mathbb{Z}$.

(ii) For w_1 close to w_0 (both outside γ^*),

$$|\operatorname{ind}(\gamma, w_1) - \operatorname{ind}(\gamma, w_0)| = \frac{1}{2\pi} \,|[|\Big]\int_\gamma \left(\frac{1}{z - w_1} - \frac{1}{z - w_0}\right)\mathrm{d}z = \frac{|w_1 - w_0|}{2\pi} \,|[|\Big]\int_\gamma \frac{\mathrm{d}z}{(z - w_0)(z - w_1)},$$

which tends to 0 as $w_1 \to w_0$ by the ML-inequality.

(iii) For $|w|$ sufficiently large (larger than $\max_t |\gamma(t)|$), we can bound $|1/(z - w)| \leq 1/(|w| - \max|\gamma|)$ and the integral tends to 0 as $|w| \to \infty$. Since $\operatorname{ind}(\gamma, w)$ is integer-valued and continuous on the unbounded component, it must be 0 everywhere on that component.

(iv) We already computed this in Example 4.11 (for $n = 1$; the general case is similar). □

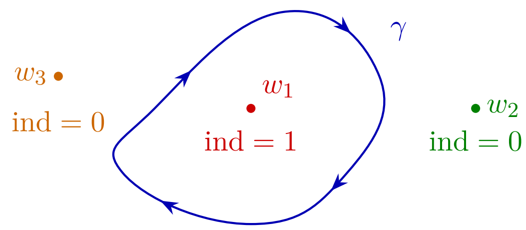


Figure 4.4: Winding numbers: the curve γ winds once around w_1 (inside the curve) and zero times around w_2 and w_3 (outside).

Remark 4.30 (Geometric interpretation). The winding number $\operatorname{ind}(\gamma, w)$ counts the number of times γ wraps around w in the counterclockwise direction. As z traverses γ, the argument of $z - w$ changes by $2\pi \cdot \operatorname{ind}(\gamma, w)$ in total.

4.6 Cauchy's Theorem — General Formulation

Theorem 4.31 (Cauchy's theorem — general version). *Let f be holomorphic on an open set $\Omega \subseteq \mathbb{C}$, and let γ be a closed piecewise smooth curve in Ω such that* $\operatorname{ind}(\gamma, w) = 0$ *for every $w \in \mathbb{C} \setminus \Omega$. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z \;=\; 0.$$

This version subsumes all previous versions: in a simply connected domain, every closed curve satisfies $\operatorname{ind}(\gamma, w) = 0$ for $w \notin \Omega$.

Corollary 4.32 (Deformation of contours). *Let f be holomorphic on Ω, and let γ_1, γ_2 be two closed piecewise smooth curves in Ω such that* $\operatorname{ind}(\gamma_1, w) = \operatorname{ind}(\gamma_2, w)$ *for all $w \notin \Omega$. Then*

$$\oint_{\gamma_1} f(z)\,\mathrm{d}z \;=\; \oint_{\gamma_2} f(z)\,\mathrm{d}z.$$

Proof. Apply Theorem 4.31 to the curve $\gamma = \gamma_1 + \gamma_2^-$, noting that $\operatorname{ind}(\gamma, w) = \operatorname{ind}(\gamma_1, w) - \operatorname{ind}(\gamma_2, w) = 0$ for $w \notin \Omega$. □

Example 4.33 (Deforming a contour around a pole). Let $\Omega = \mathbb{C} \setminus \{0\}$ and $f(z) = 1/z$. Let γ_1 be any simple closed curve in Ω that winds once counterclockwise around 0, and let $\gamma_2(t) = e^{it}$, $t \in [0, 2\pi]$, be the unit circle. Since both have winding number 1 around 0 (the only point outside Ω), we get

$$\oint_{\gamma_1} \frac{\mathrm{d}z}{z} = \oint_{\gamma_2} \frac{\mathrm{d}z}{z} = 2\pi i.$$

This is extremely useful: to evaluate the integral of $1/z$ around any curve, we need only know how many times it winds around the origin.

4.7 Applications and Consequences

Theorem 4.34 (Existence of primitives on simply connected domains)**.** *Let Ω be a simply connected domain and $f\colon \Omega \to \mathbb{C}$ holomorphic. Then f has a primitive on Ω. In particular:*

- *A branch of the logarithm exists on every simply connected subdomain of $\mathbb{C}^*$.*
- *For any $\alpha \in \mathbb{C}$, the function z^α can be defined as a single-valued holomorphic function on any simply connected subdomain of $\mathbb{C}^*$.*

Proof. By Corollary 4.27 and Theorem 4.18. For the logarithm: $1/z$ has a primitive L on Ω; then $ze^{-L(z)}$ is constant (as shown in Proposition 3.11), and adjusting the constant of integration gives $e^{L(z)} = z$. □

Proposition 4.35 (Integral of $1/z$ around a closed curve)**.** *Let γ be a closed piecewise smooth curve in $\mathbb{C}^*$. Then*

$$\oint_\gamma \frac{\mathrm{d}z}{z} = 2\pi i \cdot \operatorname{ind}(\gamma, 0).$$

Proof. This is the definition of the winding number. □

4.8 Worked Examples

Example 4.36 (Direct computation on a circle)**.** Let $\gamma(t) = 2e^{it}$, $t \in [0, 2\pi]$ (circle of radius 2, centered at the origin).

(a) $\oint_\gamma z^2 \,\mathrm{d}z = 0$ (since z^2 has primitive $z^3/3$ on $\mathbb{C}$).

(b) $\oint_\gamma \frac{\mathrm{d}z}{z-1} = 2\pi i$ (since $\operatorname{ind}(\gamma, 1) = 1$: the point 1 is inside the circle of radius 2).

(c) $\oint_\gamma \frac{\mathrm{d}z}{z-3} = 0$ (since $\operatorname{ind}(\gamma, 3) = 0$: the point 3 is outside the circle).

(d) $\oint_\gamma \frac{\mathrm{d}z}{z^2+1} = \oint_\gamma \frac{\mathrm{d}z}{(z-i)(z+i)}$. Both i and $-i$ lie inside γ. By partial fractions: $\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$. So the integral is $\frac{1}{2i}(2\pi i - 2\pi i) = 0$.

Example 4.37 (Integral of $\operatorname{Re} z$ around the unit circle)**.** Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then $\operatorname{Re} z = \frac{z+\overline{z}}{2}$ and on the unit circle $\overline{z} = 1/z$, so $\operatorname{Re} z = \frac{z+1/z}{2}$. Hence

$$\oint_\gamma \operatorname{Re} z \,\mathrm{d}z = \frac{1}{2}\oint_\gamma z \,\mathrm{d}z + \frac{1}{2}\oint_\gamma \frac{\mathrm{d}z}{z} = 0 + \frac{1}{2}\cdot 2\pi i = \pi i.$$

Example 4.38 (A non-trivial winding number computation)**.** Consider a figure-eight curve γ that winds once counterclockwise around $z = 1$ and once clockwise around $z = -1$. For any function f holomorphic on $\mathbb{C} \setminus \{-1, 1\}$, the winding numbers are $\operatorname{ind}(\gamma, 1) = 1$ and $\operatorname{ind}(\gamma, -1) = -1$. Hence

$$\oint_\gamma \frac{\mathrm{d}z}{z-1} = 2\pi i, \quad \oint_\gamma \frac{\mathrm{d}z}{z+1} = -2\pi i, \quad \oint_\gamma \frac{\mathrm{d}z}{z^2-1} = \frac{1}{2}(2\pi i - (-2\pi i)) = 2\pi i.$$

Example 4.39 (Integral of $e^{1/z}$ around the unit circle)**.** Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. The function $e^{1/z}$ is holomorphic on $\mathbb{C}^*$ but has an essential singularity at 0. Using the Laurent expansion:

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$$

By (4.1), only the $1/z$ term contributes to the integral:

$$\oint_\gamma e^{1/z}\, \mathrm{d}z = 2\pi i.$$

(This anticipates the *residue theorem* of later chapters.)

4.9 Exercises

Exercise 4.1 (Direct computations)**.** Compute the following integrals, where $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$, is the circle of radius R centered at the origin.

(a) $\displaystyle\oint_\gamma \overline{z}\, \mathrm{d}z$ (with $R = 1$).

(b) $\displaystyle\oint_\gamma |z|^2\, \mathrm{d}z$ (with $R = 1$).

(c) $\displaystyle\oint_\gamma z^n\, \mathrm{d}z$ for $n \in \mathbb{Z}$ and arbitrary $R > 0$.

(d) $\displaystyle\oint_\gamma \frac{z+1}{z-1}\, \mathrm{d}z$ with $R = 2$.

Exercise 4.2 (Path integrals)**.**

(a) Compute $\int_\gamma z^2\, \mathrm{d}z$ where γ is the segment from 0 to $1 + i$.

(b) Compute $\int_\gamma \operatorname{Re} z\, \mathrm{d}z$ along (i) the segment from 0 to 1, and (ii) the upper semicircle from 0 to 1. Are the results equal? Why or why not?

(c) Compute $\int_\gamma \frac{\mathrm{d}z}{z}$ where γ goes from -1 to 1 along (i) the upper semicircle, (ii) the lower semicircle.

Exercise 4.3 (ML-inequality applications)**.**

(a) Show that $|[|\big] \oint_{|z|=R} \frac{e^z}{z^2}\, \mathrm{d}z \leq \frac{2\pi e^R}{R}$ for $R > 0$.

(b) Show that $|[|\,\big]\oint_{|z|=R}\frac{\mathrm{d}z}{z^n+1} \to 0$ as $R \to \infty$ for $n \geq 2$.

(c) Let γ_R be the upper semicircle Re^{it}, $t \in [0,\pi]$. Show that $|[|\,\big]\int_{\gamma_R}\frac{e^{iz}}{z}\,\mathrm{d}z \to 0$ as $R \to \infty$.

Exercise 4.4 (Cauchy's theorem — applications)**.**

(a) Let f be entire. Show that $\oint_\gamma f(z)\,\mathrm{d}z = 0$ for every closed contour γ.

(b) Let f be holomorphic on $\mathbb{C} \setminus \{0\}$ and suppose f has a primitive on $\mathbb{C} \setminus \{0\}$. Show that $\oint_{|z|=1} f(z)\,\mathrm{d}z = 0$.

(c) Show that $1/z$ does *not* have a primitive on $\mathbb{C} \setminus \{0\}$.

Exercise 4.5 (Winding numbers)**.**

(a) Compute $\operatorname{ind}(\gamma, 0)$ where $\gamma(t) = e^{2it}$, $t \in [0, 2\pi]$.

(b) Let $\gamma(t) = 2\cos t + i\sin t$, $t \in [0, 2\pi]$ (an ellipse). What is $\operatorname{ind}(\gamma, 0)$? What is $\operatorname{ind}(\gamma, 3)$?

(c) Let γ be the boundary of the square with vertices $\pm 2 \pm 2i$, traversed counterclockwise. Compute $\operatorname{ind}(\gamma, 0)$, $\operatorname{ind}(\gamma, 1+i)$, and $\operatorname{ind}(\gamma, 5)$.

Exercise 4.6 (Primitives on domains)**.**

(a) Show that $z^2 e^z$ has a primitive on $\mathbb{C}$ and find it explicitly.

(b) Does $1/z^2$ have a primitive on $\mathbb{C} \setminus \{0\}$? If so, find it. Does this contradict anything?

(c) Show that $1/(z^2+1)$ has a primitive on the right half-plane $\{\operatorname{Re} z > 0\}$ but not on $\mathbb{C} \setminus \{\pm i\}$.

Exercise 4.7 (Cauchy's theorem for multiply connected domains)**.** Let $\gamma_1(t) = 3e^{it}$ and $\gamma_2(t) = e^{it}$, $t \in [0, 2\pi]$. Let f be holomorphic on the annulus $\Omega = \{1 < |z| < 3\}$.

(a) Explain why we cannot directly conclude $\oint_{\gamma_1} f\,\mathrm{d}z = 0$.

(b) Show that $\oint_{\gamma_1} f\,\mathrm{d}z = \oint_{\gamma_2} f\,\mathrm{d}z$. *Hint:* connect the two circles by cuts and apply Cauchy's theorem to a simply connected subdomain.

(c) Verify this when $f(z) = 1/z$.

Exercise 4.8 (The integral $\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+\cos\theta}$)**.** Let $a > 1$. Use the substitution $z = e^{i\theta}$ to show that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}.$$

Hint: On $|z| = 1$, $\cos\theta = (z + z^{-1})/2$ and $\mathrm{d}\theta = \mathrm{d}z/(iz)$.

Exercise 4.9 (Green's function approach)**.** Let Ω be a bounded domain with piecewise smooth boundary, and let f, g be holomorphic on an open set containing $\overline{\Omega}$. Show that

$$\oint_{\partial\Omega} f(z)g(z)\,\mathrm{d}z = 0.$$

Exercise 4.10 (An integral estimate)**.** Let f be holomorphic on $\{z : |z| > R_0\}$ and suppose $|f(z)| \leq M/|z|^2$ for $|z| > R_0$. Show that $\oint_{|z|=R} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$.

Exercise 4.11 (Deformation principle)**.** Let f be holomorphic on $\mathbb{C} \setminus \{i, -i\}$. Let γ be a simple closed curve enclosing both i and $-i$, and let γ_1 (resp. γ_2) be a small circle around i (resp. $-i$), both positively oriented. Show that

$$\oint_{\gamma} f(z)\,\mathrm{d}z = \oint_{\gamma_1} f(z)\,\mathrm{d}z + \oint_{\gamma_2} f(z)\,\mathrm{d}z.$$

Apply this to $f(z) = 1/(z^2 + 1)$ to recover the result of Example 4.36(d).

Chapter 5

Cauchy Integral Formula and Consequences

Introduction

The Cauchy integral theorem, established in the preceding chapter, tells us that the integral of a holomorphic function around a closed contour is zero. In this chapter we explore the remarkable *converse* direction: the values of a holomorphic function inside a domain are completely determined by its values on the boundary. This is the content of the **Cauchy integral formula**, one of the most powerful results in all of analysis. Its consequences are sweeping: every holomorphic function is infinitely differentiable, every holomorphic function is analytic (representable by a convergent power series), bounded entire functions are constant (Liouville's theorem), and the modulus of a holomorphic function cannot attain a local maximum in the interior of its domain.

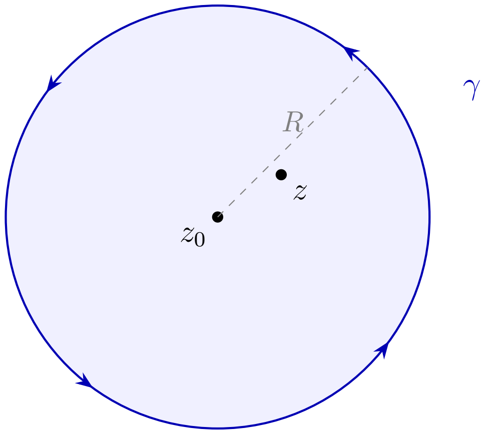
5.1 The Cauchy Integral Formula

5.1.1 Statement for a disk

We begin with the simplest and most fundamental version of the formula.

Theorem 5.1 (Cauchy Integral Formula for a Disk)**.** *Let f be holomorphic on an open set $\Omega \subseteq \mathbb{C}$, and let $\overline{D}(z_0, R) = \{z \in \mathbb{C} : |z - z_0| \leq R\} \subseteq \Omega$ be a closed disk contained in Ω. Denote by γ the positively oriented circle $\gamma(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$. Then for every z with $|z - z_0| < R$,*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta. \tag{5.1}$$



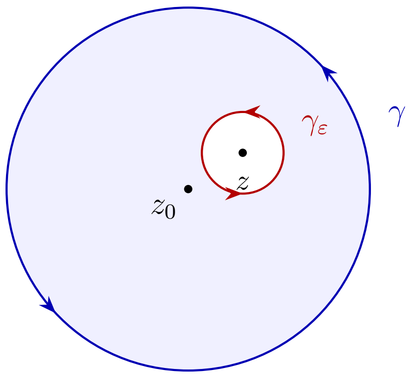
Positively oriented circle

Proof. Fix z with $|z - z_0| < R$. The key idea is to deform the contour: since the integrand $\zeta \mapsto f(\zeta)/(\zeta - z)$ is holomorphic on $\Omega \setminus \{z\}$, we may replace γ by a small circle around z without changing the integral.

More precisely, let $\varepsilon > 0$ be small enough so that $\overline{D}(z, \varepsilon) \subset D(z_0, R)$, and let $\gamma_\varepsilon(t) = z + \varepsilon e^{it}$, $t \in [0, 2\pi]$.

Step 1. *Reduction to the small circle.* The function $g(\zeta) = f(\zeta)/(\zeta - z)$ is holomorphic on the annular region between γ and γ_ε. By the Cauchy integral theorem for multiply connected domains (or by introducing a slit connecting the two circles),

$$\oint_\gamma g(\zeta)\, \mathrm{d}\zeta \;=\; \oint_{\gamma_\varepsilon} g(\zeta)\, \mathrm{d}\zeta.$$



Step 2. *Evaluation on the small circle.* On γ_ε we write $\zeta = z + \varepsilon e^{it}$, so $\mathrm{d}\zeta = i\varepsilon e^{it}\, \mathrm{d}t$ and $\zeta - z = \varepsilon e^{it}$. Then

$$\begin{aligned}\oint_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta &= \int_0^{2\pi} \frac{f(z + \varepsilon e^{it})}{\varepsilon e^{it}} \cdot i\varepsilon e^{it}\, \mathrm{d}t \\ &= i \int_0^{2\pi} f(z + \varepsilon e^{it})\, \mathrm{d}t.\end{aligned}$$

Since f is continuous at z, as $\varepsilon \to 0$ the integrand converges uniformly to $f(z)$, so

$$\oint_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \;=\; i \int_0^{2\pi} f(z + \varepsilon e^{it})\, \mathrm{d}t \;\xrightarrow{\varepsilon \to 0}\; 2\pi i\, f(z).$$

But the integral on the left equals $\oint_\gamma f(\zeta)/(\zeta - z)\,\mathrm{d}\zeta$ for all sufficiently small ε (by Step 1), hence this value is independent of ε. Therefore

$$\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 2\pi i\, f(z),$$

which gives (5.1). □

Remark 5.2 (Representation Formula). Equation (5.1) is extraordinary: the values of f in the *interior* of the disk are entirely determined by the values of f on the *boundary*. No analogous result exists for smooth functions of a real variable. This rigidity is one of the hallmarks of complex analysis.

5.1.2 General version

Theorem 5.3 (Cauchy Integral Formula — General Version). *Let $\Omega \subseteq \mathbb{C}$ be a simply connected open set, and let γ be a positively oriented, simple, piecewise smooth closed curve in Ω. If f is holomorphic on Ω and z lies in the interior of γ, then*

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Proof. The proof follows the same deformation argument as for the disk: introduce a small circle γ_ε around z contained in the interior of γ. Since $f(\zeta)/(\zeta - z)$ is holomorphic in the region between γ and γ_ε, the Cauchy integral theorem for multiply connected domains gives

$$\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \oint_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 2\pi i\, f(z),$$

where the last equality is the computation from the proof of Theorem 5.1. □

Example 5.4 (A First Application). Let us compute

$$I = \oint_{|z|=2} \frac{e^z}{z-1}\,\mathrm{d}z.$$

Here $f(z) = e^z$ is entire, and $z_0 = 1$ lies inside the circle $|z| = 2$. By the Cauchy integral formula,

$$I = 2\pi i\, f(1) = 2\pi i\, e.$$

Example 5.5 (Distinguishing Inside and Outside). Compute

$$J = \oint_{|z|=1} \frac{\cos z}{z-3}\,\mathrm{d}z.$$

Since $z_0 = 3$ lies *outside* the unit circle, the integrand $\zeta \mapsto \cos\zeta/(\zeta - 3)$ is holomorphic on and inside $|z| = 1$. By the Cauchy integral theorem, $J = 0$.

5.2 Cauchy Integral Formula for Higher Derivatives

One of the most striking consequences of the Cauchy integral formula is that holomorphic functions are automatically infinitely differentiable, and all derivatives admit integral representations.

Theorem 5.6 (Cauchy Formula for Derivatives)**.** *Let f be holomorphic on an open set Ω, and let γ be a positively oriented, simple, piecewise smooth closed curve in Ω whose interior is contained in Ω. Then f is infinitely differentiable in the interior of γ, and for every $n \geq 0$ and every z in the interior of γ,*

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta. \tag{5.2}$$

Proof. We prove the formula by induction on n. The base case $n = 0$ is the Cauchy integral formula (Theorem 5.3).

Step 1: The case $n = 1$. Let z be in the interior of γ. For $h \neq 0$ small enough so that $z + h$ also lies in the interior, the Cauchy integral formula gives

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i} \oint_\gamma f(\zeta) \cdot \frac{1}{h} \left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right) \mathrm{d}\zeta.$$

A direct computation shows

$$\frac{1}{h} \left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right) = \frac{1}{(\zeta - z - h)(\zeta - z)}.$$

Let $d = \min_{\zeta \in \gamma} |\zeta - z| > 0$. For $|h| < d/2$ we have $|\zeta - z - h| \geq d/2$, so

$$\frac{1}{(\zeta - z - h)(\zeta - z)} \xrightarrow{h \to 0} \frac{1}{(\zeta - z)^2}$$

uniformly for $\zeta \in \gamma$. Since f is continuous (hence bounded) on γ, we may pass the limit under the integral sign to obtain

$$f'(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, \mathrm{d}\zeta,$$

which is (5.2) for $n = 1$.

Step 2: Inductive step. Assume (5.2) holds for some $n \geq 1$. We differentiate under the integral sign. Set

$$g_n(\zeta, z) = \frac{f(\zeta)}{(\zeta - z)^{n+1}}.$$

The difference quotient argument analogous to Step 1 shows that for $|h|$ small,

$$\frac{g_n(\zeta, z+h) - g_n(\zeta, z)}{h} \;\to\; \frac{(n+1)f(\zeta)}{(\zeta - z)^{n+2}}$$

uniformly in $\zeta \in \gamma$. Passing the limit under the integral,

$$f^{(n+1)}(z) = \frac{n!}{2\pi i} \cdot (n+1) \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+2}}\, \mathrm{d}\zeta = \frac{(n+1)!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+2}}\, \mathrm{d}\zeta,$$

completing the induction. □

Corollary 5.7 (Infinite Differentiability)**.** *Every holomorphic function on an open set $\Omega \subseteq \mathbb{C}$ is infinitely (complex) differentiable on Ω. In particular, f' is itself holomorphic.*

Remark 5.8. This result has no analogue in real analysis: a real-valued function can be differentiable everywhere yet have a discontinuous second derivative. In complex analysis, a single complex derivative implies all higher derivatives exist and are continuous.

Example 5.9 (Computing with the Derivative Formula)**.** Compute

$$\oint_{|z|=2} \frac{e^z}{(z-1)^3}\,\mathrm{d}z.$$

By Theorem 5.6 with $f(\zeta) = e^\zeta$, $z_0 = 1$, and $n = 2$,

$$\oint_{|z|=2} \frac{e^z}{(z-1)^3}\,\mathrm{d}z = \frac{2\pi i}{2!} f''(1) = \pi i \cdot e.$$

Example 5.10. Compute

$$\oint_{|z|=1} \frac{\sin z}{z^4}\,\mathrm{d}z.$$

Here $f(z) = \sin z$ and we evaluate at $z_0 = 0$ with $n = 3$:

$$\oint_{|z|=1} \frac{\sin z}{z^4}\,\mathrm{d}z = \frac{2\pi i}{3!} f'''(0) = \frac{2\pi i}{6} \cdot (-\cos 0) = -\frac{\pi i}{3}.$$

5.3 Morera's Theorem

Theorem 5.11 (Morera's Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f\colon \Omega \to \mathbb{C}$ be continuous. If*

$$\oint_T f(z)\,\mathrm{d}z = 0$$

for every triangle T whose closure is contained in Ω, then f is holomorphic on Ω.

Proof. It suffices to prove that f is holomorphic on each open disk $D(z_0, r) \subseteq \Omega$. Fix such a disk and define

$$F(z) = \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta, \qquad z \in D(z_0, r),$$

where the integral is taken along the line segment from z_0 to z. The hypothesis that $\oint_T f = 0$ for every triangle ensures that F is well-defined and independent of path within $D(z_0, r)$.

We claim that $F'(z) = f(z)$. Indeed, for h small,

$$F(z+h) - F(z) = \int_{z}^{z+h} f(\zeta)\,\mathrm{d}\zeta,$$

where the integral is along the segment $[z, z+h]$. Since f is continuous at z, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(\zeta) - f(z)| < \varepsilon$ whenever $|\zeta - z| < \delta$. For $|h| < \delta$,

$$|F(z+h) - F(z) - hf(z)| = |*| \int_z^{z+h} \big(f(\zeta) - f(z)\big)\, \mathrm{d}\zeta \le \varepsilon\, |h|\,.$$

Hence $F'(z) = f(z)$. Since F is holomorphic on $D(z_0, r)$, Corollary 5.7 implies $F' = f$ is holomorphic as well. □

Remark 5.12. Morera's theorem is a partial converse to the Cauchy integral theorem: the Cauchy theorem says that if f is holomorphic then $\oint_\gamma f = 0$ for closed curves, while Morera's theorem says that if $\oint_T f = 0$ for all triangles then f is holomorphic. The two together characterize holomorphy among continuous functions.

5.4 Cauchy Inequalities

Theorem 5.13 (Cauchy Inequalities). *Let f be holomorphic on an open set containing $\overline{D}(z_0, R)$, and let*

$$M(R) = \max_{|z-z_0|=R} |f(z)|\,.$$

Then for all $n \ge 0$,

$$\left|f^{(n)}(z_0)\right| \;\le\; \frac{n!\, M(R)}{R^n}. \tag{5.3}$$

Proof. By Theorem 5.6 with γ the circle of radius R centered at z_0,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{|\zeta-z_0|=R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta.$$

By the standard estimate for contour integrals,

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi} \cdot \frac{M(R)}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M(R)}{R^n}.$$ □

5.5 Liouville's Theorem and the Fundamental Theorem of Algebra

Definition 5.14 (Entire Function). A function $f\colon \mathbb{C} \to \mathbb{C}$ is called **entire** if it is holomorphic on all of $\mathbb{C}$.

Theorem 5.15 (Liouville's Theorem). *Every bounded entire function is constant.*

Proof. Let f be entire with $|f(z)| \le M$ for all $z \in \mathbb{C}$. Fix any $z_0 \in \mathbb{C}$. By the Cauchy inequality (5.3) with $n = 1$,

$$|f'(z_0)| \le \frac{M}{R}$$

for every $R > 0$. Letting $R \to \infty$, we obtain $f'(z_0) = 0$. Since z_0 was arbitrary, $f' \equiv 0$ on $\mathbb{C}$, so f is constant. □

Remark 5.16. Liouville's theorem admits several generalizations. For instance, if f is entire and $|f(z)| \le C(1+|z|)^n$ for some $C > 0$ and some non-negative integer n, then f is a polynomial of degree at most n. The proof uses the Cauchy inequality for the $(n+1)$-st derivative.

Theorem 5.17 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$ with complex coefficients and $n \ge 1$ has at least one root in $\mathbb{C}$.*

Proof. Suppose for contradiction that $p(z) \ne 0$ for all $z \in \mathbb{C}$. Then $g(z) = 1/p(z)$ is entire. We show that g is bounded.

Since $a_n \ne 0$ and $n \ge 1$, we have

$$|p(z)| = |a_n|\,|z|^n\,|*|\,1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_0}{a_n z^n}.$$

For $|z| \ge R_0$ sufficiently large, the sum in the second factor has modulus at least $1/2$, so

$$|p(z)| \ge \frac{|a_n|}{2}\,|z|^n \ge \frac{|a_n|}{2} R_0^n \qquad \text{for } |z| \ge R_0.$$

Hence $|g(z)| \le 2/(|a_n|\, R_0^n)$ for $|z| \ge R_0$.

On the compact disk $\overline{D}(0, R_0)$, the continuous function g is bounded, say $|g(z)| \le M_0$.

Taking $M = \max\{M_0,\, 2/(|a_n|\, R_0^n)\}$, we see that $|g(z)| \le M$ for all $z \in \mathbb{C}$. By Liouville's theorem, g is constant, hence p is constant, contradicting $n \ge 1$. □

Corollary 5.18. *Every polynomial of degree $n \ge 1$ with complex coefficients factors completely as*

$$p(z) = a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n),$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ (not necessarily distinct) are its roots.

Proof. By Theorem 5.17, p has a root α_1. By polynomial division, $p(z) = (z-\alpha_1)q(z)$ where $\deg q = n-1$. The result follows by induction. □

5.6 Mean Value Property

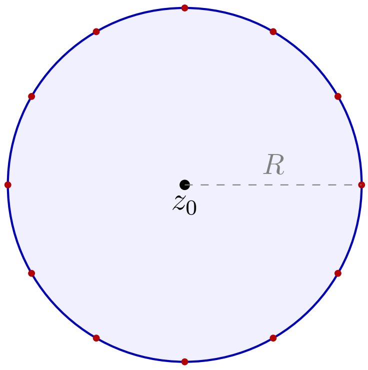
Theorem 5.19 (Mean Value Property)**.** *Let f be holomorphic on an open set containing $\overline{D}(z_0, R)$. Then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,\mathrm{d}t. \tag{5.4}$$

That is, the value of f at the center of a disk equals the average of f over the boundary circle.

Proof. By the Cauchy integral formula with $\gamma(t) = z_0 + Re^{it}$,

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{it})}{Re^{it}} \cdot iRe^{it}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{it})\,\mathrm{d}t. \quad \square$$



$f(z_0)$ equals the average of f
over the red sample points (in the limit)

5.7 Maximum and Minimum Modulus Principles

Theorem 5.20 (Maximum Modulus Principle). *Let $\Omega \subseteq \mathbb{C}$ be an open connected set, and let f be holomorphic on Ω. If $|f|$ attains a local maximum at some point $z_0 \in \Omega$, then f is constant on Ω.*

Proof. Suppose $|f(z_0)| \geq |f(z)|$ for all z in some disk $D(z_0, r) \subseteq \Omega$. We first show f is constant on $D(z_0, r)$.

If $f(z_0) = 0$, then $|f| \equiv 0$ on $D(z_0, r)$ and the claim is clear. Assume $f(z_0) \neq 0$.

By the mean value property, for any $0 < \rho < r$,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it})\,\mathrm{d}t.$$

Taking moduli and using $|f(z_0 + \rho e^{it})| \leq |f(z_0)|$,

$$|f(z_0)| = |*| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it})\,\mathrm{d}t \leq \frac{1}{2\pi} \int_0^{2\pi} \left|f(z_0 + \rho e^{it})\right| \mathrm{d}t \leq |f(z_0)|\,.$$

Hence equality holds throughout. In particular,

$$\frac{1}{2\pi} \int_0^{2\pi} \left[|f(z_0)| - \left|f(z_0 + \rho e^{it})\right|\right] \mathrm{d}t = 0.$$

Since the integrand is non-negative and continuous, it must vanish identically: $|f(z_0 + \rho e^{it})| = |f(z_0)|$ for all t and all $0 < \rho < r$. Thus $|f|$ is constant on $D(z_0, r)$.

Now, $|f|$ constant on an open disk implies f is constant there (since a holomorphic function with constant modulus on an open set is constant—this follows from the Cauchy–Riemann equations or from the open mapping theorem).

The set $S = \{z \in \Omega : f(z) = f(z_0)\}$ is closed in Ω (by continuity) and open (by the argument above). Since Ω is connected and S is non-empty, $S = \Omega$. □

Corollary 5.21 (Maximum Modulus on Compact Sets). *Let Ω be a bounded open* *connected set, and let f be holomorphic on Ω and continuous on $\overline{\Omega}$. Then*

$$\max_{z\in\overline{\Omega}} |f(z)| = \max_{z\in\partial\Omega} |f(z)| .$$

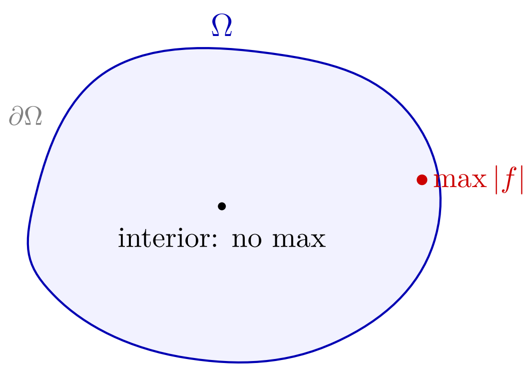
That is, $|f|$ attains its maximum on the boundary $\partial\Omega$.

Proof. Since $\overline{\Omega}$ is compact and $|f|$ is continuous, the maximum is attained at some $z_0 \in \overline{\Omega}$. If $z_0 \in \partial\Omega$, we are done. If $z_0 \in \Omega$, Theorem 5.20 implies f is constant, and the result is trivially true. □

Theorem 5.22 (Minimum Modulus Principle). *Let $\Omega \subseteq \mathbb{C}$ be an open connected set, and let f be holomorphic on Ω with $f(z) \neq 0$ for all $z \in \Omega$. If $|f|$ attains a local minimum at some $z_0 \in \Omega$, then f is constant.*

Proof. Since f is never zero on Ω, the function $g = 1/f$ is holomorphic on Ω. A local minimum of $|f|$ at z_0 is a local maximum of $|g|$ at z_0. By Theorem 5.20, g is constant, hence f is constant. □

Remark 5.23. The hypothesis $f \neq 0$ is essential. For instance, $f(z) = z$ on $D(0,1)$ has $|f(0)| = 0$, which is a minimum of $|f|$, yet f is not constant. The zeros of f are precisely the points where the minimum modulus principle can fail.



5.8 Taylor Series of Holomorphic Functions

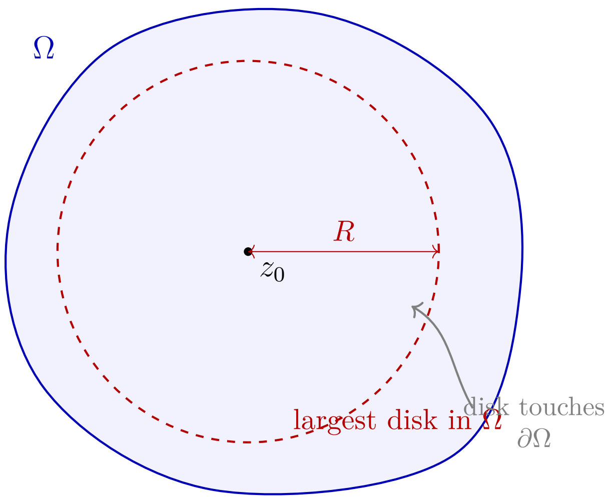
Theorem 5.24 (Taylor Series Expansion)**.** *Let f be holomorphic on an open set $\Omega \subseteq \mathbb{C}$, and let $z_0 \in \Omega$. Let R be the largest radius such that $D(z_0, R) \subseteq \Omega$ (possibly $R = \infty$). Then f has a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad |z - z_0| < R, \tag{5.5}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \tag{5.6}$$

for any $0 < \rho < R$. The convergence is absolute and uniform on each compact subset of $D(z_0, R)$.



Proof. Fix $z \in D(z_0, R)$ and choose ρ with $|z - z_0| < \rho < R$. Let γ be the circle $|\zeta - z_0| = \rho$. By the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

We expand the Cauchy kernel:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}.$$

Since $|z - z_0| < \rho = |\zeta - z_0|$ for $\zeta \in \gamma$, we have $|(z - z_0)/(\zeta - z_0)| < 1$, and the geometric series converges:

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

The convergence is uniform in $\zeta \in \gamma$ (with ratio $|z - z_0| / \rho < 1$), so we may interchange summation and integration:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\gamma} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \, d\zeta \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right] (z - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \end{aligned}$$

where the last equality uses the Cauchy formula for derivatives (5.2).

Uniform convergence. Let $K \subset D(z_0, R)$ be compact. Choose r with $\sup_{z\in K} |z - z_0| < r < R$ and set $\rho = (r + R)/2$. For $z \in K$ and ζ on $|\zeta - z_0| = \rho$,

$$|*| \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \leq \frac{r^n}{\rho^{n+1}}.$$

Since $M = \max_{|\zeta - z_0|=\rho} |f(\zeta)| < \infty$ and $\sum_n (r/\rho)^n < \infty$, the Weierstrass M-test gives uniform convergence on K. □

Corollary 5.25 (Holomorphic Implies Analytic)**.** *A function is holomorphic on an open set Ω if and only if it is complex analytic on Ω (i.e., locally representable by a convergent power series).*

Remark 5.26. The radius of convergence of the Taylor series (5.5) is *at least* $R = \operatorname{dist}(z_0, \partial\Omega)$. In fact, the radius of convergence equals the distance from z_0 to the nearest singularity of f (which may lie outside Ω if f can be analytically continued).

Example 5.27 (Taylor Series of Standard Functions)**.** (a) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, valid for all $z \in \mathbb{C}$ (radius of convergence $R = \infty$).

(b) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, valid for $|z| < 1$ (singularity at $z = 1$, $R = 1$).

(c) $\operatorname{Log}(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$, valid for $|z| < 1$ (branch point at $z = -1$, $R = 1$).

(d) $\frac{1}{z-2} = \frac{-1}{2} \cdot \frac{1}{1 - z/2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$, centered at $z_0 = 0$, valid for $|z| < 2$.

Example 5.28 (Taylor Expansion at a Non-Zero Center)**.** Expand $f(z) = 1/(1 - z)$ in a Taylor series centered at $z_0 = i$.
We write

$$\frac{1}{1-z} = \frac{1}{(1-i) - (z-i)} = \frac{1}{1-i} \cdot \frac{1}{1 - \frac{z-i}{1-i}} = \frac{1}{1-i} \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^n} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}},$$

valid for $|z - i| < |1 - i| = \sqrt{2}$, which is the distance from i to the singularity at $z = 1$.

5.9 Further Applications

Proposition 5.29 (Schwarz Lemma)**.** *Let* $f\colon D(0,1) \to D(0,1)$ *be holomorphic with* $f(0) = 0$*. Then:*

(i) $|f(z)| \le |z|$ *for all* $z \in D(0,1)$*.*

(ii) $|f'(0)| \le 1$*.*

(iii) If equality holds in (i) for some $z \neq 0$*, or in (ii), then* $f(z) = e^{i\theta} z$ *for some* $\theta \in \mathbb{R}$*.*

Proof. Since $f(0) = 0$, the function $g(z) = f(z)/z$ has a removable singularity at 0 (set $g(0) = f'(0)$) and is holomorphic on $D(0,1)$.

For $0 < r < 1$, the maximum modulus principle applied to g on $\overline{D}(0,r)$ gives

$$|g(z)| \le \max_{|\zeta|=r} |g(\zeta)| = \max_{|\zeta|=r} \frac{|f(\zeta)|}{r} \le \frac{1}{r}.$$

Letting $r \to 1^-$, we get $|g(z)| \le 1$ for all $z \in D(0,1)$. This gives (i) and (ii).

If equality holds (i.e., $|g(z_0)| = 1$ for some $z_0 \in D(0,1)$), then $|g|$ attains its maximum in the interior. By the maximum modulus principle, g is constant, say $g(z) = e^{i\theta}$, so $f(z) = e^{i\theta} z$. ☐

5.10 Exercises

Exercise 5.1 (★)**.** Use the Cauchy integral formula to evaluate:

(a) $\displaystyle\oint_{|z|=3} \frac{z^2+1}{z-2}\,\mathrm{d}z$

(b) $\displaystyle\oint_{|z|=1} \frac{e^z}{z^2}\,\mathrm{d}z$

(c) $\displaystyle\oint_{|z-i|=2} \frac{\cos z}{z^2+1}\,\mathrm{d}z$

Hint: For (c), factor the denominator and determine which poles lie inside the contour.

Exercise 5.2 (★)**.** Let (f_n) be a sequence of holomorphic functions on an open set Ω that converges uniformly on compact subsets of Ω to a function f. Use Morera's theorem to prove that f is holomorphic on Ω.

Exercise 5.3 (★★)**.** Let f be entire and suppose there exist constants $C > 0$ and $k \in \mathbb{N}$ such that $|f(z)| \le C\,|z|^k$ for all $|z| \ge 1$. Prove that f is a polynomial of degree at most k.

Exercise 5.4 (★★)**.** Let f be holomorphic on $D(0,1)$ and continuous on $\overline{D}(0,1)$, with $|f(z)| = 1$ for all $|z| = 1$. Prove that f has at least one zero in $D(0,1)$.

Hint: Apply the minimum modulus principle.

Exercise 5.5 (★★). Let $u\colon \Omega \to \mathbb{R}$ be harmonic (i.e., $u = \operatorname{Re} f$ for some holomorphic f). Prove that u satisfies the mean value property:

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\,\mathrm{d}t$$

for every $\overline{D}(z_0, r) \subseteq \Omega$.

Exercise 5.6 (★★). Let $u\colon \mathbb{C} \to \mathbb{R}$ be a harmonic function that is bounded above. Prove that u is constant.

Hint: Consider $g(z) = e^{u(z)+iv(z)}$ where v is a harmonic conjugate.

Exercise 5.7 (★★★). (Schwarz–Pick Lemma.) Let $f\colon D(0,1) \to D(0,1)$ be holomorphic. Prove that for all $z, w \in D(0,1)$,

$$|*|\, \frac{f(z) - f(w)}{1 - \overline{f(w)} f(z)} \le |*|\, \frac{z - w}{1 - \overline{w} z},$$

and $|f'(z)| \le \dfrac{1 - |f(z)|^2}{1 - |z|^2}$.

Hint: Compose f with suitable Möbius transformations and apply the Schwarz lemma.

Exercise 5.8 (★★★). (Hadamard Three-Circles Theorem.) Let f be holomorphic on the annulus $\{z : r_1 < |z| < r_2\}$ and continuous on its closure. Define $M(r) = \max_{|z|=r} |f(z)|$ for $r_1 \le r \le r_2$. Prove that $\log M(r)$ is a convex function of $\log r$. More precisely, for $r_1 \le r \le r_2$,

$$\log M(r) \le \frac{\log(r_2/r)}{\log(r_2/r_1)} \log M(r_1) + \frac{\log(r/r_1)}{\log(r_2/r_1)} \log M(r_2).$$

Hint: Apply the maximum modulus principle to $z^\alpha f(z)$ for a suitable real number α.

Exercise 5.9 (★★★). (Open Mapping Theorem.) Let f be a non-constant holomorphic function on a connected open set Ω. Prove that $f(\Omega)$ is open.

Hint: Given $w_0 = f(z_0)$, use the argument principle (or a direct approach with Rouché's theorem) to show that f takes all values in a neighborhood of w_0.

Exercise 5.10 (★★). Compute the following integrals using partial fractions and the Cauchy integral formula:

(a) $\displaystyle\oint_{|z|=3} \frac{z}{(z-1)(z-2)}\,\mathrm{d}z$

(b) $\displaystyle\oint_{|z|=2} \frac{1}{z^2(z-3)}\,\mathrm{d}z$

(c) $\displaystyle\oint_{|z|=4} \frac{e^z}{z^2-1}\,\mathrm{d}z$

Exercise 5.11 (★★). Find the Taylor series centered at $z_0 = 0$ for:

(a) $f(z) = \dfrac{z}{(1-z)^2}$

(b) $f(z) = \dfrac{1}{z^2+4}$

(c) $f(z) = z\cos(z^2)$

Determine the radius of convergence in each case.

Chapter 6

Laurent Series and Isolated Singularities

Introduction

The Taylor series expansion developed in Chapter 5 represents a holomorphic function as a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in a disk centered at z_0. However, many functions of interest are not defined at certain isolated points—for instance, $1/z$ at the origin or $e^{1/z}$ at the origin. To study such functions, we need an expansion that allows *negative* powers of $(z-z_0)$. This is the **Laurent series**, which converges in an *annulus* rather than a disk. The nature of the negative-power part (the *principal part*) leads to a classification of singularities that is fundamental to the residue calculus of the next chapter.

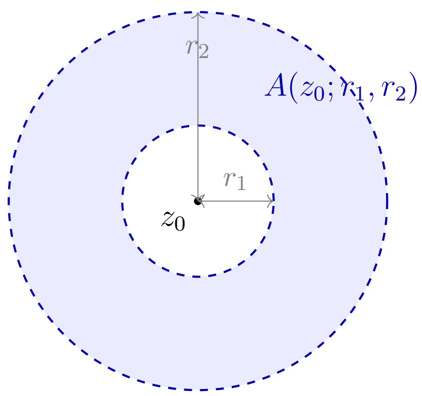
6.1 Laurent Series

6.1.1 Statement and uniqueness

Definition 6.1 (Annulus)**.** For $0 \le r_1 < r_2 \le +\infty$ and $z_0 \in \mathbb{C}$, the **open annulus** centered at z_0 with radii r_1 and r_2 is

$$A(z_0;\, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}.$$

When $r_1 = 0$ this is a *punctured disk* $D(z_0, r_2) \setminus \{z_0\}$. When $r_2 = +\infty$ this is the complement of a closed disk.



Theorem 6.2 (Laurent Series Expansion)**.** *Let f be holomorphic on the annulus $A(z_0; r_1, r_2)$. Then f has a unique expansion*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}}_{\text{principal part}}, \tag{6.1}$$

valid for all $z \in A(z_0; r_1, r_2)$. The convergence is absolute and uniform on compact subsets of the annulus. The coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta, \qquad n \in \mathbb{Z}, \tag{6.2}$$

where γ_ρ is the positively oriented circle $|\zeta - z_0| = \rho$ for any $r_1 < \rho < r_2$.

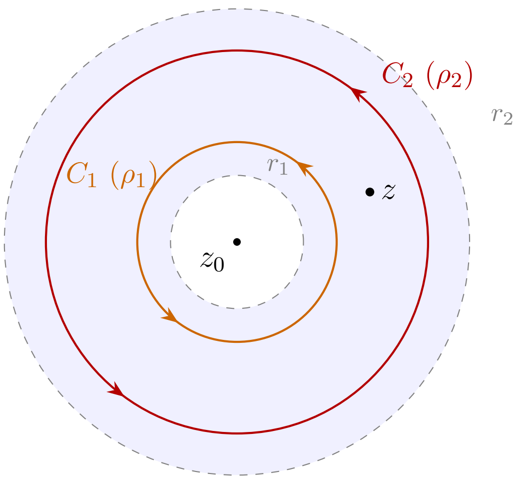
Remark 6.3 (Uniqueness). The Laurent expansion is *unique*: if $f(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$ converges on $A(z_0; r_1, r_2)$, then $b_n = a_n$ for all n. This follows from the integral formula (6.2) and the fact that

$$\frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho} (\zeta - z_0)^m \, d\zeta = \begin{cases} 1 & \text{if } m = -1, \\ 0 & \text{if } m \neq -1, \end{cases}$$

for any integer m.

6.1.2 Proof via the Cauchy formula

Proof of Theorem 6.2. Fix $z \in A(z_0; r_1, r_2)$. Choose radii ρ_1, ρ_2 with $r_1 < \rho_1 < |z - z_0| < \rho_2 < r_2$. Let C_1 be the circle $|\zeta - z_0| = \rho_1$ (positively oriented) and C_2 the circle $|\zeta - z_0| = \rho_2$ (positively oriented).



Step 1: Cauchy formula with two circles. By the Cauchy integral formula for multiply connected domains, the function $\zeta \mapsto f(\zeta)/(\zeta - z)$ is holomorphic in the annular region between C_1 and C_2 (except at $\zeta = z$, which lies in this region). We introduce a keyhole

contour (or use the standard deformation argument) to obtain:

$$f(z) = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{6.3}$$

The minus sign arises because C_1 must be traversed clockwise (negatively) in the boundary of the region between the two circles, but we have written it with positive orientation.

Step 2: Expanding the outer integral. For $\zeta \in C_2$ we have $|z - z_0| < \rho_2 = |\zeta - z_0|$, so exactly as in the proof of the Taylor theorem,

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}},$$

uniformly for $\zeta \in C_2$. Substituting into the first integral in (6.3) and interchanging sum and integral:

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta\right]}_{=a_n}(z - z_0)^n. \tag{6.4}$$

Step 3: Expanding the inner integral. For $\zeta \in C_1$ we have $|\zeta - z_0| = \rho_1 < |z - z_0|$. We write

$$\frac{1}{\zeta - z} = \frac{-1}{(z - z_0) - (\zeta - z_0)} = \frac{-1}{z - z_0}\cdot\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\sum_{m=0}^{\infty} \frac{(\zeta - z_0)^m}{(z - z_0)^{m+1}},$$

uniformly for $\zeta \in C_1$ (since the ratio $\rho_1/\,|z - z_0| < 1$). Substituting:

$$\begin{aligned} -\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta &= \sum_{m=0}^{\infty}\left[\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)^m\,\mathrm{d}\zeta\right]\frac{1}{(z - z_0)^{m+1}} \\ &= \sum_{m=0}^{\infty}\underbrace{\left[\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-m}}\,\mathrm{d}\zeta\right]}\frac{1}{(z - z_0)^{m+1}}. \end{aligned} \tag{6.5}$$

Setting $n = -(m+1)$ (so $m = -n - 1$ and n runs over $-1, -2, -3, \ldots$), the generic term becomes

$$a_n(z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta.$$

Step 4: Combining. Adding (6.4) and (6.5),

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=-\infty}^{-1} a_n(z - z_0)^n = \sum_{n=-\infty}^{+\infty} a_n(z - z_0)^n.$$

Step 5: Independence of ρ. Since $f(\zeta)/(\zeta - z_0)^{n+1}$ is holomorphic on the annulus (for each fixed n), the Cauchy theorem implies that the integral in (6.2) is independent of the choice of $\rho \in (r_1, r_2)$.

Step 6: Uniqueness. If $f(z) = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n$ converges in $A(z_0; r_1, r_2)$, multiply both sides by $(z - z_0)^{-k-1}$ and integrate over $|z - z_0| = \rho$:

$$\frac{1}{2\pi i}\oint_{|z-z_0|=\rho} \frac{f(z)}{(z - z_0)^{k+1}}\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} b_n \cdot \frac{1}{2\pi i}\oint_{|z-z_0|=\rho} (z - z_0)^{n-k-1}\,\mathrm{d}z = b_k.$$

Hence $b_k = a_k$ for all k. □

6.2 Classification of Isolated Singularities

Definition 6.4 (Isolated Singularity). A point $z_0 \in \mathbb{C}$ is an **isolated singularity** of a function f if f is holomorphic on some punctured disk $D(z_0, R) \setminus \{z_0\}$ for some $R > 0$, but f is not holomorphic at z_0 itself (either f is not defined at z_0 or not differentiable there).

At an isolated singularity z_0, the function has a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ valid in $0 < |z - z_0| < R$. The nature of the principal part $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ determines the type of singularity.

Definition 6.5 (Classification of Singularities). Let z_0 be an isolated singularity of f with Laurent expansion $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ in $0 < |z - z_0| < R$.

(i) z_0 is a **removable singularity** if $a_n = 0$ for all $n < 0$. In this case the Laurent series is just a Taylor series, and f can be extended to a holomorphic function at z_0 by setting $f(z_0) = a_0$.

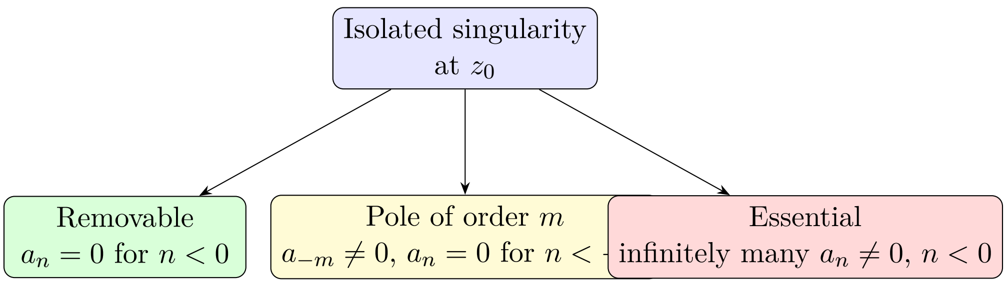
(ii) z_0 is a **pole** if there exists $m \geq 1$ such that $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$. The integer m is called the **order** (or **multiplicity**) of the pole.

(iii) z_0 is an **essential singularity** if $a_n \neq 0$ for infinitely many $n < 0$.

Definition 6.6 (Principal Part). The **principal part** of f at an isolated singularity z_0 is

$$\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}.$$

At a removable singularity, the principal part is zero. At a pole of order m, it is a finite sum $\sum_{n=1}^{m} a_{-n}(z - z_0)^{-n}$. At an essential singularity, it is an infinite series.



6.3 Removable Singularities

Theorem 6.7 (Riemann's Criterion for Removable Singularities). *Let z_0 be an isolated singularity of f. The following are equivalent:*

(i) z_0 is a removable singularity.

(ii) f is bounded in some punctured neighborhood of z_0.

(iii) $\lim_{z\to z_0}(z-z_0)f(z)=0$.

Proof.

(i)⇒(ii): If z_0 is removable, f extends to a holomorphic (hence continuous) function at z_0. A continuous function is bounded near z_0.

(ii)⇒(iii): If $|f(z)| \le M$ for $0<|z-z_0|<r$, then $|(z-z_0)f(z)| \le M|z-z_0| \to 0$ as $z \to z_0$.

(iii)⇒(i): Define $g(z)=(z-z_0)^2 f(z)$ for $z \neq z_0$ and $g(z_0)=0$. The hypothesis $\lim_{z\to z_0}(z-z_0)f(z)=0$ implies g is continuous at z_0, and in fact

$$g'(z_0)=\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}=\lim_{z\to z_0}(z-z_0)f(z)=0.$$

So g is holomorphic at z_0 (and on the whole disk). The Taylor expansion of g at z_0 starts as $g(z)=g(z_0)+g'(z_0)(z-z_0)+\cdots=0+0\cdot(z-z_0)+a_2(z-z_0)^2+\cdots$ Hence $g(z)=(z-z_0)^2h(z)$ where h is holomorphic at z_0. For $z\neq z_0$ we have $f(z)=h(z)$, so h is the holomorphic extension of f.

Alternatively, using the Laurent series: for $n<0$, the coefficient is

$$a_n=\frac{1}{2\pi i}\oint_{|\zeta-z_0|=\rho}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta.$$

The hypothesis (ii) gives $|f(\zeta)|\le M$ on the circle. For $n\le -1$ (so $n+1\le 0$),

$$|a_n|\le\frac{1}{2\pi}\cdot\frac{M}{\rho^{n+1}}\cdot 2\pi\rho=M\rho^{-n}.$$

Letting $\rho\to 0^+$, and since $-n\ge 1>0$, we get $a_n=0$ for all $n<0$.

□

Example 6.8 (A Classic Removable Singularity). Consider $f(z)=\dfrac{\sin z}{z}$ for $z\neq 0$. Since $\sin z = z - z^3/6+\cdots$, we have

$$f(z)=1-\frac{z^2}{6}+\frac{z^4}{120}-\cdots$$

The Laurent series has no negative powers, so $z=0$ is a removable singularity. Setting $f(0)=1$ gives a holomorphic extension to all of $\mathbb{C}$.

6.4 Poles

Proposition 6.9 (Characterization of Poles). *Let z_0 be an isolated singularity of f. The following are equivalent:*

(i) z_0 is a pole of order m.

(ii) $\lim_{z\to z_0}|f(z)|=+\infty$.

(iii) There exist a positive integer m and a holomorphic function h with $h(z_0) \neq 0$ such that

$$f(z) = \frac{h(z)}{(z - z_0)^m} \qquad \text{for } z \neq z_0.$$

(iv) The function $g(z) = (z - z_0)^m f(z)$ extends to a holomorphic function at z_0 with $g(z_0) \neq 0$, and m is the smallest such positive integer.

Proof.

(i)⇒(iii): If z_0 is a pole of order m, the Laurent expansion is

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

with $a_{-m} \neq 0$. Setting $h(z) = (z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots$, we see h is holomorphic at z_0 with $h(z_0) = a_{-m} \neq 0$.

(iii)⇒(ii): $|f(z)| = |h(z)| / |z - z_0|^m \to +\infty$ as $z \to z_0$, since $h(z_0) \neq 0$.

(ii)⇒(i): Since $|f(z)| \to \infty$, for $|z - z_0|$ small we have $f(z) \neq 0$, so $g(z) = 1/f(z)$ is holomorphic near z_0 with $g(z) \to 0$. By the Riemann criterion, g has a removable singularity at z_0 with $g(z_0) = 0$. Since $g \not\equiv 0$, the zero of g at z_0 has some finite order $m \geq 1$: $g(z) = (z - z_0)^m \phi(z)$ with $\phi(z_0) \neq 0$. Then $f(z) = 1/g(z) = (z - z_0)^{-m}/\phi(z)$, and $h(z) = 1/\phi(z)$ is holomorphic with $h(z_0) \neq 0$.

The equivalence with (iv) is immediate from (iii).

□

Example 6.10 (Poles of Various Orders). (a) $f(z) = \dfrac{1}{z - 1}$ has a simple pole (pole of order 1) at $z = 1$.

(b) $f(z) = \dfrac{1}{(z + i)^3}$ has a pole of order 3 at $z = -i$.

(c) $f(z) = \dfrac{e^z - 1}{z^2}$ at $z = 0$: since $e^z - 1 = z + z^2/2 + \cdots$, we have $f(z) = 1/z + 1/2 + \cdots$, so $z = 0$ is a simple pole.

(d) $f(z) = \dfrac{\sin z}{z^3}$ at $z = 0$: since $\sin z = z - z^3/6 + \cdots$, we have $f(z) = 1/z^2 - 1/6 + \cdots$, a pole of order 2.

6.5 Essential Singularities and the Casorati–Weierstrass Theorem

Essential singularities exhibit dramatically different behavior from removable singularities and poles. While the function stays bounded near a removable singularity and blows up near a pole, near an essential singularity the function takes values arbitrarily close to every complex number.

Theorem 6.11 (Casorati–Weierstrass Theorem). *Let z_0 be an essential singularity of f. Then for every $\delta > 0$, the image $f\big(D(z_0, \delta) \setminus \{z_0\}\big)$ is dense in $\mathbb{C}$. That is, for every $w \in \mathbb{C}$ and every $\varepsilon > 0$, there exists z with $0 < |z - z_0| < \delta$ and $|f(z) - w| < \varepsilon$.*

Proof. Suppose for contradiction that the image is not dense. Then there exist $w_0 \in \mathbb{C}$, $\varepsilon > 0$, and $\delta > 0$ such that $|f(z) - w_0| \geq \varepsilon$ for all z with $0 < |z - z_0| < \delta$.

Define $g(z) = \dfrac{1}{f(z) - w_0}$ for $0 < |z - z_0| < \delta$. Then g is holomorphic and $|g(z)| \leq 1/\varepsilon$, so g is bounded. By the Riemann criterion (Theorem 6.7), z_0 is a removable singularity of g, so g extends holomorphically to z_0.

Case 1: $g(z_0) \neq 0$. Then $f(z) = w_0 + 1/g(z)$ is holomorphic at z_0, contradicting the assumption that z_0 is a singularity of f (let alone an essential one).

Case 2: $g(z_0) = 0$. Then g has a zero of some order $m \geq 1$: $g(z) = (z - z_0)^m \phi(z)$ with $\phi(z_0) \neq 0$. Hence $f(z) = w_0 + 1/g(z) = w_0 + (z - z_0)^{-m}/\phi(z)$, which shows z_0 is a pole of order m, again a contradiction.

In both cases we reach a contradiction, so the image must be dense. □

Remark 6.12 (The Great Picard Theorem). The Casorati–Weierstrass theorem is a "weak" version of a much deeper result. The **Great Picard Theorem** states that in any punctured neighborhood of an essential singularity, f assumes every complex value infinitely often, with at most one exception. For instance, $e^{1/z}$ near $z = 0$ takes every value except 0 infinitely often. The proof of Picard's theorem requires more advanced tools (modular functions or Bloch–Schottky type theorems) and is beyond the scope of this course.

6.6 Worked Examples of Laurent Expansions

Example 6.13 (Laurent Expansion of $1/(z(z-1))$). We find the Laurent series of $f(z) = \dfrac{1}{z(z-1)}$ in different annular regions centered at $z_0 = 0$.

Partial fractions:

$$\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}.$$

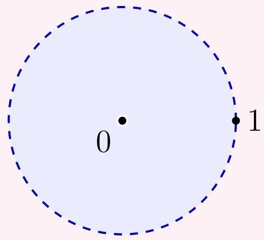
Region I: $0 < |z| < 1$. We need to expand $1/(z-1)$ in powers of z. Since $|z| < 1$:

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Therefore

$$f(z) = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n = -\frac{1}{z} - 1 - z - z^2 - \cdots$$

This is the Laurent series in $0 < |z| < 1$. The singularity at $z = 0$ is a simple pole.



Region I: $0 < |z| < 1$

Region II: $|z| > 1$**.** We need to expand $1/(z-1)$ in negative powers of z. Since $|z| > 1$, $|1/z| < 1$:

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-1/z} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \sum_{k=1}^{\infty} \frac{1}{z^k}.$$

Therefore

$$f(z) = -\frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{z^k} = \sum_{k=2}^{\infty} \frac{1}{z^k} = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

This Laurent series converges for $|z| > 1$.

Example 6.14 (Laurent Expansion of $e^{1/z}$)**.** The function $f(z) = e^{1/z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, with an isolated singularity at $z = 0$. Replacing z by $1/z$ in the exponential series:

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \frac{1}{3!\, z^3} + \cdots$$

This is the Laurent series in $0 < |z| < \infty$. Since there are infinitely many non-zero coefficients $a_{-n} = 1/n!$ for $n \geq 1$, the singularity at $z = 0$ is **essential**.
Note that $e^{1/z}$ never vanishes (consistent with the Great Picard Theorem: the value 0 is the single exceptional value).

Example 6.15 (Laurent Series Centered at a Pole)**.** Find the Laurent expansion of $f(z) = \dfrac{z}{(z-1)^2(z-2)}$ centered at $z_0 = 1$.

Partial fractions: Let $w = z - 1$, so $z = w + 1$ and $z - 2 = w - 1$.

$$f(z) = \frac{w+1}{w^2(w-1)}.$$

We decompose:

$$\frac{w+1}{w^2(w-1)} = \frac{A}{w} + \frac{B}{w^2} + \frac{C}{w-1}.$$

Multiplying through: $w + 1 = Aw(w-1) + B(w-1) + Cw^2$.

- $w = 0$: $1 = -B$, so $B = -1$.
- $w = 1$: $2 = C$.

- Comparing w^2 coefficients: $0 = A + C$, so $A = -2$.

Thus
$$f(z) = \frac{-2}{w} + \frac{-1}{w^2} + \frac{2}{w-1}, \qquad w = z - 1.$$

For $0 < |w| < 1$:
$$\frac{2}{w-1} = \frac{-2}{1-w} = -2\sum_{n=0}^{\infty} w^n.$$

So
$$f(z) = -\frac{1}{(z-1)^2} - \frac{2}{z-1} - 2\sum_{n=0}^{\infty}(z-1)^n.$$

This exhibits a pole of order 2 at $z_0 = 1$.

Example 6.16 (Laurent Expansion of $\cos(1/z)$)**.**

$$\cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\, z^{2n}} = 1 - \frac{1}{2!\, z^2} + \frac{1}{4!\, z^4} - \cdots$$

This converges for $0 < |z| < \infty$. Since infinitely many negative-power coefficients are non-zero, $z = 0$ is an essential singularity of $\cos(1/z)$.

Example 6.17 (Laurent Series in Multiple Annuli)**.** Consider $f(z) = \dfrac{1}{(z-1)(z-3)}$, centered at $z_0 = 0$. The singularities are at $z = 1$ and $z = 3$, giving three regions. By partial fractions: $f(z) = \dfrac{1}{2}\left(\dfrac{1}{z-3} - \dfrac{1}{z-1}\right)$.

Region I: $|z| < 1$**.** Both $1/(z-1)$ and $1/(z-3)$ expand in non-negative powers of z:

$$\frac{-1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$
$$\frac{1}{z-3} = \frac{-1}{3}\cdot\frac{1}{1-z/3} = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n} = -\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}.$$

So $f(z) = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{3^{n+1}} - 1\right) z^n$ (a Taylor series; this region has no singularity).

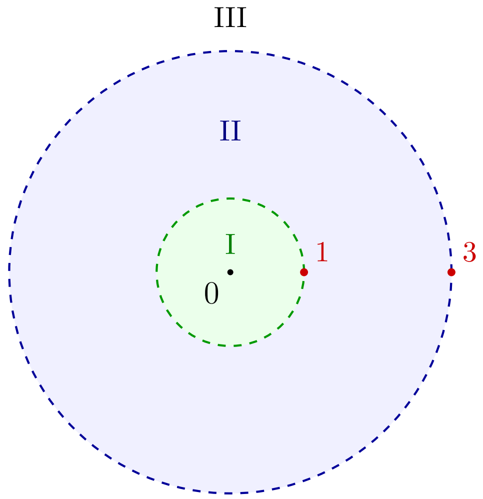
Region II: $1 < |z| < 3$**.** We expand $1/(z-1)$ in negative powers and $1/(z-3)$ in non-negative powers:

$$\frac{-1}{z-1} = \frac{-1}{z}\cdot\frac{1}{1-1/z} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\sum_{k=1}^{\infty}\frac{1}{z^k},$$
$$\frac{1}{z-3} = -\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}.$$

Region III: $|z| > 3$**.** Both terms expand in negative powers:

$$\frac{-1}{z-1} = -\sum_{k=1}^{\infty} \frac{1}{z^k},$$
$$\frac{1}{z-3} = \frac{1}{z} \cdot \frac{1}{1-3/z} = \sum_{n=0}^{\infty} \frac{3^n}{z^{n+1}} = \sum_{k=1}^{\infty} \frac{3^{k-1}}{z^k}.$$

So $f(z) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{3^{k-1}-1}{z^k}$.



6.7 Singularity at Infinity

Definition 6.18 (Singularity at Infinity)**.** A function f is said to have an **isolated singularity at** ∞ if f is holomorphic on $\{z : |z| > R\}$ for some $R > 0$. The nature of this singularity is defined by studying $g(w) = f(1/w)$ near $w = 0$.

(i) f has a **removable singularity at** ∞ if g has a removable singularity at 0 (equivalently, f is bounded for $|z|$ large).

(ii) f has a **pole of order** m **at** ∞ if g has a pole of order m at 0 (equivalently, $f(z)/z^m$ has a finite non-zero limit as $|z| \to \infty$).

(iii) f has an **essential singularity at** ∞ if g has an essential singularity at 0.

Remark 6.19. If f is holomorphic for $|z| > R$, it has a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ converging for $|z| > R$. The singularity at ∞ is:

- removable if $a_n = 0$ for all $n \geq 1$;
- a pole of order m if $a_m \neq 0$ and $a_n = 0$ for $n > m$;
- essential if $a_n \neq 0$ for infinitely many $n > 0$.

Note the reversal: at ∞, it is the *positive* powers that constitute the "singular part."

Example 6.20 (Singularities at Infinity). (a) $f(z) = \dfrac{1}{z}$: $g(w) = f(1/w) = w$ has a zero at 0, so f has a removable singularity at ∞ (with $f(\infty) = 0$).

(b) $f(z) = z^3 + 2z$: $g(w) = 1/w^3 + 2/w$ has a pole of order 3 at 0. So f has a pole of order 3 at ∞.

(c) $f(z) = e^z$: $g(w) = e^{1/w}$ has an essential singularity at 0. So e^z has an essential singularity at ∞.

(d) A rational function $p(z)/q(z)$ with $\deg p > \deg q$ has a pole at ∞ of order $\deg p - \deg q$.

Proposition 6.21. *An entire function f has a pole of order m at ∞ if and only if f is a polynomial of degree m. An entire function has a removable singularity at ∞ if and only if it is constant.*

Proof. An entire function has Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converging for all z.

If f has a pole of order m at ∞, then $a_n = 0$ for all $n > m$ (with $a_m \neq 0$), so f is a polynomial of degree m.

If f has a removable singularity at ∞, then $a_n = 0$ for all $n \geq 1$, so $f = a_0$ is constant. (Alternatively, a removable singularity at ∞ means f is bounded, and the conclusion follows from Liouville's theorem.) $\square$

6.8 Meromorphic Functions

Definition 6.22 (Meromorphic Function). A function f is **meromorphic** on an open set $\Omega \subseteq \mathbb{C}$ if f is holomorphic on Ω except for isolated singularities, all of which are poles.

Example 6.23. (a) Every rational function $p(z)/q(z)$ is meromorphic on $\mathbb{C}$, with poles at the zeros of q.

(b) $\tan z = \sin z / \cos z$ is meromorphic on $\mathbb{C}$, with simple poles at $z = (n + 1/2)\pi$, $n \in \mathbb{Z}$.

(c) $e^{1/z}$ is *not* meromorphic on $\mathbb{C}$ (the singularity at 0 is essential).

Remark 6.24. The set of meromorphic functions on a connected open set Ω forms a *field* under pointwise addition and multiplication (with the convention that poles are handled in the natural way). This field contains the field of holomorphic functions as a subring.

6.9 Further Results on Singularities

Proposition 6.25 (Behavior Near Different Singularities — Summary). *Let z_0 be an isolated singularity of f. Then:*

(i) z_0 *is removable* $\iff$ f *is bounded near* z_0 $\iff$ $\lim_{z\to z_0} f(z)$ *exists in* $\mathbb{C}$.

(ii) z_0 *is a pole* $\iff$ $\lim_{z\to z_0} |f(z)| = +\infty$.

(iii) z_0 *is essential* $\iff$ $\lim_{z\to z_0} f(z)$ *does not exist in* $\mathbb{C}\cup\{\infty\}$ $\iff$ $f(D(z_0,\delta)\setminus\{z_0\})$ *is dense in* $\mathbb{C}$ *for all* $\delta > 0$.

Proof. Parts (i) and (ii) are Theorem 6.7 and Proposition 6.9. Part (iii): if z_0 is essential, it is neither removable nor a pole, so f neither has a limit in $\mathbb{C}$ nor tends to ∞. The density statement is the Casorati–Weierstrass theorem (Theorem 6.11). ☐

Proposition 6.26 (Counting Negative Laurent Coefficients). *If z_0 is a pole of order m of f, then the Laurent expansion at z_0 is*

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

The coefficient a_{-1} is called the ***residue*** *of f at z_0, denoted* $\operatorname{Res}(f,z_0)$ *or* $\operatorname{Res}_{z=z_0} f(z)$. *For a simple pole ($m=1$),*

$$\operatorname{Res}(f,z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

For a pole of order m,

$$\operatorname{Res}(f,z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\Big[(z-z_0)^m f(z)\Big].$$

Proof. The Laurent expansion with finitely many negative terms is immediate from the definition of a pole of order m. For the residue formula: $(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + \cdots$ is holomorphic at z_0. Taking the $(m-1)$-st derivative and evaluating at z_0 extracts the coefficient of $(z-z_0)^{m-1}$, which is $(m-1)!\,a_{-1}$. ☐

Example 6.27 (Computing a Residue)**.** Find the residue of $f(z) = \dfrac{e^z}{(z-1)^2}$ at $z_0 = 1$.

Here $z_0 = 1$ is a pole of order 2. Using the formula:

$$\operatorname{Res}(f,1) = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\Big[(z-1)^2 f(z)\Big] = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}(e^z) = e.$$

6.10 Exercises

Exercise 6.1 (★)**.** Find the Laurent series of $f(z) = \dfrac{1}{z^2-z}$ centered at $z_0 = 0$ in the regions:

(a) $0 < |z| < 1$

(b) $|z| > 1$

Identify the type of singularity at $z = 0$.

Exercise 6.2 (★). Classify the singularities (removable, pole with order, or essential) of:

(a) $\dfrac{z^2 - 1}{z - 1}$ at $z = 1$

(b) $\dfrac{1 - \cos z}{z^2}$ at $z = 0$

(c) $z^2 e^{1/z}$ at $z = 0$

(d) $\dfrac{1}{\sin z}$ at $z = 0$

(e) $\dfrac{z}{\sin^2 z}$ at $z = 0$

Exercise 6.3 (★). Find the Laurent expansion of $f(z) = e^{z+1/z}$ centered at $z_0 = 0$, valid for $0 < |z| < \infty$. Express the coefficient a_n as a series.

Hint: Write $e^{z+1/z} = e^z \cdot e^{1/z}$ and multiply the two series.

Exercise 6.4 (★★). Let $f(z) = e^{1/z}$.

(a) Show directly (without invoking the Casorati–Weierstrass theorem) that for every $w \in \mathbb{C} \setminus \{0\}$ and every $\delta > 0$, there exists z with $0 < |z| < \delta$ and $f(z) = w$.

(b) Why is $w = 0$ not attained?

Exercise 6.5 (★★). Find the Laurent expansions of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ in the annuli:

(a) $0 < |z| < 1$

(b) $1 < |z| < 2$

(c) $|z| > 2$

Exercise 6.6 (★★). Let f be holomorphic on $D(z_0, R) \setminus \{z_0\}$ and suppose that $\lim_{z \to z_0} (z - z_0)^\alpha f(z) = 0$ for some $\alpha \in (0, 1)$. Prove that z_0 is a removable singularity.

Hint: Show that the hypothesis implies $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then apply Riemann's criterion.

Exercise 6.7 (★★). Compute the residues at all singularities of:

(a) $\dfrac{z^3}{z^2 + 1}$

(b) $\dfrac{e^z}{z^2(z^2 + \pi^2)}$

(c) $\dfrac{1}{z^2 \sin z}$

Exercise 6.8 (★★). Classify the singularity at ∞ for each of the following:

(a) $f(z) = \dfrac{z^3+1}{z^2-1}$

(b) $f(z) = e^{1/z}$

(c) $f(z) = \sin z$

(d) $f(z) = \dfrac{z}{z+1}$

Exercise 6.9 (★★★). Let f have an essential singularity at z_0. Prove that for every $n \geq 1$, the function f^n (i.e., f raised to the n-th power) also has an essential singularity at z_0.

Exercise 6.10 (★★★). (Identity Theorem.) Let f be holomorphic on a connected open set Ω, and suppose f is not identically zero. Prove that the zeros of f are isolated: for every $z_0 \in \Omega$ with $f(z_0) = 0$, there exists $\delta > 0$ such that $f(z) \neq 0$ for $0 < |z - z_0| < \delta$.

Hint: Use the Taylor expansion of f at z_0.

Exercise 6.11 (★★★). Let f be meromorphic on $\mathbb{C}$ with only finitely many poles, and suppose f has a pole (or removable singularity) at ∞. Prove that f is a rational function.

Hint: Subtract the principal parts at each pole to reduce to an entire function with a pole at ∞.

Exercise 6.12 (★★★). Prove that if f is meromorphic on $D(z_0, R) \setminus \{z_0\}$ (i.e., f is holomorphic except for poles in the punctured disk), and z_0 is not an accumulation point of the poles, then f has an isolated singularity at z_0 (which may be removable, a pole, or essential).

Chapter 7

Residue Theorem and Applications

7.1 Residues: Definition and Computation

Throughout this chapter, we let $\Omega \subset \mathbb{C}$ be an open set and $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ a function holomorphic on the punctured neighbourhood of an isolated singularity z_0.

Definition 7.1 (Residue)**.** Let z_0 be an isolated singularity of f. The *residue* of f at z_0 is the coefficient a_{-1} in the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

i.e.

$$\operatorname{Res}_{z=z_0} f(z) \;=\; a_{-1} \;=\; \frac{1}{2\pi i} \oint_{\gamma} f(z)\,\mathrm{d}z,$$

where γ is any positively oriented simple closed curve in Ω enclosing z_0 and no other singularity of f.

Remark 7.2 (Residue is intrinsic)*.* The residue does not depend on the choice of the curve γ: by the deformation theorem for holomorphic functions, any two such curves yield the same integral.

7.1.1 Residue at a simple pole

Proposition 7.3 (Residue at a simple pole)**.** *If f has a simple pole at z_0, then*

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof. Since z_0 is a simple pole, the Laurent series has the form

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

Multiplying both sides by $(z - z_0)$ and letting $z \to z_0$ gives $(z - z_0)f(z) \to a_{-1}$. □

Proposition 7.4 (Quotient rule for simple poles)**.** *If $f(z) = g(z)/h(z)$ where g and h are holomorphic near z_0, $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$ (so z_0 is a simple zero of h), then*

$$\operatorname{Res}_{z=z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}.$$

Proof. We have

$$\operatorname{Res}_{z=z_0} \frac{g(z)}{h(z)} = \lim_{z\to z_0} (z-z_0)\frac{g(z)}{h(z)} = \lim_{z\to z_0} g(z) \cdot \frac{z-z_0}{h(z)-h(z_0)} = \frac{g(z_0)}{h'(z_0)}.$$

□

7.1.2 Residue at a pole of order n

Proposition 7.5 (Residue at a pole of order n)**.** *If f has a pole of order $n \geq 1$ at z_0, then*

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \Big[(z-z_0)^n f(z)\Big].$$

Proof. Write the Laurent expansion

$$f(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^k.$$

Then $(z-z_0)^n f(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^{k+n} = \sum_{j=0}^{\infty} a_{j-n}(z-z_0)^j$, which is holomorphic at z_0. Its Taylor coefficients satisfy

$$a_{j-n} = \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\Big[(z-z_0)^n f(z)\Big]\bigg|_{z=z_0}.$$

Setting $j = n-1$ gives $a_{-1} = \frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\Big[(z-z_0)^n f(z)\Big]\Big|_{z=z_0}$. □

Example 7.6 (Computing residues)**.** Compute the residues of $f(z) = \dfrac{e^z}{(z-1)^2 z}$.

Solution. The singularities are $z = 0$ (simple pole) and $z = 1$ (pole of order 2).
At $z = 0$: Using Proposition 7.3,

$$\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} z \cdot \frac{e^z}{(z-1)^2 z} = \frac{e^0}{(0-1)^2} = 1.$$

At $z = 1$: Using Proposition 7.5 with $n = 2$,

$$\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2 \cdot \frac{e^z}{(z-1)^2 z}\right] = \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\frac{e^z}{z} = \lim_{z\to 1} \frac{e^z(z-1)}{z^2} = 0.$$

Example 7.7 (Residue via Laurent series)**.** Find $\operatorname{Res}_{z=0} \dfrac{1}{z^2 \sin z}$.

Solution. We know $\sin z = z - z^3/6 + z^5/120 - \cdots$, so

$$\frac{1}{z^2 \sin z} = \frac{1}{z^3(1 - z^2/6 + \cdots)} = \frac{1}{z^3}\left(1 + \frac{z^2}{6} + \cdots\right) = \frac{1}{z^3} + \frac{1}{6z} + \cdots$$

Therefore $\operatorname{Res}_{z=0} \dfrac{1}{z^2 \sin z} = \dfrac{1}{6}$.

7.2 The Residue Theorem

Theorem 7.8 (Residue Theorem). *Let $\Omega \subset \mathbb{C}$ be a simply connected open set and let $z_1, \ldots, z_N$ be finitely many distinct points in Ω. If $f\colon \Omega \setminus \{z_1, \ldots, z_N\} \to \mathbb{C}$ is holomorphic and γ is a positively oriented simple closed contour in Ω enclosing exactly the singularities $z_1, \ldots, z_N$, then*

$$\boxed{\frac{1}{2\pi i} \oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{N} \operatorname{Res}_{z=z_k} f(z).}$$

Proof. For each $k = 1, \ldots, N$ let γ_k be a small positively oriented circle of radius ε_k centred at z_k, where the radii are chosen small enough so that the disks $\overline{D(z_k, \varepsilon_k)}$ are pairwise disjoint and contained in the interior of γ.

Define the region $G = (\text{interior of } \gamma) \setminus \bigcup_{k=1}^{N} \overline{D(z_k, \varepsilon_k)}$. Then f is holomorphic on $\overline{G}$, and ∂G consists of γ (traversed positively) together with the circles γ_k (each traversed negatively, as seen from G).

By Cauchy's theorem applied to the multiply connected region G,

$$\oint_\gamma f(z)\,\mathrm{d}z - \sum_{k=1}^{N} \oint_{\gamma_k} f(z)\,\mathrm{d}z = 0.$$

Hence

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{N} \oint_{\gamma_k} f(z)\,\mathrm{d}z = \sum_{k=1}^{N} 2\pi i \operatorname{Res}_{z=z_k} f(z),$$

where the last equality follows from the definition of the residue (Definition 7.1). □

Remark 7.9 (Winding number version). More generally, if γ is a closed curve (not necessarily simple) that does not pass through any z_k, the residue theorem takes the form

$$\frac{1}{2\pi i} \oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{N} \operatorname{ind}(\gamma, z_k) \operatorname{Res}_{z=z_k} f(z),$$

where $\operatorname{ind}(\gamma, z_k) = \frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}z}{z - z_k}$ is the winding number (index) of γ about z_k.

7.3 Applications to Real Integrals

One of the most striking applications of the residue theorem is the evaluation of definite real integrals that are difficult or impossible to compute by real-variable methods alone. The general strategy is:

(i) embed the real integral as part of a complex contour integral,

(ii) close the contour so that it encloses certain poles,

(iii) apply the residue theorem,

(iv) show that the contributions from the added arcs vanish (or can be evaluated) in a suitable limit.

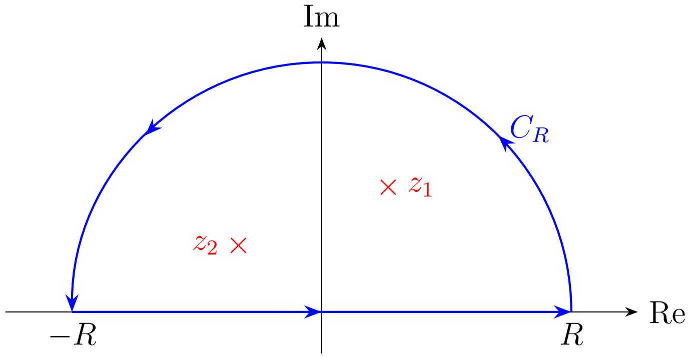
7.3.1 Rational integrals on the real line

Proposition 7.10 (Rational integrals). *Let $R(x) = P(x)/Q(x)$ be a rational function where $\deg Q \geq \deg P + 2$ and Q has no real zeros. Then*

$$\int_{-\infty}^{+\infty} R(x)\,\mathrm{d}x = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} R(z),$$

where the sum is over all poles of R in the upper half-plane $\{z : \operatorname{Im} z > 0\}$.

Proof. Consider the semicircular contour Γ_R consisting of the segment $[-R, R]$ on the real axis and the upper semicircle $C_R = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$, traversed counter-clockwise.



For R large enough, Γ_R encloses all poles of $R(z)$ in the upper half-plane. By the residue theorem,

$$\oint_{\Gamma_R} R(z)\,\mathrm{d}z = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z=z_k} R(z).$$

Now we split:

$$\oint_{\Gamma_R} R(z)\,\mathrm{d}z = \int_{-R}^{R} R(x)\,\mathrm{d}x + \int_{C_R} R(z)\,\mathrm{d}z.$$

On C_R with $z = Re^{i\theta}$, since $\deg Q \geq \deg P + 2$, we have $|R(z)| \leq M/R^2$ for large R. Therefore

$$\left|\int_{C_R} R(z)\,\mathrm{d}z\right| \leq \frac{M}{R^2} \cdot \pi R = \frac{M\pi}{R} \to 0 \quad \text{as } R \to \infty.$$

Taking $R \to \infty$ yields the result. □

Example 7.11 (A basic rational integral). Evaluate $I = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^2}$.

Solution. The function $f(z) = 1/(1+z^2) = 1/((z+i)(z-i))$ has simple poles at

$z = \pm i$. Only $z = i$ lies in the upper half-plane.

$$\operatorname{Res}_{z=i} \frac{1}{1+z^2} = \frac{1}{2i}.$$

Therefore $I = 2\pi i \cdot \frac{1}{2i} = \pi$.

Example 7.12 (Degree condition is essential)**.** Evaluate $I = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^2}$.

Solution. Here $f(z) = 1/(1+z^2)^2$ has a pole of order 2 at $z = i$ in the upper half-plane. We compute

$$\operatorname{Res}_{z=i} \frac{1}{(z^2+1)^2} = \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{(z+i)^2} = \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}.$$

Therefore $I = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$.

7.3.2 Trigonometric integrals

Proposition 7.13 (Trigonometric integrals)**.** *Let $R(\cos\theta, \sin\theta)$ be a rational function of $\cos\theta$ and $\sin\theta$ with no singularities on the unit circle. Then*

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,\mathrm{d}\theta = \oint_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{\mathrm{d}z}{iz}.$$

The right-hand side is evaluated by the residue theorem, summing over poles inside the unit disk $\mathbb{D}$.

Proof. Set $z = e^{i\theta}$, so $\mathrm{d}z = ie^{i\theta}\mathrm{d}\theta = iz\,\mathrm{d}\theta$, hence $\mathrm{d}\theta = \mathrm{d}z/(iz)$. Also

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}.$$

As θ ranges over $[0, 2\pi)$, z traverses the unit circle $|z| = 1$ once counter-clockwise. □

Example 7.14 (Trigonometric integral)**.** Evaluate $I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2+\cos\theta}$.

Solution. Substitute $z = e^{i\theta}$:

$$I = \oint_{|z|=1} \frac{1}{2+\frac{z+z^{-1}}{2}} \frac{\mathrm{d}z}{iz} = \oint_{|z|=1} \frac{1}{\frac{4+z+z^{-1}}{2}} \frac{\mathrm{d}z}{iz} = \oint_{|z|=1} \frac{2\,\mathrm{d}z}{iz(4+z+z^{-1})}.$$

Multiply numerator and denominator by z:

$$I = \oint_{|z|=1} \frac{2\,\mathrm{d}z}{i(z^2+4z+1)}.$$

The roots of $z^2+4z+1 = 0$ are $z = -2 \pm \sqrt{3}$. Only $z_0 = -2+\sqrt{3}$ lies inside $\mathbb{D}$ (since $\left|-2+\sqrt{3}\right| = 2-\sqrt{3} < 1$).

$$\operatorname{Res}_{z=z_0} \frac{2}{i(z^2+4z+1)} = \frac{2}{i\cdot 2z_0 + i\cdot 4}\bigg|_{z=z_0} = \frac{2}{i(2(-2+\sqrt{3})+4)} = \frac{2}{i\cdot 2\sqrt{3}} = \frac{1}{i\sqrt{3}}.$$

Therefore

$$I = 2\pi i \cdot \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

Example 7.15 (Another trigonometric integral)**.** Evaluate $I = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{5-4\cos\theta}$.

Solution. With $z = e^{i\theta}$:

$$I = \oint_{|z|=1} \frac{\mathrm{d}z}{iz(5-2(z+z^{-1}))} = \oint_{|z|=1} \frac{\mathrm{d}z}{i(5z-2z^2-2)} = \oint_{|z|=1} \frac{\mathrm{d}z}{-i(2z^2-5z+2)}.$$

Factor: $2z^2-5z+2 = (2z-1)(z-2)$. The roots are $z=1/2$ and $z=2$. Only $z=1/2$ lies inside $\mathbb{D}$.

$$\operatorname{Res}_{z=1/2} \frac{1}{-i(2z-1)(z-2)} = \frac{1}{-i\cdot 2\cdot(1/2-2)} = \frac{1}{-i\cdot 2\cdot(-3/2)} = \frac{1}{3i}.$$

Therefore $I = 2\pi i \cdot \dfrac{1}{3i} = \dfrac{2\pi}{3}$.

7.3.3 Jordan's lemma and Fourier-type integrals

Lemma 7.16 (Jordan's Lemma)**.** *Let f be continuous on $\{z : \operatorname{Im} z \geq 0,\ |z| \geq R_0\}$ for some $R_0 > 0$, and suppose $\left|f(Re^{i\theta})\right| \to 0$ uniformly in $\theta \in [0,\pi]$ as $R \to \infty$. Then for every $a > 0$,*

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\,\mathrm{d}z = 0,$$

where $C_R = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$ is the upper semicircle.

Proof. Set $M(R) = \max_{0\leq\theta\leq\pi}\left|f(Re^{i\theta})\right|$. On C_R, $z = Re^{i\theta}$, so $e^{iaz} = e^{iaR\cos\theta - aR\sin\theta}$ and $|e^{iaz}| = e^{-aR\sin\theta}$. Therefore

$$\begin{aligned}\left|\int_{C_R} f(z)e^{iaz}\,\mathrm{d}z\right| &\leq M(R)\int_0^{\pi} e^{-aR\sin\theta}\,R\,\mathrm{d}\theta \\ &= 2M(R)R\int_0^{\pi/2} e^{-aR\sin\theta}\,\mathrm{d}\theta.\end{aligned}$$

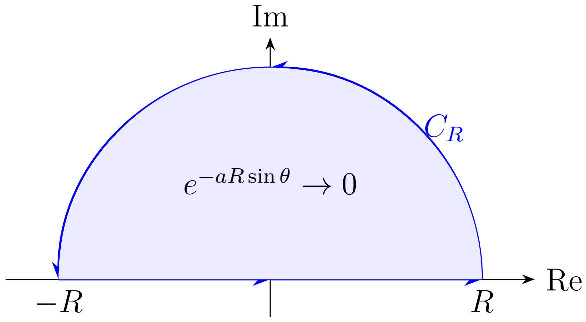
By Jordan's inequality, $\sin\theta \geq 2\theta/\pi$ for $\theta\in[0,\pi/2]$, so

$$\int_0^{\pi/2} e^{-aR\sin\theta}\,\mathrm{d}\theta \leq \int_0^{\pi/2} e^{-2aR\theta/\pi}\,\mathrm{d}\theta = \frac{\pi}{2aR}(1-e^{-aR}) \leq \frac{\pi}{2aR}.$$

Hence

$$\left|\int_{C_R} f(z)e^{iaz}\,\mathrm{d}z\right| \leq 2M(R)R\cdot\frac{\pi}{2aR} = \frac{\pi M(R)}{a} \to 0.$$

□



Example 7.17 (Fourier-type integral)**.** Evaluate $I = \displaystyle\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}\,\mathrm{d}x$.

Solution. Since $\cos x = \mathrm{Re}(e^{ix})$, we compute

$$J = \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}\,\mathrm{d}x$$

and take the real part. Using the upper semicircular contour with $f(z) = 1/(z^2+1)$, Jordan's lemma gives $\int_{C_R} f(z)e^{iz}\mathrm{d}z \to 0$. The only pole in the upper half-plane is $z = i$:

$$\mathrm{Res}_{z=i} \frac{e^{iz}}{z^2+1} = \frac{e^{i\cdot i}}{2i} = \frac{e^{-1}}{2i}.$$

Therefore $J = 2\pi i \cdot \dfrac{e^{-1}}{2i} = \dfrac{\pi}{e}$, and $I = \mathrm{Re}(J) = \dfrac{\pi}{e}$.

Example 7.18 (Integral with $\sin x$ in the numerator)**.** Evaluate $I = \displaystyle\int_{0}^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$.

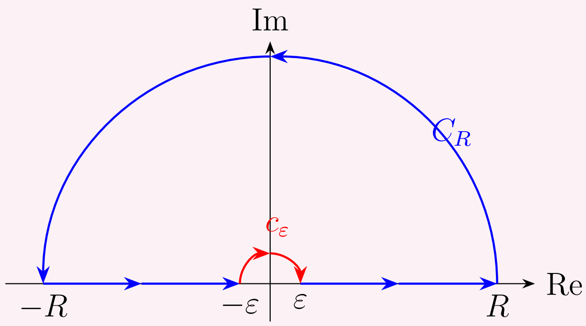
Solution. The integrand $(\sin x)/x$ extends continuously to $x = 0$ (with value 1), so there is no issue at the origin in the real sense. However, the complex function e^{iz}/z has a simple pole at $z = 0$. We use an *indented contour*: let Γ consist of

(i) the segment $[\varepsilon, R]$ on the real axis,

(ii) the upper semicircle C_R from R to $-R$,

(iii) the segment $[-R, -\varepsilon]$ on the real axis,

(iv) the small semicircle c_ε from $-\varepsilon$ to ε, traversed clockwise (i.e. in the upper half-plane).



Since e^{iz}/z is holomorphic inside Γ (the pole at 0 has been excluded), the residue theorem gives

$$\oint_\Gamma \frac{e^{iz}}{z}\,\mathrm{d}z = 0.$$

By Jordan's lemma, $\int_{C_R} \frac{e^{iz}}{z}\mathrm{d}z \to 0$ as $R \to \infty$. For the small semicircle, parametrise $z = \varepsilon e^{i\theta}$, $\theta : \pi \to 0$ (clockwise):

$$\int_{c_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z = \int_\pi^0 \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,\mathrm{d}\theta = i\int_\pi^0 e^{i\varepsilon e^{i\theta}}\,\mathrm{d}\theta \xrightarrow{\varepsilon\to 0} i\int_\pi^0 1\,\mathrm{d}\theta = -i\pi.$$

The real-axis contributions give

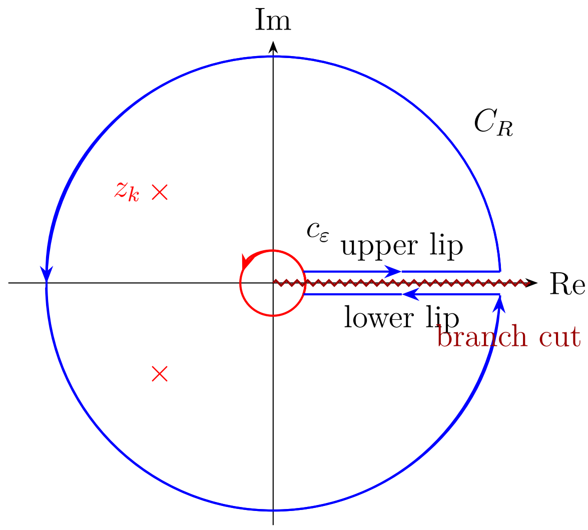
$$\int_\varepsilon^R \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,\mathrm{d}x = \int_\varepsilon^R \frac{e^{ix} - e^{-ix}}{x}\,\mathrm{d}x = 2i\int_\varepsilon^R \frac{\sin x}{x}\,\mathrm{d}x.$$

Combining and letting $R \to \infty$, $\varepsilon \to 0$:

$$2i\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x - i\pi = 0 \implies \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

7.3.4 Integrals with a logarithmic branch cut

A powerful technique for evaluating integrals of the form $\int_0^\infty x^{\alpha-1}R(x)\,\mathrm{d}x$ (where $0 < \alpha < 1$ and R is rational with no poles on $[0,\infty)$) uses the *keyhole contour*.



Proposition 7.19 (Keyhole contour method). *Let $0 < \alpha < 1$ and let $R(z) = P(z)/Q(z)$ be a rational function with $\deg Q \geq \deg P + 1$, no poles on $[0,\infty)$. Define $z^{\alpha-1} = e^{(\alpha-1)\operatorname{Log} z}$ using the principal branch with the branch cut on $[0,\infty)$ (i.e. $0 < \operatorname{Arg} z < 2\pi$). Then*

$$\int_0^\infty x^{\alpha-1}R(x)\,\mathrm{d}x = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum_k \operatorname{Res}_{z=z_k}\left[z^{\alpha-1}R(z)\right],$$

where the sum is over all poles z_k of R.

Proof. Integrate $f(z) = z^{\alpha-1}R(z)$ over the keyhole contour. On the upper lip of the cut, $\operatorname{Arg} z = 0^+$, so $z^{\alpha-1} = x^{\alpha-1}$. On the lower lip, $\operatorname{Arg} z = 2\pi^-$, so $z^{\alpha-1} = x^{\alpha-1}e^{2\pi i(\alpha-1)}$.

By the residue theorem:

$$\oint_{\text{keyhole}} f(z)\,\mathrm{d}z = 2\pi i \sum_k \operatorname{Res}_{z=z_k} f(z).$$

The contributions from the large circle C_R and small circle c_ε vanish as $R \to \infty$ and $\varepsilon \to 0$ (using $\deg Q \geq \deg P + 1$ and $0 < \alpha < 1$). The remaining contributions are:

$$\begin{aligned}
&\text{Upper lip: } \int_0^\infty x^{\alpha-1}R(x)\,\mathrm{d}x, \\
&\text{Lower lip: } -\int_0^\infty x^{\alpha-1}e^{2\pi i(\alpha-1)}R(x)\,\mathrm{d}x = -e^{2\pi i(\alpha-1)}\int_0^\infty x^{\alpha-1}R(x)\,\mathrm{d}x.
\end{aligned}$$

(The minus sign comes from the reversed direction of traversal.) Therefore

$$(1 - e^{2\pi i(\alpha-1)})\int_0^\infty x^{\alpha-1}R(x)\,\mathrm{d}x = 2\pi i \sum_k \operatorname{Res}_{z=z_k} f(z).$$

Since $e^{2\pi i(\alpha-1)} = e^{2\pi i\alpha} \cdot e^{-2\pi i} = e^{2\pi i\alpha}$, the result follows. ☐

Example 7.20 (Integral via keyhole contour)**.** Evaluate $I = \displaystyle\int_0^\infty \frac{x^{-1/2}}{1+x}\,\mathrm{d}x$.

Solution. Here $\alpha = 1/2$, $R(x) = 1/(1+x)$. The only pole of R is at $z = -1$ (not on $[0,\infty)$). With $z^{-1/2} = e^{-\frac{1}{2}\operatorname{Log} z}$ where $0 < \operatorname{Arg} z < 2\pi$, at $z = -1$ we have $\operatorname{Arg}(-1) = \pi$, so $(-1)^{-1/2} = e^{-i\pi/2} = -i$.

$$\operatorname{Res}_{z=-1} \frac{z^{-1/2}}{1+z} = (-1)^{-1/2} \cdot 1 = -i.$$

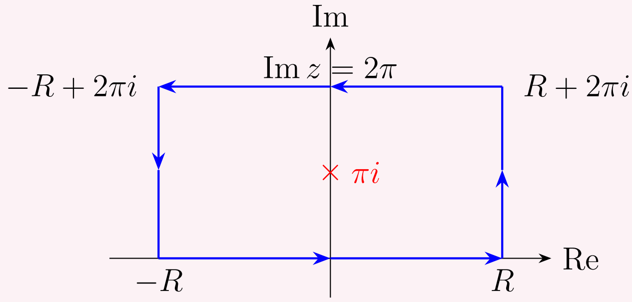
Therefore

$$I = \frac{2\pi i}{1 - e^{i\pi}} \cdot (-i) = \frac{2\pi i}{1-(-1)} \cdot (-i) = \frac{2\pi i}{2} \cdot (-i) = \pi i \cdot (-i) = \pi.$$

7.3.5 Rectangular contour technique

Example 7.21 (Gaussian-type integral with rectangular contour)**.** Evaluate $I = \displaystyle\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x}\,\mathrm{d}x$ for $0 < a < 1$.

Solution. Consider the rectangular contour Γ with vertices at $-R$, R, $R + 2\pi i$, $-R + 2\pi i$, traversed counter-clockwise.



The function $f(z) = e^{az}/(1+e^z)$ has poles where $e^z = -1$, i.e. $z = (2k+1)\pi i$ for $k \in \mathbb{Z}$. Inside Γ, the only pole is $z = \pi i$.

$$\mathrm{Res}_{z=\pi i} \frac{e^{az}}{1+e^z} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}.$$

On the bottom side: $\int_{-R}^{R} \frac{e^{ax}}{1+e^x} \mathrm{d}x$.
On the top side ($z = x + 2\pi i$, x from R to $-R$): $\int_{R}^{-R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} \mathrm{d}x = -e^{2\pi i a} \int_{-R}^{R} \frac{e^{ax}}{1+e^x} \mathrm{d}x$.
The integrals on the vertical sides tend to 0 as $R \to \infty$ (using $0 < a < 1$).
Therefore:

$$(1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} \mathrm{d}x = 2\pi i \cdot (-e^{a\pi i})$$

$$I = \frac{-2\pi i\, e^{a\pi i}}{1 - e^{2\pi i a}} = \frac{-2\pi i\, e^{a\pi i}}{e^{a\pi i}(e^{-a\pi i} - e^{a\pi i})} = \frac{-2\pi i}{-2i \sin(a\pi)} = \frac{\pi}{\sin(a\pi)}.$$

7.4 Further Worked Examples

Example 7.22 (Integral involving x^4)**.** Evaluate $I = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^4+1}$.

Solution. The poles of $f(z) = 1/(z^4+1)$ are the fourth roots of -1: $z_k = e^{i(\pi+2k\pi)/4}$ for $k = 0, 1, 2, 3$. Those in the upper half-plane are

$$z_0 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}, \qquad z_1 = e^{3i\pi/4} = \frac{-1+i}{\sqrt{2}}.$$

Since each is a simple pole of $1/(z^4+1)$ and $\frac{\mathrm{d}}{\mathrm{d}z}(z^4+1) = 4z^3$:

$$\operatorname{Res}_{z=z_k} \frac{1}{z^4+1} = \frac{1}{4z_k^3} = \frac{z_k}{4z_k^4} = \frac{z_k}{4(-1)} = -\frac{z_k}{4}.$$

Therefore

$$\begin{aligned} I &= 2\pi i \left(-\frac{z_0}{4} - \frac{z_1}{4}\right) = 2\pi i \cdot \left(-\frac{1}{4}\right)(z_0 + z_1) \\ &= 2\pi i \cdot \left(-\frac{1}{4}\right)\left(\frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}}\right) = 2\pi i \cdot \left(-\frac{1}{4}\right) \cdot \frac{2i}{\sqrt{2}} \\ &= 2\pi i \cdot \frac{-i}{2\sqrt{2}} = 2\pi \cdot \frac{1}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

Example 7.23 (Integral with exponential)**.** Evaluate $I = \int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+a^2}\,\mathrm{d}x$ for $a > 0$.

Solution. We write $\sin x = \operatorname{Im}(e^{ix})$ and compute

$$J = \int_{-\infty}^{+\infty} \frac{xe^{ix}}{x^2+a^2}\,\mathrm{d}x.$$

Then $I = \operatorname{Im}(J)$. The function $g(z) = ze^{iz}/(z^2+a^2)$ has poles at $z = \pm ia$. Only $z = ia$ is in the upper half-plane.
By Jordan's lemma, the semicircular arc contribution vanishes.

$$\operatorname{Res}_{z=ia} \frac{ze^{iz}}{z^2+a^2} = \frac{ia\, e^{i(ia)}}{2ia} = \frac{e^{-a}}{2}.$$

Therefore $J = 2\pi i \cdot \frac{e^{-a}}{2} = i\pi e^{-a}$, and $I = \operatorname{Im}(J) = \pi e^{-a}$.

Example 7.24 (A more challenging integral)**.** Evaluate $I = \int_0^{\infty} \frac{\ln x}{(1+x^2)^2}\,\mathrm{d}x$.

Solution. Consider $f(z) = (\operatorname{Log} z)/(1+z^2)^2$ with $\operatorname{Log} z$ taken with branch cut on $(-\infty, 0]$ (principal branch, $-\pi < \operatorname{Arg} z \leq \pi$).
We integrate over the contour consisting of $[\varepsilon, R]$ on the real axis, upper semicircle C_R, $[-R, -\varepsilon]$, and indented semicircle c_ε above the origin.

The only pole in the upper half-plane is $z = i$ (order 2). With $\operatorname{Log}(i) = i\pi/2$:

$$\begin{aligned}\operatorname{Res}_{z=i} \frac{\operatorname{Log} z}{(1+z^2)^2} &= \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z} \frac{\operatorname{Log} z}{(z+i)^2} \\ &= \lim_{z\to i} \frac{\frac{1}{z}(z+i)^2 - 2(z+i)\operatorname{Log} z}{(z+i)^4} \\ &= \frac{\frac{1}{i}(2i)^2 - 2(2i)\cdot\frac{i\pi}{2}}{(2i)^4} \\ &= \frac{\frac{-4}{i} + 2\pi}{16} = \frac{4i + 2\pi}{16} = \frac{\pi + 2i}{8}.\end{aligned}$$

On $[\varepsilon, R]$: $\operatorname{Log} x = \ln x$ (real). On $[-R, -\varepsilon]$ (write $x = -t$, $t > 0$): $\operatorname{Log}(-t) = \ln t + i\pi$. As $R \to \infty$, $\varepsilon \to 0$, the arc contributions vanish, and

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\mathrm{d}x + \int_0^\infty \frac{\ln t + i\pi}{(1+t^2)^2}\mathrm{d}t = 2\pi i \cdot \frac{\pi + 2i}{8}.$$

So $2\int_0^\infty \frac{\ln x}{(1+x^2)^2}\mathrm{d}x + i\pi\int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^2} = \frac{2\pi i(\pi+2i)}{8} = \frac{\pi(\pi i - 2)}{4}$.
We already know (Example 7.12) $\int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^2} = \pi/4$. Equating real parts:

$$2I = -\frac{\pi}{2} \implies I = -\frac{\pi}{4}.$$

7.5 Argument Principle and Rouché's Theorem

Theorem 7.25 (Argument Principle)**.** *Let f be meromorphic inside and on a simple closed contour γ, with zeros $a_1, \ldots, a_p$ (counted with multiplicity) and poles $b_1, \ldots, b_q$ (counted with multiplicity) inside γ, and no zeros or poles on γ. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = p - q.$$

Proof. Near a zero a_j of order m_j, write $f(z) = (z - a_j)^{m_j} g(z)$ with $g(a_j) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m_j}{z - a_j} + \frac{g'(z)}{g(z)},$$

so $\operatorname{Res}_{z=a_j} \frac{f'}{f} = m_j$. Similarly, near a pole b_k of order n_k, $f(z) = (z - b_k)^{-n_k} h(z)$ with $h(b_k) \neq 0$, giving $\operatorname{Res}_{z=b_k} \frac{f'}{f} = -n_k$. By the Residue Theorem,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_j m_j - \sum_k n_k = p - q.$$

□

Theorem 7.26 (Rouché's Theorem)**.** *Let f and g be holomorphic inside and on a simple closed contour γ. If $|g(z)| < |f(z)|$ for all $z \in \gamma$, then f and $f + g$ have the same number of zeros (counted with multiplicity) inside γ.*

Proof. Set $h_t(z) = f(z) + tg(z)$ for $t \in [0, 1]$. On γ, $|h_t(z)| \geq |f(z)| - t\,|g(z)| > 0$, so h_t has no zeros on γ. The function

$$N(t) = \frac{1}{2\pi i} \oint_\gamma \frac{h_t'(z)}{h_t(z)}\,\mathrm{d}z$$

is continuous in t and integer-valued by the Argument Principle. Hence $N(t)$ is constant, so $N(0) = N(1)$, i.e. the number of zeros of f equals the number of zeros of $f + g$ inside γ. □

Example 7.27 (Application of Rouché's theorem)**.** Show that the polynomial $p(z) = z^5 + 3z + 1$ has exactly one zero in the disk $|z| < 1$.

Solution. On $|z| = 1$, let $f(z) = 3z$ and $g(z) = z^5 + 1$. Then $|f(z)| = 3$ and $|g(z)| \leq |z|^5 + 1 = 2 < 3 = |f(z)|$. By Rouché's theorem, $f + g = p$ has the same number of zeros as $f(z) = 3z$ inside $|z| < 1$, namely one zero.

7.6 Exercises

Exercise 7.1 (Residue computations)**.** Compute the residues of the following functions at all their isolated singularities:

(a) $\dfrac{z^2}{(z-1)(z+2)^3}$

(b) $\dfrac{e^{1/z}}{z^2}$

(c) $\dfrac{\cos z}{z^2(z-\pi)}$

(d) $\dfrac{1}{z^2 \sin z}$

(e) $\dfrac{z}{(e^z - 1)^2}$

Exercise 7.2 (Real integrals via residues I)**.** Evaluate the following integrals using the residue theorem:

(a) $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^4 + 4}$

(b) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2\,\mathrm{d}x}{(x^2+1)(x^2+4)}$

(c) $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+a^2)^3} \qquad (a > 0)$

(d) $\displaystyle\int_{0}^{2\pi} \frac{\mathrm{d}\theta}{3 + 2\cos\theta + \sin\theta}$

Exercise 7.3 (Fourier-type integrals)**.** Evaluate:

(a) $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos(3x)}{x^2+4}\,\mathrm{d}x$

(b) $\displaystyle\int_{0}^{\infty} \frac{x\sin(2x)}{x^2+9}\,\mathrm{d}x$

(c) $\displaystyle\int_{-\infty}^{+\infty} \frac{x\cos x}{(x^2+1)^2}\,\mathrm{d}x$

Exercise 7.4 (Keyhole contour)**.** Using a keyhole contour, evaluate:

(a) $\displaystyle\int_{0}^{\infty} \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x$

(b) $\displaystyle\int_{0}^{\infty} \frac{\ln x}{x^2+1}\,\mathrm{d}x$

(c) $\displaystyle\int_{0}^{\infty} \frac{x^{a-1}}{1+x}\,\mathrm{d}x \quad (0<a<1)$ [*Euler's reflection formula:* $\Gamma(a)\Gamma(1-a)=\pi/\sin(\pi a)$]

Exercise 7.5 (Argument principle and Rouché)**.** (a) Show that z^4-5z+1 has exactly one zero in $|z|<1$.

(b) Show that z^7-5z^3+12 has no zeros in $|z|<1$ and exactly three zeros in $1<|z|<2$.

(c) Let $p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_0$ with $|a_0|>|a_1|+\cdots+|a_{n-1}|+1$. Show that p has no zeros in $|z|\le 1$.

Exercise 7.6 (Summation by residues)**.** Use the residue theorem with the function $\pi\cot(\pi z)/g(z)$ to show:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi}{2a}\coth(\pi a)-\frac{1}{2a^2} \quad (a>0)$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=G$ (Catalan's constant; express as a contour integral)

[*Hint:* For (a), integrate $\pi\cot(\pi z)/z^2$ over a square contour with vertices at $\pm(N+\frac{1}{2})\pm(N+\frac{1}{2})i$.]

Exercise 7.7 (Integral challenge)**.** Evaluate, using an appropriate contour:

(a) $\displaystyle\int_{0}^{\infty} \frac{\mathrm{d}x}{(1+x^2)(1+x^3)}$

(b) $\displaystyle\int_{0}^{\infty} \frac{(\ln x)^2}{1+x^2}\,\mathrm{d}x$

(c) $\displaystyle\int_{0}^{\infty} \frac{\cos(x^2)\,\mathrm{d}x}{}$ (Fresnel integral; use the contour consisting of $[0,R]$, an arc of angle $\pi/4$, and the line $te^{i\pi/4}$, $t\in[R,0]$).

Chapter 8

Conformal Mappings

8.1 Conformal Maps: Definition and First Properties

Throughout this chapter, we study holomorphic maps that preserve angles and local geometry. Conformal mappings are indispensable in potential theory, fluid dynamics, and electrostatics, where they allow one to transfer solutions of Laplace's equation between domains.

Definition 8.1 (Conformal map)**.** Let $\Omega \subset \mathbb{C}$ be an open set. A holomorphic function $f\colon \Omega \to \mathbb{C}$ is said to be *conformal* at a point $z_0 \in \Omega$ if $f'(z_0) \neq 0$. We say f is *conformal on* Ω if $f'(z) \neq 0$ for all $z \in \Omega$.
A *conformal equivalence* (or *biholomorphism*) between open sets Ω_1 and Ω_2 is a bijective holomorphic map $f\colon \Omega_1 \to \Omega_2$ whose inverse $f^{-1}\colon \Omega_2 \to \Omega_1$ is also holomorphic.

Theorem 8.2 (Angle preservation)**.** *If f is conformal at z_0, then f preserves angles between curves at z_0: if two smooth curves γ_1, γ_2 meet at z_0 with angle α, then their images $f \circ \gamma_1$ and $f \circ \gamma_2$ meet at $f(z_0)$ with the same angle α, both in magnitude and orientation.*

Proof. Let $\gamma_j(t)$ be smooth curves with $\gamma_j(0) = z_0$ and $\gamma_j'(0) \neq 0$ for $j = 1, 2$. The tangent direction of $f \circ \gamma_j$ at $t = 0$ is

$$(f \circ \gamma_j)'(0) = f'(z_0)\,\gamma_j'(0).$$

Since $f'(z_0) \neq 0$, write $f'(z_0) = re^{i\phi}$ with $r > 0$. Then

$$\arg(f \circ \gamma_j)'(0) = \phi + \arg \gamma_j'(0).$$

The angle between the image curves is

$$\arg(f \circ \gamma_2)'(0) - \arg(f \circ \gamma_1)'(0) = \arg \gamma_2'(0) - \arg \gamma_1'(0),$$

which equals the angle between the original curves. □

Remark 8.3 (Conformal ⇔ locally bijective). By the inverse function theorem for holomorphic functions, f is conformal at z_0 if and only if f is locally biholomorphic near z_0. A holomorphic bijection between open sets is automatically conformal (its inverse is holomorphic).

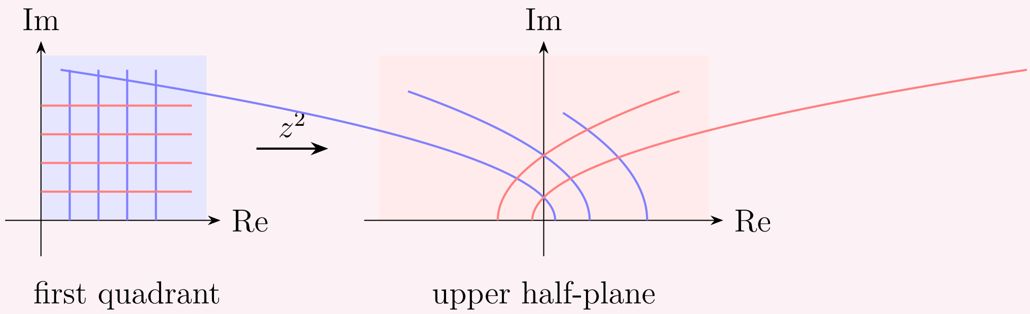
Proposition 8.4 (Anti-conformal maps). *The map $z \mapsto \overline{z}$ preserves angle magnitudes but reverses orientation. Composing a conformal map with conjugation yields an* anti-conformal *map.*

8.2 Basic Examples of Conformal Maps

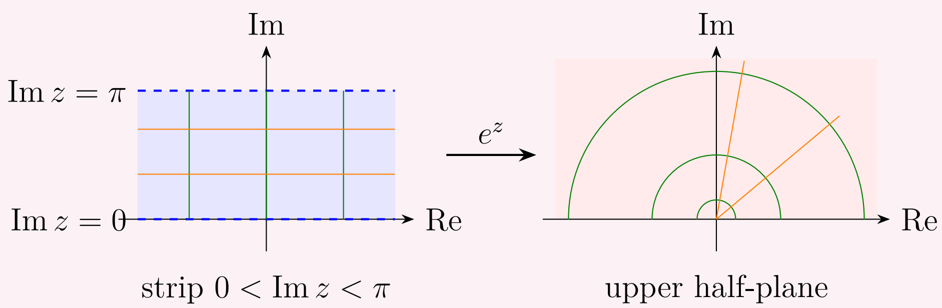
Example 8.5 (Translation). $f(z) = z + b$ for $b \in \mathbb{C}$. This is conformal everywhere ($f'(z) = 1$) and maps every region to a translated copy.

Example 8.6 (Rotation and dilation). $f(z) = az$ for $a \in \mathbb{C}\setminus\{0\}$. Writing $a = re^{i\theta}$, this is a rotation by θ and dilation by factor r. It is conformal everywhere ($f'(z) = a \neq 0$).

Example 8.7 (The power map). $f(z) = z^n$ for $n \geq 2$. This is conformal except at $z = 0$ (where $f'(0) = 0$) and maps a sector of opening angle α to a sector of opening angle $n\alpha$. In particular, z^2 maps the first quadrant $\{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ conformally onto the upper half-plane.



Example 8.8 (The exponential map). $f(z) = e^z$ is conformal everywhere ($f'(z) = e^z \neq 0$). It maps the horizontal strip $S = \{z : 0 < \operatorname{Im} z < \pi\}$ conformally onto the upper half-plane $\{w : \operatorname{Im} w > 0\}$. Vertical lines $\operatorname{Re} z = c$ map to semicircles $|w| = e^c$ in the upper half-plane, and horizontal lines $\operatorname{Im} z = d$ map to rays $\arg w = d$.



Example 8.9 (Inversion). $f(z) = 1/z$ is conformal on $\mathbb{C} \setminus \{0\}$. It maps circles and lines to circles and lines (treating lines as circles through ∞). The unit disk $\mathbb{D}$ maps to its exterior $\{|z| > 1\} \cup \{\infty\}$ and the unit circle $\mathbb{T}$ maps to itself.

8.3 Möbius Transformations

Definition 8.10 (Möbius transformation). A *Möbius transformation* (or *fractional linear transformation*) is a map of the form

$$T(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

The condition $ad - bc \neq 0$ ensures that T is not constant. We extend T to the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by setting $T(-d/c) = \infty$ and $T(\infty) = a/c$ (when $c \neq 0$; if $c = 0$, $T(\infty) = \infty$).

Proposition 8.11 (Properties of Möbius transformations). *(i) Every Möbius transformation is a conformal bijection $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*

(ii) The set of all Möbius transformations forms a group under composition, isomorphic to $\mathrm{PGL}(2, \mathbb{C})$.

(iii) The inverse of $T(z) = \frac{az+b}{cz+d}$ is $T^{-1}(z) = \frac{dz-b}{-cz+a}$.

(iv) Composition corresponds to matrix multiplication:

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix}.$$

(v) Every Möbius transformation is a composition of translations, rotations, dilations, and inversions.

Proof. We prove (v). If $c = 0$, then $T(z) = (a/d)z + b/d$, a dilation-rotation followed by a translation. If $c \neq 0$, perform polynomial long division:

$$T(z) = \frac{az + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c^2} \cdot \frac{1}{z + d/c}.$$

This is the composition of four elementary maps:

(1) $z \mapsto z + d/c$ (translation),

(2) $z \mapsto 1/z$ (inversion),

(3) $z \mapsto -(ad - bc)/c^2 \cdot z$ (rotation-dilation),

(4) $z \mapsto z + a/c$ (translation).

Each is conformal ($ad - bc \neq 0$ ensures step (3) is non-degenerate). □

Theorem 8.12 (Circles to circles)**.** *Möbius transformations map generalised circles (circles and lines in $\mathbb{C}$, viewed as circles on $\hat{\mathbb{C}}$) to generalised circles.*

Proof. By Proposition 8.11(v), it suffices to check each elementary type. Translations, rotations, and dilations clearly preserve the class of generalised circles.

For the inversion $w = 1/z$: a generalised circle has equation $A|z|^2 + \mathrm{Re}(\overline{B}z) + D = 0$ with $A, D \in \mathbb{R}$ and $B \in \mathbb{C}$ (a line if $A = 0$, a circle if $A \neq 0$). Under $z = 1/w$, $|z|^2 = 1/|w|^2$ and $\mathrm{Re}(\overline{B}z) = \mathrm{Re}(\overline{B}/w) = \mathrm{Re}(B\overline{w})/|w|^2$. Multiplying through by $|w|^2$:

$$A + \mathrm{Re}(\overline{\overline{B}}\,\overline{w}) + D|w|^2 = 0,$$

which is again a generalised circle equation (with A and D swapped, and B replaced by $\overline{B}$). □

8.3.1 Fixed points and classification

Definition 8.13 (Fixed points)**.** A *fixed point* of a Möbius transformation T is a point $z_0 \in \hat{\mathbb{C}}$ with $T(z_0) = z_0$.

Proposition 8.14 (Number of fixed points)**.** *A Möbius transformation $T \neq \mathrm{id}$ has exactly 1 or 2 fixed points in $\hat{\mathbb{C}}$.*

Proof. If $c \neq 0$, the fixed points in $\mathbb{C}$ satisfy $cz^2 + (d-a)z - b = 0$. Since $T \neq \mathrm{id}$, this quadratic is non-trivial, yielding 0, 1, or 2 finite roots. Including $z = \infty$ (which is fixed iff $c = 0$), the total is at most 2. It is at least 1 because a quadratic over $\mathbb{C}$ always has a root. □

Definition 8.15 (Classification of Möbius transformations)**.** Let $T \neq \mathrm{id}$ with matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ normalised so that $ad - bc = 1$. The *trace* $\tau = a + d$ determines the type (up to conjugation in the Möbius group):

- **Parabolic:** $\tau^2 = 4$ (one fixed point). Conjugate to $z \mapsto z + 1$.
- **Elliptic:** $\tau^2 \in [0, 4)$ with $\tau \in \mathbb{R}$ (two fixed points). Conjugate to $z \mapsto e^{i\theta}z$ (rotation).
- **Hyperbolic:** $\tau^2 \in (4, +\infty)$ with $\tau \in \mathbb{R}$ (two fixed points). Conjugate to $z \mapsto \lambda z$, $\lambda > 0$, $\lambda \neq 1$ (dilation).
- **Loxodromic:** $\tau \notin \mathbb{R}$ (two fixed points). Conjugate to $z \mapsto \lambda z$ with $|\lambda| \neq 1$, $\lambda \notin \mathbb{R}$ (spiral).

Example 8.16 (Classifying a Möbius transformation)**.** Classify $T(z) = \dfrac{3z-4}{z-1}$.

Solution. Here $a = 3$, $b = -4$, $c = 1$, $d = -1$, so $ad - bc = -3 + 4 = 1$. The trace is $\tau = a + d = 3 + (-1) = 2$, and $\tau^2 = 4$. Therefore T is **parabolic**.

The unique fixed point: $z^2 - 4z + 4 = 0$, i.e. $(z-2)^2 = 0$, so $z = 2$.

8.3.2 Cross-ratio

Definition 8.17 (Cross-ratio)**.** Given four distinct points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, the *cross-ratio* is

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

If one point is ∞, the definition is obtained by taking the appropriate limit (e.g. $(z_1, z_2; z_3, \infty) = (z_1 - z_3)/(z_2 - z_3)$).

Theorem 8.18 (Cross-ratio is a Möbius invariant)**.** *If T is a Möbius transformation, then*

$$(T(z_1), T(z_2); T(z_3), T(z_4)) = (z_1, z_2; z_3, z_4).$$

Proof. Define $S(z) = (z, z_2; z_3, z_4)$, a Möbius transformation with $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$. Similarly, $S'(w) = (w, Tz_2; Tz_3, Tz_4)$ sends $Tz_2 \mapsto 1$, $Tz_3 \mapsto 0$, $Tz_4 \mapsto \infty$. Then $S' \circ T$ and S are Möbius transformations agreeing at three points z_2, z_3, z_4, hence they are identical. Evaluating at z_1 gives the result. □

Proposition 8.19 (Three-point determination)**.** *Given two ordered triples of distinct points (z_1, z_2, z_3) and (w_1, w_2, w_3) in $\hat{\mathbb{C}}$, there exists a unique Möbius transformation T with $T(z_j) = w_j$ for $j = 1, 2, 3$. It is determined implicitly by*

$$(w, w_1; w_2, w_3) = (z, z_1; z_2, z_3).$$

Example 8.20 (Finding a Möbius transformation)**.** Find the Möbius transformation T sending $0 \mapsto 1$, $1 \mapsto i$, $\infty \mapsto -1$.

Solution. Write $T(z) = (az + b)/(cz + d)$. From the conditions:

$$\begin{aligned} T(\infty) = a/c = -1 &\implies a = -c, \\ T(0) = b/d = 1 &\implies b = d, \\ T(1) = (a + b)/(c + d) = i &\implies (-c + d)/(c + d) = i. \end{aligned}$$

From the last equation: $-c + d = ic + id$, so $d(1 - i) = c(1 + i)$, giving $c = d(1 - i)/(1 + i) = d \cdot (-i) = -id$. Take $d = 1$: then $c = -i$, $a = i$, $b = 1$.

$$\boxed{T(z) = \frac{iz + 1}{-iz + 1} = \frac{iz + 1}{1 - iz}.}$$

Verification: $T(0) = 1/1 = 1$; $T(1) = (i + 1)/(1 - i) = (1 + i)^2/2 = 2i/2 = i$; $T(\infty) = i/(-i) = -1$. ✓

8.4 The Cayley Transform

Definition 8.21 (Cayley transform)**.** The *Cayley transform* is the Möbius transformation

$$\varphi(z) = \frac{z-i}{z+i}.$$

Its inverse is $\varphi^{-1}(w) = i\,\frac{1+w}{1-w}$.

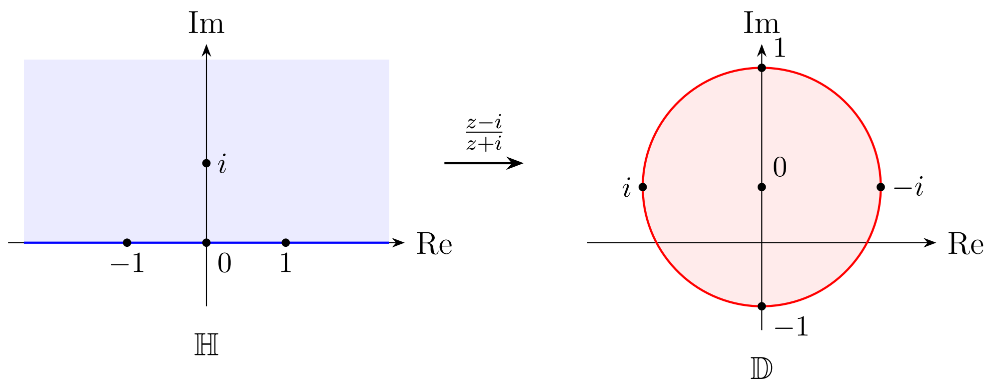
Theorem 8.22 (Cayley maps upper half-plane to disk)**.** *The Cayley transform φ is a conformal equivalence between the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and the unit disk $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$. It maps the real line $\mathbb{R} \cup \{\infty\}$ bijectively onto the unit circle $\mathbb{T}$.*

Proof. For $z = x + iy$ with $z \neq -i$:

$$|\varphi(z)|^2 = \frac{|z-i|^2}{|z+i|^2} = \frac{x^2 + (y-1)^2}{x^2 + (y+1)^2}.$$

This quantity is less than 1 iff $(y-1)^2 < (y+1)^2$, i.e. $y > 0$. Thus $|\varphi(z)| < 1 \Leftrightarrow \operatorname{Im} z > 0$. Similarly, $|\varphi(z)| = 1 \Leftrightarrow y = 0$.

Since φ is a Möbius transformation, it is a bijection $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, restricting to a bijection $\mathbb{H} \to \mathbb{D}$. It is holomorphic with $\varphi'(z) = 2i/(z+i)^2 \neq 0$, hence conformal. ☐



Remark 8.23 (Key correspondences under the Cayley transform).

$\mathbb{H}$	$\mathbb{D}$
i	0
0	-1
1	$-i$
-1	i
∞	1

The upper half of the imaginary axis maps to $(-1, 0)$ on the real axis in $\mathbb{D}$, and the real axis maps to $\mathbb{T}$.

Remark 8.24 (Automorphisms of $\mathbb{D}$). Every conformal automorphism of the unit disk has the form

$$T(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}, \qquad \theta \in \mathbb{R},\ |a| < 1.$$

These are *Blaschke factors*. The group $\mathrm{Aut}(\mathbb{D})$ is isomorphic to $\mathrm{PSU}(1,1)$.

Remark 8.25 (Automorphisms of $\mathbb{H}$). Every conformal automorphism of the upper half-plane has the form

$$T(z) = \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R}, \quad ad-bc > 0.$$

The group $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2,\mathbb{R})$.

8.5 The Riemann Mapping Theorem

Theorem 8.26 (Riemann Mapping Theorem)**.** *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected open set with $\Omega \neq \mathbb{C}$. Then there exists a conformal equivalence $f\colon \Omega \to \mathbb{D}$. Moreover, for any $z_0 \in \Omega$, there is a unique such f satisfying $f(z_0) = 0$ and $f'(z_0) > 0$.*

Remark 8.27 (Sketch of proof). The proof is one of the great achievements of complex analysis. We outline the main steps.

Step 1: **Family.** Let $\mathcal{F}$ be the family of all injective holomorphic functions $g\colon \Omega \to \mathbb{D}$ with $g(z_0) = 0$. One shows $\mathcal{F} \neq \emptyset$ using the hypothesis $\Omega \neq \mathbb{C}$ and a square-root construction.

Step 2: **Extremal problem.** Consider $\sup\{|g'(z_0)| : g \in \mathcal{F}\}$. This is finite by the Schwarz–Pick lemma.

Step 3: **Normal families.** By Montel's theorem, $\mathcal{F}$ is a normal family. A subsequence converges uniformly on compact subsets to some $f \in \mathcal{F}$ achieving the supremum.

Step 4: **Surjectivity.** If $f(\Omega) \neq \mathbb{D}$, one constructs $\tilde{f} \in \mathcal{F}$ with $\left|\tilde{f}'(z_0)\right| > |f'(z_0)|$, contradicting maximality.

Uniqueness follows from the Schwarz lemma: if f_1, f_2 are two such maps, then $f_1 \circ f_2^{-1}$ is an automorphism of $\mathbb{D}$ fixing 0 with positive derivative, so it must be the identity.

Corollary 8.28 (Equivalence of simply connected domains)**.** *Any two simply connected proper open subsets of $\mathbb{C}$ are conformally equivalent.*

Proof. If $\Omega_1, \Omega_2 \subsetneq \mathbb{C}$ are simply connected, let $f_j\colon \Omega_j \to \mathbb{D}$ be Riemann maps. Then $f_2^{-1} \circ f_1\colon \Omega_1 \to \Omega_2$ is a biholomorphism. ∎

8.6 Classical Conformal Maps

We catalogue the most important conformal maps between standard domains. These are the building blocks for solving boundary value problems.

8.6.1 Half-plane, disk, and strips

Example 8.29 (Upper half-plane → unit disk)**.** The Cayley transform (Definition 8.21): $w = \dfrac{z-i}{z+i} : \mathbb{H} \xrightarrow{\sim} \mathbb{D}$.

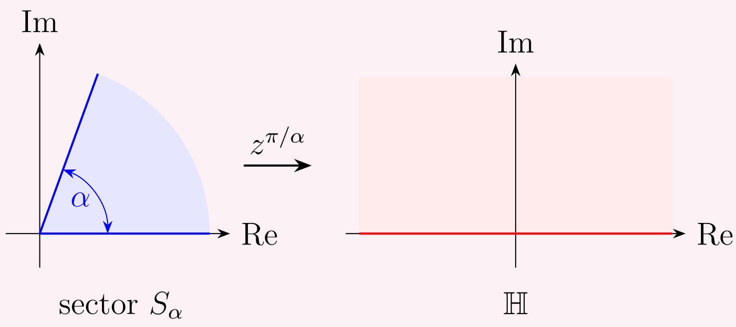
Example 8.30 (Strip → upper half-plane)**.** The horizontal strip $S = \{z : 0 < \operatorname{Im} z < \pi\}$ is mapped to $\mathbb{H}$ by $w = e^z$. If $z = x + iy$ with $0 < y < \pi$, then $w = e^x e^{iy}$ has $\operatorname{Im} w = e^x \sin y > 0$.

Example 8.31 (Strip → unit disk)**.** Composing, $w = \dfrac{e^z - i}{e^z + i}$ maps $S = \{0 < \operatorname{Im} z < \pi\}$ to $\mathbb{D}$. More generally, $w = \tanh\!\left(\frac{\pi z}{4a}\right)$ maps $\{-a < \operatorname{Im} z < a\}$ to $\mathbb{D}$.

Example 8.32 (Right half-plane → unit disk)**.** $w = \dfrac{z-1}{z+1}$ maps $\{\operatorname{Re} z > 0\} \to \mathbb{D}$. This follows from the Cayley transform by a 90° rotation: if $\operatorname{Re} z > 0$, then $\operatorname{Im}(iz) > 0$, so $\frac{iz-i}{iz+i} = \frac{z-1}{z+1} \in \mathbb{D}$.

8.6.2 Sectors and wedges

Example 8.33 (Sector → upper half-plane)**.** The sector $S_\alpha = \{z : 0 < \arg z < \alpha\}$, $0 < \alpha \leq 2\pi$, is mapped to $\mathbb{H}$ by $w = z^{\pi/\alpha}$ (principal branch).



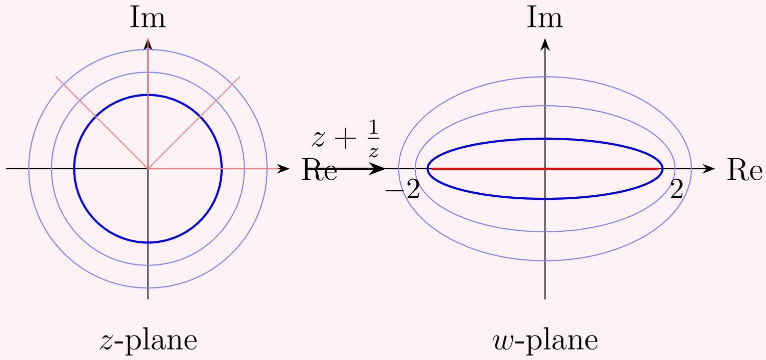
Special cases:

- $\alpha = \pi/2$ (first quadrant): $w = z^2$ maps to $\mathbb{H}$.
- $\alpha = \pi$ (upper half-plane to itself): $w = z$ (identity).
- $\alpha = 2\pi$ (slit plane): $w = z^{1/2}$ maps to $\mathbb{H}$.

8.6.3 The Joukowski map

Example 8.34 (Joukowski (Zhukovsky) map)**.** The map $J(z) = z + 1/z$ is conformal on $\{|z| > 1\}$ (since $J'(z) = 1 - 1/z^2 \neq 0$ there). It maps:

- the exterior of the unit disk $\{|z| > 1\}$ conformally onto $\mathbb{C} \setminus [-2, 2]$;
- the unit circle to the segment $[-2, 2]$, traversed twice;
- circles $|z| = r > 1$ to ellipses with semi-axes $r + 1/r$ and $r - 1/r$;
- rays $\arg z = \theta$ to branches of hyperbolas.



This map is fundamental in aerodynamics: by applying J to a circle passing close to $z = -1$, one obtains airfoil profiles (Joukowski airfoils).

8.6.4 Lens domains

Example 8.35 (Lens to half-plane)**.** A *lens* (or *lune*) is a region bounded by two circular arcs meeting at angles α at their intersection points p and q. To map a lens with vertices at ± 1 and interior angle α to the upper half-plane:

(1) Apply $T(z) = (z-1)/(z+1)$, sending $1 \mapsto 0$, $-1 \mapsto \infty$. The lens becomes a sector of angle α.

(2) Apply $w \mapsto w^{\pi/\alpha}$ to open the sector to a half-plane.

The composition $w = \left(\frac{z-1}{z+1}\right)^{\pi/\alpha}$ maps the lens conformally to $\mathbb{H}$.

8.6.5 The logarithm and related maps

Example 8.36 (Slit plane $\to$ strip)**.** The principal logarithm $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ (with $-\pi < \operatorname{Arg} z < \pi$) maps $\mathbb{C} \setminus (-\infty, 0]$ conformally onto the horizontal strip $\{w : -\pi < \operatorname{Im} w < \pi\}$.

Example 8.37 (Upper half-plane $\to$ strip via Log)**.** The map $w = \operatorname{Log} z$ sends $\mathbb{H}$ to the strip $\{0 < \operatorname{Im} w < \pi\}$. This is the inverse of $e^w : \{0 < \operatorname{Im} w < \pi\} \to \mathbb{H}$.

8.6.6 Summary table of standard conformal maps

Domain	**Map**	**Image**
$\mathbb{H}$	$\dfrac{z-i}{z+i}$	$\mathbb{D}$
$\{0 < \operatorname{Im} z < \pi\}$	e^z	$\mathbb{H}$
$\{0 < \arg z < \alpha\}$	$z^{\pi/\alpha}$	$\mathbb{H}$
$\{\operatorname{Re} z > 0\}$	$\dfrac{z-1}{z+1}$	$\mathbb{D}$
$\mathbb{C} \setminus (-\infty, 0]$	$\operatorname{Log} z$	$\{-\pi < \operatorname{Im} w < \pi\}$
$\mathbb{C} \setminus [-2, 2]$	$\frac{1}{2}(z + \sqrt{z^2 - 4})$	$\{\lvert w\rvert > 1\}$
$\mathbb{D}$	$\dfrac{z}{(1-z)^2}$ (Koebe)	$\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$

8.7 Images of Grids Under Conformal Maps

A powerful way to visualise a conformal map f is to draw the images of a Cartesian grid. Since f is conformal, the image curves intersect at right angles wherever $f' \neq 0$.

Example 8.38 (Grid image under $f(z) = z^2$)**.** Under $w = z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$:

- Vertical lines $x = c$ map to the parabolas $u = c^2 - v^2/(4c^2)$, opening to the left.
- Horizontal lines $y = c$ map to the parabolas $u = v^2/(4c^2) - c^2$, opening to the right.

These two families of parabolas are orthogonal, confirming conformality.

Example 8.39 (Grid image under $f(z) = e^z$)**.** Under $w = e^z = e^x(\cos y + i \sin y)$:

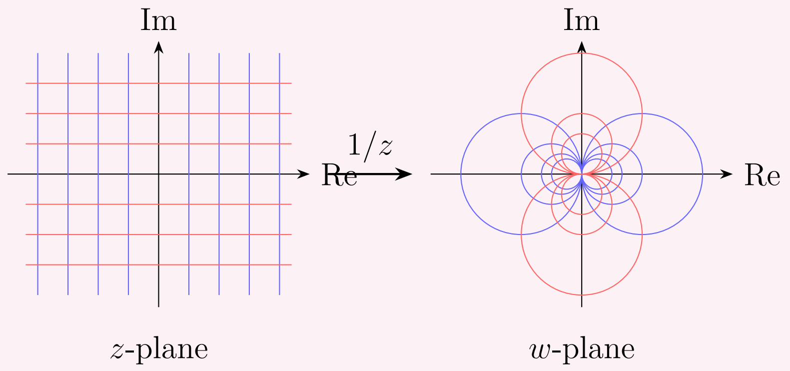
- Vertical lines $x = c$ map to circles $|w| = e^c$.
- Horizontal lines $y = c$ map to rays $\arg w = c$.

The Cartesian grid becomes a polar grid.

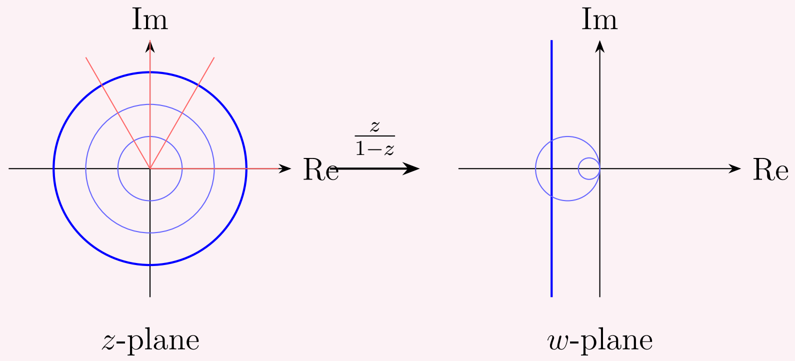
Example 8.40 (Grid image under $f(z) = 1/z$)**.** Under $w = 1/z = \bar{z}/|z|^2$, with $z = x + iy$:

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

- Vertical lines $x = c \neq 0$ map to circles $\left(u - \frac{1}{2c}\right)^2 + v^2 = \frac{1}{4c^2}$ (passing through the origin).
- Horizontal lines $y = c \neq 0$ map to circles $u^2 + \left(v + \frac{1}{2c}\right)^2 = \frac{1}{4c^2}$.



Example 8.41 (Möbius transformation action on circles)**.** Consider $T(z) = \dfrac{z}{1 - z}$, which sends $0 \mapsto 0$, $\infty \mapsto -1$, $1 \mapsto \infty$. It maps the unit circle $|z| = 1$ to the vertical line $\operatorname{Re} w = -1/2$ (since T maps the boundary of $\mathbb{D}$ to a generalised circle through $T(1) = \infty$).
The family of circles $|z| = r$ for $0 < r < 1$ maps to a family of circles in $\{\operatorname{Re} w > -1/2\}$ that shrink to the point 0 as $r \to 0$.



8.8 The Schwarz–Christoffel Formula

We briefly state the Schwarz–Christoffel formula, which gives an explicit conformal map from the upper half-plane to any polygon.

Theorem 8.42 (Schwarz–Christoffel formula)**.** *Let P be a polygon with vertices $w_1, \dots, w_n$ (in order) and interior angles $\alpha_1\pi, \dots, \alpha_n\pi$ (where $0 < \alpha_k < 2$ and $\sum \alpha_k = n-2$). Then a conformal map $f\colon \mathbb{H} \to \operatorname{int}(P)$ has the form*

$$f(z) = A\int_{z_0}^{z} \prod_{k=1}^{n} (\zeta - x_k)^{\alpha_k - 1}\, \mathrm{d}\zeta + B,$$

where $x_1 < x_2 < \cdots < x_n$ are points on $\mathbb{R}$ with $f(x_k) = w_k$, and $A, B \in \mathbb{C}$ are constants.

Example 8.43 (Map to a half-strip)**.** The infinite vertical half-strip $\{w : \operatorname{Re} w \in (-\pi/2, \pi/2), \operatorname{Im} w > 0\}$ is the image of $\mathbb{H}$ under $f(z) = \arcsin z$. This is a Schwarz–Christoffel map with two finite vertices at $w = \pm\pi/2$ (angles $\pi/2$ each) and one vertex at ∞.

8.9 Exercises

Exercise 8.1 (Basic conformal maps)**.** (a) Show that $f(z) = z + 1/z$ maps $\mathbb{D} \setminus \{0\}$ to $\mathbb{C} \setminus [-2, 2]$ (but is 2-to-1 on $\mathbb{D}$).

(b) Find a conformal map from the first quadrant $\{\operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ to the unit disk $\mathbb{D}$.

(c) Find a conformal map from $\{z : |z| < 1, \operatorname{Im} z > 0\}$ (upper half of the disk) to $\mathbb{D}$.

(d) Find a conformal map from the vertical strip $\{z : 0 < \operatorname{Re} z < 1\}$ to $\mathbb{H}$.

Exercise 8.2 (Möbius transformations)**.** (a) Find the Möbius transformation sending $1 \mapsto 0$, $i \mapsto 1$, $-1 \mapsto \infty$.

(b) Show that the Möbius transformations preserving $\mathbb{D}$ are exactly those of the form $e^{i\theta}\frac{z-a}{1-\bar{a}z}$.

(c) Prove that a Möbius transformation mapping $\mathbb{R} \cup \{\infty\}$ to itself has the form $(az+b)/(cz+d)$ with $a, b, c, d \in \mathbb{R}$.

(d) Let $T(z) = (2z+i)/(iz+2)$. Find the image of $\mathbb{D}$ under T.

Exercise 8.3 (Cross-ratio). (a) Compute $(1, i; -1, -i)$.

(b) Show that four points lie on a generalised circle if and only if their cross-ratio is real.

(c) Show that the cross-ratio takes only six values under permutations of the four points. If $\lambda = (z_1, z_2; z_3, z_4)$, list all six values in terms of λ.

Exercise 8.4 (Conformal maps to standard domains). Find explicit conformal maps from the following regions to the unit disk $\mathbb{D}$:

(a) The sector $\{z : 0 < \arg z < \pi/3\}$.

(b) The region $\{z : |z| > 1, \operatorname{Im} z > 0\}$ (upper half of the complement of $\mathbb{D}$).

(c) The lens-shaped region bounded by $|z - i| = \sqrt{2}$ and $|z + i| = \sqrt{2}$.

(d) The region $\{z : \operatorname{Re} z > 0\} \setminus [0, 1]$ (right half-plane with a slit).

Exercise 8.5 (Schwarz–Christoffel). (a) Verify that $f(z) = \sqrt{z}$ (principal branch) is the Schwarz–Christoffel map from $\mathbb{H}$ to the first quadrant.

(b) Show that $f(z) = \cosh^{-1}(z)$ maps $\mathbb{H}$ to the semi-infinite strip $\{w : 0 < \operatorname{Im} w < \pi, \operatorname{Re} w > 0\}$.

(c) Using the Schwarz–Christoffel formula, find the conformal map from $\mathbb{H}$ to an equilateral triangle. Express the answer as an integral.

Exercise 8.6 (Geometric properties). (a) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Show that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ (Schwarz lemma) and deduce that $|f'(0)| \leq 1$.

(b) Prove that $\operatorname{Aut}(\mathbb{C}) = \{z \mapsto az + b : a \neq 0\}$.

(c) Prove that $\operatorname{Aut}(\mathbb{C} \setminus \{0\})$ consists of $z \mapsto az$ and $z \mapsto a/z$ for $a \neq 0$.

(d) Show that there is no conformal equivalence between $\mathbb{C}$ and $\mathbb{D}$. (*Hint:* Liouville's theorem.)

Exercise 8.7 (Visualisation). Describe (or sketch) the image of the Cartesian grid $\{x + iy : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$ under the following maps:

(a) $f(z) = z^2$

(b) $f(z) = e^z$

(c) $f(z) = \sin z$

(d) $f(z) = (z-1)/(z+1)$

For (c), note that $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$, so vertical lines map to hyperbolas and horizontal lines map to ellipses (all confocal with foci at ± 1).

Chapter 9

Rouché's Theorem and the Argument Principle

9.1 Zeros of Holomorphic Functions

We begin by establishing the fundamental algebraic structure of zeros of holomorphic functions, which is considerably more rigid than in the real-variable setting.

Definition 9.1 (Zero of order m)**.** Let f be holomorphic on a domain $\Omega \subseteq \mathbb{C}$ and let $z_0 \in \Omega$. We say that f has a **zero of order** (or **multiplicity**) $m \geq 1$ at z_0 if

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0, \qquad f^{(m)}(z_0) \neq 0.$$

Equivalently, f can be written as

$$f(z) = (z - z_0)^m \, g(z),$$

where g is holomorphic on Ω and $g(z_0) \neq 0$.

Proposition 9.2 (Characterisation of zero order)**.** *Let f be holomorphic on a domain Ω with $f \not\equiv 0$, and suppose $f(z_0) = 0$ for some $z_0 \in \Omega$. Then there exists a unique integer $m \geq 1$ such that $f(z) = (z - z_0)^m g(z)$ with g holomorphic on Ω and $g(z_0) \neq 0$.*

Proof. Since $f \not\equiv 0$, the Taylor series of f about z_0,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!},$$

is not identically zero. Let m be the smallest index with $a_m \neq 0$. Then $m \geq 1$ (since $a_0 = f(z_0) = 0$) and

$$f(z) = (z - z_0)^m \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^n = (z - z_0)^m \, g(z).$$

The function $g(z) = \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^n$ is holomorphic on the disc of convergence and satisfies $g(z_0) = a_m \neq 0$. Extending g to all of Ω by $g(z) = f(z)/(z - z_0)^m$ for $z \neq z_0$ gives the result. Uniqueness of m follows from the identity theorem. □

Theorem 9.3 (Isolation of zeros). *Let f be holomorphic on a domain Ω with $f \not\equiv 0$. Then the zeros of f are isolated: for every zero z_0, there exists $r > 0$ such that $f(z) \neq 0$ for $0 < |z - z_0| < r$.*

Proof. Write $f(z) = (z - z_0)^m g(z)$ as in Proposition 9.2. Since $g(z_0) \neq 0$ and g is continuous, there exists $r > 0$ such that $g(z) \neq 0$ for $|z - z_0| < r$. For $0 < |z - z_0| < r$, we then have $f(z) = (z - z_0)^m g(z) \neq 0$. □

Corollary 9.4 (Identity theorem). *If f and g are holomorphic on a domain Ω and $f(z_n) = g(z_n)$ for a sequence (z_n) with a limit point in Ω, then $f \equiv g$ on Ω.*

Proof. Apply Theorem 9.3 to $h = f - g$. If $h \not\equiv 0$, then its zeros are isolated, contradicting the existence of an accumulation point of zeros in Ω. □

Definition 9.5 (Meromorphic function — zeros and poles). Let f be meromorphic on Ω. For $z_0 \in \Omega$, define the **order** of f at z_0, denoted $\operatorname{ord}_{z_0}(f)$, by:

- $\operatorname{ord}_{z_0}(f) = m > 0$ if f has a zero of order m at z_0;
- $\operatorname{ord}_{z_0}(f) = -m < 0$ if f has a pole of order m at z_0;
- $\operatorname{ord}_{z_0}(f) = 0$ if f is holomorphic and nonzero at z_0.

Remark 9.6 (Logarithmic derivative). If f is meromorphic and not identically zero, then the **logarithmic derivative** f'/f is meromorphic, and at each point z_0 where $\operatorname{ord}_{z_0}(f) = m$ (possibly negative), we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \text{holomorphic terms near } z_0.$$

Hence $\operatorname{Res}_{z_0}\left(\frac{f'}{f}\right) = m = \operatorname{ord}_{z_0}(f)$. This observation is the key to the argument principle.

9.2 The Argument Principle

Theorem 9.7 (Argument Principle). *Let f be meromorphic on a domain containing $\overline{D}$, where D is a bounded domain with piecewise-smooth boundary ∂D, oriented positively. Suppose f has no zeros or poles on ∂D. Let Z denote the number of zeros and P the number of poles of f inside D, counted with multiplicity. Then*

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)}\, \mathrm{d}z = Z - P. \tag{9.1}$$

Proof. Since f is meromorphic on a domain containing $\overline{D}$ and has no zeros or poles on ∂D, the function f'/f is meromorphic on a neighbourhood of $\overline{D}$ with poles only at the zeros and poles of f.

Let $z_1, \ldots, z_N$ be the zeros of f in D, with respective orders $m_1, \ldots, m_N$, and let $w_1, \ldots, w_M$ be the poles of f in D, with respective orders $p_1, \ldots, p_M$. Near a zero z_k of order m_k, we can write $f(z) = (z - z_k)^{m_k} g_k(z)$ with g_k holomorphic and nonvanishing, so

$$\frac{f'(z)}{f(z)} = \frac{m_k}{z - z_k} + \frac{g_k'(z)}{g_k(z)}.$$

The second term is holomorphic near z_k, so $\operatorname{Res}_{z_k}\left(\frac{f'}{f}\right) = m_k$.

Similarly, near a pole w_j of order p_j, writing $f(z) = (z - w_j)^{-p_j} h_j(z)$ with h_j holomorphic and nonvanishing:

$$\frac{f'(z)}{f(z)} = \frac{-p_j}{z - w_j} + \frac{h_j'(z)}{h_j(z)},$$

so $\operatorname{Res}_{w_j}\left(\frac{f'}{f}\right) = -p_j$.

By the Residue Theorem,

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)}\, \mathrm{d}z = \sum_{k=1}^{N} m_k + \sum_{j=1}^{M} (-p_j) = Z - P. \qquad \square$$

Remark 9.8 (Topological interpretation). The integral on the left-hand side of (9.1) can be interpreted as the **winding number** of the image curve $f \circ \gamma$ around the origin:

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)}\, \mathrm{d}z = \operatorname{ind}(f \circ \gamma, 0),$$

where γ parametrises ∂D. Indeed, writing $w = f(z)$ gives $\mathrm{d}w = f'(z)\, \mathrm{d}z$, so the integral becomes $\frac{1}{2\pi i} \oint_{f \circ \gamma} \frac{\mathrm{d}w}{w}$. Thus the argument principle states that the number of times the image $f(\gamma)$ winds around the origin equals $Z - P$.

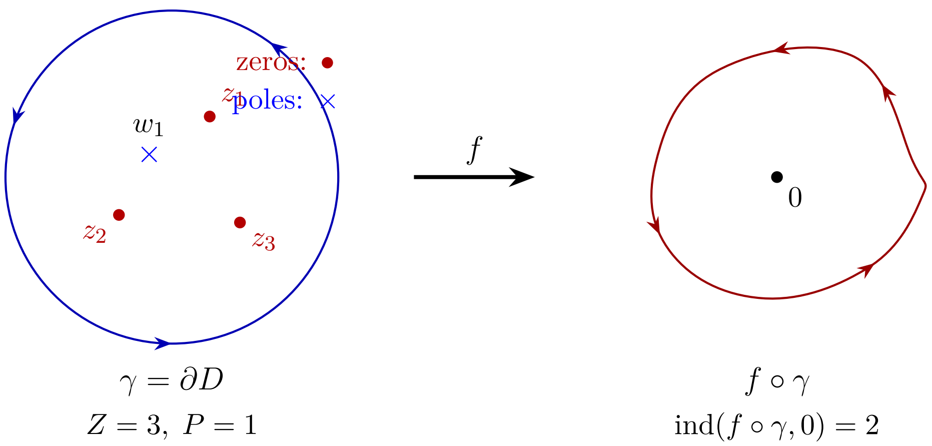


Figure 9.1: The argument principle: f maps the boundary γ to a curve winding $Z - P = 3 - 1 = 2$ times around the origin.

Example 9.9 (Counting zeros of a polynomial)**.** Let $f(z) = z^4 + 6z + 3$. We want to determine how many zeros f has in the disc $|z| < 2$. On $|z| = 2$:

$$\left|z^4\right| = 16, \qquad |6z + 3| \leq 6 \cdot 2 + 3 = 15 < 16.$$

Since f is a polynomial (no poles), the argument principle gives $Z = \frac{1}{2\pi i} \oint_{|z|=2} \frac{f'(z)}{f(z)} \, dz$. By the estimate above and Rouché's theorem (which we shall prove shortly), f has the same number of zeros in $|z| < 2$ as z^4, namely 4. Since $\deg f = 4$, all zeros of f lie in $|z| < 2$.

Proposition 9.10 (Argument principle for $f - a$)**.** *Under the hypotheses of Theorem 9.7, if $a \in \mathbb{C}$ is not in the image $f(\partial D)$, then*

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z) - a} \, dz = N(f, a, D) - P,$$

where $N(f, a, D)$ denotes the number of solutions of $f(z) = a$ in D, counted with multiplicity, and P is the number of poles.

Proof. Apply Theorem 9.7 to $g(z) = f(z) - a$, noting that g has the same poles as f and zeros exactly where $f(z) = a$. □

9.3 Rouché's Theorem

Rouché's theorem is one of the most powerful tools for locating zeros of holomorphic functions. It allows one to compare the zeros of a "complicated" function with those of a simpler one.

Theorem 9.11 (Rouché's Theorem)**.** *Let f and g be holomorphic on a domain containing $\overline{D}$, where D is a bounded domain with piecewise-smooth positively oriented boundary ∂D. If*

$$|g(z)| < |f(z)| \quad \text{for all } z \in \partial D, \tag{9.2}$$

then f and $f + g$ have the same number of zeros (counted with multiplicity) inside D.

Proof. The condition $|g(z)| < |f(z)|$ on ∂D implies in particular that $f(z) \neq 0$ on ∂D, and also that $(f + g)(z) \neq 0$ on ∂D (since $|f + g| \geq |f| - |g| > 0$).

Consider the family of functions

$$f_t(z) = f(z) + t\, g(z), \qquad t \in [0, 1].$$

For each $t \in [0, 1]$ and $z \in \partial D$:

$$|f_t(z)| = |f(z) + tg(z)| \geq |f(z)| - t\,|g(z)| \geq |f(z)| - |g(z)| > 0.$$

Thus f_t has no zeros on ∂D for any t. By the argument principle, the number of zeros of f_t inside D is

$$N(t) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f_t'(z)}{f_t(z)} \, dz = \frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z) + t\, g'(z)}{f(z) + t\, g(z)} \, dz.$$

The integrand depends continuously on t (uniformly on ∂D, since the denominator is bounded away from zero). Therefore $N(t)$ is a continuous function of t. But $N(t)$ is always a non-negative integer, so it must be constant. Hence $N(0) = N(1)$, i.e., f and $f + g$ have the same number of zeros in D. □

Remark 9.12 (Symmetric form of Rouché's theorem). A stronger, symmetric version states: if $|f(z) - g(z)| < |f(z)| + |g(z)|$ on ∂D, then f and g have the same number of zeros in D. The classical form follows by setting $g_{\text{sym}} = f + g$ and noting that $|(f + g) - f| = |g| < |f| \leq |f| + |f + g|$ whenever $|g| < |f|$.

9.3.1 Applications of Rouché's Theorem

Theorem 9.13 (Fundamental Theorem of Algebra via Rouché). *Every non-constant polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ with $a_n \neq 0$ has exactly n zeros in $\mathbb{C}$ (counted with multiplicity).*

Proof. Set $f(z) = a_n z^n$ and $g(z) = a_{n-1} z^{n-1} + \cdots + a_0$. On $|z| = R$:

$$|g(z)| \leq |a_{n-1}|\, R^{n-1} + \cdots + |a_0| \leq C\, R^{n-1},$$

where $C = |a_{n-1}| + \cdots + |a_0|$. Also $|f(z)| = |a_n|\; R^n$. For $R > C/\,|a_n|$ we have $|g(z)| < |f(z)|$ on $|z| = R$. By Rouché's theorem, $p = f + g$ has the same number of zeros in $|z| < R$ as $f(z) = a_n z^n$, which has exactly n zeros (all at the origin). Hence p has n zeros in $\mathbb{C}$. □

Example 9.14 (Counting zeros in a specific region). Determine the number of roots of $p(z) = z^5 + 3z^3 + 7$ in the annulus $1 < |z| < 2$.
Step 1: Zeros in $|z| < 1$. On $|z| = 1$, set $f(z) = 7$ and $g(z) = z^5 + 3z^3$. Then $|g(z)| \leq 1 + 3 = 4 < 7 = |f(z)|$. By Rouché, p has the same number of zeros in $|z| < 1$ as $f(z) = 7$, which is 0.
Step 2: Zeros in $|z| < 2$. On $|z| = 2$, set $f(z) = z^5$ and $g(z) = 3z^3 + 7$. Then $|f(z)| = 32$ and $|g(z)| \leq 3 \cdot 8 + 7 = 31 < 32$. By Rouché, p has 5 zeros in $|z| < 2$.
Conclusion. There are $5 - 0 = 5$ zeros in $1 \leq |z| < 2$. Since $|p(z)|$ on $|z| = 1$ satisfies $|p(z)| \geq 7 - 4 = 3 > 0$, there are no zeros on $|z| = 1$, so all 5 zeros lie in $1 < |z| < 2$.

Example 9.15 (Perturbation of zeros). Show that for small $\varepsilon > 0$, the equation $e^z = 1 + z + \varepsilon z^2$ has exactly one root in $|z| < 1$.
Set $f(z) = 1 + z$ and $g(z) = e^z - (1 + z) - \varepsilon z^2$. Note that $f + g = e^z - \varepsilon z^2$. We use a different decomposition: let $F(z) = e^z$ and $G(z) = -1 - z - \varepsilon z^2$, so $F + G = e^z - 1 - z - \varepsilon z^2$.
Actually, we apply Rouché differently. Consider $h(z) = e^z - 1 - z$ and $k(z) = \varepsilon z^2$. On $|z| = 1$, we have $|h(z)| = |\sum_{n=2}^{\infty} z^n/n!| \leq \sum_{n=2}^{\infty} 1/n! = e - 2 \approx 0.718$. Also $|k(z)| = \varepsilon$. We want to compare e^z and $1 + z + \varepsilon z^2$. Let $f(z) = e^z$ and $g(z) = -(1 + z + \varepsilon z^2)$. On $|z| = 1$: $|f(z)| = e^{\operatorname{Re} z} \geq e^{-1} \approx 0.368$ and $|g(z)| \leq 1 + 1 + \varepsilon = 2 + \varepsilon$. This does not directly work.
Instead, set $f(z) = 1 + z$ and $g(z) = \varepsilon z^2$, so $f + g = 1 + z + \varepsilon z^2$. On $|z| = r$ for suitable r: we need $|f(z)| > |g(z)|$. On $|z| = 1/2$: $|f(z)| \geq 1 - 1/2 = 1/2$ and $|g(z)| = \varepsilon/4$.

For $\varepsilon < 2$, Rouché gives that $1+z+\varepsilon z^2$ has 1 zero in $|z| < 1/2$ (same as $1+z$, which has its zero at $z=-1$, outside the disc). This gives 0 zeros.
Let us reconsider the problem. The equation is $e^z = 1+z+\varepsilon z^2$. Set $h(z) = e^z - 1 - z = \frac{z^2}{2} + \frac{z^3}{6} + \cdots$, so h has a zero of order 2 at $z=0$. Now $e^z - (1+z+\varepsilon z^2) = h(z) - \varepsilon z^2 = (\frac{1}{2} - \varepsilon)z^2 + \frac{z^3}{6} + \cdots$. On $|z| = \delta$ small: set $f(z) = (\frac{1}{2}-\varepsilon)z^2$ and $g(z) = \frac{z^3}{6} + \cdots$, so $|f(z)| = \left|\frac{1}{2} - \varepsilon\right| \delta^2$ and $|g(z)| \le C\delta^3$. For δ small enough, $|g| < |f|$, giving 2 zeros of $e^z - 1 - z - \varepsilon z^2$ near the origin.
On $|z| = 1$: let $f(z) = e^z - 1 - z$ and $g(z) = -\varepsilon z^2$. Then $|g(z)| = \varepsilon$. We need $|f(z)| > \varepsilon$ on $|z| = 1$. Since $f(z) = z^2/2 + z^3/6 + \cdots$ and $|f(z)|$ achieves a minimum on $|z| = 1$ that is strictly positive (as f has no zeros on $|z| = 1$; its only zero is at $z = 0$), for ε sufficiently small, Rouché applies and $e^z - 1 - z - \varepsilon z^2$ has 2 zeros in $|z| < 1$ (same as $e^z - 1 - z$).
But the original equation asks for roots of $e^z - 1 - z - \varepsilon z^2 = 0$. Since $e^z - 1 - z$ has a double zero at 0, and the perturbation εz^2 splits it, we get exactly 2 roots near the origin (and hence in $|z| < 1$) for small ε. The claim of "exactly one root" holds only under specific conditions on the region or the perturbation parameter.
For a cleaner application: the equation $z^7 - 5z^3 + 12 = 0$. On $|z| = 1$: $|12| > |z^7 - 5z^3| \le 1 + 5 = 6$, so 0 zeros in $|z| < 1$. On $|z| = 2$: $|-5z^3| = 40$ and $|z^7 + 12| \le 128 + 12 = 140$. This is not sharp. Instead: $|z^7| = 128$ vs $|-5z^3 + 12| \le 40 + 12 = 52 < 128$, giving 7 zeros in $|z| < 2$. So 7 zeros in $1 \le |z| < 2$.

9.4 Hurwitz's Theorem

Theorem 9.16 (Hurwitz's Theorem). *Let (f_n) be a sequence of holomorphic functions on a domain Ω converging uniformly on compact subsets to f. If $f \not\equiv 0$ and $f(z_0) = 0$ for some $z_0 \in \Omega$, then for every sufficiently small $r > 0$ such that $\overline{D(z_0, r)} \subseteq \Omega$ and f has no other zeros in $\overline{D(z_0, r)}$, there exists N such that for all $n \ge N$, f_n has exactly as many zeros in $D(z_0, r)$ as f does (i.e., the order of the zero of f at z_0).*
More precisely, if $f \not\equiv 0$, then either:

(i) every $z_0 \in \Omega$ with $f(z_0) = 0$ attracts zeros of f_n with the correct multiplicity, or

(ii) f has no zeros in Ω and, for n large enough, neither does f_n.

Proof. Choose $r > 0$ small enough that $\overline{D(z_0, r)} \subseteq \Omega$ and $f(z) \ne 0$ for $0 < |z - z_0| \le r$. This is possible by the isolation of zeros. Let

$$\delta = \min_{|z-z_0|=r} |f(z)| > 0.$$

Since $f_n \to f$ uniformly on the circle $|z - z_0| = r$, for n large enough we have $|f_n(z) - f(z)| < \delta \le |f(z)|$ on this circle. By Rouché's theorem (Theorem 9.11), $f_n = f + (f_n - f)$ has the same number of zeros in $D(z_0, r)$ as f, namely the order of the zero of f at z_0. □

Corollary 9.17 (Hurwitz — injective limit). *If (f_n) is a sequence of injective holomorphic functions on a domain Ω converging uniformly on compact subsets to f, then f is either injective or constant.*

Proof. Suppose f is not constant and $f(a) = f(b)$ for distinct $a, b \in \Omega$. Set $c = f(a) = f(b)$ and apply Hurwitz's theorem to $g_n(z) = f_n(z) - c$ converging to $g(z) = f(z) - c$. Since $g(a) = 0$, for large n, g_n has a zero a_n near a (with $a_n \to a$). Similarly, g_n has a zero b_n near b (with $b_n \to b$). For large n, $a_n \neq b_n$, so $f_n(a_n) = f_n(b_n) = c$, contradicting injectivity of f_n. □

9.5 The Open Mapping Theorem

Theorem 9.18 (Open Mapping Theorem). *If f is a non-constant holomorphic function on a domain Ω, then f is an* ***open map****: for every open set $U \subseteq \Omega$, the image $f(U)$ is open in $\mathbb{C}$.*

Proof. Let $w_0 = f(z_0)$ for some $z_0 \in U \subseteq \Omega$. We must show that w_0 is an interior point of $f(U)$, i.e., some disc centred at w_0 is contained in $f(U)$.

Since f is non-constant, the function $f(z) - w_0$ has an isolated zero at z_0 (of some order $m \geq 1$). Choose $r > 0$ such that $\overline{D(z_0, r)} \subseteq U$ and $f(z) - w_0 \neq 0$ for $0 < |z - z_0| \leq r$. Set

$$\delta = \min_{|z - z_0| = r} |f(z) - w_0| > 0.$$

Now let w be any point with $|w - w_0| < \delta$. On the circle $|z - z_0| = r$:

$$|(f(z) - w) - (f(z) - w_0)| = |w - w_0| < \delta \leq |f(z) - w_0|.$$

By Rouché's theorem, $f(z) - w$ has the same number of zeros in $D(z_0, r)$ as $f(z) - w_0$, namely $m \geq 1$. In particular, $f(z) = w$ for some $z \in D(z_0, r) \subseteq U$, so $w \in f(U)$.

This shows $D(w_0, \delta) \subseteq f(U)$, hence $f(U)$ is open. □

Corollary 9.19 (Maximum modulus principle — alternate proof). *If f is holomorphic and non-constant on a domain Ω, then $|f|$ has no local maximum in Ω.*

Proof. If $|f|$ had a local maximum at $z_0 \in \Omega$, then $f(z_0)$ would be in the boundary of $f(U)$ for any small neighbourhood U of z_0. But by the open mapping theorem, $f(U)$ is open, so $f(z_0)$ is an interior point — a contradiction. □

Corollary 9.20 (Inverse function theorem for holomorphic functions). *If f is holomorphic on Ω with $f'(z_0) \neq 0$, then f is a local biholomorphism near z_0: there exist neighbourhoods U of z_0 and V of $f(z_0)$ such that $f\colon U \to V$ is bijective and $f^{-1}\colon V \to U$ is holomorphic.*

Proof. Since $f'(z_0) \neq 0$, the zero of $f(z) - f(z_0)$ at z_0 is simple (order $m = 1$). Choose $r > 0$ as in the proof of the open mapping theorem. Then for w near $f(z_0)$, the equation $f(z) = w$ has exactly 1 solution in $D(z_0, r)$, so f is injective on some neighbourhood U of z_0. By the open mapping theorem, $V = f(U)$ is open. The inverse $f^{-1} : V \to U$ is continuous (as f is open and bijective) and holomorphic by the formula $(f^{-1})'(w) = 1/f'(f^{-1}(w))$. □

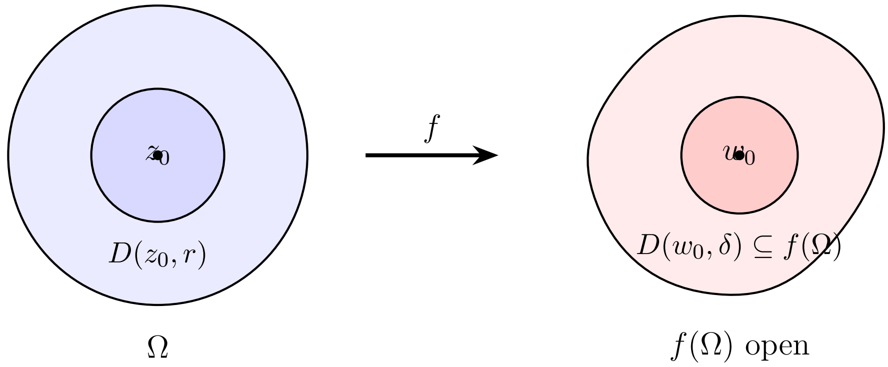


Figure 9.2: The open mapping theorem: f maps open sets to open sets.

9.6 Further Applications

Proposition 9.21 (Local mapping theorem). *If f is holomorphic at z_0 with a zero of order m at z_0 (i.e., $f(z) - f(z_0) = (z - z_0)^m h(z)$ with $h(z_0) \neq 0$), then f is locally m-to-1 near z_0: for w close to $w_0 = f(z_0)$ (but $w \neq w_0$), the equation $f(z) = w$ has exactly m solutions near z_0.*

Proof. This follows directly from Rouché's theorem, as in the proof of the open mapping theorem: for $|w - w_0| < \delta$, the function $f(z) - w$ has exactly m zeros in $D(z_0, r)$, counted with multiplicity. When $w \neq w_0$ and δ is small enough, these m zeros are distinct (as the multiplicity phenomenon occurs only at w_0). □

Theorem 9.22 (Counting zeros via the argument principle). *Under the hypotheses of Theorem 9.7, if f is holomorphic (no poles) on $\overline{D}$, then the number of zeros in D (with multiplicity) is*

$$Z = \frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi} \Delta_{\partial D} \arg f(z),$$

where $\Delta_{\partial D} \arg f(z)$ denotes the total change in argument of $f(z)$ as z traverses ∂D.

Example 9.23 (Zeros of $e^z - 3z$ in the unit disc). Let $f(z) = e^z - 3z$. On $|z| = 1$, set $g(z) = -3z$ and $h(z) = e^z$. We have $|g(z)| = 3$ and $|h(z)| = e^{\operatorname{Re} z} \leq e$. Since $e \approx 2.718 < 3$, we get $|h(z)| < |g(z)|$ on $|z| = 1$. By Rouché, $f = g + h = e^z - 3z$ has the same number of zeros in $|z| < 1$ as $g(z) = -3z$, which has exactly 1 zero. So $e^z - 3z$ has exactly 1 zero in the unit disc.

Example 9.24 (Zeros of $z^4 - 5z + 1$ in the annulus $1 < |z| < 2$). Let $p(z) = z^4 - 5z + 1$.
On $|z| = 1$: $|-5z| = 5$ and $|z^4 + 1| \leq 2 < 5$. By Rouché, p has 1 zero in $|z| < 1$ (same as $-5z$).
On $|z| = 2$: $|z^4| = 16$ and $|-5z + 1| \leq 11 < 16$. By Rouché, p has 4 zeros in $|z| < 2$ (same as z^4).

Therefore p has $4-1=3$ zeros in the annulus $1<|z|<2$ (after verifying p has no zeros on $|z|=1$: $|p(z)|\geq 5-2=3>0$).

9.7 Exercises

Exercise 9.1 (Zeros of z^6+z^3+9)**.** Show that all zeros of $p(z)=z^6+z^3+9$ lie in the annulus $1<|z|<2$.

Hint: On $|z|=1$, compare 9 with z^6+z^3. On $|z|=2$, compare z^6 with z^3+9.

Exercise 9.2 (Zeros near a simple zero)**.** Let $f_n(z)=z+1/n+z^2/n$. Show that for n large enough, f_n has exactly one zero in $|z|<1/2$, and find the limit of this zero as $n\to\infty$.

Exercise 9.3 (Fixed points of holomorphic maps)**.** Let $f\colon\overline{\mathbb{D}}\to\mathbb{D}$ be continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$. Show that f has a unique fixed point in $\mathbb{D}$.

Hint: Apply Rouché's theorem to $g(z)=z-f(z)$ and $h(z)=z$.

Exercise 9.4 (Argument principle computation)**.** Let $f(z)=\dfrac{z^3-1}{z^2+1}$. Compute $\dfrac{1}{2\pi i}\displaystyle\oint_{|z|=2}\frac{f'(z)}{f(z)}\,\mathrm{d}z$.

Solution sketch: In $|z|<2$, f has zeros at the three cube roots of unity (each simple) and poles at $z=\pm i$ (each simple). So $Z-P=3-2=1$.

Exercise 9.5 (Image of the boundary)**.** Let $f(z)=z^2+z$ and γ the circle $|z|=2$ traversed once counterclockwise. Determine $\operatorname{ind}(f\circ\gamma,0)$ and $\operatorname{ind}(f\circ\gamma,-1/4)$.

Hint: The zeros of f in $|z|<2$ are $z=0$ and $z=-1$ (both simple), so $\operatorname{ind}(f\circ\gamma,0)=2$. For $-1/4$: solve $z^2+z=-1/4$, i.e., $(z+1/2)^2=0$, double root at $z=-1/2$.

Exercise 9.6 (Rouché for exponentials)**.** Show that the equation $3e^{-z}=z-1$ has exactly one root in the half-plane $\operatorname{Re}(z)>0$.

Hint: Consider the rectangle with vertices $\pm Ri$ and $M\pm Ri$ for large R,M and apply the argument principle.

Exercise 9.7 (Hurwitz and univalence)**.** Let $f_n(z)=z+z^n/n^2$ for $n\geq 1$. Show that each f_n is injective on $\mathbb{D}$ for n large enough, and that the limit $f(z)=z$ is injective on $\mathbb{D}$, consistent with Corollary 9.17.

Exercise 9.8 (Open mapping and boundary behaviour)**.** Let f be holomorphic and non-constant on a bounded domain Ω. Show that if $|f|$ extends continuously to $\overline{\Omega}$, then $\max_{\overline{\Omega}}|f|$ is achieved on $\partial\Omega$. Deduce the maximum modulus principle from the open mapping theorem.

Exercise 9.9 (Number of preimages)**.** Let $f(z)=z^3-3z$. Using the argument principle, find the number of solutions of $f(z)=w$ in $|z|<2$ for: (a) $w=0$; (b) $w=2$; (c) $w=10$.

Exercise 9.10 (Generalised Rouché)**.** Prove the symmetric version of Rouché's theorem (Remark 9.12): if f and g are holomorphic on $\overline{D}$ with no zeros on ∂D and $|f(z)-g(z)|<|f(z)|+|g(z)|$ on ∂D, then f and g have the same number of zeros in D.

Hint: Consider $h_t(z)=(1-t)f(z)+tg(z)$ and show h_t has no zeros on ∂D for $t\in[0,1]$.

Chapter 10

Introduction to Entire and Meromorphic Functions

10.1 Entire Functions

Definition 10.1 (Entire function)**.** A function $f: \mathbb{C} \to \mathbb{C}$ is called **entire** if it is holomorphic on all of $\mathbb{C}$. Equivalently, f has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with infinite radius of convergence.

Example 10.2 (Classical entire functions)**.** The following are entire functions:

(i) **Polynomials:** $p(z) = a_n z^n + \cdots + a_0$.

(ii) **Exponential:** $e^z = \sum_{n=0}^{\infty} z^n/n!$.

(iii) **Trigonometric:** $\sin z = \sum_{n=0}^{\infty}(-1)^n z^{2n+1}/(2n+1)!$ and $\cos z = \sum_{n=0}^{\infty}(-1)^n z^{2n}/(2n)!$.

(iv) **Hyperbolic:** $\sinh z$ and $\cosh z$.

(v) **Compositions:** e^{e^z}, $\sin(z^2)$, e^{-z^2}, etc.

Note that $1/z$, $\tan z$, and $\Gamma(z)$ are *not* entire (they have singularities in $\mathbb{C}$).

Definition 10.3 (Transcendental entire function)**.** An entire function that is not a polynomial is called **transcendental**. Equivalently, f is transcendental entire if and only if its Taylor series has infinitely many nonzero terms, i.e., $z = \infty$ is an essential singularity.

Remark 10.4 (Polynomial vs. transcendental)*.* By the Fundamental Theorem of Algebra, a polynomial of degree n has exactly n zeros (with multiplicity). A transcendental entire function can have any number of zeros: finitely many (like $e^z + 1$, with zeros at $z = (2k+1)\pi i$... actually infinitely many), or none (like e^z), or infinitely many (like $\sin z$). In fact, the only entire functions with no zeros are those of the form $e^{g(z)}$ for some entire g (since $\mathbb{C}$ is simply connected).

10.2 Liouville's Theorem Revisited

We recall the fundamental theorem constraining bounded entire functions.

Theorem 10.5 (Liouville's Theorem)**.** *Every bounded entire function is constant.*

Proof. Let $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By Cauchy's inequality applied to the disc $D(0, R)$:

$$\left|f^{(n)}(0)\right| \leq \frac{n!\, M}{R^n} \quad \text{for all } R > 0.$$

Letting $R \to \infty$ gives $f^{(n)}(0) = 0$ for all $n \geq 1$. Hence $f(z) = f(0)$ is constant. □

Theorem 10.6 (Generalised Liouville)**.** *If f is entire and $|f(z)| \leq C(1 + |z|)^k$ for some constants $C > 0$ and $k \geq 0$, then f is a polynomial of degree at most $\lfloor k \rfloor$.*

Proof. By Cauchy's inequality for the n-th derivative on $D(0, R)$:

$$\left|f^{(n)}(0)\right| \leq \frac{n!}{R^n} \max_{|z|=R} |f(z)| \leq \frac{n!\, C(1 + R)^k}{R^n}.$$

For $n > k$, as $R \to \infty$ the right-hand side tends to 0, so $f^{(n)}(0) = 0$ for all $n > k$. The Taylor series terminates: f is a polynomial of degree at most $\lfloor k \rfloor$. □

Corollary 10.7 (Entire functions with polynomial growth)**.** *If f is an entire function satisfying $|f(z)| = o(|z|^n)$ as $|z| \to \infty$ for some positive integer n, then f is a polynomial of degree at most $n - 1$.*

10.3 Picard's Theorems

The little Picard theorem dramatically strengthens Liouville's theorem.

Theorem 10.8 (Little Picard Theorem)**.** *Every non-constant entire function takes every complex value with at most one exception. That is, if f is entire and non-constant, then $f(\mathbb{C}) \supseteq \mathbb{C} \setminus \{a\}$ for at most one point $a \in \mathbb{C}$.*

The proof of Picard's theorem requires the theory of modular functions and covering spaces, which is beyond the scope of this introductory course. We state it without proof but illustrate its content.

Example 10.9 (Picard's theorem in action)**.**

(i) e^z omits exactly one value: 0. It takes every other value $w \neq 0$, since $e^z = w$ has solutions $z = \operatorname{Log} w + 2\pi i k$.

(ii) $\sin z$ takes *every* complex value (no exceptions). Indeed, for any $w \in \mathbb{C}$, $\sin z = w$

is equivalent to $e^{iz} - 2iwe^{-iz} = e^{-iz}$, which always has solutions.

(iii) A polynomial of degree $n \geq 1$ takes every value (by the FTA), so there are zero exceptions.

(iv) There is no entire function omitting two or more values. For instance, there is no entire function with range $\mathbb{C} \setminus \{0, 1\}$.

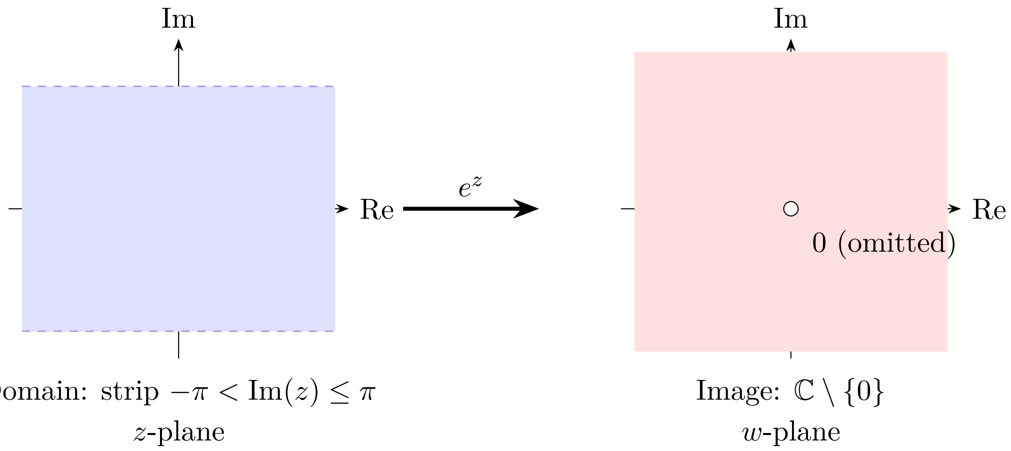


Figure 10.1: Little Picard: e^z omits exactly one value (0). Every other value is attained.

Theorem 10.10 (Great Picard Theorem). *If f has an essential singularity at z_0, then in every punctured neighbourhood of z_0, f takes every complex value with at most one exception.*

This generalises the Casorati–Weierstrass theorem (which only gives density of the image) and will not be proved here.

10.4 Order of Growth

Definition 10.11 (Order of an entire function)**.** The **order** of an entire function f is

$$\rho = \rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r},$$

where $M(r) = \max_{|z|=r} |f(z)|$ is the **maximum modulus** of f.

Remark 10.12 (Interpretation of order). An entire function has order ρ roughly if $M(r) \sim e^{r^\rho}$. More precisely, ρ is the infimum of $\alpha > 0$ such that $M(r) \leq e^{r^\alpha}$ for all sufficiently large r.

- Polynomials have order $\rho = 0$.
- e^z has order $\rho = 1$ (since $M(r) = e^r$).
- e^{z^2} has order $\rho = 2$.

- e^{e^z} has infinite order.
- $\sin z$ and $\cos z$ have order $\rho = 1$ (since $M(r) \sim e^r/2$).

Proposition 10.13 (Order from Taylor coefficients). *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is entire and of finite order, then*

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{-\log |a_n|}.$$

Proof. This follows from a careful comparison of the maximum modulus with the coefficient estimates. For the upper bound, from $M(r) \le \sum |a_n| r^n$ and optimising in r, one obtains the formula. For the lower bound, $|a_n| \le M(r)/r^n$ (Cauchy inequality), and choosing $r = n^{1/\rho}$ leads to the reverse inequality. We refer to Boas [8] or Conway [2] for the complete argument. □

Definition 10.14 (Type of an entire function). If f has finite order ρ, the **type** of f is

$$\sigma = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho}.$$

We say f is of **minimal type** if $\sigma = 0$, **normal type** if $0 < \sigma < \infty$, and **maximal type** if $\sigma = \infty$.

Example 10.15 (Orders and types). (i) e^z: order 1, type 1 (normal).

(ii) e^{3z^2}: order 2, type 3.

(iii) $\sin z$: order 1, type 1.

(iv) $e^{z^{1/2}}$ (defined on an appropriate domain, or its entire extension): order 1/2, type 1.

10.5 Weierstrass Products

A fundamental question in function theory is: given a sequence of complex numbers (the desired zero set), can we construct an entire function having exactly those zeros? The Weierstrass factorisation theorem answers this affirmatively.

Definition 10.16 (Elementary factors). The **elementary factors** are defined by

$$E_0(z) = 1 - z, \qquad E_p(z) = (1 - z)\exp\Big(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\Big) \quad \text{for } p \ge 1.$$

Lemma 10.17 (Properties of elementary factors). *For $|z| \le 1$ and $p \ge 0$:*

$$|1 - E_p(z)| \le |z|^{p+1}. \tag{10.1}$$

In particular, $E_p(z)$ vanishes only at $z = 1$, with a simple zero.

Proof. Write $\log E_p(z) = \log(1-z) + z + z^2/2 + \cdots + z^p/p = -\sum_{n=p+1}^{\infty} z^n/n$ for $|z| < 1$. Then $|\log E_p(z)| \le \sum_{n=p+1}^{\infty} |z|^n/n \le |z|^{p+1} \sum_{n=0}^{\infty} |z|^n/(n+p+1) \le |z|^{p+1}/(1-|z|)$ for $|z| < 1$. The bound $|1 - E_p(z)| \le |z|^{p+1}$ follows from $|e^w - 1| \le |w|\, e^{|w|}$ and a more careful estimate (see, e.g., Stein–Shakarchi [3]). □

Theorem 10.18 (Weierstrass Factorisation Theorem)**.** *Let f be an entire function with $f(0) \neq 0$, and let $a_1, a_2, \dots$ be the nonzero zeros of f (listed with multiplicity), with $0 < |a_1| \le |a_2| \le \cdots$ and $|a_n| \to \infty$. Then there exist non-negative integers p_n and an entire function g such that*

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_{p_n}\Big(\frac{z}{a_n}\Big), \tag{10.2}$$

where $m = \operatorname{ord}_0(f)$ is the order of the zero at the origin (possibly 0).
If the p_n are chosen so that $\sum_n (r/|a_n|)^{p_n+1}$ converges for all $r > 0$, the product converges uniformly on compact sets and defines an entire function with the prescribed zeros.

The proof, which we omit, proceeds by showing that the product $\prod E_{p_n}(z/a_n)$ converges, and then the entire function g is determined by $f/\big(z^m \prod E_{p_n}(z/a_n)\big)$, which is entire and zero-free, hence of the form $e^{g(z)}$.

Definition 10.19 (Genus and canonical product)**.** If the integers p_n in (10.2) can all be taken equal to some fixed p, and this is the smallest such p, then p is called the **genus** of the canonical product. The **genus of** f is defined as the maximum of p and the degree of g (if g is a polynomial).

10.5.1 Factorisation of $\sin z$

Theorem 10.20 (Product formula for sine)**.**

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big). \tag{10.3}$$

Proof. The function $\sin(\pi z)$ is entire with simple zeros at $z = n$ for $n \in \mathbb{Z}$. We aim to write

$$\sin(\pi z) = \pi z\, e^{g(z)} \prod_{n=1}^{\infty} E_1\Big(\frac{z}{n}\Big) \cdot E_1\Big(\frac{z}{-n}\Big).$$

Now, using $E_1(z) = (1-z)e^z$:

$$E_1\Big(\frac{z}{n}\Big) \cdot E_1\Big(\frac{z}{-n}\Big) = \Big(1 - \frac{z}{n}\Big)e^{z/n} \cdot \Big(1 + \frac{z}{n}\Big)e^{-z/n} = 1 - \frac{z^2}{n^2}.$$

Since $\sum_{n=1}^{\infty} 1/n^2$ converges (equals $\pi^2/6$), the product $\prod_{n=1}^{\infty}(1 - z^2/n^2)$ converges uniformly on compact sets.

Denoting $P(z) = \pi z \prod_{n=1}^{\infty}(1 - z^2/n^2)$, we must show $g(z) = 0$. The quotient $h(z) = \sin(\pi z)/P(z)$ is entire and nonvanishing. To show $h \equiv 1$, we use:

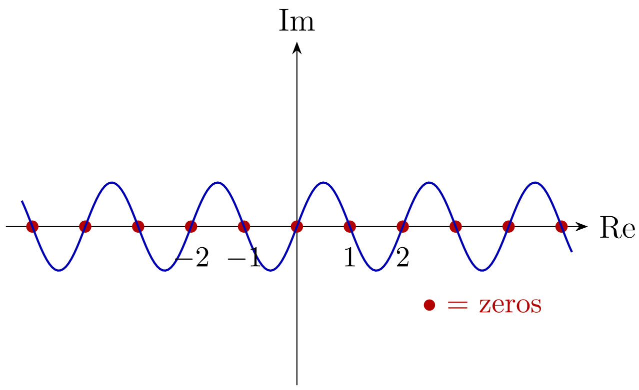
Boundedness argument. Consider the logarithmic derivative:

$$\frac{\pi\cos(\pi z)}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

This is the partial fraction expansion of $\pi\cot(\pi z)$. Similarly, the logarithmic derivative of $P(z)$ is

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

Thus $h'(z)/h(z) = g'(z) = 0$, so g is constant. Evaluating at $z = 0$: $\lim_{z\to 0}\frac{\sin(\pi z)}{\pi z} = 1$ and $\lim_{z\to 0}\prod(1 - z^2/n^2) = 1$, so $e^{g(0)} = 1$, giving $g \equiv 0$. □



Zeros of $\sin(\pi z)$: $z = n,\ n \in \mathbb{Z}$

Figure 10.2: The zeros of $\sin(\pi z)$ are precisely the integers. The Weierstrass product reflects this by having factors $(1 - z^2/n^2)$.

Corollary 10.21 (Euler's identity)**.**

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. Expand both sides of (10.3) in Taylor series about $z = 0$. The left-hand side gives

$$\sin(\pi z) = \pi z - \frac{(\pi z)^3}{3!} + \cdots = \pi z\left(1 - \frac{\pi^2 z^2}{6} + \cdots\right).$$

The right-hand side gives

$$\pi z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) = \pi z\left(1 - z^2\sum_{n=1}^{\infty}\frac{1}{n^2} + \cdots\right).$$

Comparing the coefficients of z^3: $-\pi^2/6 = -\sum_{n=1}^{\infty}1/n^2$, giving the result. □

10.6 Meromorphic Functions and Partial Fractions

Definition 10.22 (Meromorphic function on $\mathbb{C}$)**.** A function f is **meromorphic on** $\mathbb{C}$ if it is holomorphic on $\mathbb{C} \setminus S$, where S is a discrete subset of $\mathbb{C}$ consisting entirely of poles of f.

Remark 10.23 (Field of meromorphic functions). The set $\mathcal{M}(\mathbb{C})$ of meromorphic functions on $\mathbb{C}$ forms a **field** under the usual operations of addition and multiplication. The entire functions $\mathcal{O}(\mathbb{C})$ form an integral domain within this field. Every meromorphic function on $\mathbb{C}$ can be written as a quotient of two entire functions: $f = g/h$ with g, h entire and $h \not\equiv 0$.

The Mittag-Leffler theorem is the analogue of the Weierstrass factorisation theorem: whereas Weierstrass prescribes *zeros*, Mittag-Leffler prescribes *principal parts* at poles.

Theorem 10.24 (Mittag-Leffler Theorem)**.** *Let $(b_n)_{n\geq 1}$ be a sequence of distinct complex numbers with $|b_n| \to \infty$, and let $p_n(z)$ be polynomials in $1/(z - b_n)$ without constant term (**principal parts**). Then there exists a meromorphic function f on $\mathbb{C}$ whose poles are exactly the b_n and whose principal part at each b_n is $p_n(z)$.*
Moreover, f can be written as

$$f(z) = \sum_{n=1}^{\infty} \Big[p_n(z) - q_n(z) \Big] + g(z),$$

where $q_n(z)$ are polynomials chosen to ensure convergence and g is entire. The function is unique up to addition of an entire function.

Example 10.25 (Partial fraction of $\pi\cot(\pi z)$)**.** The function $\pi\cot(\pi z)$ is meromorphic with simple poles at $z = n$ ($n \in \mathbb{Z}$), each with residue 1. The Mittag-Leffler expansion gives:

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{10.4}$$

Here the "correction polynomials" q_n are the constants $1/n$ and $-1/n$ combined in pairs to achieve convergence.

Example 10.26 (Partial fraction of $\pi/\sin(\pi z)$)**.** Similarly, $\pi/\sin(\pi z)$ has simple poles at $z = n$ with residues $(-1)^n$. Its Mittag-Leffler expansion is:

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2z}{z^2 - n^2}.$$

Example 10.27 (Partial fraction of $1/\sin^2(\pi z)$)**.** The function $\pi^2/\sin^2(\pi z)$ has double poles at each integer. The principal part at $z = n$ is $1/(z-n)^2$. Its expansion is:

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

This series converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$ (since $\sum 1/n^2 < \infty$, no correction polynomials are needed).

10.7 Introduction to Elliptic Functions

We give a brief overview of elliptic functions, one of the crowning achievements of 19th-century complex analysis.

Definition 10.28 (Elliptic function)**.** An **elliptic function** is a meromorphic function $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ that is doubly periodic: there exist $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, such that

$$f(z + \omega_1) = f(z), \qquad f(z + \omega_2) = f(z) \quad \text{for all } z \in \mathbb{C}.$$

The **period lattice** is $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and the **fundamental period parallelogram** (or **fundamental domain**) is

$$\Pi = \{t_1\omega_1 + t_2\omega_2 : 0 \le t_1, t_2 < 1\}.$$

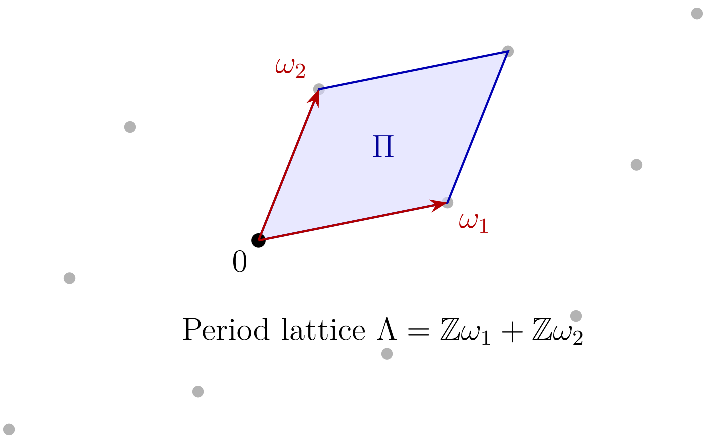


Figure 10.3: The period lattice and fundamental parallelogram of an elliptic function.

Theorem 10.29 (Basic properties of elliptic functions)**.** *Let f be an elliptic function with period lattice Λ.*

(i) If f is entire (i.e., has no poles), then f is constant.

(ii) The sum of the residues of f in any fundamental parallelogram (whose boundary contains no poles) is zero.

(iii) The number of zeros equals the number of poles in any fundamental parallelogram (counted with multiplicity). This common number is called the ***order*** *of f.*

(iv) A non-constant elliptic function has order at least 2.

Proof. (i) An entire elliptic function is bounded on Π (continuous on compact $\overline{\Pi}$), hence bounded on all of $\mathbb{C}$ by periodicity. By Liouville's theorem, f is constant.

(ii) By the residue theorem, the sum of residues in Π equals $\frac{1}{2\pi i}\oint_{\partial\Pi} f(z)\,\mathrm{d}z$. By periodicity, the integrals over opposite sides cancel, giving 0.

(iii) Apply the argument principle to f over $\partial\Pi$. Again, opposite sides cancel, giving $Z - P = 0$.

(iv) If f had order 1, it would have a single simple pole in Π, with residue (by (ii)) equal to 0 — contradicting simplicity. □

Definition 10.30 (Weierstrass $\wp$-function)**.** The **Weierstrass $\wp$-function** associated to a lattice Λ is defined by

$$\wp(z) = \wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

Proposition 10.31 (Properties of $\wp$)**.** *(i) $\wp$ is an elliptic function of order 2 with period lattice Λ.*

(ii) $\wp$ has a double pole at each lattice point and no other poles.

(iii) $\wp$ is an even function: $\wp(-z) = \wp(z)$.

(iv) $\wp$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{10.5}$$

*where $g_2 = 60\sum_{\omega\neq 0}\omega^{-4}$ and $g_3 = 140\sum_{\omega\neq 0}\omega^{-6}$ are the **Eisenstein series**.*

Proof sketch. Convergence of the series defining $\wp$ follows from $\sum_{\omega\neq 0}|\omega|^{-3} < \infty$ (which holds for any lattice in $\mathbb{C}$). Periodicity: $\wp(z+\omega_k) - \wp(z)$ is entire and bounded, hence constant by Liouville; checking at $z = -\omega_k/2$ shows the constant is 0. The differential equation is obtained by computing the Laurent expansion of both sides about $z = 0$ and verifying that the difference is an entire elliptic function vanishing at $z = 0$, hence identically zero. □

Remark 10.32 (Connection to elliptic curves)*.* The differential equation (10.5) defines an **elliptic curve** $y^2 = 4x^3 - g_2x - g_3$ in the (x, y)-plane. The map $z \mapsto (\wp(z), \wp'(z))$ establishes an isomorphism between the complex torus $\mathbb{C}/\Lambda$ and (the projective completion of) this elliptic curve. This profound connection between complex analysis, algebraic geometry, and number theory is one of the most beautiful themes in modern mathematics.

10.8 The Hadamard Factorisation Theorem

For entire functions of finite order, the Weierstrass product can be made more precise.

Theorem 10.33 (Hadamard Factorisation Theorem). *Let f be an entire function of finite order ρ, with zeros $a_1, a_2, \ldots$ (listed with multiplicity, $a_n \neq 0$) and a zero of order m at the origin. Set $p = \lfloor \rho \rfloor$. Then*

$$f(z) = e^{Q(z)} z^m \prod_{n=1}^{\infty} E_p\Big(\frac{z}{a_n}\Big), \tag{10.6}$$

where Q is a polynomial of degree at most p.

The key improvement over the general Weierstrass theorem is that the integers p_n can all be taken equal to $p = \lfloor \rho \rfloor$, and the function $g(z)$ is a polynomial (not just an arbitrary entire function).

Example 10.34 (Hadamard factorisation of $\sin(\pi z)$). $\sin(\pi z)$ has order $\rho = 1$, so $p = 1$. Its zeros are $a_n = n$ ($n \in \mathbb{Z} \setminus \{0\}$) and it has a simple zero at 0. By Hadamard:

$$\sin(\pi z) = e^{Q(z)}\, z \prod_{n=1}^{\infty} E_1(z/n)\, E_1(-z/n) = e^{Q(z)}\, z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big),$$

where $\deg Q \leq 1$. From the product formula (Theorem 10.20), we know the answer is $\pi z \prod(1 - z^2/n^2)$, so $e^{Q(z)} = \pi$, i.e., $Q(z) = \log \pi$ (a constant). This is consistent with $\deg Q \leq 1$.

Example 10.35 (Hadamard factorisation of $1/\Gamma(z)$). The reciprocal gamma function $1/\Gamma(z)$ is entire of order 1, with simple zeros at $z = 0, -1, -2, \ldots$. The Hadamard (= Weierstrass) product gives:

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n}\Big) e^{-z/n},$$

where $\gamma = \lim_{n\to\infty}(1 + 1/2 + \cdots + 1/n - \log n)$ is the Euler–Mascheroni constant.

10.9 Jensen's Formula

Jensen's formula connects the distribution of zeros of a holomorphic function to its growth, providing a bridge between the material of this chapter and the preceding one.

Theorem 10.36 (Jensen's Formula). *Let f be holomorphic on a neighbourhood of $\overline{D(0,R)}$, with $f(0) \neq 0$. Let $a_1, \ldots, a_N$ be the zeros of f in $D(0,R)$ (counted with multiplicity). Then*

$$\log|f(0)| = -\sum_{k=1}^{N} \log \frac{R}{|a_k|} + \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(Re^{i\theta})\right| \, d\theta. \tag{10.7}$$

Proof. First assume f has no zeros on $|z| = R$. Define

$$g(z) = f(z) \prod_{k=1}^{N} \frac{R^2 - \overline{a_k} z}{R(z - a_k)}.$$

Each factor $\frac{R^2 - \overline{a_k} z}{R(z-a_k)}$ is a (rotated) Blaschke factor with modulus 1 on $|z| = R$ and a simple zero at $z = R^2/\overline{a_k}$ (outside $\overline{D(0,R)}$) and a simple pole at $z = a_k$. Thus g is holomorphic and nonvanishing on $\overline{D(0,R)}$, and $|g(z)| = |f(z)|$ on $|z| = R$.

Since g is holomorphic and nonzero on $\overline{D(0,R)}$, $\log |g|$ is harmonic there, so the mean value property gives:

$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left|g(Re^{i\theta})\right| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log \left|f(Re^{i\theta})\right| \, d\theta.$$

Now $g(0) = f(0) \prod_{k=1}^{N} \frac{R^2}{R(-a_k)} = f(0) \prod_{k=1}^{N} \frac{-R}{a_k}$, so

$$|g(0)| = |f(0)| \prod_{k=1}^{N} \frac{R}{|a_k|}.$$

Taking logarithms: $\log |f(0)| + \sum_k \log(R/|a_k|) = \frac{1}{2\pi} \int_0^{2\pi} \log \left|f(Re^{i\theta})\right| \, d\theta$. □

Remark 10.37 (Counting function). Jensen's formula can be rewritten using the **counting function** $n(r) = \#\{k : |a_k| \le r\}$ as

$$\log |f(0)| = -\int_0^R \frac{n(t)}{t} \, dt + \frac{1}{2\pi} \int_0^{2\pi} \log \left|f(Re^{i\theta})\right| \, d\theta.$$

This form makes it clear that faster growth of f on circles forces more zeros inside those circles — a quantitative version of the principle that entire functions of large order must have many zeros.

10.10 Exercises

Exercise 10.1 (Entire functions omitting values)**.** Let f be entire. Show that if f omits two distinct values $a, b \in \mathbb{C}$, then f is constant.

Hint: Consider $g(z) = (f(z) - a)/(b - a)$, which omits 0 and 1. Apply Little Picard.

Exercise 10.2 (Order computation)**.** Compute the order of each of the following entire functions:

(a) $f(z) = e^{z^3 + z}$;

(b) $f(z) = \cos(\sqrt{z})$ (interpret as the entire function $\sum_{n=0}^{\infty} (-1)^n z^n/(2n)!$);

(c) $f(z) = \sum_{n=0}^{\infty} z^n/(n!)^2$.

Exercise 10.3 (Weierstrass product for cos)**.** Derive the product formula

$$\cos(\pi z) = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right)$$

from the product formula for $\sin(\pi z)$ using $\cos(\pi z) = \sin(2\pi z)/(2\sin(\pi z))$.

Exercise 10.4 (Mittag-Leffler for $1/\sin^2$). Verify the partial fraction expansion $\dfrac{\pi^2}{\sin^2(\pi z)} = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ by showing that the difference of the two sides is an entire bounded function, hence constant, and evaluate the constant.

Exercise 10.5 (Jensen's formula application). Let $f(z) = \sin(\pi z)$. Apply Jensen's formula with $R = n+1/2$ (where n is a positive integer) to deduce the asymptotic formula $n(r) \sim 2r$ for the number of zeros of $\sin(\pi z)$ in $|z| \le r$.

Exercise 10.6 (Elliptic function of order 2). Let f be an elliptic function of order 2 with periods ω_1, ω_2.

(a) Show that f takes every value in $\mathbb{C} \cup \{\infty\}$ exactly twice (counted with multiplicity) in each fundamental parallelogram.

(b) Show that if f has a single double pole at $z = 0 \pmod{\Lambda}$, then $f(z) = A\,\wp(z) + B$ for constants A, B.

Exercise 10.7 (Growth and zeros). Let f be an entire function of order $\rho < \infty$, and let (a_n) be its nonzero zeros. Show that $\sum_{n=1}^{\infty} |a_n|^{-\alpha}$ converges for every $\alpha > \rho$.

Hint: Use Jensen's formula to relate $n(r)$ to $\log M(r)$, then integrate by parts.

Exercise 10.8 (Liouville for meromorphic functions). Let f be a meromorphic function on $\mathbb{C}$ satisfying $|f(z)| \le C\,|z|^N$ for $|z|$ large and some constants C, N. Show that f is a rational function of degree at most N.

Hint: If f has poles $b_1, \ldots, b_M$ in $\mathbb{C}$, consider $g(z) = f(z)\prod(z - b_j)^{n_j}$, which is entire.

Exercise 10.9 (Hadamard for cos). Use the Hadamard factorisation theorem to show that

$$\cos(z) = \prod_{n=0}^{\infty}\Big(1 - \frac{4z^2}{(2n+1)^2\pi^2}\Big).$$

Verify that this is consistent with $\cos(z)$ having order 1 and zeros at $z = (n+1/2)\pi$, $n \in \mathbb{Z}$.

Exercise 10.10 (The Weierstrass σ-function). The **Weierstrass σ-function** is defined by

$$\sigma(z) = z \prod_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \Big(1 - \frac{z}{\omega}\Big) \exp\Big(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\Big).$$

Show that σ is an entire function with simple zeros at the lattice points and that $\wp(z) = -(\log\sigma(z))''$.

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